CATALAN WORDS AVOIDING A PATTERN OF LENGTH FOUR

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Abstract. Let $\mathcal{C}_n$ denote the set of Catalan words of length n and $\mathcal{C}_n(p)$ the subset of $\mathcal{C}_n$ whose members avoid the pattern p. In this paper, we enumerate members of $\mathcal{C}_n(p)$ where p is a permutation pattern of length four. This extends recent work on Catalan words concerning the avoidance of classical, consecutive or vincular patterns of length at most three. Indeed, we determine the generating function of the distribution for the number of descents on each of the corresponding avoidance classes. We make use of the symbolic counting method to establish our results, together with a variety of other enumerative techniques, including use of strategic decompositions, introduction of auxiliary generating functions and analysis of active sites.

1. Introduction

A *Catalan* word is a positive integral sequence starting with 1 where each letter is either less than or equal to its predecessor or is its predecessor incremented by one. More formally, it is a word $w = w_1 \cdots w_n$ with positive integer entries, where $w_1 = 1$ and $w_{i+1} \leq w_i + 1$ for $1 \leq i \leq n-1$. Let $\mathcal{C}_n$ denote the set of all Catalan words of length n (i.e., that contain n letters). It is well known that the cardinality of $\mathcal{C}_n$ is given by the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$; see, e.g., [15, Exercise 80]. For example, we have $C_4 = 14$ and

$$\mathcal{C}_4 = \{1111, 1112, 1121, 1122, 1123, 1211, 1212, 1221, 1222, 1223, 1231, 1232, 1233, 1234\}.$$

Catalan words have been widely studied in the context of various combinatorial structures enumerated by the Catalan numbers, such as rooted binary trees [8] and Dyck paths [13]. For example, members of $\mathcal{C}_n$ encode the heights of the up steps in Dyck paths of semilength n, i.e., lattice paths in $\mathbb{N}^2$ running from the origin to $(2n, 0)$ consisting of up $(1, 1)$ and down $(1, -1)$ steps. See also [11], where Catalan words arise in the context of the exhaustive generation of Gray codes for growth-restricted words.

Recall that a *pattern* p is a finite sequence in $[\ell] = \{1, \ldots, \ell\}$ for some $\ell \geq 1$ in which each element of $[\ell]$ occurs at least once. We say that a word $w = w_1 \cdots w_n$ *contains* the pattern $p = p_1 \cdots p_k$ if there exist indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $w_{i_r}\, x\, w_{i_s}$ if and only if $p_r\, x\, p_s$ for all $r, s \in [k]$ and each $x \in \{<, >, =\}$, i.e., if there exists a subsequence of w that is order-isomorphic to p. If no such subsequence exists, then w is said to avoid the pattern p. For example, the Catalan word 12123423412 contains four occurrences of the pattern 321 (as witnessed by the subsequences 321, 421, 431 and 432), but is seen to avoid 4213. We will say that p is a *permutation pattern* whenever all of its letters are distinct. Given $n \geq 0$, let $\mathcal{C}_n(p)$ denote the set of Catalan words of length n avoiding the pattern p and let $\boldsymbol{c}_p(n) = |\mathcal{C}_n(p)|$.

Baril et al. [4] studied the distribution of descents on $\mathcal{C}_n(p)$, where p is a classical pattern of length two or three, with these results being extended to pairs of such patterns in [3].

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Ramírez and Rojas [12] considered the distribution of descents for Catalan words avoiding consecutive patterns of length at most three, and analogous results were found for pairs of partial order relations [5]. Comparable results involving the last letter statistic instead of descents were obtained for Catalan words avoiding a classical [2] or consecutive [1] pattern. Further combinatorial parameters on the polyominoes associated with words in $\mathcal{C}_n$ were studied in [6, 9], and in [13], the generating function for the distribution on $\mathcal{C}_n$ of any consecutive pattern of length two or three was found. Finally, Mansour and Shattuck [10] established closed form expressions and/or generating function formulas for the number of Catalan words avoiding any vincular pattern of type $(1,2)$ or $(2,1)$.

The goal of the current paper is to extend this recent work to Catalan words avoiding a permutation pattern of length four. For each such pattern p, we compute its bivariate generating function $\boldsymbol{C}_p(x,y)$ whose coefficient of $x^n y^k$ is given by the number of members of $\mathcal{C}_n(p)$ with k descents avoiding the pattern p. As a consequence, we may deduce in several cases explicit formulas for $\boldsymbol{c}_p(n)$ and the total number of descents in $\mathcal{C}_n(p)$.

We now introduce some further notation as follows. Let $\mathsf{des}(w)$ denote the number of descents in the word $w = w_1 \cdots w_n$, i.e., the number of indices $i \in [n-1]$ such that $w_i > w_{i+1}$. We denote the set of Catalan words $w \in \mathcal{C}_n(p)$ such that $\mathsf{des}(w) = k$ by $\mathcal{C}_{n,k}(p)$. Let $\boldsymbol{c}_p(n,k) := |\mathcal{C}_{n,k}(p)|$ for $0 \leq k \leq n-1$. Hence, $\boldsymbol{c}_p(n) = \sum_{k=0}^{n-1} \boldsymbol{c}_p(n,k)$ for all $n \geq 1$, with $\boldsymbol{c}_p(0) = 1$, by the definitions.

We consider here several cases of the bivariate generating function

$$\boldsymbol{C}_p(x,y) := \sum_{w \in \mathcal{C}(p)} x^{|w|} y^{\mathsf{des}(w)} = 1 + \sum_{0 \leq k < n} \boldsymbol{c}_p(n,k) x^n y^k,$$

where $\mathcal{C}(p) := \bigcup_{n \geq 0} \mathcal{C}_n(p)$. Let $\boldsymbol{d}_p(n)$ denote the total number of descents in all the members of $\mathcal{C}_n(p)$. Note that the generating function for the sequence $\boldsymbol{d}_p(n)$ is given by

$$\boldsymbol{D}_p(x) := \left. \frac{\partial \boldsymbol{C}_p(x,y)}{\partial y} \right|_{y=1}.$$

Remark 1.1. Let $\mathcal{C}'(p) = \mathcal{C}(p, \underline{11})$ denote the set of all Catalan words avoiding the pattern p and containing no levels (i.e., identical adjacent symbols). We introduce the bivariate generating function

$$\boldsymbol{C}'_p(x,y) := \sum_{w \in \mathcal{C}'(p)} x^{|w|} y^{\mathsf{des}(w)}.$$

Then we have the dual relations

$$\boldsymbol{C}_p(x,y) = \boldsymbol{C}'_p\left(\frac{x}{1-x}, y\right) \quad \text{and} \quad \boldsymbol{C}'_p(x,y) = \boldsymbol{C}_p\left(\frac{x}{1+x}, y\right).$$

In several of the proofs below, we will make use of the formulas for $\boldsymbol{C}_p(x,y)$ found in [4]. Indeed, these formulas for a pattern p of length three will allow us to obtain $\boldsymbol{C}'_p(x,y) = \boldsymbol{C}_p\left(\frac{x}{1+x}, y\right)$ where it is needed.

The organization of this paper is as follows. In the second section, we treat each permutation pattern of length four that starts with 1. These cases may be obtained by combining the first return decomposition (i.e., position of the second 1, if it exists) with prior results from [4], taken together with Remark 1.1. In these cases, without too much additional work, one may also ascertain explicit formulas for $\boldsymbol{c}_p(n)$ and $\boldsymbol{d}_p(n)$ for the various patterns p. Furthermore, combinatorial proofs are provided for the explicit formulas for $\boldsymbol{c}_p(n)$ in the cases

Class	p	$\boldsymbol{C}_p(x)$	Reference/OEIS #
1	1234	$\frac{1-2x}{1-3x+x^2}$	Cor. 2.2/A001519
2	1243	$\frac{(1-x)(1-5x+7x^2-x^3)}{(1-2x)^2(1-3x+x^2)}$	Cor. 2.5/A244885
3	1324 1423	$\frac{1-7x+16x^2-12x^3+x^4}{(1-2x)(1-3x)(1-3x+x^2)}$	Cor. 2.8, Thm. 2.10
4	1342	$\frac{1-6x+12x^2-9x^3+3x^4}{(1-x)(1-3x+x^2)^2}$	Cor. 2.12/A116845
5	1432	$\frac{1-11x+49x^2-112x^3+136x^4-78x^5+9x^6+6x^7-x^8}{(1-x)(1-2x)^2(1-3x+x^2)(1-4x+3x^2+x^3)}$	Cor. 2.15
6	2134 3412	$\frac{(1-x)(1-3x)}{1-5x+6x^2-x^3}$	Thm. 3.1/A080937
7	2143 4312	$\frac{(1-3x+x^2)(1-7x+17x^2-17x^3+6x^4-x^5)}{(1-x)(1-2x)^2(1-6x+10x^2-4x^3+x^4)}$	Thms. 4.2 and 3.7
8	2314 2413 3124 4123	$\frac{1-5x+6x^2-x^3}{(1-2x)(1-4x+2x^2)}$	Thm. 3.2
9	2341	$\frac{1-3x+x^2}{(1-x)(1-3x)}$	Thm. 3.3/A024175
10	2431	$\frac{1-8x+23x^2-27x^3+8x^4+5x^5-x^6}{(1-2x)^2(1-5x+6x^2-x^4)}$	Thm. 3.5
11	3142	$\frac{1-13x+69x^2-192x^3+297x^4-244x^5+82x^6+9x^7-11x^8+x^9}{(1-x)(1-3x+x^2)(1-10x+37x^2-61x^3+39x^4+x^5-5x^6)}$	Thm. 4.4
12	3214 4132 4213	$\frac{(1-2x)(1-4x+2x^2)}{(1-x)(1-6x+9x^2-x^3)}$	Thm. 3.8/A080938
13	3241 4231	$\frac{1-8x+21x^2-18x^3+x^5}{(1-3x)(1-6x+10x^2-3x^3-x^4)}$	Thm. 3.9
14	3421	$\frac{(1-x)(1-6x+10x^2-2x^3-2x^4)}{(1-3x+x^2)(1-5x+6x^2-x^4)}$	Thm. 3.10
15	4321	Too lengthy to state here	Thm. 4.6

TABLE 1. Generating function $\boldsymbol{C}_p(x)$ for all permutation patterns p of length four.

when $p = 1234$ or 1243 and a bijection is given demonstrating the equivalence of 1324 and 1423. In the third section, we consider most of the remaining cases and compute $\boldsymbol{C}_p(x, y)$ for each p. We reserve for the fourth section the patterns 2143, 3142 and 4321, which are apparently more difficult than the others. In the final section, we make some concluding remarks and raise a general question. Setting $y = 1$ in the formulas found below for $\boldsymbol{C}_p(x, y)$ for the various patterns p yields the entries in Table 1 for the univariate generating function $\boldsymbol{C}_p(x) := \sum_{n\geq 0} \boldsymbol{c}_p(n)x^n$. Further, it should be noted that each of the Wilf equivalences seen in Table 1 respects the descents statistic.

In addition to utilizing the symbolic generating function method (see, e.g., [7]) as a basic tool, we draw upon a variety of other techniques in finding $\boldsymbol{C}_p(x, y)$ for each p, including strategic decompositions of words and analysis of active sites. For the latter, we consider inserting certain sequences of 1's and 2's into precursor Catalan words on the alphabet $\{2, 3, \ldots\}$

at various sites such that the avoidance of a given pattern is preserved. It will also be convenient to consider certain refinements or subsets of $\mathcal{C}(p)$, which will require the use of auxiliary generating functions in writing a system of equations from which $\boldsymbol{C}_p(x,y)$ may be deduced. For a permutation pattern p, it suffices to restrict to the case of no levels, by Remark 1.1, and compute $\boldsymbol{C}'_p(x,y)$ prior to replacing x with $\frac{x}{1-x}$. Here, we found it more convenient to work with the members of $\mathcal{C}'(p)$ instead of $\mathcal{C}(p)$ when considering the various decompositions or performing an active site analysis. Finally, we remark that as a consequence of our results, new combinatorial interpretations of several sequences from the OEIS [14] are found, see Table 1.

2. Descent distributions for patterns starting with 1

In this section, we focus on permutation patterns p starting with the letter 1. For these cases, it is possible to deduce without too much extra difficulty closed-form expressions for $\boldsymbol{c}_p(n)$ and $\boldsymbol{d}_p(n)$.

2.1. The case 1234.

Theorem 2.1. *We have*

$$\boldsymbol{C}_{1234}(x,y)=\frac{1-2x+2x^2(1-y)}{1-3x+x^2(3-2y)-x^3(1-y)}.$$

Proof. In light of Remark 1.1, we first count Catalan words w avoiding 1234 and $\underline{11}$. Such a nonempty word is either (i) 1; or (ii) $1(w'+1)w''$, with $w'\in\mathcal{C}'(123)\backslash\{\epsilon\}$ and $w''\in\mathcal{C}'(1234)$, where $w'+1$ denotes the word obtained by increasing each entry of w' by one. If $f:=\boldsymbol{C}'_{1234}(x,y)$ and $g:=\boldsymbol{C}'_{123}(x,y)=\boldsymbol{C}_{123}\left(\frac{x}{1+x},y\right)$, then we obtain the functional equation

$$f=1+x+x\,(g-1)\,(y(f-1)+1),$$

where $\boldsymbol{C}_{123}(x,y)$ is given in [4]. Solving this equation for f, and considering Remark 1.1, we obtain the desired result. □

The first few terms in the series expansion of $\boldsymbol{C}_{1234}(x,y)$ are

$$1+x+2x^2+(4+y)x^3+(7+6y)x^4+(11+21y+2y^2)x^5+(16+56y+17y^2)x^6$$
$$+(22+126y+81y^2+4y^3)x^7+(29+252y+285y^2+44y^3)x^8+O(x^9).$$

Let F_n denote the n-th Fibonacci number satisfying $F_n=F_{n-1}+F_{n-2}$ for $n\geq 2$, with $F_0=0$ and $F_1=1$. Letting $y=1$ in Theorem 2.1, and extracting the coefficient of x^n, yields the following result.

Corollary 2.2. *We have*

$$\boldsymbol{C}_{1234}(x)=\frac{1-2x}{1-3x+x^2}.$$

Moreover, $\boldsymbol{c}_{1234}(n)=F_{2n-1}$ *for* $n\geq 1$.

Differentiation of the expression for $\boldsymbol{C}_{1234}(x,y)$ from Theorem 2.1 with respect to y, and setting $y=1$, leads to the following formulas for $\boldsymbol{D}_{1234}(x)$ and $\boldsymbol{d}_{1234}(n)$.

Corollary 2.3. *We have*

$$\boldsymbol{D}_{1234}(x) = \frac{x^3}{(1-3x+x^2)^2}.$$

Moreover,

$$\boldsymbol{d}_{1234}(n) = \frac{1}{5}\left((4n-10)F_{2n-4} + (3n-6)F_{2n-5}\right), \quad n \geq 3.$$

We remark that $\boldsymbol{d}_{1234}(n)$ corresponds to the OEIS sequence A001871.

2.2. **The case 1243.**

Theorem 2.4. *The bivariate generating function* $\boldsymbol{C}_{1243}(x,y)$ *is given by*

$$\frac{1-6x+3x^2(5-y)-4x^3(5-3y)+x^4(15-16y+2y^2)-2x^5(3-5y+2y^2)+x^6(1-y)^2}{(1-2x)(1-2x+x^2(1-y))(1-3x+x^2(3-2y)-x^3(1-y))}.$$

Proof. Again, we first count Catalan words w avoiding 1243 and $\underline{11}$. Such a nonempty word w is either (i) 1; or (ii) $1(w'+1)w''$, with $w' \in \mathcal{C}'(123)\backslash\{\epsilon\}$ and $w'' \in \mathcal{C}'(1243)$; or (iii) $1(23)^k 4\cdots \ell w'$, $k \geq 1$, $\ell \geq 4$, with $w' \in \mathcal{C}'(123)$; or (iii') $1(23)^k 4 \cdots \ell 2 w'$, $k \geq 1$, $\ell \geq 4$, with $w' \in \mathcal{C}'(123)$. If $f := \boldsymbol{C}'_{1243}(x,y)$ and $g := \boldsymbol{C}_{123}\left(\frac{x}{1+x}, y\right)$, then we obtain the functional equation

$$f = 1 + x + x(g-1)(y(f-1)+1) + \left(\frac{x^4}{(1-x)(1-x^2y)} + \frac{x^5y}{(1-x)(1-x^2y)}\right)(y(g-1)+1),$$

where $\boldsymbol{C}_{123}(x,y)$ is given in [4]. Solving this equation for f, and considering Remark 1.1, we obtain the desired result. □

The first few terms in the series expansion of $\boldsymbol{C}_{1243}(x,y)$ are

$$1 + x + 2x^2 + (4+y)x^3 + (8+6y)x^4 + (16+23y+2y^2)x^5 + (32+71y+18y^2)x^6 + (64+192y+94y^2+4y^3)x^7 + (128+475y+371y^2+47y^3)x^8 + O(x^9).$$

Corollary 2.5. *We have*

$$\boldsymbol{C}_{1243}(x) = \frac{(1-x)(1-5x+7x^2-x^3)}{(1-2x)^2(1-3x+x^2)}.$$

Moreover, $\boldsymbol{c}_{1243}(n) = F_{2n+1} - (n+1)2^{n-2}$ *for* $n \geq 1$.

Corollary 2.6. *We have*

$$\boldsymbol{D}_{1243}(x) = \frac{x^3(1-6x+14x^2-15x^3+6x^4-x^5)}{(1-2x)^3(1-3x+x^2)^2}.$$

Moreover,

$$\boldsymbol{d}_{1243}(n) = 2^{n-5}(26+5n-n^2) + \frac{1}{5}\left((n-8)F_{2n} - (5-2n)F_{2n-1}\right), \quad n \geq 3.$$

The first few values of the sequence $\boldsymbol{d}_{1243}(n)$ for $n \geq 3$ are

$$1, \quad 6, \quad 27, \quad 107, \quad 392, \quad 1358, \quad 4509, \quad 14481, \ldots$$

We conclude this subsection by providing combinatorial proofs of the closed form expressions for $\boldsymbol{c}_p(n)$ in Corollaries 2.2 and 2.5.

Combinatorial proofs of formulas for $c_p(n)$, where p is 1234 or 1243:

Let $a_n = c_{1234}(n)$ for $n \geq 1$ and note $a_1 = 1$, $a_2 = 2$. Let $a_n^{(m)}$ for $1 \leq m \leq 3$ denote the cardinality of the subset of $\mathcal{C}_n(1234)$ whose members end in m. We show $a_n = F_{2n-1}$ for all n by arguing combinatorially that a_n satisfies the recurrence $a_n = 3a_{n-1} - a_{n-2}$ for $n \geq 3$. Note $a_n^{(1)} = a_n^{(2)} = a_{n-1}$, upon appending 1 or 2 to an arbitrary member of $\mathcal{C}_{n-1}(1234)$. On the other hand, members of $\mathcal{C}_n(1234)$ ending in 3 must have penultimate letter 2 or 3 and hence have cardinality $a_{n-1} - a_{n-2}$, by subtraction. Combining the prior cases yields the desired recurrence for a_n.

It is also possible to show $a_n = F_{2n-1}$ by defining a bijection between an object enumerated by F_{2n-1} and $\mathcal{C}_n(1234)$ as follows. Let $\mathcal{F}_n$ denote the set of square-and-domino tilings of length n and recall $|\mathcal{F}_n| = F_{n+1}$ for $n \geq 0$. We denote an individual square or domino within a tiling by s or d, respectively. Note that members of $\mathcal{F}_n$ may be viewed as sequences in $\{s, d\}$ consisting of m d's and $n - 2m$ s's for some $0 \leq m \leq \lfloor n/2 \rfloor$. Let $\widetilde{\mathcal{F}}_n$ denote the subset of $\mathcal{F}_n$ whose members start with d and hence $|\widetilde{\mathcal{F}}_n| = F_{n-1}$ for $n \geq 1$. To show $a_n = F_{2n-1}$, it suffices to define a bijection between $\widetilde{\mathcal{F}}_{2n}$ and $\mathcal{C}_n(1234)$.

To do so, first note that each member $\pi \in \mathcal{F}_{2n}$ may be obtained uniquely as a sequence of n steps each being one of the following three operations performed recursively on tilings of increasing length starting with the empty tiling: (i) add a d to the end of the present tiling, (ii) add s^2 to the end or (iii) if the present tiling ends in s, then insert d directly prior to the final s. Put 1, 2 or 3 for the i-th entry of a ternary sequence, which we will denote by α_π, depending on which of (i), (ii) or (iii) is performed in the i-th step of the procedure for $1 \leq i \leq n$. Note that π is recoverable from α_π for all π and that no 3 can directly follow a 1 within α_π or occur at the very beginning, by construction. Further, it is seen that α_π starts with 1 if and only if π starts with d, and hence α_π belongs to $\mathcal{C}_n(1234)$ for such π. Thus, we have that $\pi \mapsto \alpha_\pi$ defines a bijection between $\widetilde{\mathcal{F}}_{2n}$ and $\mathcal{C}_n(1234)$ for all $n \geq 1$, as desired.

Now let $b_n = c_{1243}(n)$ and we wish to show $b_n = F_{2n+1} - (n+1)2^{n-2}$ for $n \geq 1$. As the formula is seen to hold for $1 \leq n \leq 3$, we may assume $n \geq 4$. By the preceding, there are F_{2n-1} members of $\mathcal{C}_n(1243)$ that do not contain 4, i.e., belong to $\mathcal{C}_n(1234)$. So assume $\pi \in \mathcal{C}_n(1243) - \mathcal{C}_n(1234)$, which we decompose as

$$\pi = \pi' \alpha \pi'',$$

where π' is ternary and ends in 3, α starts with 4 and has letters in $\{4, 5, \ldots\}$ and π'' is empty or starts with a letter in $\{1, 2, 3\}$ if nonempty. Since π avoids 1243, we have that α is weakly increasing and π'' is binary. Note that such π may then be obtained by selecting $\rho \in \mathcal{C}_i(1234) - \mathcal{C}_i(123)$ for some $3 \leq i \leq n-1$ and inserting the letters comprising a sequence of the form α of length $n - i$ directly following the rightmost 3 in ρ. From this, it is seen that $\mathcal{C}_n(1243) - \mathcal{C}_n(1234)$ has cardinality

$$\sum_{i=3}^{n-1} (F_{2i-1} - 2^{i-1}) 2^{n-i-1} = \sum_{i=1}^{n-1} 2^{n-i-1} F_{2i-1} - (n-1)2^{n-2}.$$

Thus, to complete the proof of the formula for b_n, we provide a combinatorial argument for

$$\sum_{i=1}^{n-1} 2^{n-i-1} F_{2i-1} = F_{2n} - 2^{n-1}, \qquad n \geq 1. \tag{1}$$

To do so, let $\mathcal{F}'_{2n-1}$ denote the subset of $\mathcal{F}_{2n-1}$ consisting of those tilings with the property that if $2j+1$ is covered by a square, then so is $2j$ for all $1 \le j \le n-1$. Note $\beta \in \mathcal{F}'_{2n-1}$ implies $\beta = s\beta'$, where β' is a sequence in $\{s^2, d\}$ of length $n-1$, and hence $|\mathcal{F}'_{2n-1}| = 2^{n-1}$. Thus, to complete the proof of (1), we argue

$$|\mathcal{F}_{2n-1} - \mathcal{F}'_{2n-1}| = \sum_{i=1}^{n-1} 2^{n-i-1} F_{2i-1}, \qquad n \ge 1.$$

Let $\rho \in \mathcal{F}_{2n-1} - \mathcal{F}'_{2n-1}$ and we consider the largest index i such that s covers $2i+1$, and is preceded by a d. That is, ρ can be decomposed as $\rho = \rho' d s \sigma$, where $\rho' \in \mathcal{F}_{2i-2}$ and σ is a sequence in $\{s^2, d\}$ of length $n-i-1$. Then there are $2^{n-i-1}F_{2i-1}$ possibilities for ρ for each i and considering all i implies (1), as desired. □

2.3. The cases 1324, 1423.

Theorem 2.7. *The bivariate generating function* $\boldsymbol{C}_{1324}(x,y)$ *is given by*

$$\frac{R(x,y)}{(1-2x+x^2(1-y))(1-3x+2x^2(1-y))(1-3x+x^2(3-2y)-x^3(1-y))},$$

where

$$R(x,y) = 1 - 7x + x^2(21-5y) - x^3(35-23y) + x^4(35-42y+8y^2) \\ - 3x^5(7-13y+6y^2) + x^6(1-y)^2(7-4y) - x^7(1-y)^3.$$

Proof. We first count Catalan words w avoiding 1324 and $\underline{11}$. Such a nonempty word w is either (i) 1; or (ii) $12w'$, with $w' \in \mathcal{C}'(1324)$; or (iii) $1(w'+1)w''$, with $w' \in \mathcal{C}'(213)\backslash\{\epsilon, 1\}$ and $w'' \in \mathcal{C}'(1234)$. If $f := \boldsymbol{C}'_{1324}(x,y)$, $g := \boldsymbol{C}_{213}\left(\frac{x}{1+x}, y\right)$ and $h := \boldsymbol{C}_{1234}\left(\frac{x}{1+x}, y\right)$, then we obtain

$$f = 1 + x + x^2 + x^2y(f-1) + x\,(g-1-x)\,(1+y\,(h-1)),$$

where $\boldsymbol{C}_{213}(x,y)$ and $\boldsymbol{C}_{1234}(x,y)$ are given in [4] and Theorem 2.1, respectively. Solving this last equation for f, and making use of Remark 1.1, we obtain the desired result. □

The first few terms in the series expansion of $\boldsymbol{C}_{1324}(x,y)$ are

$$1 + x + 2x^2 + (4+y)x^3 + (8+6y)x^4 + (16+24y+2y^2)x^5 + (32+79y+20y^2)x^6 \\ + (64+230y+117y^2+5y^3)x^7 + (128+615y+518y^2+67y^3)x^8 + O(x^9).$$

Corollary 2.8. *We have*

$$\boldsymbol{C}_{1324}(x) = \frac{1-7x+16x^2-12x^3+x^4}{(1-2x)(1-3x)(1-3x+x^2)}.$$

Moreover, $\boldsymbol{c}_{1324}(n) = 2^{n-1} + 3^{n-1} - F_{2n}$ *for* $n \ge 1$.

Corollary 2.9. *We have*

$$\boldsymbol{D}_{1324}(x) = \frac{x^3(1-10x+40x^2-79x^3+75x^4-28x^5+5x^6)}{(1-2x)^2(1-3x)^2(1-3x+x^2)^2}.$$

Moreover,

$$\boldsymbol{d}_{1324}(n) = 2^{n-3}(n-2) + 3^{n-3}(2n-1) + \frac{1}{5}\left((1-2n)F_{2n} + nF_{2n-1}\right), \quad n \ge 3.$$

The first few values of the sequence $\boldsymbol{d}_{1324}(n)$ for $n \geq 3$ are

$$1, \quad 6, \quad 28, \quad 119, \quad 479, \quad 1852, \quad 6930, \quad 25232, \ldots$$

Theorem 2.10. *We have* $\boldsymbol{C}_{1423}(x, y) = \boldsymbol{C}_{1324}(x, y)$.

Proof. We first count Catalan words w avoiding 1423 and $\underline{11}$. Such a nonempty word w is either (i) 1; or (ii) $1(w'+1)w''$, with $w' \in \mathcal{C}'(123)\backslash\{\epsilon\}$ and $w'' \in \mathcal{C}'(1423)$; or (iii) $1(w'+1)w''$, with $w' \in \mathcal{C}'(312)\backslash\mathcal{C}'(123)$ and $w'' \in \mathcal{C}'(123)$. If $f := \boldsymbol{C}'_{1423}(x, y)$, $g := \boldsymbol{C}_{123}\left(\frac{x}{1+x}, y\right)$ and $h := \boldsymbol{C}_{312}\left(\frac{x}{1+x}, y\right)$, then we obtain

$$f = 1 + x + x(y(f-1)+1)\,(g-1) + x\,(y\,(g-1)+1)\,(h-g)\,,$$

where $\boldsymbol{C}_{312}(x, y)$ and $\boldsymbol{C}_{123}(x, y)$ are already known from [4]. Solving for f in this equation, making use of Remark 1.1 and comparing with Theorem 2.7 yields the stated equality. □

A bijection between $\mathcal{C}_n(1324)$ and $\mathcal{C}_n(1423)$:

To exhibit a bijection between $\mathcal{C}_n(1324)$ and $\mathcal{C}_n(1423)$ for all $n \geq 1$, it is enough to exhibit a bijection between $\mathcal{C}'_n(1324)$ and $\mathcal{C}'_n(1423)$, since one may increase the run lengths by inserting extra copies of letters without introducing an occurrence of either pattern.

We first define a bijection g_n between $\mathcal{C}'_n(213)$ and $\mathcal{C}'_n(312)$ recursively for $n \geq 1$ as follows, the cases $n = 1, 2$ being clear. So assume $n \geq 3$ and let $\pi \in \mathcal{C}'_n(213)$. If $\pi = 1\pi'$, where π' does not contain 1, then let $g_n(\pi) = 1g_{n-1}(\pi')$, where it is understood that the domain and range of g_{n-1} here consists of Catalan words on the alphabet $\{2, 3, \ldots\}$. Otherwise, $\pi = 1\alpha 1\beta$, where α is nonempty and does not contain 1 and β may be empty. Note that α avoids 213 if and only if α does, with β not containing 3 (i.e., β is binary). Let $m = |\alpha|$ so that $1 \leq m \leq n-2$ and define g_n in this case by

$$g_n(\pi) = \begin{cases} 1\beta 1 g_m(\alpha), & \text{if } \beta \neq \varnothing; \\ 1 g_m(\alpha) 1, & \text{if } \beta = \varnothing. \end{cases}$$

One may verify $g_n(\pi)$ avoids 312 for all π and that g_n is indeed a bijection between $\mathcal{C}'_n(213)$ and $\mathcal{C}'_n(312)$.

We now define a bijection between $\mathcal{C}'_n(1324)$ and $\mathcal{C}'_n(1423)$, which will be seen to preserve the number of descents. To do so, we form members of either set by inserting 1's into 213- or 312-avoiding sequences on the alphabet $\{2, 3, \ldots\}$. Let $\tau \in \mathcal{C}'_\ell(213)$ for some $2 \leq \ell \leq n-1$ and suppose τ contains exactly j 1's for some $1 \leq j \leq \lfloor(\ell+1)/2\rfloor$. Consider the decomposition $\tau = \tau^{(1)} \cdots \tau^{(j)}$, where $j \geq 1$ and each section $\tau^{(i)}$ for $1 \leq i \leq j$ starts with 1 and contains no other 1's.

Let π be a Catalan word obtained from τ having the form

$$\pi = \alpha^{(1)}(\tau^{(1)}+1)\alpha^{(2)}(\tau^{(2)}+1)\cdots\alpha^{(j)}(\tau^{(j)}+1)\alpha^{(j+1)}, \tag{2}$$

where (a) $\alpha^{(1)}$ is nonempty and is given by $\alpha^{(1)} = 1(21)^{a_1}$ for some $a_1 \geq 0$, (b) $\alpha^{(i)}$ for $2 \leq i \leq j$ is either empty or of the form $\alpha^{(i)} = (12)^{a_i}1$ or $2(12)^{a_i}1$ for some $a_i \geq 0$ and (c) $\alpha^{(j+1)}$ is empty or is given by $\alpha^{(j+1)} = (12)^{a_{j+1}}1$ or $2(12)^{a_{j+1}}1$ for some $a_{j+1} \geq 0$ if $|\tau^{(j)}| \geq 2$, with $\alpha^{(j+1)}$ empty if $|\tau^{(j)}| = 1$. Note that if $j \geq 2$ in (2), then τ avoiding 213 implies that each section $\tau^{(i)}+1$ for $2 \leq i < j$ equals 23, with $\tau^{(j)}+1$ equal 2 or 23. Further, the section $\tau^{(1)}+1$ contains 3 for all $j \geq 1$, with this holding when $j = 1$ by the assumption $\ell \geq 2$. From the preceding, it is seen that each $\pi \in \mathcal{C}'_n(1324)$ containing 3 can be expressed as in (2), upon allowing τ to vary.

Let $\sigma^{(i)} = g_{k_i}(\tau^{(i)})$, where $\tau_i \in \mathcal{C}'_{k_i}(213)$ for $1 \le i \le j$, and note that each $\sigma^{(i)}$, like $\tau^{(i)}$, starts with 1 and contains no other 1's. Given $\pi \in \mathcal{C}'_n(1324)$ expressed as in (2), let

$$f_n(\pi) = \begin{cases} \alpha^{(1)}(\sigma^{(j)}+1)\alpha^{(2)}(\sigma^{(j-1)}+1)\cdots\alpha^{(j)}(\sigma^{(1)}+1)\alpha^{(j+1)}, & \text{if } |\tau^{(j)}| \ge 2; \\ \alpha^{(1)}(\sigma^{(j-1)}+1)\alpha^{(2)}(\sigma^{(j-2)}+1)\cdots\alpha^{(j-1)}(\sigma^{(1)}+1)\alpha^{(j)}2, & \text{if } |\tau^{(j)}| = 1, \end{cases}$$

with $f_n(\pi) = \pi$ for $\pi = 1212\cdots$. Note that in the second part of the definition of f_n, we must have $j \ge 2$ since $|\tau^{(j)}| = 1$ and $\ell \ge 2$. One may verify $f_n(\pi) \in \mathcal{C}'_n(1423)$ for all π and that f_n furnishes the desired bijection between $\mathcal{C}'_n(1324)$ and $\mathcal{C}'_n(1423)$, which completes the proof. □

2.4. **The case 1342.**

Theorem 2.11. *The bivariate generating function $\boldsymbol{C}_{1342}(x,y)$ is given by*

$$\frac{1-6x+3x^2(5-y)-x^3(20-11y)+x^4(15-14y+2y^2)-2x^5(3-4y+y^2)+x^6(1-y)^2}{(1-x)(1-3x+x^2(2-y))(1-3x+x^2(3-2y)-x^3(1-y))}.$$

Proof. We first count Catalan words w avoiding 1342 and $\underline{11}$. Such a nonempty word w is either (i) 1; or (ii) $1(w'+1)w''$, with $w' \in \mathcal{C}'(123)\backslash\{\epsilon\}$ and $w'' \in \mathcal{C}'(1342)$; or (iii) $1(w'+1)w''$, with $w' \in \mathcal{C}'(231)\backslash\mathcal{C}'(123)$ and $w'' \in \{\epsilon, 1\}$. If $f := \boldsymbol{C}'_{1342}(x,y)$, $g := \boldsymbol{C}_{123}\left(\frac{x}{1+x}, y\right)$ and $h := \boldsymbol{C}_{231}\left(\frac{x}{1+x}, y\right)$, then we obtain

$$f = 1 + x + x\,(g-1)\,(y(f-1)+1) + \; + x(xy+1)\,(h-g)\,,$$

where $\boldsymbol{C}_{123}(x,y)$ and $\boldsymbol{C}_{231}(x,y)$ are given in [4]. Solving this equation and making use of Remark 1.1, we obtain the desired result. □

The first few terms in the series expansion of $\boldsymbol{C}_{1342}(x,y)$ are

$$1 + x + 2x^2 + (4+y)x^3 + (8+6y)x^4 + (16+23y+2y^2)x^5 + (32+71y+18y^2)x^6$$
$$+ 64 + 193y + 94y^2 + 4y^3)x^7 + (128 + 484y + 373y^2 + 47y^3)x^8 + O(x^9).$$

Corollary 2.12. *We have*

$$\boldsymbol{C}_{1342}(x) = \frac{1-6x+12x^2-9x^3+3x^4}{(1-x)(1-3x+x^2)^2}.$$

Moreover,

$$\boldsymbol{c}_{1342}(n) = 1 + \frac{1}{5}\left((1+3n)F_{2n} - 4nF_{2n-1}\right), \quad n \ge 1.$$

Corollary 2.13. *We have*

$$\boldsymbol{D}_{1342}(x) = \frac{x^3(1-x)^3}{(1-3x+x^2)^3}.$$

Moreover,

$$\boldsymbol{d}_{1342}(n) = \frac{1}{50}\left((-34+33n-5n^2)F_{2n-2} + (16-21n+5n^2)F_{2n-1}\right), \quad n \ge 3.$$

The first terms of the sequence $\boldsymbol{d}_{1342}(n)$ for $n \ge 3$ are

$$1, \quad 6, \quad 27, \quad 107, \quad 393, \quad 1371, \quad 4607, \quad 15045, \ldots$$

2.5. **The case 1432.**

Theorem 2.14. *The bivariate generating function* $\boldsymbol{C}_{1432}(x,y)$ *is given by* P/Q, *where*

$$\begin{aligned}P &= 1-11x+x^2(53-4y)-7x^3(21-5y)+x^4(259-128y+5y^2)\\ &- x^5(301-255y+32y^2)+x^6(231-301y+81y^2-2y^3)-x^7(113-213y+102y^2-8y^3)\\ &+x^8(32-86y+65y^2-12y^3)-x^9(4-17y+20y^2-7y^3)-x^{10}y(1-y)^2\end{aligned}$$

and

$$Q=(1-x)(1-2x)(1-2x+x^2(1-y))(1-3x+x^2(3-2y)-x^3(1-y))(1-4x+x^2(4-y)+x^3y)).$$

Proof. We first count Catalan words w avoiding 1432 and $\underline{11}$. Such a nonempty word w is either (i) $1(w'+1)$, with $w'\in\mathcal{C}'(321)$; or (ii) $1(w'+1)1$, with $w'\in\mathcal{C}'(321)\backslash\{\epsilon\}$; or (iii) $1(w'+1)w''$, with $w'=(12)^a1$, $a\geq 0$, $w''\in\mathcal{C}'(1432)\backslash\{\epsilon,1\}$; or (iii') $1(w'+1)w''$, with $w'=(12)^a$, $a\geq 1$, $w''\in\mathcal{C}'(1432)\backslash\{\epsilon,1\}$; or (iv) $1(w'+1)w''$, with $w'=(12)^a123\cdots k1$, $a\geq 0$, $k\geq 3$, $w''=(12)^b12\cdots \ell v$, $b\geq 0$, $\ell\geq 2$ and $v\in\{\epsilon,1\}$; or (iv') $1(w'+1)w''$, with $w'=(12)^a123\cdots k$, $a\geq 0$, $k\geq 3$, $w''=(12)^b12\cdots\ell v$, $b\geq 0$, $\ell\geq 2$ and $v\in\{\epsilon,1\}$; or (v) $1(w'+1)w''$, with $w'=(12)^a123\cdots k1$, $a\geq 0$, $k\geq 3$, $w''=(12)^b123\cdots(k-1)(w'''+k-1)v$, $b\geq 0$, $v\in\{\epsilon,1\}$ and $w'''\in\mathcal{C}'(321)\backslash\{\epsilon,12\cdots m,m\geq 1\}$; or (v') $1(w'+1)w''$, with $w'=(12)^a123\cdots k$, $a\geq 0$, $k\geq 3$, $w''=(12)^b123\cdots(k-1)(w'''+k-1)v$, $b\geq 0$, $v\in\{\epsilon,1\}$ and $w'''\in\mathcal{C}'(321)\backslash\{\epsilon,12\cdots m,m\geq 1\}$.

If $f:=\boldsymbol{C}'_{1432}(x,y)$ and $g:=\boldsymbol{C}_{321}\left(\frac{x}{1+x},y\right)$, then we obtain

$$\begin{aligned}f &= 1+xg+x^2y\,(g-1)+\left(\frac{x^2y}{1-x^2y}+\frac{x^3y}{1-x^2y}\right)(f-1-x)+\frac{x^7y^2(1+xy)}{(1-x)^2(1-x^2y)^2}\\ &+\frac{x^6y(1+xy)}{(1-x)^2(1-x^2y)^2}+\frac{x^7y^2(1+xy)}{(1-x^2)(1-x^2y)^2}\left(g-\frac{1}{1-x}\right)\\ &+\frac{x^6y(1+xy)}{(1-x^2)(1-x^2y)^2}\left(g-\frac{1}{1-x}\right),\end{aligned}$$

where $\boldsymbol{C}_{321}(x,y)$ is given in [4]. Solving for f, and making use of Remark 1.1, we obtain the desired result. □

The first few terms in the series expansion of $\boldsymbol{C}_{1432}(x,y)$ are

$$\begin{aligned}&1+x+2x^2+(4+y)x^3+(8+6y)x^4+(16+24y+2y^2)x^5+(32+80y+19y^2)x^6\\ &+(64+240y+107y^2+4y^3)x^7+(128+672y+465y^2+50y^3)x^8+O(x^9).\end{aligned}$$

Corollary 2.15. *We have*

$$\boldsymbol{C}_{1432}(x)=\frac{1-11x+49x^2-112x^3+136x^4-78x^5+9x^6+6x^7-x^8}{(1-x)(1-2x)^2(1-3x+x^2)(1-4x+3x^2+x^3)}.$$

Theorem 2.16. *We have*

$$\boldsymbol{D}_{1432}(x)=\frac{p(x)}{(1-2x)^3(1-7x+16x^2-12x^3+x^5)^2},$$

where

$$p(x)=x^3(1-14x+85x^2-290x^3+598x^4-736x^5+477x^6-78x^7-67x^8+21x^9+2x^{10}-x^{11}).$$

The first few values of the sequence $\boldsymbol{d}_{1432}(n)$ for $n \geq 3$ are

$$1, \quad 6, \quad 28, \quad 118, \quad 466, \quad 1752, \quad 6333, \quad 22170, \quad 75588, \ldots$$

3. Other descent distributions

Theorem 3.1. *The bivariate generating functions $\boldsymbol{C}_{2134}(x,y)$ and $\boldsymbol{C}_{3412}(x,y)$ are each given by*

$$\frac{1-4x+3x^2(2-y)-4x^3(1-y)+x^4(1-y)^2}{1-5x+3x^2(3-y)-x^3(7-6y)+x^4(1-y)(2-y)}.$$

Proof. Let $f = \boldsymbol{C}'_{2134}(x,y)$ and $h = \boldsymbol{C}'_{1234}(x,y)$. By Theorem 2.1, we have

$$h = \frac{(1+x)(1+x^2(1-2y))}{1-2x^2y-x^3y}.$$

A nonempty $\pi \in \mathcal{C}'(2134)$ may be expressed as (i) $\pi = 1(\alpha+1)$, where α may be empty, or (ii) $\pi = 1(\alpha+1)1\beta$, where α is nonempty and β is possibly empty. Note that π avoiding 2134 and $\alpha+1$ containing 2 in case (ii) implies β cannot contain 4 (i.e., β is ternary). Further, the section $\alpha+1$ in (ii) is then unaffected by β with respect to the avoidance of 2134 since β is ternary. This implies

$$f = 1 + xf + xy(f-1)(h-1) = 1 + xf + \frac{x^2y(1+x+x^2(1-y))}{1-2x^2y-x^3y}(f-1),$$

and solving for f gives

$$f = \frac{1-3x^2y-2x^3y-x^4y(1-y)}{1-x-3x^2y+x^4y^2}.$$

Applying Remark 1.1 now leads to the stated formula for $\boldsymbol{C}_{2134}(x,y)$.

Now let $f = \boldsymbol{C}'_{3412}(x,y)$ and we consider the following cases on a nonempty $\pi \in \mathcal{C}'(3412)$: (i) $1(\alpha+1)$, (ii) $1(23)^i u1\alpha$, where $i \geq 0$, $u \in \{\epsilon, 2\}$ such that $u=2$ if $i=0$ and α is possibly empty, or (iii) $1(23)^j 4\sigma 1$, where $j \geq 1$ and σ may be empty and does not contain 1. Note that if π is not of form (i) such that its first block of non-1 letters contains 4, then π cannot contain any letters beyond its second 1 so as to avoid 3412, which implies the form in (iii). Observe further that the subsequence $(23)^j 4\sigma$ of π in (iii) is accounted for by $f-1-\frac{x(1+x)}{1-x^2y}$, by subtraction, since it represents a member of $\mathcal{C}'(3412)$ on the alphabet $\{2,3,\ldots\}$ that contains at least three distinct letters. Combining the cases (i)–(iii) for π then yields

$$f = 1 + xf + \frac{x^2y(1+x)}{1-x^2y}(f-1) + x^2y\left(f-1-\frac{x(1+x)}{1-x^2y}\right).$$

Solving for f gives the same expression for f as in the case 2134, which implies the result for 3412. □

Theorem 3.2. *The bivariate generating functions $\boldsymbol{C}_{2314}(x,y)$, $\boldsymbol{C}_{2413}(x,y)$, $\boldsymbol{C}_{3124}(x,y)$ and $\boldsymbol{C}_{4123}(x,y)$ are each given by*

$$\frac{(1-x)^5-x^2(4-9x+8x^2-2x^3)y+x^4(3-x)y^2}{((1-x)^2-x^2y)(1-4x+5x^2-2x^3-x^2(3-2x)y)}.$$

Proof. Let $f = \boldsymbol{C}'_\tau(x,y)$ for the pattern τ under current consideration. We first treat $\tau = 2314$. Note that a nonempty $\pi \in \mathcal{C}'(2314)$ has the form (i) $1(\alpha+1)$, (ii) 121β or (iii) $1(\alpha+1)1\beta$, where $|\alpha| \geq 2$, with β possibly empty in (ii) and (iii). Note that $\alpha+1$ containing 3 in (iii)

implies β must be ternary in this case, with no such restriction on β in (ii). Further, β is seen not to restrict α in (iii), since β does not contain 4. Thus, from (i)–(iii), we have

$$f = 1 + xf + x^2y(f-1) + xy(f-1-x)(h-1),$$

where h is as in the proof of Theorem 3.1. Solving for f gives

$$f = \frac{1-4x^2y-3x^3y+x^4y(3y-2)+x^5y(2y-1)}{(1-x^2y)(1-x-3x^2y-x^3y)},$$

and replacing x with $\frac{x}{1-x}$ implies the stated formula for $\boldsymbol{C}_\tau(x,y)$ when $\tau = 2314$. Note that the same argument is seen to apply to f when $\tau = 3124$, and hence the result in this case follows as well.

Now let $\tau = 2413$. In this case, first note that a nonempty $\pi \in \mathcal{C}'(2413)$ is of the form (i) $1(\alpha+1)$, (ii) $1(23)^i u 1\alpha$, where $i \geq 0$, $u \in \{\epsilon, 2\}$ such that $u = 2$ if $i = 0$ and α may be empty, or (iii) $1(23)^i 4\sigma(12)^j v$, where $i \geq 1$, $j \geq 0$, σ does not contain 1 and may be empty and $v \in \{\epsilon, 1\}$ such that $v = 1$ if $j = 0$. Note that the section $(23)^i 4\sigma$ of π in case (iii) is a member of $\mathcal{C}'(2413)$ on $\{2, 3, \ldots\}$ containing at least three distinct letters, and hence is enumerated by $f - 1 - \frac{x(1+x)}{1-x^2y}$. Further, the subsequent section $(12)^j v$ is accounted for by $\frac{xy(1+x)}{1-x^2y}$ since there is a single descent if $j = 0$ or if $j = 1$ with $v = \epsilon$ (caused by the last letter of 234σ), and incrementally increasing j by one in each of these cases is seen to add a descent (with each such increase thus contributing a factor of x^2y). Hence, the π in case (iii) have generating function given by

$$\frac{x^2y(1+x)}{1-x^2y}\left(f - 1 - \frac{x(1+x)}{1-x^2y}\right).$$

Combining with the other two cases for π, we get

$$f = 1 + xf + \frac{x^2y(1+x)}{1-x^2y}(f-1) + \frac{x^2y(1+x)}{1-x^2y}\left(f - 1 - \frac{x(1+x)}{1-x^2y}\right),$$

which leads to the same formula for f as before and establishes the result for $\tau = 2413$. The same argument is seen to apply in the case $\tau = 4123$, which completes the proof. $\square$

Theorem 3.3. *We have*

$$\boldsymbol{C}_{2341}(x,y) = \frac{1-3x+x^2(3-2y)-x^3(1-y)}{(1-x)(1-3x+2x^2(1-y))}.$$

Proof. Let $f = \boldsymbol{C}'_{2341}(x,y)$ and $\omega = \boldsymbol{C}'_{123}(x,y)$. Note

$$\omega = 1 + x + x^2 + x^3y + x^4y + x^5y^2 + x^6y^2 + \cdots = 1 + \frac{x(1+x)}{1-x^2y}.$$

A nonempty $\pi \in \mathcal{C}'(2341)$ must be of the form (i) $\pi = 1(\alpha+1)$ or (ii) $\pi = 1(\alpha+1)1\beta$, where $\alpha \in \mathcal{C}'(123)$ in (ii). Thus, we have that f satisfies

$$f = 1 + xf + xy(\omega-1)(f-1),$$

which implies $f = \frac{1-2x^2y-x^3y}{1-x-2x^2y}$. Applying Remark 1.1 now leads to the stated formula for $\boldsymbol{C}_{2341}(x,y)$. $\square$

Now let $f = \boldsymbol{C}'_{2431}(x,y)$. To assist in finding f, we introduce the auxiliary generating functions $f_m = f_m(x,y)$ for $m \geq 3$ that count members of $\mathcal{C}'(2431)$ with prefix $12\cdots m1$ according to the number of descents. The f_m satisfy the following recurrence.

Lemma 3.4. *If $m \geq 4$, then*

$$f_m = \frac{x^{m+1}y}{1-x^2y} + \frac{x^2}{1-x^2y} f_{m-1}, \tag{3}$$

with $f_3 = x^3y(f-1)$.

Proof. We first observe that $\pi \in \mathcal{C}'(2431)$ enumerated by f_m where $m \geq 4$ is expressible as either (a) $\pi = 12\cdots m1(21)^i$ or (b) $\pi = 12\cdots m1(21)^i 2\sigma$, where σ if nonempty starts with 3 and $i \geq 0$ in both cases. Note that the π in (a) contribute $\frac{x^{m+1}y}{1-x^2y}$ towards f_m, as each additional pair of letters in the sequence $(21)^i$ yields a factor of x^2y. On the other hand, if π is of form (b), then the subsequence $23\cdots m2\sigma$ is seen to be accounted for by $\frac{1}{y}f_{m-1}$, as the second 2 of the subsequence is preceded by a 1 in π and hence does not contribute a descent. The remaining letters of π contribute $\frac{x^2y}{1-x^2y}$, where the factor of y in the numerator arises from the second 1 of π being directly preceded by m. Thus, the π of form (b) are enumerated by $\frac{x^2}{1-x^2y}f_{m-1}$, and combining with those in (a) gives (3). Finally, the initial condition when $m = 3$ follows from the fact that Catalan words enumerated by f_3 are of the form 1231σ, with the prefix 123 imposing no restriction on the nonempty Catalan word 1σ with respect to the avoidance of 2431. □

Theorem 3.5. *The bivariate generating function $\boldsymbol{C}_{2431}(x,y)$ is given by*

$$\frac{(1-x)^4(1-2x)^2 - x^2(1-x)^2(3-11x+10x^2-x^3)y + x^4(2-7x+7x^2-x^3)y^2}{(1-2x)((1-x)^2-x^2y)(1-5x+8x^2-4x^3-x^2(2-4x+x^2)y)}.$$

Proof. A nonempty $\pi \in \mathcal{C}'(2431)$ must have the form (i) $1(\alpha+1)$, (ii) $1(\alpha+1)1\beta$, where α does not contain the letter 3 and β is possibly empty, (iii) $1(23)^i 45\cdots mu(12)^j 1$ or (iv) $1(23)^i 45\cdots mu1(21)^j 2\sigma$, where σ starts with 3 if nonempty in (iv) and $i \geq 1$, $j \geq 0$, $m \geq 4$ and $u \in \{\epsilon, 2\}$ in both (iii) and (iv). Observe that σ cannot contain 1 in (iv), for otherwise 2431 would occur, whence the subsequence $23\cdots m2\sigma$ in (iv) where $m \geq 4$ is accounted for by $\frac{1}{y}\sum_{m\geq 4} f_{m-1}$. Combining the cases (i)–(iv) then implies f satisfies

$$f = 1 + xf + \frac{x^2y(1+x)}{1-x^2y}(f-1) + \frac{x(1+xy)}{(1-x^2y)^2}\sum_{m\geq 4} x^m y + \frac{x^2(1+xy)}{(1-x^2y)^2}\sum_{m\geq 4} f_{m-1}, \tag{4}$$

where f_m is given recursively by (3).

To aid in solving (4), let $F(t) = \sum_{m\geq 3} f_m t^m$. Then, by (3), we have

$$F(t) - x^3y(f-1)t^3 = \frac{x^5yt^4}{(1-x^2y)(1-xt)} + \frac{x^2t}{1-x^2y}F(t),$$

which gives

$$F(t) = \frac{x^3y(1-x^2y)(1-xt)(f-1)t^3 + x^5yt^4}{(1-xt)(1-x^2y-x^2t)}.$$

Thus, by the fact $\sum_{m\geq 3} f_m = F(1)$, we have

$$\begin{aligned} f = 1 + xf &+ \frac{x^2y(1+x)}{1-x^2y}(f-1) + \frac{x^5y(1+xy)}{(1-x)(1-x^2y)^2} \\ &+ \frac{x^2(1+xy)}{(1-x^2y)^2} \cdot \frac{x^3y(1-x)(1-x^2y)(f-1) + x^5y}{(1-x)(1-x^2-x^2y)}. \end{aligned}$$

Solving for f, we obtain

$$f(x,y)=\frac{x^7y^2-x^6y^2-x^5y^2-x^5y+2x^4y^2+3x^4y+2x^3y+x^3-3x^2y-x^2-x+1}{(1-x)(1-x^2y)(x^4y+x^3-2x^2y-x^2-x+1)}.$$

Replacing x with $\frac{x}{1-x}$ in the last formula yields the desired result. □

A comparable idea applies to the pattern 4312. Let g_m be defined the same way f_m was above, but in conjunction with 4312 instead, and let $g=\boldsymbol{C}'_{4312}(x,y)$. Considering the following cases on $\pi\in\mathcal{C}'(4312)$ leads to the recurrence for g_m where $m\geq 4$: (a) $\pi=12\cdots m1$ or $12\cdots m12$, (b) $\pi=12\cdots m121\sigma$, where σ may be empty, or (c) $\pi=12\cdots m123\sigma u$, where σ does not contain 1 and $u\in\{\epsilon,1\}$.

Lemma 3.6. *If $m\geq 4$, then*

$$g_m=x^{m+1}(1+x)y+x^2yg_m+x^2(1+xy)(g_{m-1}-x^my), \tag{5}$$

with $g_3=x^3y(g-1)$.

We can now establish the formula for $\boldsymbol{C}_{4312}(x,y)$.

Theorem 3.7. *We have $\boldsymbol{C}_{4312}(x,y)=A/B$, where*

$$\begin{aligned}A&=(1-x)^6(1-2x)^2-x^2(1-x)^4(4-13x+9x^2+x^3)y\\&\quad+x^4(1-x)(4-15x+16x^2-2x^3-2x^4)y^2-x^6(1-2x-x^2+x^3)y^3,\\B&=(1-x)(1-2x)((1-x)^2-x^2y)((1-3x+2x^2)^2-x^2(1-x)(3-5x-x^2)y\\&\quad+x^4(1+x)y^2).\end{aligned}$$

Proof. Let $h=\boldsymbol{C}'_{1234}(x,y)$. Before proceeding, we will need the restriction of h to Catalan words ending in a specific letter. Let $h^{(i)}=h^{(i)}(x,y)$ for $i=1,2,3$ denote the restriction of the generating function h to the members of $\mathcal{C}'(1234)$ ending in i. Note that $h=1+h^{(1)}+h^{(2)}+h^{(3)}$, by the definitions. Considering the penultimate letter of members of $\mathcal{C}'(1234)$ implies the following system of equations: (a) $h^{(1)}=x+xy(h^{(2)}+h^{(3)})$, (b) $h^{(2)}=xh^{(1)}+xyh^{(3)}$, (c) $h^{(3)}=xh^{(2)}$. Solving (a)–(c) implies

$$h^{(1)}=\frac{x(1-x^2y)}{1-2x^2y-x^3y},\quad h^{(2)}=\frac{x^2}{1-2x^2y-x^3y},\quad h^{(3)}=\frac{x^3}{1-2x^2y-x^3y}.$$

We consider the following cases on $\pi\in\mathcal{C}'(4312)$: (i) π ternary, (ii) $\pi=\alpha 23\cdots m\beta u$, where β starts with a letter in $[2,m-1]$ if nonempty and does not contain 1 and $u\in\{\epsilon,1\}$, or (iii) $\pi=\alpha 23\cdots mv1\beta$, where β is nonempty and $v\in\{\epsilon,2\}$, with $m\geq 4$ and α ternary in both (ii) and (iii). Note that the section $23\cdots m\beta$ in case (ii) has generating function $(1-x^2y)g-1-x-x^2(1-y)$, by subtraction. Taking into account the contributions towards g for the π in cases (i)–(iii) then yields

$$g=h+(h^{(1)}+yh^{(3)})(1+xy)\big((1-x^2y)g-1-x-x^2(1-y)+\frac{1}{x}\sum_{m\geq 4}(g_m-x^{m+1}y)\big), \tag{6}$$

where g_m is given by (5). To solve (6), let $G(t)=\sum_{m\geq 3}g_mt^m$. Then, by (5),

$$(1-x^2y)G(t)-x^3y(1-x^2y)(g-1)t^3=\frac{x^5y(1+x)t^4}{1-xt}+x^2(1+xy)t\left(G(t)-\frac{x^4yt^3}{1-xt}\right),$$

which implies
$$G(t)=\frac{x^3yt^3(x^2y-1)(xtg-x^2t-xt-g+1)}{(xt-1)(x^3yt+x^2t+x^2y-1)}.$$
Note $\sum_{m\geq 4}\left(g_m-x^{m+1}y\right)=G(1)-x^3y(g-1)-\frac{x^5y}{1-x}$ so that
$$\begin{aligned}g=&\frac{(1+x)(1+x^2(1-2y))}{1-2x^2y-x^3y}+\frac{x(1+xy)}{1-2x^2y-x^3y}\left((1-x^2y)g-1-x-x^2(1-y)\right)\\&+\frac{1+xy}{1-2x^2y-x^3y}\left(\frac{x^3y(x^2y-1)(xg-x^2-x-g+1)}{(x-1)(x^3y+x^2+x^2y-1)}-x^3y(g-1)-\frac{x^5y}{1-x}\right).\end{aligned}$$
Solving for g, we obtain a formula of
$$\frac{(1+x)(1-x)^2-x^2(4-x-5x^2+x^3)y+x^4(4+x-5x^2+x^3+x^4)y^2-x^6(1+x-2x^2-x^3)y^3}{(1-x)(1-x^2y)((1+x)(1-x)^2-x^2(3+x-3x^2)y+x^4(1+2x)y^2)}.$$
Making use of Remark 1.1 now yields the desired formula for $\boldsymbol{C}_{4312}(x,y)$. □

Theorem 3.8. *The bivariate generating functions* $\boldsymbol{C}_{3214}(x,y)$*,* $\boldsymbol{C}_{4132}(x,y)$ *and* $\boldsymbol{C}_{4213}(x,y)$ *are each given by*
$$\frac{(1-2x+x^2(1-y))(1-4x+5x^2-2x^3-x^2(3-2x)y)}{(1-x)(1-6x+13x^2-12x^3+4x^4-x^2(4-11x+7x^2)y+3x^4y^2)}.$$

Proof. Let $f=\boldsymbol{C}'_\tau(x,y)$, where τ is the pattern in question. To establish the formula for 3214, we first compute f and consider the following cases on nonempty $\pi\in\mathcal{C}'(3214)$: (i) π binary, (ii) $\pi=(12)^i123\sigma$, (iii) $\pi=(12)^i123\sigma1\rho$, where ρ is ternary and possibly empty, or (iv) $\pi=(12)^i123\sigma1\rho\beta$, where ρ contains 4 but not 1 and β if nonempty has first letter 1, with $i\geq 0$ and σ not containing 1 in all cases. Note that π avoiding 3214 implies β must be ternary in (iv), with no 2 occurring in σ in this case since ρ contains 4. Thus, the subsequence $23\sigma\rho$ of π in (iv) is a 3214-avoiding Catalan word with no levels on the alphabet $\{2,3,\ldots\}$ such that 4 occurs to the right of the second 2. Note that the generating function for such Catalan words is given by $(1-x)f-1-\frac{x^2y(1+x)}{1-x^2y}(f-1)$, by subtraction.

Combining the contributions towards f from cases (i)–(iv) above then gives
$$\begin{aligned}f=&1+\frac{x(1+x)}{1-x^2y}+\frac{x}{1-x^2y}(f-1-x)+\frac{xy}{1-x^2y}(f-1-x)(h-1)\\&+\frac{x^2}{1-x^2y}(1+y(h-1))\left((1-x)f-1-\frac{x^2y(1+x)}{1-x^2y}(f-1)\right),\end{aligned}$$
and solving for f yields
$$f=\frac{(1+x)(1-x^2y)(1-x-3x^2y-x^3y)}{1-x-x^2(1+4y)+x^3(1-y)+3x^4y(1+y)+3x^5y^2}.$$
Replacing x with $\frac{x}{1-x}$ then gives the desired formula for $\boldsymbol{C}_{3214}(x,y)$.

For the pattern 4132, first let $\rho\in\mathcal{C}'_i(4132)$ for some $i\in[4,n]$ of the form $\rho=\alpha\rho'$, where the section α is ternary and $\rho'=234\beta$, with β not containing 1. We wish to make various insertions of letters into these "precursors" ρ of the stated form so as to obtain $\pi\in\mathcal{C}'_n(4132)$ that contain 4. We consider the following cases on ρ according to the section β: (a) β does not contain 2, (b) β contains only a terminal 2 or (c) β contains a 2 that is not terminal. Based on these cases for ρ, we make insertions of letters as follows. In cases (b) and (c), we may insert a sequence of letters of the form $(21)^j$ or $1(21)^j$ for some $j\geq 0$ directly prior to the rightmost 2 in β so as to obtain various π from ρ. Further, in all three cases, one may add a single 1 to the end of β (or do nothing in this regard). Note that inserting a 1 or a sequence of letters as described elsewhere into β is seen to introduce an occurrence of 4132. Moreover, this process of insertion may be reversed by considering the following decomposition of $\lambda \in \mathcal{C}'_n(4132)$ containing 4 for some $\ell \geq 0$:

$$\lambda = \lambda^{(0)}\alpha^{(1)}\lambda^{(1)} \cdots \alpha^{(\ell)}\lambda^{(\ell)}\alpha^{(\ell+1)},$$

where $\lambda^{(0)}$ contains 4 and ends in a letter ≥ 3, $\alpha^{(j)}$ for $1 \leq j \leq \ell$ is binary and ends in 2, $\lambda^{(j)}$ for $1 \leq j \leq \ell$ starts with 3 and contains no 1 or 2 and $\alpha^{(\ell+1)}$ if nonempty is binary.

Note that the section ρ' of a precursor in case (a) where β does not contain 2 is accounted for by the generating function $x(f-1-x)$. Further, by subtraction, all of the possible ρ' from cases (a)–(c) combined are seen to be enumerated by $(1-x^2y)f-1-x-x^2(1-y)$. Hence, again by subtraction, the ρ' in cases (b) and (c) combined are counted by

$$(1-x^2y)f-1-x-x^2(1-y)-x(f-1-x) = (1-x-x^2y)f-1+x^2y.$$

Therefore, upon considering the insertions to obtain the various π from the precursors ρ as described above in accordance with the cases (a)–(c), we obtain

$$\begin{aligned} f &= h + (h^{(1)} + yh^{(3)})\left(x(1+xy)(f-1-x) + \frac{(1+x)(1+xy)}{1-x^2y}((1-x-x^2y)f-1+x^2y)\right) \\ &= \frac{(1+x)(1+x^2(1-2y))}{1-2x^2y-x^3y} \\ &\quad + \frac{x(1+xy)}{1-2x^2y-x^3y}\left(x(f-1-x) + \frac{1+x}{1-x^2y}((1-x-x^2y)f-1+x^2y)\right), \end{aligned}$$

where h and the $h^{(i)}$ are as in Theorem 3.7. Solving for f in the last equation, we obtain the same expression as before, which implies the result for 4132.

A similar argument applies to the case 4213, wherein we add sequences of letters to precursors $\rho = \alpha\rho' \in \mathcal{C}_i(4213)$ to obtain members of $\mathcal{C}_n(4213)$, where α and $\rho' = 234\beta$ are as before. We again consider the cases (a)–(c) above concerning the section β. Here, we may append $(12)^j$ or $(12)^j1$ for some $j \geq 0$ to any ρ with β of the form (a)–(c), and, in addition, for those ρ where β is of the form (c), we may insert a single 1 directly prior to the first 2 in β. Note that inserting a sequence of the form $(21)^j$ or $1(21)^j$ where $j \geq 1$ directly prior to any non-terminal 2 in β within ρ (or inserting a 1 directly prior to any non-terminal 2 that is not the first 2 of β) introduces 4213, and hence the insertions here are as described. By subtraction, the generating function for ρ' in case (c) is given by

$$(1-x^2y)f-1-x-x^2(1-y)-x(1+xy)(f-1-x) = (1-x-2x^2y)f-1+x^2(2+x)y.$$

Thus, making the insertions into ρ to obtain the various $\pi \in \mathcal{C}'_n(4213)$ as described implies that f here satisfies

$$\begin{aligned} f = h + (h^{(1)} + yh^{(3)})\Bigg(&\frac{x(1+xy)^2}{1-x^2y}(f-1-x) \\ &+ \frac{(1+x)(1+xy)}{1-x^2y}((1-x-2x^2y)f-1+x^2(2+x)y)\Bigg). \end{aligned}$$

Solving for f, we obtain the same expression as before, which yields the result for 4213 and completes the proof. □

Theorem 3.9. *The bivariate generating functions $\boldsymbol{C}_{3241}(x,y)$ and $\boldsymbol{C}_{4231}(x,y)$ are each given by*

$$\frac{(1-x)^4(1-2x)^2-x^2(1-x)^3(5-11x+x^2)y+2x^4(1-x)(4-6x+x^2)y^2-x^6(4-x)y^3}{(1-3x+2x^2-2x^2y)((1-3x+2x^2)^2-x^2(1-x)(3-6x+x^2)y+x^4(2-x)y^2)}.$$

Proof. First, let $f=\boldsymbol{C}'_{3241}(x,y)$ and $g=\boldsymbol{C}'_{213}(x,y)$. Note $g=\frac{1-2x^2y-x^3y}{1-x-2x^2y}$, upon replacing x with $\frac{x}{1+x}$ in [4, Theorem 8]. A nonempty $\pi\in\mathcal{C}'(3241)$ has the form (i) 1α, where α does not contain 1, (ii) 121α, where α may contain 1, (iii) $123\sigma(23)^iu1\rho$ or (iv) $123\sigma(23)^iu1\rho234\tau$, where $i\geq 0$, $u\in\{\epsilon,2\}$, σ does not contain 1 or 2 and ρ is ternary in both (iii) and (iv). Note that π avoiding 3241 implies τ cannot contain 1 in (iv). Further, a 1 occurring to the right of the section 3σ in (iii) implies it is counted by $g-1$, with 1ρ counted by $h-1$, where h is as in the proof of Theorem 3.7. Thus, the π from (iii) make a contribution towards f of $\frac{x^2y(1+xy)}{1-x^2y}(g-1)(h-1)$.

By subtraction, the subsequence $s=23\sigma234\tau$ of π in (iv) has generating function

$$f-1-xf-x^2y(1+x)(g-1)-x^2y(f-1-xf),$$

where the last two subtracted terms correspond respectively to excluded sequences of the form $23\sigma2v$ and $23\sigma232\sigma'$, where σ does not contain 1 or 2, σ' does not contain 1 and $v\in\{\epsilon,3\}$. Note that while the descent between the section 3σ and 2 in s does not actually occur in π, it is implicitly counted by the preceding generating function for s. However, the aforementioned descent is replaced by the one occurring between the second 1 of π and its predecessor and hence no extra factor of y is required for it, as it ordinarily would be. Thus, π of the form in (iv) are enumerated by

$$\frac{x(1+xy)}{1-x^2y}(h^{(1)}+yh^{(3)})((1-x^2y)(f-1-xf)-x^2y(1+x)(g-1)),$$

where the $h^{(i)}$ are as in Theorem 3.7. Combining cases (i)–(iv) then implies f satisfies

$$\begin{aligned}f&=1+xf+x^2y(f-1)+\frac{x^2y(1+xy)}{1-x^2y}(g-1)(h-1)\\&\quad+\frac{x^2(1+xy)}{(1-x^2y)(1-2x^2y-x^3y)}((1-x^2y)(f-1-xf)-x^2y(1+x)(g-1)),\end{aligned}$$

which may be simplified and rewritten as

$$\begin{aligned}\left(1-x-x^2y-\frac{x^2(1-x)(1+xy)}{1-2x^2y-x^3y}\right)f&=\frac{1-x^2(1+3y)-2x^3y+2x^4y^2+x^5y^2}{1-2x^2y-x^3y}\\&\quad+\frac{x^4y(1+xy)(1-x^2y)}{(1-x-2x^2y)(1-2x^2y-x^3y)}.\end{aligned}$$

Solving for f in the last equation gives

$$f=\frac{3x^7y^3+4x^6y^3+2x^6y^2-4x^5y^2-8x^4y^2-5x^4y-x^3y-x^3+5x^2y+x^2+x-1}{(2x^2y+x-1)(x^5y^2+2x^4y^2+2x^4y+x^3-3x^2y-x^2-x+1)},$$

and replacing x with $\frac{x}{1-x}$ yields the stated formula for $\boldsymbol{C}_{3241}(x,y)$.

Now let $f=\boldsymbol{C}'_{4231}(x,y)$ and $g=\boldsymbol{C}'_{312}(x,y)$. Note that g is as before, by Theorem [4, Theorem 8]. We consider the following cases on a nonempty $\pi\in\mathcal{C}'(4231)$: (i) 1α, where α does not contain 1, (ii) $1\alpha1\beta$, where α does not contain 1 or 4 and β is possibly empty, (iii)

$1(23)^i4\sigma u1\rho$, where ρ if nonempty is binary, or (iv) $1(23)^i4\sigma u1(21)^j23\tau$ with $j\geq 0$, where $i\geq 1$, σ does not contain 1 or 2 and $u\in\{\epsilon,2\}$ in (iii) and (iv). Observe that τ may be empty and does not contain 1 in (iv) since π avoids 4231.

Note further that the subsequence 34σ of π in (iii) is counted by $g-1-x$, since it must avoid 312 by virtue of a 1 lying to its right. Thus, the π in (iii) make a contribution towards f of

$$\frac{x^2(g-1-x)}{1-x^2y}\cdot\frac{xy(1+x)(1+xy)}{1-x^2y}=\frac{x^5y(1+x)(1+xy)^2}{(1-x^2y)^2(1-x-2x^2y)}.$$

For π of the form (iv), we first extract the subsequence $234\sigma23\tau$, which by subtraction is counted by $f-1-xf-x^2y(f-1)-x^2y(g-1-x)$, where the last two subtracted terms account respectively for excluded words on $\{2,3,\ldots\}$ of the form $232\sigma'$ and $234\sigma2$, where σ' may contain 2. This implies π of the form (iv) make a contribution towards f of

$$\frac{x^2(1+xy)}{(1-x^2y)^2}\left((1-x-x^2y)f-1+2x^2y+x^3y-\frac{x^2y(1-2x^2y-x^3y)}{1-x-2x^2y}\right).$$

Combining cases (i)–(iv) then implies f satisfies

$$f=1+xf+\frac{x^2y(1+x)}{1-x^2y}(f-1)+\frac{x^5y(1+x)(1+xy)^2}{(1-x^2y)^2(1-x-2x^2y)}+\frac{x^2(1+xy)(1-x-x^2y)}{(1-x^2y)^2}f$$
$$-\frac{x^2(1+xy)(1-x-3x^2y+x^3y+x^4y(1+2y)+x^5y^2)}{(1-x^2y)^2(1-x-2x^2y)},$$

which may be simplified and rewritten as

$$\left(1-\frac{x(1+xy)}{1-x^2y}-\frac{x^2(1+xy)(1-x-x^2y)}{(1-x^2y)^2}\right)f=\frac{1-2x^2y-x^3y}{1-x^2y}$$
$$-\frac{x^2(1+xy)(1-x-3x^2y+x^4y^2)}{(1-x^2y)^2(1-x-2x^2y)}.$$

Solving for f in the last equation gives the same expression for f as before and implies the stated formula in the case 4231, which completes the proof. □

Theorem 3.10. *We have*

$$\boldsymbol{C}_{3421}(x,y)=\frac{(1-x)(1-6x+13x^2-12x^3+4x^4-x^2(3-10x+8x^2-x^3)y+x^4(2-x)y^2)}{(1-3x+x^2(2-y))(1-5x+8x^2-4x^3-x^2(2-4x+x^2)y)}.$$

Proof. Let $f=\boldsymbol{C}'_{3421}(x,y)$ and $r=\boldsymbol{C}'_{231}(x,y)$. Replacing x with $\frac{x}{1+x}$ in [4, Theorem 9] gives $r=\frac{1-x^2y}{1-x-x^2y}$. Note that a nonempty $\pi\in\mathcal{C}'(3421)$ can be expressed as (i) $1(\alpha+1)$, (ii) $1(\alpha+1)1\beta$, where α does not contain 3, (iii) $1(23)^i4\sigma1$ or (iv) $1(23)^i4\sigma1\rho$, where $i\geq 1$ and σ does not contain 1 or 2 in (iii) and (iv) and ρ is nonempty in (iv). Note that π avoiding 3421 implies ρ cannot contain 1. Further, the subsequence 34σ in (iii) must avoid 231, and hence the π in this case are enumerated by $\frac{x^3y}{1-x^2y}(r-1-x)$.

For π of the form (iv), first note that the subsequence $s=(23)^i4\sigma\rho$ is a member of $\mathcal{C}'(3421)$ on $\{2,3,\ldots\}$ in which there are at least three distinct letters and a 2 occurs somewhere to the right of the first 4. By subtraction, such s are enumerated by

$$f-1-\frac{x(1+x)}{1-x^2y}-\frac{x}{1-x^2y}(f-1-x)=f-1-\frac{x}{1-x^2y}f,$$

upon excluding members of $\mathcal{C}'(3421)$ that fail to contain three distinct letters or are of the form $(23)^i4\sigma$, where σ does not contain 2. Combining cases (*i*)–(*iv*), we then have

$$f = 1 + xf + \frac{x^2y(1+x)}{1-x^2y}(f-1) + \frac{x^3y}{1-x^2y}(r-1-x) + x^2\left(f - 1 - \frac{x}{1-x^2y}f\right).$$

Solving for f gives

$$f = \frac{1-x-x^2(1+3y)+x^3(1+y)+x^4y(3+2y)+x^5y^2}{(1-x-x^2y)(1-x-x^2(1+2y)+x^3+x^4y)},$$

and making use of Remark 1.1 completes the proof. □

4. The patterns 2143, 3142 and 4321

These three cases entail a more involved analysis than the others and are treated below.

4.1. **The case 2143.**

We first consider the generating functions $k_m = k_m(x,y)$ for $m \geq 1$ enumerating the members of $\mathcal{C}'(2143)$ ending in m according to the number of descents. Let $k = \boldsymbol{C}'_{2143}(x,y)$ and note $k = 1 + \sum_{m\geq 1} k_m$, by the definitions. Let k_m^* for $m \geq 3$ denote the restriction of k_m to those words containing exactly two 1's such that the block of letters occurring to the right of the second 1 is $23\cdots m$. Then we have the following recurrences involving k_m and k_m^*.

Lemma 4.1. *The k_m and k_m^* are given by*

$$k_m = \frac{1}{1-2x^2y-x^3y}k_m^* + xk_{m-1}, \quad m \geq 3, \tag{7}$$

with $k_1 = \frac{x}{1+xy}(yk+1-y)$ *and* $k_2 = \frac{x}{(1+xy)^2}((1+x)yk + x - y - 2xy)$, *and*

$$k_m^* = x^{m+2}y + \frac{x^2(1+xy)(1-x^2y)}{1-2x^2y-x^3y}k_{m-1}^*, \quad m \geq 4, \tag{8}$$

with $k_3^* = x^4y(k-1)$.

Proof. The initial conditions for k_1 and k_2 follow from observing the equalities

$$k_1 = x + xy(k - k_1 - 1) \quad \text{and} \quad k_2 = xk_1 + xy(k - k_1 - k_2 - 1).$$

If $m \geq 3$, then $\pi \in \mathcal{C}'(2143)$ ending in m may be decomposed as $\pi = 1\pi^{(1)}\cdots 1\pi^{(j)}$ for some $j \geq 1$, where each section $\pi^{(i)}$ for $1 \leq i \leq j$ does not contain 1. If $j = 1$, then we clearly get a contribution of xk_{m-1} towards the enumeration of π. On the other hand, if $j \geq 2$, then π ending in $m \geq 3$ (and hence $\pi^{(j)}$ contains 3) implies $\pi^{(i)}$ for $2 \leq i \leq j-1$ is an alternating sequence in $\{2,3\}$ starting with 2. Further, we must have $\pi^{(j)} = (23)^i4\cdots m$ for some $i \geq 1$ since no descent tops to the right of the second 1 can be ≥ 4. To see this, note that otherwise there would be an occurrence of 2143 witnessed by the first 2 in π, the second 1, z and m if $z \geq m+1$ or by $2,1,z,z-1$ if $4 \leq z \leq m$, where z denotes a descent top of π greater than or equal 4. This implies when $j \geq 2$ that π is expressible as $\pi = 1\pi^{(1)}\rho 1(23)^i4\cdots m$ for some $i \geq 1$, where ρ is possibly empty and belongs to $\mathcal{C}'_{1234}$. Note that the suitable ρ in this case are accounted for by $1 + y(h^{(2)} + h^{(3)}) = \frac{1-x^2y}{1-2x^2y-x^3y}$, where the $h^{(i)}$ are as in Theorem 3.7. Since any $i \geq 1$ is permitted in the decomposition of π above, it follows that such π contribute $\frac{1}{1-2x^2y-x^3y}k_m^*$ towards k_m, and combining with the prior case when $j = 1$ yields recurrence (7).

For (8), note first that the initial condition when $m = 3$ follows from the fact that π enumerated by k_3^* have the form $\pi = 1\alpha 123$, where $\alpha - 1$ is an arbitrary nonempty member of $\mathcal{C}'(2143)$. Now suppose $\pi \in \mathcal{C}'(2143)$ is given by $\pi = 1\sigma 12\cdots m$, where σ does not contain 1 and $m \geq 4$. We decompose σ as $2\sigma^{(1)}\cdots 2\sigma^{(\ell)}u$ for some $\ell \geq 0$, where $u \in \{\epsilon, 2\}$ such that $u = 2$ if $\ell = 0$ and each section $\sigma^{(i)}$ for $1 \leq i \leq \ell$ is nonempty and does not contain 1 or 2. If $\ell = 0$, then there is a single possibility for π, which has weight $x^{m+2}y$.

If $\ell \geq 1$, then it is seen that u and both 1's within π may be deleted as they are extraneous concerning the avoidance of the pattern 2143. We claim that each section $2\sigma^{(i)}$ for $2 \leq i \leq \ell$ is also extraneous and may be deleted. To see this, first note that no σ_i for $2 \leq i \leq \ell$ can contain 5, for if one did, then there would be an occurrence of 2143 witnessed by the first 3 of σ, the second 2 of σ, the aforementioned 5 and the rightmost 4 of π. Then $2\sigma^{(i)}$ not containing 5 for $i \geq 2$ implies that, within the subsequence $s = 2\sigma^{(1)}\cdots 2\sigma^{(\ell)}23\cdots m$, no letter in $2\sigma^{(2)}\cdots 2\sigma^{(\ell)}$ can play the role of a 2, 1 or 4 in a possible occurrence of 2143 in s. Further, the section $2\sigma^{(2)}\cdots 2\sigma^{(\ell)}$ is redundant with regard to any of its letters playing the role of a 3 in a 2143 due to it being followed by $23\cdots m$. Thus, after deleting u, both 1's and the extraneous sections $2\sigma^{(i)}$ for $2 \leq i \leq \ell$ of π, one is left with a Catalan word of the form enumerated by k_{m-1}^*, which implies π for which $\ell \geq 1$ are accounted for by

$$\frac{x^2(1+xy)}{1-\frac{x^2y(1+x)}{1-x^2y}}k_{m-1}^* = \frac{x^2(1+xy)(1-x^2y)}{1-2x^2y-x^3y}k_{m-1}^*.$$

Combining the cases when $\ell = 0$ and $\ell \geq 1$ then implies (8) and completes the proof. □

Theorem 4.2. *We have $\boldsymbol{C}_{2143}(x,y) = A/B$, where A and B are as in Theorem 3.7.*

Proof. Let $K(t) = \sum_{m\geq 1} k_m t^m$ and $K^*(t) = \sum_{m\geq 3} k_m^* t^m$. By (7), we then have

$$K(t) - k_1 t - k_2 t^2 = \frac{1}{1-2x^2y-x^3y}K^*(t) + xt(K(t) - k_1 t),$$

with $k_1 = \frac{x}{1+xy}(yK(1)+1)$ and $k_2 = \frac{x}{(1+xy)^2}((1+x)yK(1)+x-xy)$. By (8), we also have

$$K^*(t) - x^4yK(1)t^3 = \frac{x^2(1+xy)(1-x^2y)t}{1-2x^2y-x^3y}K^*(t) + \frac{x^6yt^4}{1-xt}.$$

Solving for $K(1)$ and $K^*(1)$ in the last two equations yields

$$K(1) = \frac{x((1+x)(1-x)^2 - x^2(x^3-4x^2-x+3)y + x^4(x^3-2x^2+x+2)y^2 + x^8y^3)}{(1-x)((1+x)(1-x)^2 - x^2(x^3-4x^2+4)y - x^4(3x^2-3x-4)y^2 - x^6(2x+1)y^3)}.$$

Noting $k(x,y) = K(1)+1$, we find that k is given by the same expression as $g = \boldsymbol{C}'_{4312}(x,y)$ in the proof of Theorem 3.7 above was, and hence the result for 2143 follows. □

4.2. **The case 3142.**

Let $f = \boldsymbol{C}'_{3142}(x,y)$ and f_i for $i = 1,2$ denote the restriction of f to the members of $\mathcal{C}'(3142)$ whose members end in 1 or 2, respectively. To write a system of equations involving these generating functions, we will also need to consider the restriction of f_2 to members of $\mathcal{C}'(3142)$ whose last two letters are 1, 2, which will be denoted by f_2^*. There is the following formula for f in terms of f_2 and f_2^*.

Lemma 4.3. *We have*

$$\left(1-\frac{x(1+x^2y)}{1-x^2y}-\frac{x^2(1+y+2xy-x^2y^2-x^3y^2)(1-x-2x^2y+x^3y+x^4y^2)}{(1+xy)(1-x^2y)^2(1-2x^2y-x^3y)}\right.$$
$$\left.-\frac{x^3y(1+y+2xy)}{(1+xy)(1-x^2y)^2}\right)f=\frac{1-x^2y-x^3y}{1-x^2y}+\frac{x^5y(1+y+2xy-x^2y^2-x^3y^2)}{(1-x^2y)^2(1-2x^2y-x^3y)}$$
$$-\frac{x^2(1+y+2xy-x^2y^2-x^3y^2)}{(1+xy)(1-2x^2y-x^3y)}$$
$$+\frac{x^4y(1+xy)(2+y+x(1+4y)+2x^2y)}{(1-x^2y)^2(1-2x^2y-x^3y)}(f_2-f_2^*)-\frac{x^3y(1+y+2xy)}{(1-x^2y)^2}\left(f_2^*+\frac{1+x}{1+xy}\right). \tag{9}$$

Proof. We consider several cases on nonempty $\pi\in\mathcal{C}'(3142)$ and obtain their respective contributions towards the generating function f. First note that π in which no 1 appears to the right of the first 3 (including the binary case) are enumerated by $\frac{x}{1-x^2y}f$. Further, observe that π of the form $\pi=\alpha2\beta1$, where α is binary and β is nonempty and does not contain 1 or 2 are counted by $\frac{x^3y}{1-x^2y}(f-1)$.

To complete the equation for f, we must enumerate the subset $\mathcal{T}$ consisting of those members of $\mathcal{C}'(3142)$ which contain 3 such that both 1 and 2 occur somewhere to the right of the first 3. To obtain $\pi\in\mathcal{T}$, consider inserting sequences of letters into precursors p on $\{2,3,\ldots\}$ that contain more than one 2 such that $p-1\in\mathcal{C}'(3142)$. Note that in all cases we must insert a sequence of the form $(12)^i1$ for some $i\geq0$ directly prior to the first letter of p, as the resulting word must start with 1. Note that this insertion is always accounted for by a factor of $\frac{x}{1-x^2y}$.

Regarding other insertions (which we will describe as *non-initial*), it is convenient to write the precursor p as $p=2\alpha^{(1)}\cdots2\alpha^{(r)}$ for some $r\geq2$, where the $\alpha^{(i)}$ for $1\leq i\leq r-1$ are nonempty, with $\alpha^{(r)}$ possibly empty, and no $\alpha^{(i)}$ contains 2. We will refer to each section $2\alpha^{(i)}$ as a *unit* of p. First suppose p does not end in 2 (i.e., $\alpha^{(r)}$ is nonempty). We consider subcases of this case based on whether or not 4 occurs prior to the final unit of p. First assume 4 does occur prior to the final unit of p, which itself need not contain 4. We then decompose p as $p=\alpha(23)^\ell2\beta$, where $\ell\geq0$, the final unit of α contains 4 and β is nonempty and does not contain 2.

We first enumerate the precursors p of the stated form with $\ell=0$ and denote their generating function by $k=k(x,y)$. By subtraction, we have

$$k=f-1-xf-(f_1-x)-x^2y(f-f_1-1),$$

upon excluding precursors containing only a single 2, that end in 2 (and contain more than one 2) or that do not end in 2 but have penultimate unit 2, 3. Note $f_1=x+xy(f-f_1-1)$, by the definitions, and hence $f_1=\frac{x}{1+xy}(yf+1-y)$. Therefore, we have

$$\begin{aligned}k&=(1-x-x^2y)f-(1-x^2y)f_1-1+x+x^2y\\&=\left(1-x-x^2y-\frac{xy(1-x^2y)}{1+xy}\right)f-\frac{x(1-y)(1-x^2y)}{1+xy}-1+x+x^2y\\&=\frac{1-x-2x^2y}{1+xy}f-\frac{1-2x^2y-x^3y}{1+xy}.\end{aligned}$$

Additionally, for p of the form $\alpha(23)^\ell 2\beta$, one may make a non-initial insertion of a possibly empty binary sequence with no levels ending in 1 prior to each of the ℓ units $2,3$ as well as directly preceding the section 2β and also add an optional 1 to the very end of p. Note that a 1 (or any binary sequence containing 1, as described) may not be inserted anywhere else into a precursor p of the stated form. For otherwise, there would be an occurrence of 3142 in which the roles of the respective letters are played by the first 3 in p, the inserted 1, the rightmost 4 in α and the 2 directly preceding β. Further, we require that at least a single 1 be added somewhere beyond the section α, as we have already enumerated the case in which no 1 occurs beyond the first 3. Moreover, an insertion made directly preceding a non-initial 2 of p is seen to have weight $\omega := 1 + \frac{x(1+xy)}{1-x^2y} = \frac{1+x}{1-x^2y}$.

Thus, the letters of the section $(23)^\ell$ for some $\ell \geq 0$ within p, together with the insertions as described (including the one that is required at the very beginning of p), are accounted for by $S = S(x,y)$ given by

$$S := \sum_{\ell\geq 0}(x^2y)^\ell\left(\frac{x^2y}{1-x^2y}\cdot\omega^{\ell+1} + \frac{x}{1-x^2y}(\omega^{\ell+1}-1)\right),$$

upon considering whether or not the optional 1 is added to the end. Note that in the case when it is added, the insertions before or after a unit $2,3$ together yield $\omega^{\ell+1}$, with these insertions yielding $\omega^{\ell+1}-1$ if no terminal 1 is added. Simplifying S, we get

$$S = \frac{x(1+x)(1+xy)}{(1-x^2y)(1-2x^2y-x^3y)} - \frac{x}{(1-x^2y)^2} = \frac{x^2(1+y+2xy-x^2y^2-x^3y^2)}{(1-x^2y)^2(1-2x^2y-x^3y)}.$$

Combining the preceding observations, we have that $\pi \in \mathcal{T}$ containing 4 prior to the rightmost 2 and not ending in 2 or 21 make a contribution of kS towards f, where k and S are as given.

Now suppose that the precursor p does not end in 2 or contain 4 prior to the final 2. In this case, we have $p = (23)^j 2\beta$, where $j \geq 1$ and β nonempty does not contain 2. Then the section 2β is accounted for by $x(f-1)$ since it is unaffected by the letters preceding it with regard to the avoidance of 3142. We make insertions as described above to the beginning of p and prior to each subsequent unit $2,3$ or the section 2β as well as possibly appending an optional 1. This implies that the section $(23)^j$ of p where $j \geq 1$, together with the insertions such that at least a single 1 is added somewhere to the right of the first 3, are accounted for by

$$\sum_{j\geq 1}(x^2y)^j\left(\frac{x^2y}{1-x^2y}\cdot\omega^j + \frac{x}{1-x^2y}(\omega^j-1)\right) = x^2yS.$$

Thus, $\pi \in \mathcal{T}$ not ending in 2 or 21 and not containing 4 prior to the rightmost 2 contribute $x^3yS(f-1)$ towards f. Note that combining the prior two cases, we get $(k + x^3y(f-1))S$, where

$$k + x^3y(f-1) = \frac{1-x-2x^2y+x^3y+x^4y^2}{1+xy}f - \frac{1-2x^2y+x^4y^2}{1+xy}.$$

Next suppose p ends in 2 and we make use of the same subcases as before. First assume $p = \alpha(23)^\ell 2$ for some $\ell \geq 0$, where the final unit of α contains 4. Since the terminal unit of p now contains only 2, we must differentiate the cases when $\ell = 0$ and $\ell > 0$. If $\ell = 0$, then the non-initial insertions are limited to a possibly empty binary sequence ending in 1 with no levels placed directly prior to the terminal 2 of p or an optional single terminal 1. To see this, note that for $p = \alpha 2$ in which the final unit of α contains 4, the addition of a 1 (or a sequence containing 1) prior to any 2 of p other than the first or last is seen to introduce

3142. Since a 1 must occur to the right of the first 3 after the insertions are made, we have that the non-initial insertions are accounted for by $\frac{xy(1+x)}{1-x^2y}$ when the optional terminal 1 is added and by $\frac{x(1+xy)}{1-x^2y}$ when it is not. Further, the precursor $p=\alpha 2$ is seen to be enumerated by $f_1-x-xyf_2^*$, by subtraction, since the final unit of α contains 4 (and hence is not $2,3$). Combining the prior observations, it follows that $\pi\in\mathcal{T}$ ending in 2 or 21 and containing 4 such that no string 23 occurs between the rightmost 4 and 2 make a contribution towards f of

$$\begin{aligned}&\frac{x}{1-x^2y}(f_1-x-xyf_2^*)\left(\frac{xy(1+x)}{1-x^2y}+\frac{x(1+xy)}{1-x^2y}\right)\\&=\frac{x^3y(1+y+2xy)}{(1+xy)(1-x^2y)^2}f-\frac{x^3y(1+y+2xy)}{(1-x^2y)^2}\left(f_2^*+\frac{1+x}{1+xy}\right),\end{aligned}$$

upon making use of the fact $f_1=\frac{x}{1+xy}(yf+1-y)$.

Now suppose $p=\alpha(23)^\ell 2$, where $\ell\geq 1$ and the final unit of α contains 4. We account for the subsequence $\alpha 2$ of p in this case as follows. To do so, first note that 3 occurring beyond α imposes an extra restriction on α when p ends in 2 and implies α (on the alphabet $\{2,3,\ldots\}$) is such that $\alpha 23$ avoids 3142. To enumerate such α, let τ be a Catalan word on $\{2,3,\ldots\}$ of the form such that $\tau-1$ has generating function $f_2-f_2^*$. Let t denote the final letter of α. If $t=3$, then α is exactly of the form τ and $\alpha 2$ is enumerated by $xy(f_2-f_2^*)$. On the other hand, if $t>3$, then replacing the terminal 3 of τ with 2 is seen to yield all of the possibilities for the subsequence $\alpha 2$ in this case, and hence it is enumerated by $f_2-f_2^*$. Combining the two prior cases implies $\alpha 2$ is accounted for by $(1+xy)(f_2-f_2^*)$ when $\ell\geq 1$. We now insert letters just as in the case above when p did not end in 2 and α contained 4. Then the section $(23)^\ell$ of p, together with the inserted letters, are accounted for by

$$\begin{aligned}&\sum_{\ell\geq 1}(x^2y)^\ell\left(\frac{x^2y}{1-x^2y}\cdot\omega^{\ell+1}+\frac{x}{1-x^2y}(\omega^{\ell+1}-1)\right)\\&=S-\frac{x(1+x)(1+xy)}{(1-x^2y)^2}+\frac{x}{1-x^2y}=\frac{x^4y(2+y+x(1+4y)+2x^2y)}{(1-x^2y)^2(1-2x^2y-x^3y)}.\end{aligned}$$

Note that the $\pi\in\mathcal{T}$ which arise from making insertions into p as described are those ending in 2 or 21 and containing 4 such that the string 23 occurs between the rightmost 4 and 2. Combining the prior observations implies that they make a contribution towards f of

$$\frac{x^4y(1+xy)(2+y+x(1+4y)+2x^2y)}{(1-x^2y)^2(1-2x^2y-x^3y)}(f_2-f_2^*).$$

Finally, suppose that the precursor p is given by $p=(23)^j 2$ for some $j\geq 1$. Then one may make insertions as previously described prior to any 2 as well as append an optional terminal 1. This implies $\pi\in\mathcal{T}$ not containing 4 and ending in 2 or 21 make a contribution towards f of

$$x^3yS=\frac{x^5y(1+y+2xy-x^2y^2-x^3y^2)}{(1-x^2y)^2(1-2x^2y-x^3y)}.$$

Combining all of the previous cases regarding nonempty $\pi\in\mathcal{C}'(3142)$, and rewriting somewhat the resulting equation for f, yields (9). □

Theorem 4.4. *The bivariate generating function* $\boldsymbol{C}_{3142}(x,y)$ *is given by* $f\left(\frac{x}{1-x},y\right)$*, where* $f(x,y)=C/D$*, with*

$$\begin{aligned}C &= 1-x+x^2(x^3+3x-8)y+x^4(4x^2+x+24)y^2-x^6(x^4+3x^3+6x^2+9x+34)y^3\\ &\quad -x^8(x^3-9x-23)y^4-x^{10}(x^2+4x+6)y^5,\\ D &= (1-x)^2-x^2(x^4-x^3+3x^2-10x+8)y-x^4(3x^3-5x^2+17x-24)y^2\\ &\quad -x^6(x^3+4x^2-12x+34)y^3-x^8(2x^2+2x-23)y^4-2x^{10}(x+3)y^5.\end{aligned}$$

In particular, we have

$$\boldsymbol{C}_{3142}(x)=\frac{x^9-11x^8+9x^7+82x^6-244x^5+297x^4-192x^3+69x^2-13x+1}{(x-1)(x^2-3x+1)(5x^6-x^5-39x^4+61x^3-37x^2+10x-1)}.$$

Proof. We first write equations for f_2^* and f_2. For f_2^*, let c denote the antepenultimate letter within $\rho\in\mathcal{C}'(3142)$ ending in $1,2$ and of length at least three. If $c=2$, then we get a contribution towards f_2^* of x^2yf_2. On the other hand, if $c>2$, then such ρ are seen to be enumerated by $x(f_2-f_2^*)$, as they may be obtained by inserting a 1 between the final two letters of a member of $\mathcal{C}'(3142)$ ending in $c,2$. Thus, we have $f_2^*=x^2+x^2yf_2+x(f_2-f_2^*)$, which implies

$$f_2^*=\frac{x^2}{1+x}+\frac{x(1+xy)}{1+x}f_2. \tag{10}$$

To write an equation for f_2, first observe that members of $\mathcal{C}'(3142)$ that end in 2 in which no 1 occurs beyond the first 3 (including binary words) are enumerated by

$$\frac{x}{1-x^2y}f_1=\frac{x^2}{(1+xy)(1-x^2y)}(yf+1-y).$$

To complete the equation for f_2, we must enumerate the subset $\mathcal{T}'$ of $\mathcal{T}$ whose members end in 2, where $\mathcal{T}$ is as in the proof of Lemma 4.3. We proceed similarly as before and insert sequences of letters into precursors p on $\{2,3,\dots\}$ containing more than one 2 such that $p-1\in\mathcal{C}'(3142)$. We first consider the case when $p=\alpha(23)^\ell 2$ for some $\ell\geq 0$, where the final unit of α contains 4. Again, we must differentiate the cases $\ell=0$ and $\ell>0$. Note that one may proceed as in the proof of Lemma 4.3 in determining the contributions towards f_2 in these cases. The chief difference is that now no terminal 1 can be added to a precursor, as the final resulting word after all of the insertions are made must end in 2. Recall that in order to create a member of $\mathcal{T}$, and, in particular, of $\mathcal{T}'$, at least one 1 must be inserted beyond the first 2 of p in all cases.

Thus, when $\ell=0$, the members of $\mathcal{T}'$ resulting from the insertions into precursors p of the stated form make a contribution towards f_2 of

$$\frac{x^2(1+xy)}{(1-x^2y)^2}(f_1-x-xyf_2^*)=\frac{x^3y}{(1-x^2y)^2}f-\frac{x^3y(1+xy)}{(1-x^2y)^2}\left(f_2^*+\frac{1+x}{1+xy}\right).$$

If $\ell>0$, then the members of $\mathcal{T}'$ that arise from the insertions contribute

$$\frac{x(1+xy)(f_2-f_2^*)}{1-x^2y}\sum_{\ell\geq1}(x^2y)^\ell(\omega^{\ell+1}-1)=\frac{x^4y(1+xy)(2+x(1+2y)+x^2y)}{(1-x^2y)^2(1-2x^2y-x^3y)}(f_2-f_2^*).$$

The remaining possibility is for a precursor to have the form $p = (23)^j2$ for some $j \geq 1$. Insertion of letters into such p yields the remaining members of $\mathcal{T}'$, which are thus accounted for by

$$\frac{x^2}{1-x^2y}\sum_{j\geq 1}(x^2y)^j(\omega^j-1) = \frac{x^5y(1+xy)}{(1-x^2y)^2(1-2x^2y-x^3y)}.$$

Combining the contributions from the various cases above yields

$$\begin{aligned} f_2 = {} & \frac{x^5y(1+xy)}{(1-x^2y)^2(1-2x^2y-x^3y)} + \frac{x^2(1-y)}{(1+xy)(1-x^2y)} + \frac{x^2y(1+x)}{(1+xy)(1-x^2y)^2}f \\ & + \frac{x^4y(1+xy)(2+x(1+2y)+x^2y)}{(1-x^2y)^2(1-2x^2y-x^3y)}(f_2-f_2^*) - \frac{x^3y(1+xy)}{(1-x^2y)^2}\left(f_2^* + \frac{1+x}{1+xy}\right). \end{aligned} \tag{11}$$

Solving the system of equations (9), (10) and (11) for $f = f(x,y)$ yields the desired formula for $\boldsymbol{C}'_{3142}(x,y)$. We omit writing the full expression for $\boldsymbol{C}_{3142}(x,y) = f\left(\frac{x}{1-x}, y\right)$, which is rather lengthy, and state only the $y = 1$ case. □

4.3. **The case 4321.**

Let $f = \boldsymbol{C}'_{4321}(x,y)$ and $g = \boldsymbol{C}'_{321}(x,y)$. Replacing x with $\frac{x}{1+x}$ in [4, Theorem 14] gives $g = \frac{1-x-x^2(1+y)+x^3(1+y)+x^4y}{(1-x)(1-x-x^2(1+y)+x^3)}$. To aid in finding f, let $f_m = f_m(x,y)$ for $m \geq 2$ denote the restriction of f to those members of $\mathcal{C}'(4321)$ with prefix $12\cdots m1$ and let $g_m = g_m(x,y)$ be defined comparably in conjunction with the pattern 321. The f_m and g_m satisfy the following recursive relations.

Lemma 4.5. *We have*

$$f_m = \frac{x^{m+1}y(1-x^2)}{1-x^2y} + \frac{x^2(1+x)}{1-x^2y}f_{m-1} - \frac{x^5}{1-x^2y}(f_{m-2}-yg_{m-2}), \quad m \geq 4, \tag{12}$$

with $f_2 = x^2y(f-1)$ *and* $f_3 = x^3y(f-1)$, *and*

$$g_m = x^{m+1}y + x^2g_{m-1}, \quad m \geq 3, \tag{13}$$

with $g_2 = x^2y(g-1)$.

Proof. The initial conditions of both recurrences follow readily from the definitions. To show (13), note that π enumerated by g_m where $m \geq 3$ is either given by $12\cdots m1$ or has the form $12\cdots m1\sigma$, where σ is nonempty and does not contain 1. Then both of the 1's in π may be deleted in the latter case leaving a Catalan word on $\{2,3,\ldots\}$ of the form enumerated by g_{m-1}, which implies (13). Note that the formula for g above may also be obtained using (13) by noting the equality $g = 1 + xg + \sum_{m\geq 2} g_m$. To show (12), consider inserting letters into a 4321-avoiding precursor $p = 23\cdots m2\sigma$, where $m \geq 4$ and σ does not contain 1, so as to obtain Catalan words enumerated by f_m containing more than one letter 2. We consider the following cases on the section σ of p: (i) $\sigma = \varnothing$, (ii) $\sigma \neq \varnothing$ and does not have a 2, with $m\sigma$ not containing an occurrence of the pattern 321, (iii) σ does not have a 2, with $m\sigma$ containing an occurrence of 321, or (iv) σ has a 2.

If (i) holds, to obtain π enumerated by f_m from p, in addition to prepending a 1 to p, we insert a sequence of letters into p of the form $(12)^i1$ for some $i \geq 0$ between m and 2 as well as append an optional single 1 to the end of p. This is seen to yield a contribution of $\frac{x^2(1+xy)}{1-x^2y} \cdot x^my$ towards f_m in this case. If (ii), first note that the subsequence $34\cdots m\sigma$

of p is of the form enumerated by g_{m-2}, and hence p is accounted for by x^2g_{m-2} in this case, upon inserting two additional letters 2. By making the same insertions as described in (i), one obtains a contribution of $\frac{x^4(1+xy)}{1-x^2y}g_{m-2}$ from precursors in (ii). If (iii) holds, then the subsequence $34\cdots m\sigma$ of p is now accounted for by $f_{m-2}-g_{m-2}$. In this case, one can make the same insertions as in (i) and (ii) above except that the optional 1 added to the end is not permitted due to $m\sigma$ containing 321. This yields a contribution from (iii) of $\frac{x^4}{1-x^2y}(f_{m-2}-g_{m-2})$.

Finally, in (iv), we write $p=23\cdots m2\alpha2\beta$, where α does not contain 2 and β may contain 2 and is possibly empty. In this case, we add a single 1 at the beginning of p, $(12)^i1$ for some $i\geq 0$ between m and 2 and an optional 1 directly preceding the section 2β. Note that a 1 may not be inserted elsewhere into p without introducing 4321 and that the precursor p in this case is seen to have generating function $f_{m-1}-x^my-x^2f_{m-2}$, by subtraction. Thus, we obtain a contribution of $\frac{x^2(1+x)}{1-x^2y}(f_{m-1}-x^my-x^2f_{m-2})$ from (iv). One may verify that each member of $\mathcal{C}'(4321)$ enumerated by f_m with the exception of $12\cdots m1$ is seen to arise uniquely as one makes the various insertions into precursors of the forms (i)–(iv). Thus, combining the contributions of (i)–(iv) obtained above, we have

$$\begin{aligned}f_m&=x^{m+1}y+\frac{x^{m+2}y(1+xy)}{1-x^2y}+\frac{x^4(1+xy)}{1-x^2y}g_{m-2}+\frac{x^4}{1-x^2y}(f_{m-2}-g_{m-2})\\&\quad+\frac{x^2(1+x)}{1-x^2y}(f_{m-1}-x^my-x^2f_{m-2}),\qquad m\geq 4,\end{aligned}$$

which may be simplified to give (12). □

Theorem 4.6. *The bivariate generating function $\boldsymbol{C}_{4321}(x,y)$ is given by $f\left(\frac{x}{1-x},y\right)$, where $f(x,y)=K/L$, with*

$$\begin{aligned}K&=(x+1)^3(x-1)^6(x^2+x+1)+x^2(x+2)(x+1)^2(x-1)^4(x^3-2)y\\&\quad+x^4(x+1)(x-1)^2(2x^5+4x^4+2x^3-2x^2+5)y^2-x^6(x+2)(x^3-x+1)y^3,\\L&=(1-x)((x+1)(x-1)^2-x^2y)\cdot\\&\quad\cdot((x+1)^2(x-1)^4(x^2+x+1)-x^2(x+1)(x-1)^2(x^2+2x+3)y-x^4(x^3+2x^2-2)y^2).\end{aligned}$$

In particular, we have that $\boldsymbol{C}_{4321}(x)$ is given by

$$\frac{x^{12}+3x^{11}-11x^{10}-21x^9+79x^8-39x^7-211x^6+477x^5-471x^4+259x^3-82x^2+14x-1}{(2x-1)(x^3+3x^2-4x+1)(x^8+3x^7-6x^6-2x^5+28x^4-45x^3+30x^2-9x+1)}.$$

Proof. We first write an equation for f. Let π denote a nonempty member of $\mathcal{C}'(4321)$. First suppose π is of the form (i) 1α, where α does not contain 1, (ii) $1(23)^iu1\alpha$, where $i\geq 0$, $u\in\{\epsilon,2\}$ such that $u=2$ if $i=0$ and α may contain 1, or (iii) $1(23)^i4\sigma1$, where $i\geq 1$ and σ does not contain 1 or 2. Note that the subsequence 34σ in (iii) is counted by $g-1-x$, since it must avoid 321. It is then seen that the π from (i)–(iii) make respective contributions towards f of xf, $\frac{x^2y(1+x)}{1-x^2y}(f-1)$ and $\frac{x^3y}{1-x^2y}(g-1-x)$.

Let $\mathcal{S}$ denote the subset of $\mathcal{C}'(4321)$ whose members contain at least two 1's such that a 4 occurs somewhere between the first and second 1 and 2 occurs somewhere to the right of the first 4. To complete the equation for f, we need to determine a formula for the generating function that enumerates members of $\mathcal{S}$ in terms of f, f_m and g_m. To obtain members of $\mathcal{S}$, consider inserting letters into precursors $p \in \mathcal{C}'(4321)$ of the form

$$p = 1(23)^i 4 \cdots m\alpha 2\beta\gamma,$$

where $m \geq 4$, $i \geq 1$, p contains only a single 1, α if nonempty starts with a letter in $[3, m-1]$ and does not contain 2, β may be empty and does not contain 2 and γ if nonempty starts with 2 (with γ assumed to be empty if β is empty). First assume $\alpha \neq \varnothing$. In this case, we insert a single 1 directly prior to the section 2β of p, which is the only possible place where a 1 may be inserted without introducing 4321, as the first letter of α belongs to $[3, m-1]$. Note that, by subtraction, the section $23 \cdots m\alpha 2\beta\gamma$ of p has generating function given by $f - 1 - xf - x^2y(f-1) - \sum_{m\geq 3} f_m$. Hence, one gets a contribution towards f in this case of

$$\frac{x^2}{1-x^2y}\left((1-x-x^2y)f - 1 + x^2y - \sum_{m\geq 3} f_m\right).$$

So assume $\alpha = \varnothing$ in p. We consider the following subcases: (a) $\beta = \varnothing$, (b) $\beta \neq \varnothing$ and $\gamma = \varnothing$, with $m\beta$ containing an occurrence of the pattern 321, (c) $\beta \neq \varnothing$ and $\gamma = \varnothing$, with $m\beta$ not containing an occurrence of 321, or (d) $\gamma \neq \varnothing$. We wish to determine the contributions towards f coming from the subcases (a)–(d).

If (a) holds, then $p = 1(23)^i 4 \cdots m2$, and we insert $(12)^j 1$ or $(21)^j$ for some $j \geq 0$ between m and 2 as well as add an optional 1 to the end of p such that at least one 1 occurs to the right of m in the resulting Catalan word. If the optional 1 is appended to p, then all of the letters added to p are accounted for by $\frac{xy(1+x)}{1-x^2y}$, whereas if not, then the added letters are accounted for by $\frac{x(1+xy)}{1-x^2y}$. Considering all m implies the possible precursors p have weight $\sum_{m\geq 4} \frac{x^{m+1}y}{1-x^2y} = \frac{x^5y}{(1-x)(1-x^2y)}$. Combining the preceding observations, one obtains a contribution of $\frac{x^6y(1+y+2xy)}{(1-x)(1-x^2y)^2}$ towards f from the precursors in (a). If (b) holds, then the subsequence $3 \cdots m\beta$ of p is counted by $\sum_{i\geq 2}(f_i - g_i)$ since β starts with 3, with $m\beta$ containing an occurrence of 321, and hence p is enumerated by $\frac{x^3}{1-x^2y}\sum_{i\geq 2}(f_i - g_i)$. In this case, we insert $(12)^j 1$ or $(21)^{j+1}$ for some $j \geq 0$ between m and 2 within p, leading to a contribution towards f of $\frac{x^4(1+xy)}{(1-x^2y)^2}\sum_{i\geq 2}(f_i - g_i)$.

If (c) holds, then the subsequence $3 \cdots m\beta$ of p is accounted for by $\sum_{i\geq 2} g_i$, and hence the precursor by $\frac{x^3}{1-x^2y}\sum_{i\geq 2} g_i$. In additional to making an insertion of letters between m and 2 within p as described in (b), one may append an optional 1 to the end of p. Moreover, if the optional 1 is added, we need not insert any letters between m and 2. Thus, the addition of letters in this case is accounted for by $\frac{xy(1+x)}{1-x^2y}$ if the optional 1 is added, and by $\frac{x(1+xy)}{1-x^2y}$ if it isn't. Combining these cases yields a contribution of $\frac{x^4(1+y+2xy)}{(1-x^2y)^2}\sum_{i\geq 2} g_i$ towards f from precursors in (c).

Finally, if (d) holds, then the subsequence $23 \cdots m2\beta\gamma$ of p is accounted for by $\sum_{m\geq 3}(f_m - x^{m+1}y) - x^2\sum_{i\geq 2} f_i$, and hence the precursor by

$$\frac{x}{1-x^2y}\left(\sum_{m\geq 3} f_m - x^2\sum_{i\geq 2} f_i - \frac{x^4y}{1-x}\right).$$

To obtain $\pi \in \mathcal{S}$ from p in this case, we may insert $(12)^j 1$ or $(21)^j$ for some $j \geq 0$ between m and 2 as well as insert an optional 1 directly preceding the section γ such that at least one

1 is inserted to the right of m in p. Then the inserted letters are accounted for by $\frac{x(2+x+xy)}{1-x^2y}$ and one obtains a contribution towards f of

$$\frac{x^2(2+x+xy)}{(1-x^2y)^2}\left(\sum_{m\geq 3} f_m - x^2\sum_{i\geq 2} f_i - \frac{x^4y}{1-x}\right).$$

Note that all members of $\mathcal{S}$ arise uniquely by making the insertions as described to the various precursors in (a)–(d). To see this, we decompose $\pi \in \mathcal{S}$ as

$$\pi = 1(23)^k 4\cdots m\lambda^{(0)}\alpha^{(1)}\lambda^{(1)}\cdots\alpha^{(\ell)}\lambda^{(\ell)}, \tag{14}$$

for some $k \geq 1$, $\ell \geq 0$ and $m \geq 4$, where each $\lambda^{(i)}$ is binary and each $\alpha^{(i)}$ contains letters in $\{3,4,\ldots\}$ such that the $\lambda^{(i)}$ for $1 \leq i \leq \ell-1$ are nonempty, all $\alpha^{(i)}$ are nonempty and $\cup_{i=0}^{\ell}\lambda^{(i)}$ contains both 1 and 2. Observe that each $\lambda^{(i)}$ for $1 \leq i \leq \ell-1$ must end in 2, with $\lambda^{(0)}$ doing so if nonempty and $\ell \geq 1$ (note $\lambda^{(\ell)}$ may possibly end in 1). It is seen that precursors p with $\alpha \neq \varnothing$ give rise to π in (14) for which $\ell \geq 1$ with $\lambda^{(0)} = \varnothing$. Moreover, one may verify that the subcases (a)–(d) of p above wherein $\alpha = \varnothing$ give rise to π in (14) for which $\lambda^{(0)} \neq \varnothing$ and the following further respective conditions are satisfied: (a') $\ell = 0$, (b') $\ell = 1$, with $m\alpha^{(1)}$ containing 321 (and hence $\lambda^{(1)} = \varnothing$), (c') $\ell = 1$, with $m\alpha^{(1)}$ not containing 321 and $\lambda^{(1)} \in \{\epsilon, 1\}$, (d') $\ell = 1$, with $\lambda^{(1)} \in \{2, 12\}$, or $\ell \geq 2$.

Combining all of the cases above concerning the form of π (including those in $\mathcal{S}$), we have that f satisfies

$$\begin{aligned} f = 1 + xf + \frac{x^2y(1+x)}{1-x^2y}(f-1) + \frac{x^3y}{1-x^2y}(g-1-x) + \frac{x^2}{1-x^2y}\Bigg((1-x-x^2y)f - 1 + x^2y \\ - \sum_{m\geq 3} f_m\Bigg) + \frac{x^6y(1+y+2xy)}{(1-x)(1-x^2y)^2} + \frac{x^4(1+xy)}{(1-x^2y)^2}\sum_{i\geq 2}(f_i - g_i) + \frac{x^4(1+y+2xy)}{(1-x^2y)^2}\sum_{i\geq 2} g_i \\ + \frac{x^2(2+x+xy)}{(1-x^2y)^2}\left(\sum_{m\geq 3} f_m - x^2\sum_{i\geq 2} f_i - \frac{x^4y}{1-x}\right), \end{aligned}$$

which may be simplified and rewritten as

$$\begin{aligned} &\left(1 - \frac{x(1+xy)}{1-x^2y} - \frac{x^2(1-x-x^2y)}{1-x^2y}\right) f \\ &= \frac{1-x^2(1+2y)-x^3y+x^4y}{1-x^2y} + \frac{x^3y}{1-x^2y}(g-1-x) + \frac{x^6y(y-1)(1+x)}{(1-x)(1-x^2y)^2} \\ &\quad + \frac{x^2(1+x)(1+xy)}{(1-x^2y)^2}\sum_{m\geq 3} f_m - \frac{x^4(1+x)}{(1-x^2y)^2}\sum_{i\geq 2}(f_i - yg_i). \end{aligned} \tag{15}$$

Define $\widetilde{f} = \sum_{m\geq 2} f_m$ and $\widetilde{g} = \sum_{m\geq 2} g_m$. By summing (12) over $m \geq 4$ and (13) over $m \geq 3$, and then solving the resulting system for $\widetilde{f}$ and $\widetilde{g}$, we obtain

$$\begin{aligned} \widetilde{f} &= \frac{x^2y(1+x)(x^2y+x^2-1)(1-x-x^2(1+y)+x^4+(1-x-x^2(1+y)+x^3)f)}{(1-x^2(1+y)-x^3+x^5)(1-x-x^2(1+y)+x^3)}, \\ \widetilde{g} &= \frac{x^3y(1+x)}{1-x-x^2(1+y)+x^3}. \end{aligned}$$

Note (15) may be written as

$$\left(1-\frac{x(1+xy)}{1-x^2y}-\frac{x^2(1-x-x^2y)}{1-x^2y}\right)f$$
$$=\frac{1-x^2(1+2y)-x^3y+x^4y}{1-x^2y}+\frac{x^3y}{1-x^2y}(g-1-x)+\frac{x^6y(y-1)(1+x)}{(1-x)(1-x^2y)^2}$$
$$+\frac{x^2(1+x)(1+xy)}{(1-x^2y)^2}(\widetilde{f}-f_2)-\frac{x^4(1+x)}{(1-x^2y)^2}(\widetilde{f}-y\widetilde{g}).$$

Thus, by solving for $f=f(x,y)$ in the last equation, we obtain the desired formula for $\boldsymbol{C}'_{4321}(x,y)$ (and omit stating the full expression for $\boldsymbol{C}_{4321}(x,y)=f\left(\frac{x}{1-x},y\right)$, which is rather lengthy). □

5. Concluding remarks

In this paper, employing a variety of techniques, we have found an explicit formula for the generating function $\boldsymbol{C}_p(x,y)$ enumerating members of $\mathcal{C}(p)$ according to the number of descents for all permutation patterns p of length four. In each case, $\boldsymbol{C}_p(x,y)$ was rational and hence $\boldsymbol{C}_p(x)$ is as well for such p. Taken together with the comparable results for length three patterns from [4], we believe that this is evidence that $\boldsymbol{C}_p(x)$ might be rational for all permutation patterns p. Future work will focus on patterns of length four with one or more repeated letters and on generalized classes of patterns.

Additional evidence of the possible rationality of $\boldsymbol{C}_p(x)$ for all permutation patterns p is provided by the fact that there are several infinite families of patterns each of whose members give rise to a rational generating function. Let $g_p(x,y)$ denote the bivariate generating function that enumerates members of $\mathcal{C}'(p)$ according to the number of 1's and let $p'=1(p+1)$. Then it can be shown (we omit the details) that the rationality of $g_p(x,y)$ implies that of $g_{p'}(x,y)$. Note that setting $y=1$ in a rational generating function $g_p(x,y)$ implies the rationality of $\boldsymbol{C}'_p(x,1)$, and hence that of $\boldsymbol{C}_p(x)$, by Remark 1.1. From the preceding observations, it follows for example that such patterns as $12\cdots k$ and $12\cdots(k-2)k(k-1)$ have rational generating functions for all $k\geq 3$ since the rationality of $g_p(x,y)$ is readily established when $p=123$ or 132. Using the rationality of $12\cdots k$ for all k, one can show in turn that additional families of patterns are rational such as $(\ell+1)(\ell+2)\cdots k12\cdots\ell$, where $1\leq\ell<k$. Further results in this direction may be obtained by extending arguments from particular cases above. For example, generalizing the proof given for the formula of $\boldsymbol{C}_{3421}(x)$, it is possible to show that $\boldsymbol{C}_p(x)$ is rational where $p=34\cdots k21$ for all $k\geq 4$. We leave the problem of proving or disproving the rationality of $\boldsymbol{C}_p(x)$ for all permutation patterns p as an open question.

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