Rooted planar maps modulo some patterns

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Abstract

We provide generating functions for the number of equivalence classes of rooted planar maps where two maps are equivalent whenever their representations in shuffles of Dyck words coincide on all occurrences of a given pattern.

Keywords: Rooted planar map, shuffle of Dyck words, equivalence relation, pattern.

1 Introduction and notations

A *map* is defined topologically as a 2-cell imbedding of a connected graph, loops and multiple edges allowed, in a 2-dimensional surface. A *rooted planar map* is a map on a sphere with a distinguished edge, called the *root*, assigned with an orientation. Figure $1(a)$ shows a rooted planar map with seven edges. We refer to [1, 16, 17] for the enumeration of rooted planar maps with respect to the number of edges. Planar maps have been widely studied for their combinatorial structure and their links with other domains such as theoretical physics where they appear in some models of 2-dimensional quantum gravity for instance. From a combinatorial point of view, it is proved in [8] that rooted planar maps are in one-to-one correspondence with shuffles of Dyck words that avoid a specific pattern. Then, it becomes natural to extend to maps statistical studies on Dyck words (see [7, 9, 12, 10, 11, 13] for instance). It is one of the main purpose of this paper.

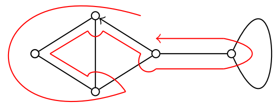
A *Dyck word* on the alphabet $\{a, \bar{a}\}$ is a word generated by the context-free grammar $S \to \varepsilon \mid aS\bar{a}S$ where ε is the empty word. Let $\mathcal{D}_a$ be the set of all Dyck words on the alphabet $\{a, \bar{a}\}$, and let $\mathcal{D}_b$ be the set of all Dyck words on the alphabet $\{b, \bar{b}\}$. It is well known that Dyck words of semilength n are counted by Catalan numbers (A000108 in the on-line encyclopedia of integer sequences [14]), and that any non-empty Dyck word $w \in \mathcal{D}_a$ has a unique decomposition of the form $w = a\alpha\bar{a}\beta$ where α and β are two Dyck words in $\mathcal{D}_a$ (see [7]). Also, a word w on the alphabet $\{a, \bar{a}\}$ belongs to $\mathcal{D}_a$ if and only if the following two properties hold: (i) the number of letters a is equal to the number of letters $\bar{a}$ in w, and (ii) in any prefix of w, the number of letters a is greater than or equal to the number of letters $\bar{a}$.

We say that an occurrence of the letter a *matches* an occurrence of $\bar{a}$ located to its right in $w \in \mathcal{D}_a$, whenever the subword of w consisting of all letters between these two occurrences also belongs to $\mathcal{D}_a$. In this case, the pair $(a, \bar{a})$ is called a *matching* in w. For instance, if $w = a\bar{a}aa\bar{a}aa\bar{a}\bar{a}\bar{a}$, then the second occurrence of the letter a matches the last occurrence of $\bar{a}$ since $a\bar{a}aa\bar{a}\bar{a}$ is a Dyck word.

A *shuffle* of two Dyck words $v \in \mathcal{D}_a$ and $w \in \mathcal{D}_b$ is a word s on the alphabet $\{a, \bar{a}, b, \bar{b}\}$ where s is constructed by interspersing the letters of v and w. Let $\mathcal{S}$ be the set of all shuffles of two Dyck words of $\mathcal{D}_a$ and $\mathcal{D}_b$. Shuffles of semilength n, $n \geq 0$, in $\mathcal{S}$ are enumerated by the sequence A005568 in [14]. The first values for $n \geq 0$ are $1, 2, 10, 70, 588, 5544, 56628$. For instance, $s = aab\bar{a}\bar{a}\bar{b}ab\bar{b}\bar{a}$ is a shuffle of the two Dyck words $aa\bar{a}\bar{a}a\bar{a} \in \mathcal{D}_a$ and $b\bar{b}b\bar{b} \in \mathcal{D}_b$.

For any shuffle s, we denote by $w_a(s)$ (resp. $w_b(s)$) the Dyck word in $\mathcal{D}_a$ (resp. $\mathcal{D}_b$) obtained from s by deleting the letters b and $\bar{b}$ (resp. a and $\bar{a}$). In the following, we will extend the definition of $w_a(s)$ and $w_b(s)$ for any word in $\{a, \bar{a}, b, \bar{b}\}^*$. In particular, if s is a prefix of shuffles, $w_a(s)$ (resp. $w_b(s)$) becomes a prefix of a Dyck word in $\mathcal{D}_a$ (resp. $\mathcal{D}_b$). For instance, if $s = aab\bar{a}\bar{a}\bar{b}ab\bar{b}\bar{a}$, then $w_a(s) = aa\bar{a}\bar{a}a\bar{a}$ and $w_b(s) = b\bar{b}b\bar{b}$; and if $s = aab\bar{a}$, it is a prefix of a shuffle, and $w_a(s) = aa\bar{a}$ and $w_b(s) = b$ are prefixes of Dyck words. Then, *a word s is a shuffle in $\mathcal{S}$ if and only if $w_a(s) \in \mathcal{D}_a$ and $w_b(s) \in \mathcal{D}_b$.*

A shuffle s of two Dyck words $v \in \mathcal{D}_a$ and $w \in \mathcal{D}_b$ will be called *crossing* whenever there exists a matching $(a, \bar{a})$ in v and a matching $(b, \bar{b})$ in w such that s can be written $s = \alpha b\beta a\gamma\bar{b}\delta\bar{a}\eta$ where $\alpha, \beta, \gamma, \delta$ and η belong to $\{a, \bar{a}, b, \bar{b}\}^*$. Then the occurrence $ba\bar{b}\bar{a}$ will be called a *pair of cross-*

(a) 

(b)

$aaabb\bar{a}\bar{a}\bar{b}\bar{a}\bar{b}abb\bar{b}\bar{a}$

Figure 1: A rooted planar map with 7 edges and its non-crossing shuffle representation.

ing matchings. Let $\mathcal{NCS} \subset \mathcal{S}$ be the subset of non-crossing shuffles in $\mathcal{S}$, *i.e.*, shuffles with no pair of crossing matchings. For instance, $a\bar{a}b\bar{b}a\bar{a}b\bar{b}a\bar{a}$ is in $\mathcal{NCS}$, while $a\bar{a}bb\bar{b}a\bar{b}\bar{a}$ is not in $\mathcal{NCS}$. Notice that non-crossing shuffles are called *canonical parenthesis-bracket systems* in [18]. The shuffles of semilength $n \geq 0$ in $\mathcal{NCS}$ are enumerated by the sequence A000168 in [14] whose first values for $n \geq 0$ are $1, 2, 9, 54, 378, 2916, 24057, 208494$. They are in one-to-one correspondence with the rooted planar maps with n edges [5, 6, 8, 18]. See Figure 1 for an example of rooted planar map with its representation as a non-crossing shuffle in $\mathcal{NCS}$. This one-to-one correspondence is obtained by the following process. Starting with the root edge, we follow or cross all edges of the map by making its tour in counter-clockwise direction. Each edge in the map must be reached twice. If the final vertex of the encountered edge has not yet been considered, then we follow this edge and we write the letter a; otherwise, if the edge is reached for the first time, then we write the letter b and we cross it; in the other cases the edge is reached for the second time, and we write $\bar{a}$ (resp. $\bar{b}$) and we follow the edge (resp. we cross the edge) whenever the first assignment of the edge was the letter a (resp. b). Table 1 gives the correspondence between some patterns of length at most two in a shuffle with their meaning in terms of map. For instance, a pattern $b\bar{b}$ in a shuffle of $\mathcal{NCS}$ corresponds to an empty loop on a vertex of the corresponding rooted planar map. Also, the pattern $a\bar{b}$ cannot occur in the shuffle representation of a rooted planar map because it necessarily creates a pair of crossing matchings.

In a recent paper [3] (for equivalence classes of permutations see [2]), the authors introduced an equivalence relation on the set of Dyck words where *two Dyck words are equivalent whenever the positions of the occurrences of a given pattern are the same in both words.* They provided generating functions for the numbers of equivalence classes whenever the patterns are

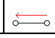
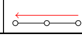
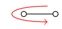
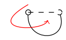
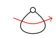
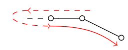
Shuffle	a	aa	$a\bar{a}$	b	$b\bar{b}$	$\bar{a}a$	$a\bar{b}$
Map							Does not appear

Table 1: Correspondence between patterns in a shuffle and patterns in a rooted planar map.

of length two: aa, $a\bar{a}$, $\bar{a}\bar{a}$ and $\bar{a}a$. See also [4] for a study of this equivalence relation on Motzkin words.

The motivation of this paper is to present a similar study for shuffles of Dyck words and for rooted planar maps considered with their representation by shuffles in $\mathcal{NCS}$. So, we define the equivalence relation on $\mathcal{S}$ and $\mathcal{NCS}$ for any pattern π of length at most two:

two shuffles with the same semilength are π-equivalent whenever they coincide on the positions of occurrences of the pattern π.

For instance, the shuffle $s = aaba\bar{b}\bar{a}b\bar{a}\bar{b}\bar{a}$ is $\bar{b}\bar{a}$-equivalent to $s' = a\bar{a}ab\bar{b}\bar{a}ab\bar{b}\bar{a}$ since the occurrences of $\bar{b}\bar{a}$ appear in positions 5 and 9 in s and s'.

In Section 2, we show that the problem of the enumeration of the π-equivalence classes in $\mathcal{NCS}$ is the same as in $\mathcal{S}$ whenever π is a pattern of length at most two that does not belong to the set $\{\bar{b}a, b\bar{a}, a\bar{b}, \bar{a}b\}$. In Sections 3-6, we present enumerative results by providing generating functions for the number of π-equivalence classes when $\pi \notin \{\bar{b}a, b\bar{a}, a\bar{b}, \bar{a}b\}$. Using the one-to-one correspondence between rooted planar maps and non-crossing shuffles of Dyck words, this induces enumerative results for equivalence classes of rooted planar maps relatively to the positions of some patterns. See Table 2 for an overview of these results. Notice that the pattern $a\bar{b}$ does not appear in any shuffle of a rooted planar map, and that we did not succeed to obtain the number of π-equivalence classes in $\mathcal{S}$ and $\mathcal{NCS}$ for $\pi \in \{\bar{a}b, \bar{b}a, b\bar{a}\}$. So, we leave these cases as open problems.

2 Some preliminary results

In this section, we prove that for some specific patterns π the numbers of π-equivalence classes in $\mathcal{S}$ and $\mathcal{NCS}$ are the same.

Lemma 1 *Let $w = \alpha\bar{a}\beta\gamma$ be a Dyck word in $\mathcal{D}_a$ with $\beta\gamma$ in $\{a, \bar{a}\}^*$. Then the word $w' = \alpha\beta\bar{a}\gamma$ is also in $\mathcal{D}_a$.*

Proof. The Dyck word w' is obtained from w by a shift on the right of a letter $\bar{a}$. So, the number of a and the number of $\bar{a}$ remains unchanged in

Pattern	Sequence	Sloane	$a_n, 1 \leq n \leq 9$
$\{a\}, \{\bar{a}\}, \{b\}, \{\bar{b}\}$	$\frac{1}{\sqrt{1-4x}}$	Central binomial coeff. $A000984$	$2, 6, 20, 70, 252, 924, 3432, 12870, 48620$
$\{a\bar{a}\}, \{b\bar{b}\}$	$\frac{1-x}{1-3x+x^2}$	Shift of $A001519$	$2, 5, 13, 34, 89, 233, 610, 1597, 4181$
$\{\bar{a}a\}, \{\bar{b}b\}$	$\frac{1-2x}{1-3x+x^2}$	$A001519$	$1, 2, 5, 13, 34, 89, 233, 610, 1597$
$\{aa\}, \{\bar{a}\bar{a}\}, \{bb\}, \{\bar{b}\bar{b}\}$	$\frac{1+x-\sqrt{1-2x-3x^2}}{2x^2+3x-1+\sqrt{1-2x-3x^2}}$	Bisection of $A191385$	$1, 2, 5, 12, 31, 81, 216, 583, 1590$
$\{ab\}, \{ba\}, \{\bar{a}\bar{b}\}, \{\bar{b}\bar{a}\}$	$\frac{-2\sqrt{3}\sin\left(1/3 \arcsin\left(3/2\sqrt{3}x\right)\right)}{4\left(\sin\left(1/3 \arcsin\left(3/2\sqrt{3}x\right)\right)\right)^2-3x}$	$A138164$	$1, 2, 4, 9, 20, 47, 109, 262, 622$

Table 2: Enumeration of π-equivalence classes for shuffles and rooted planar maps.

w'. Moreover, in any prefix of w the number of a is greater than or equal to the number of $\bar{a}$ and this property remains satisfied in w'. Using the characterization of a Dyck word given in the introduction, w' is in $\mathcal{D}_a$. □

Lemma 2 *Let $w = \alpha a \beta \bar{a} \gamma$ be a Dyck word in $\mathcal{D}_a$. If $\beta = \beta_1\beta_2 \ldots \beta_k$ is in $\mathcal{D}_a$ then for any i, $1 \leq i \leq k$, the word $w' = \alpha a \beta_1 \ldots \beta_{i-1}\bar{a}\beta_i \ldots \beta_k \gamma$ is also in $\mathcal{D}_a$.*

Proof. Since w is in $\mathcal{D}_a$, (i) the number of a and the number of $\bar{a}$ are equal in w, and in any prefix (ii) the number of a is greater than or equal to the number of $\bar{a}$. Since β is in $\mathcal{D}_a$, the number of a in any prefix of $\alpha a \beta_1 \ldots \beta_{i-1}$ is strictly greater than the number of $\bar{a}$. This means that $\alpha a \beta_1 \ldots \beta_{i-1}\bar{a}$ satisfies Condition (ii). Then, w' satisfies (ii) which implies that w' belongs to $\mathcal{D}_a$. □

Of course, the last two lemmas also hold if we replace the letter a with b. So, we use them indifferently for a and b.

Theorem 1 *Let s be a shuffle in $\mathcal{S}$ and π be a pattern of length at most two not in $\{a\bar{b}, \bar{a}b, \bar{b}a, b\bar{a}\}$. Then, there exists a shuffle s' in $\mathcal{NCS}$ with the same semilength as s so that s and s' are π-equivalent.*

Proof. Let s be a shuffle in $\mathcal{S}$ that does not lie in $\mathcal{NCS}$, *i.e.* s contains a pair of crossing matchings. Then, s can be decomposed $s = \alpha b \beta a \gamma \bar{b} \delta \bar{a} \eta$ so that $\alpha, \beta, \gamma, \delta$ and η are in $\{a, \bar{a}, b, \bar{b}\}^*$ and the crossing matching $ba\bar{b}\bar{a}$ is chosen the leftmost possible in s.

We distinguish three cases: (i) $\pi \in \{a, aa, ab, ba\}$; (ii) $\pi = a\bar{a}$; and (iii) $\pi = \bar{a}a$. The other cases are easily obtained using classical symmetries on shuffles ($a \leftrightarrow b$ and mirror).

Case (i): Let us consider the word $s' = \alpha b\beta a\gamma\bar{a}\delta\bar{b}\eta$ obtained from s by an exchange of $\bar{a}$ and $\bar{b}$. With Lemma 1 and Lemma 2, s' is also a shuffle of two Dyck words. The two shuffles s and s' belong to the same π-equivalence class since this operation cannot create or delete a pattern in $\{a, aa, ab, ba\}$. Moreover the leftmost pair of crossing matchings in s' is shifted on the right compared to s. We repeat the process for as long as required, and we obtain a shuffle in $\mathcal{NCS}$ with the same semilength as s, which is π-equivalent to s.
Case (ii): Let us assume that $s = s_1s_2\dots s_{2k}$ for some $k \geq 1$ with $s_i \in \{a, \bar{a}, b, \bar{b}\}$, $1 \leq i \leq 2k$. We obtain s' from s by the following process: we preserve the positions of all occurrences of the pattern $a\bar{a}$; the other letters of s' are chosen so that the restriction of s' to them is the Dyck word of the form $b^\ell\bar{b}^\ell$ for some $\ell \geq 0$. It is straightforward to see that $s' \in \mathcal{NCS}$ and that the two shuffles s and s' belong to the same $a\bar{a}$-equivalence class.
Case (iii): Let us assume that $s = s_1s_2\dots s_{2k}$ for some $k \geq 1$ with $s_i \in \{a, \bar{a}, b, \bar{b}\}$, $1 \leq i \leq 2k$. We obtain s' from s by the following process: we preserve the positions of all occurrences of the pattern $\bar{a}a$ and we set $s'_1 = a$ and $s'_{2k} = \bar{a}$; the other letters of s' are chosen so that the restriction to them is the Dyck word of the form $a^\ell\bar{a}^\ell$ for some $\ell \geq 0$. It is straightforward to see that $s' \in \mathcal{D}_a \subset \mathcal{NCS}$ and that the two shuffles s and s' belong to the same $\bar{a}a$-equivalence class. □

Corollary 1 *Let π be a pattern of length at most two not in $\{\bar{b}a, b\bar{a}, a\bar{b}, \bar{a}b\}$. For each word length the number of π-equivalence classes in $\mathcal{S}$ is also the number of π-equivalence classes in $\mathcal{NCS}$.*

This corollary allows us to limit our study of the general case to shuffles in $\mathcal{S}$ and patterns π not in $\{\bar{b}a, b\bar{a}, a\bar{b}, \bar{a}b\}$. All results in the following sections will hold for both sets $\mathcal{S}$ and $\mathcal{NCS}$.

3 Equivalence modulo $\pi \in \{a, b, \bar{a}, \bar{b}\}$

The results for $\pi \in \{b, \bar{a}, \bar{b}\}$ are deduced from the ones for $\pi = a$ by using the classical symmetries (mirror and $a \leftrightarrow b$) on shuffles. So, we set $\pi = a$ in this section.

Let $\mathcal{A}$ be the set of shuffles in $\mathcal{S}$ defined by the grammar $\mathcal{A} \to \mathcal{D}_a \mid \mathcal{D}_a b\mathcal{D}_a\bar{b}\mathcal{A}$ where $\mathcal{D}_a$ is the set of Dyck words on the alphabet $\{a, \bar{a}\}$.

Lemma 3 *The set $\mathcal{A}$ and the set of a-equivalence classes of $\mathcal{S}$ are in length-preserving one-to-one correspondence.*

Proof. For $k \geq 1$ let $s = s_1 s_2 \dots s_{2k}$ be a shuffle in $\mathcal{S} \backslash \mathcal{A}$ with $s_i \in \{a, \bar{a}, b, \bar{b}\}$ for $1 \leq i \leq 2k$. Let us prove that there exists a shuffle $s' \in \mathcal{A}$ (with the same semilength as s) such that s and s' belong to the same a-equivalence class. We define the word s' by performing the following process on s for i from 1 to $2k$:

- If $s_i = a$ then we set $s'_i = a$;
- Otherwise, we distinguish three cases:
  (*i*) If $w_a(s'_1 \dots s'_{i-1}) \notin \mathcal{D}_a$ then we set $s'_i = \bar{a}$;
  (*ii*) If $w_a(s'_1 \dots s'_{i-1}) \in \mathcal{D}_a$ and $w_b(s'_1 \dots s'_{i-1}) \notin \mathcal{D}_b$ then we set $s'_i = \bar{b}$;
  (*iii*) If $w_a(s'_1 \dots s'_{i-1}) \in \mathcal{D}_a$ and $w_b(s'_1 \dots s'_{i-1}) \in \mathcal{D}_b$ then we set $s'_i = b$.

It is worth noticing that $w_a(s'_1 \dots s'_i)$ (resp. $w_b(s'_1 \dots s'_i)$) is a prefix of a Dyck word in $\mathcal{D}_a$ (resp. $\mathcal{D}_b$) for all i, $1 \leq i \leq 2k$. Moreover, at the end of the process we necessarily have $w_a(s'_1 \dots s'_{2k}) \in \mathcal{D}_a$ and $w_b(s'_1 \dots s'_{2k}) \in \mathcal{D}_b$ which means that s' belongs to $\mathcal{S}$.

Since the process preserves the occurrences of the letter a in s and do not introduce other letters a in s', s and s' belong to the same a-equivalence class. Moreover, s' necessarily lies in $\mathcal{A}$. Indeed, the process sets $s'_i = \bar{a}$ whenever $w_a(s'_1 \dots s'_{i-1})$ is not a Dyck word of $\mathcal{D}_a$; in all other cases, it sets $s'_i = b$ or $s'_i = \bar{b}$ alternatively. If s' does not contain any occurrence of b, then $s' \in \mathcal{D}_a \subset \mathcal{A}$; otherwise, the previous construction ensures that s' has a prefix of the form $\alpha b \beta \bar{b}$ with α and β in $\mathcal{D}_a$. This means that s' satisfies the grammar $\mathcal{A} \rightarrow \mathcal{D}_a \mid \mathcal{D}_a b \mathcal{D}_a \bar{b} \mathcal{A}$. Thus, we have $s' \in \mathcal{A}$.

Now, it suffices to prove that two distinct shuffles s and s' with the same semilength in $\mathcal{A}$ are not a-equivalent. For a contradiction, let us assume that s and s' belong to the same class.

We distinguish two cases:

- If $s \in \mathcal{D}_a$ or $s' \in \mathcal{D}_a$ with the same positions of the occurrences of a then we have $s = s'$.

- Otherwise, s can be written $s = \alpha b \beta \bar{b} \gamma$ with α and β in $\mathcal{D}_a$ and γ in $\mathcal{A}$. Also, s' can be written $s' = \alpha' b \beta' \bar{b} \gamma'$ with α' and β' in $\mathcal{D}_a$ and γ' in $\mathcal{A}$.

Since α and α' belong to $\mathcal{D}_a$ and have the same positions of occurrences of the letter a, we necessarily have $\alpha = \alpha'$. The same argument for β implies that $\beta = \beta'$. We complete the proof by induction on the length for γ and γ' in $\mathcal{A}$ and we conclude $s = s'$. □

Theorem 2 *The generating function for the set of a-equivalence classes in $\mathcal{S}$ (or in $\mathcal{NCS}$) with respect to the semilength is given by*

$$\frac{1}{\sqrt{1-4x}} = \sum_{n \geq 0} \binom{2n}{n} x^n.$$

Proof. By Lemma 3, it suffices to provide the generating function $A(x)$ of the set $\mathcal{A}$, with respect to the semilength. Every nonempty shuffle $s \in \mathcal{A}$ is obtained by the grammar $\mathcal{A} \to \mathcal{D}_a \mid \mathcal{D}_a b \mathcal{D}_a \bar{b} \mathcal{A}$ where $\mathcal{D}_a$ is the set of Dyck words on the alphabet $\{a, \bar{a}\}$. This induces the functional equation:

$$A(x) = D(x) + xD(x)^2 A(x)$$

where $D(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the Catalan generating function for the set $\mathcal{D}_a$, which implies the required result. □

4 Equivalence modulo $\pi \in \{a\bar{a}, b\bar{b}, \bar{a}a, \bar{b}b\}$

The results for $\pi \in \{b\bar{b}, \bar{b}b\}$ are deduced from the ones for $\pi = a\bar{a}$ and $\pi = \bar{a}a$ by using the classical symmetries. Then, we study in this section the two cases $\pi = a\bar{a}$ and $\pi = \bar{a}a$.

Let $\mathcal{B}$ be the subset of $\mathcal{S}$ defined by the grammar $\mathcal{B} \to a\bar{a}\mathcal{B} \mid b\mathcal{B}'\bar{b}\mathcal{B} \mid \varepsilon$ where $\mathcal{B}' \to a\bar{a}\mathcal{B}' \mid \varepsilon$. It is worth noticing that the set $\mathcal{B}'$ consists of words of the form $(a\bar{a})^k$ for all $k \geq 0$. Moreover, it is straightforward to see that for any $s \in \mathcal{B}$ we have $w_b(s) = (b\bar{b})^i$ for some $i \geq 0$ and $w_a(s) = (a\bar{a})^j$ for some $j \geq 0$. Finally, if $X = a\bar{a}$ then the set $\mathcal{B}$ consists of all shuffles of words X^i, $i \geq 0$, with Dyck words of the form $(b\bar{b})^j$, $j \geq 0$, where the occurrences of $X = a\bar{a}$ are never split. For instance, $ba\bar{a}a\bar{a}\bar{b}a\bar{a}b\bar{b}$ belongs to $\mathcal{B}$, while $ba\bar{a}a\bar{b}\bar{a}a\bar{a}b\bar{b} \notin \mathcal{B}$.

Lemma 4 *The set $\mathcal{B}$ and the set of $a\bar{a}$-equivalence classes of $\mathcal{S}$ are in length-preserving one-to-one correspondence.*

Proof. For $k \geq 1$ let $s = s_1 s_2 \dots s_{2k}$ be a shuffle in $\mathcal{S} \backslash \mathcal{B}$ with $s_i \in \{a, \bar{a}, b, \bar{b}\}$ for $1 \leq i \leq 2k$. Let us prove that there exists a shuffle $s' \in \mathcal{B}$ (with the same semilength as s) such that s and s' belong to the same class. We define the word s' by performing the following process on s:

- all occurrences of the pattern $a\bar{a}$ in s are preserved in s';

- for i from 1 to $2k$ such that s_i does not belong to a pattern $a\bar{a}$ in s, we distinguish two cases:

(*i*) if $w_b(s'_1 \dots s'_{i-1}) = (b\bar{b})^j$ for some $j \geq 0$, then we set $s'_i = b$,

(*ii*) if $w_b(s'_1 \dots s'_{i-1}) = (b\bar{b})^j b$ for some $j \geq 0$, then we set $s'_i = \bar{b}$.

For all i $(1 \leq i \leq 2k)$ we have either $w_b(s'_1 \dots s'_i) = (b\bar{b})^j$ or $w_b(s'_1 \dots s'_i) = (b\bar{b})^j b$ for some $j \geq 0$. Moreover, we have $w_a(s'_1 \dots s'_i) = (a\bar{a})^\ell$ for some $\ell \geq 0$ so that a and $\bar{a}$ appear necessarily adjacent in s' in an occurrence $a\bar{a}$. This means that s' is a shuffle that lies in $\mathcal{B}$. Since the process preserves the occurrences of the pattern $a\bar{a}$, s and s' belong to the same $a\bar{a}$-equivalence class.

Less formally, s' is obtained from s by fixing the occurrences of $a\bar{a}$ and by replacing all other letters by a Dyck word of the form $(b\bar{b})^j$ for some $j \geq 0$. Therefore, the definition of $\mathcal{B}$ ensures us that s' is the unique element in $\mathcal{B}$ that lies into the same $a\bar{a}$-equivalence class as s. □

Theorem 3 *The generating function for the set of $a\bar{a}$-equivalence classes in $\mathcal{S}$ with respect to the semilength is given by*

$$\frac{1-x}{1-3x+x^2} = \sum_{n\geq 0} F_{2n+1}x^n$$

where F_m is the m-th Fibonacci number (defined by $F_m = F_{m-1} + F_{m-2}$ with $F_0 = 0$ and $F_1 = 1$).

Proof. By Lemma 4, it suffices to provide the generating function $B(x)$ of the set $\mathcal{B}$, with respect to the semilength. Every shuffle $s \in \mathcal{B}$ is obtained by the grammar $\mathcal{B} \to a\bar{a}\mathcal{B} \mid b\mathcal{B}'\bar{b}\mathcal{B} \mid \varepsilon$ where $\mathcal{B}' \to a\bar{a}\mathcal{B}' \mid \varepsilon$. This induces the functional equation:

$$B(x) = 1 + xB(x) + xB(x)B'(x)$$

where $B'(x) = 1 + xB'(x)$ is the generating function for the set $\mathcal{B}'$.

A simple calculation provides the result. □

Now we consider the equivalence relation for the pattern $\pi = \bar{a}a$. Let $\mathcal{B}''$ be the set $\mathcal{B}'' = a\mathcal{B}\bar{a} \mid \varepsilon$ where $\mathcal{B}$ is defined at the beginning of this section.

Lemma 5 *The set $\mathcal{B}''$ and the set of $\bar{a}a$-equivalence classes of $\mathcal{S}$ are in length-preserving one-to-one correspondence.*

Proof. Let $s = s_1s_2\dots s_{2k}$ $(k \geq 1)$ be a nonempty shuffle in $\mathcal{S}\backslash\mathcal{B}''$ with $s_i \in \{a, \bar{a}, b, \bar{b}\}$ for $1 \leq i \leq 2k$. Let us prove that there exists a shuffle $s' \in \mathcal{B}''$ (with the same semilength as s) such that s and s' belong to the same $\bar{a}a$-equivalence class.

We define the shuffle s' as follows:

- we set $s'_1 = a$ and $s'_{2k} = \bar{a}$;

- all occurrences of the pattern $\bar{a}a$ in s are preserved in s';

- for i from 2 to $2k-1$ such that s_i does not belong to a pattern $\bar{a}a$, we distinguish two cases:

(i) if $w_b(s'_1 \dots s'_{i-1}) = (b\bar{b})^j$ for some $j \geq 0$, then we set $s'_i = b$,
(ii) if $w_b(s'_1 \dots s'_{i-1}) = (b\bar{b})^j b$ for some $j \geq 0$, then we set $s'_i = \bar{b}$.

For all i, $1 \leq i \leq 2k$ we have either $w_b(s'_1 \dots s'_i) = (b\bar{b})^j$ or $w_b(s'_1 \dots s'_i) = (b\bar{b})^j b$ for some $j \geq 0$. Moreover, we have $w_a(s'_1 \dots s'_i) = (a\bar{a})^\ell$ for some $\ell \geq 0$ so that a and $\bar{a}$ appear necessarily adjacent in s' in an occurrence $\bar{a}a$, except for $s'_1 = a$ and $s'_{2k} = \bar{a}$. This means that s' belongs to $\mathcal{B}''$. Since the process preserves the occurrences of the pattern $\bar{a}a$, s and s' belong to the same $\bar{a}a$-equivalence class.

This process ensures us that s' is the unique element in $\mathcal{B}''$ that lies into the same $\bar{a}a$-equivalence class as s. □

Theorem 4 *The generating function for the set of $\bar{a}a$-equivalence classes in $\mathcal{S}$ with respect to the semilength is given by*

$$\frac{1-2x}{1-3x+x^2} = \sum_{n \geq 0} F_{2n-1} x^n$$

where F_m is the m-th Fibonacci number (with $F_{-1} = 1$).

Proof. By Lemma 5, it suffices to provide the generating function for $\mathcal{B}'' = a\mathcal{B}\bar{a} \mid \varepsilon$ which is $B''(x) = xB(x) + 1 = \frac{1-2x}{1-3x+x^2}$. □

5 Equivalence modulo $\pi \in \{aa, \bar{a}\bar{a}, bb, \bar{b}\bar{b}\}$

The results for $\pi \in \{\bar{a}\bar{a}, bb, \bar{b}\bar{b}\}$ are deduced from the ones for $\pi = aa$ by using the classical symmetries.

Let $\mathcal{C}$ be the subset of $\mathcal{S}$ defined by the grammar $\mathcal{C} \to \mathcal{C}' \mid \mathcal{C}'b\mathcal{C}'\bar{b}\mathcal{C}$ where $\mathcal{C}' \to aa\bar{a}\mathcal{C}'\bar{a}\mathcal{C}' \mid a(\mathcal{C}'\backslash\varepsilon)\bar{a}\mathcal{C}' \mid \varepsilon$. In fact, the set $\mathcal{C}'$ consists of Dyck words in $\mathcal{D}_a$ such that every occurrence of the letter a is contiguous with another occurrence of a.

Lemma 6 *The set $\mathcal{C}$ and the set of aa-equivalence classes of $\mathcal{S}$ are in length-preserving one-to-one correspondence.*

Proof. Let $s = s_1 s_2 \dots s_{2k}$ ($k \geq 1$) be a shuffle in $\mathcal{S}\backslash\mathcal{C}$ with $s_i \in \{a, \bar{a}, b, \bar{b}\}$ for $1 \leq i \leq 2k$. Let us prove that there exists a shuffle $s' \in \mathcal{C}$ (with the same semilength as s) such that s and s' belong to the same class. We define the word s' by performing the following process on s:

- all occurrences of the pattern aa in s are preserved in s';

- for i from 1 to $2k$ such that s_i does not belong to a pattern aa, we distinguish three cases:

(*i*) If $w_a(s'_1s'_2\dots s'_{i-1}) \in \mathcal{D}_a$ and $w_b(s'_1s'_2\dots s'_{i-1}) = (b\bar{b})^j b$ for some $j \geq 0$, then we set $s'_i = \bar{b}$;

(*ii*) If $w_a(s'_1s'_2\dots s'_{i-1}) \in \mathcal{D}_a$ and $w_b(s'_1s'_2\dots s'_{i-1}) = (b\bar{b})^j$ for some $j \geq 0$, then we set $s'_i = b$;

(*iii*) If $w_a(s'_1s'_2\dots s'_{i-1}) \notin \mathcal{D}_a$ then $s'_i = \bar{a}$.

For all i, $1 \leq i \leq 2k$ we have either $w_b(s'_1s'_2\dots s'_i) = (b\bar{b})^j$ or $w_b(s'_1s'_2\dots s'_i) = (b\bar{b})^j b$ for some $j \geq 0$; $w_a(s'_1s'_2\dots s'_i)$ is a prefix of a Dyck word in $\mathcal{D}_a$, $w_a(s'_1s'_2\dots s'_{2k}) \in \mathcal{D}_a$ and $w_b(s'_1s'_2\dots s'_{2k}) = (b\bar{b})^j$ for some $j \geq 0$. So, the word s' is a shuffle in $\mathcal{S}$. Moreover, any occurrence of the letter a in s' is always contiguous with another occurrence of a. Let $i_1 \geq 1$ (resp. $i_2 > i_1$) be the position of the leftmost b (resp. $\bar{b}$) in s'; then the prefix $w_a(s'_1s'_2\dots s'_{i_1-1})$ (resp. $w_a(s'_1s'_2\dots s'_{i_2-1})$) lies into $\mathcal{D}_a$ which implies that the shuffle s' is of the form $\alpha b \beta \bar{b} \gamma$ with α and β in $\mathcal{C}'$ and γ in $\mathcal{C}$; thus we have $s' \in \mathcal{C}$.

The process preserves the positions of the occurrences of aa. So, s and s' belong to the same aa-equivalence class.

The proof for the unicity of s' in $\mathcal{C}$ is obtained *mutatis mutandis* from the proof of Lemma 3. □

Theorem 5 *The generating function for the set of aa-equivalence classes in $\mathcal{S}$ with respect to the semilength is given by*

$$\frac{1+x-\sqrt{1-2x-3x^2}}{2x^2+3x-1+\sqrt{1-2x-3x^2}}.$$

This sequence corresponds to the values of even ranks in A191385 ([14]), and the first values for $n \geq 1$ are $1, 2, 5, 12, 31, 81, 216, 583, 1590$.

Proof. By Lemma 6, it suffices to provide the generating function $C(x)$ of the set $\mathcal{C}$, with respect to the semilength. Every nonempty shuffle $s \in \mathcal{C}$ is obtained by the grammar $\mathcal{C} \to \mathcal{C}' \mid \mathcal{C}'b\mathcal{C}'\bar{b}\mathcal{C}$ where $\mathcal{C}' \to aa\bar{a}\mathcal{C}'\bar{a}\mathcal{C}' \mid a(\mathcal{C}'\backslash\varepsilon)\bar{a}\mathcal{C}' \mid \varepsilon$. Let $C'(x)$ be the generating function for the set $\mathcal{C}'$. From the above grammar, we deduce the functional equation:

$$C'(x) = x^2C'(x)^2 + x(C'(x)-1)C'(x) + 1$$

and

$$C(x) = C'(x) + xC'(x)^2C(x)$$

which provides $C'(x) = \frac{x+1-\sqrt{1-2x-3x^2}}{2x(x+1)}$ and $C(x) = \frac{1+x-\sqrt{1-2x-3x^2}}{2x^2+3x-1+\sqrt{1-2x-3x^2}}$. □

6 Equivalence modulo $\pi \in \{ab, ba, \bar{a}\bar{b}, \bar{b}\bar{a}\}$

The results for $\pi \in \{ba, \bar{a}\bar{b}, \bar{b}\bar{a}\}$ are deduced from the ones for $\pi = ab$ by using the classical symmetries.

Let $\mathcal{E}$ be the set of shuffles in $\mathcal{S}$ defined by the grammar $\mathcal{E} \to \mathcal{E}' \mid \mathcal{E}'b\mathcal{E}'\bar{b}\mathcal{E}$ where $\mathcal{E}' \to ab\mathcal{E}'\bar{b}\mathcal{E}'\bar{a}\mathcal{E}' \mid \varepsilon$. In fact, the set $\mathcal{E}'$ consists of shuffles s in $\mathcal{S}$ such that every occurrence of a letter $x \in \{a, b\}$ appears in a pattern ab, and there is no matchings $(a, \bar{a})$ and $(b, \bar{b})$ such that $ab\bar{a}\bar{b}$ appears in s.

Lemma 7 *The set $\mathcal{E}$ and the set of ab-equivalence classes of $\mathcal{S}$ are in length-preserving one-to-one correspondence.*

Proof. Let $s = s_1s_2\dots s_{2k}$ $(k \geq 1)$ be a shuffle in $\mathcal{S}\backslash\mathcal{E}$ with $s_i \in \{a, \bar{a}, b, \bar{b}\}$ for $1 \leq i \leq 2k$. Let us prove that there exists a shuffle $s' \in \mathcal{E}$ (with the same semilength as s) such that s and s' belong to the same class. We define the shuffle s' by performing the following process on s:

- all occurrences of the pattern ab in s are preserved in s';

- for i from 1 to $2k$ such that s_i does not belong to a pattern ab, we distinguish two cases:

(*i*) If $w_b(s'_1s'_2\dots s'_{i-1}) \in \mathcal{D}_b$ and $w_a(s'_1s'_2\dots s'_{i-1}) \in \mathcal{D}_a$, then we set $s'_i = b$;

(*ii*) If $w_b(s'_1s'_2\dots s'_{i-1}) \notin \mathcal{D}_b$ or $w_a(s'_1s'_2\dots s'_{i-1}) \notin \mathcal{D}_a$ then there is a letter $x \in \{a, b\}$ which is not matched in the prefix $s'_1s'_2\dots s'_{i-1}$ (we choose x the rightmost possible). If $x = a$, we set $s'_i = \bar{a}$, otherwise we set $s'_i = \bar{b}$.

Now, let us prove that this process produces a shuffle in $\mathcal{E}$. We distinguish two cases.

- Let us assume that any letter $x \in \{a, b\}$ belongs to an occurrence ab in s'. If $s'_i \notin \{a, b\}$ then $s'_i = \bar{a}$ (resp. $s'_i = \bar{b}$) whenever the rightmost non-matched $x \in \{a, b\}$ in $s'_1s'_2\dots s'_{i-1}$ is $x = a$ (resp. $x = b$). So, $s'_1s'_2\dots s'_{i-1}s'_i$ can be written either $ab\alpha\bar{b}\beta\bar{a}$ or $ab\alpha\bar{b}$ according to the value of s'_i ($\bar{a}$ or $\bar{b}$), such that $w_a(\alpha) \in \mathcal{D}_a$, $w_a(\beta) \in \mathcal{D}_a$, $w_b(\alpha) \in \mathcal{D}_b$ and $w_b(\beta) \in \mathcal{D}_b$. Moreover, α and β also satisfy the property that any $x \in \{a, b\}$ belongs to an occurrence ab. Then, s' can be generated by the grammar $\mathcal{E}' \to ab\mathcal{E}'\bar{b}\mathcal{E}'\bar{a}\mathcal{E}' \mid \varepsilon$.

- Now, let us assume that there exists an occurrence b that does not lie in a pattern ab (the process ensures that any occurrence of the letter a lies into a pattern ab in s'). We choose the leftmost b with this property. Then, s' can be written $s' = \alpha b\beta\bar{b}\gamma$ where $\alpha \in \mathcal{E}'$, $\beta \in \mathcal{E}'$ and $\gamma \in \{a, b, \bar{a}, \bar{b}\}^*$. By induction, we have $\gamma \in \mathcal{E}$ and s' can be generated by the grammar $\mathcal{E} \to \mathcal{E}'b\mathcal{E}'\bar{b}\mathcal{E}$.

Considering the two cases, s' can be generated by the grammar $\mathcal{E} \to \mathcal{E}' \mid \mathcal{E}' b \mathcal{E}' \bar{b} \mathcal{E}$ where $\mathcal{E}' \to ab\mathcal{E}'\bar{b}\mathcal{E}'\bar{a}\mathcal{E}' \mid \varepsilon$.

It is clear that two shuffles s and s' in $\mathcal{E}'$ lying into the same ab-equivalence class are necessarily identical. Using a recursive argument, this induces the unicity of a shuffle in $\mathcal{E}$ for a given ab-equivalence class. □

Theorem 6 *The generating function for the set of ab-equivalence classes in $\mathcal{S}$ with respect to the semilength is given by*

$$\frac{-2\sqrt{3}\sin\left(1/3\ \arcsin\left(3/2\sqrt{3}x\right)\right)}{4\left(\sin\left(1/3\ \arcsin\left(3/2\sqrt{3}x\right)\right)\right)^2 - 3x}.$$

This is the sequence A138164 in [14], and the first values for $n \geq 1$ are $1, 2, 4, 9, 20, 47, 109, 262, 622$.

Proof. By Lemma 7, it suffices to provide the generating function $E(x)$ of the set $\mathcal{E}$, with respect to the semilength. Every nonempty shuffle $s \in \mathcal{E}$ is obtained by the grammar $\mathcal{E} \to \mathcal{E}' \mid \mathcal{E}' b \mathcal{E}' \bar{b} \mathcal{E}$ where $\mathcal{E}' \to ab\mathcal{E}'\bar{b}\mathcal{E}'\bar{a}\mathcal{E}' \mid \varepsilon$.

Let $E'(x)$ be the generating function for the set $\mathcal{E}'$. From the above grammar, we deduce the functional equation

$$E'(x) = 1 + x^2 E'(x)^3$$

and

$$E(x) = E'(x) + xE'(x)^2 E(x).$$

The generating function $E'(x)$ is known (see A001764 in [14]) and given by

$$E'(x) = \frac{2\sqrt{3}\sin\left(1/3\ \arcsin\left(3/2\sqrt{3}x\right)\right)}{3x}.$$

A simple calculation gives

$$E(x) = \frac{-2\sqrt{3}\sin\left(1/3\ \arcsin\left(3/2\sqrt{3}x\right)\right)}{4\left(\sin\left(1/3\ \arcsin\left(3/2\sqrt{3}x\right)\right)\right)^2 - 3x}.$$

□

7 Going further

Our study focused on the enumeration of the π-equivalence classes of rooted planar maps in shuffle representation for a pattern π of length at most two not in $\{a\bar{b}, \bar{a}b, b\bar{a}, \bar{b}a\}$. Since the pattern $a\bar{b}$ does not occur in any shuffle of a rooted planar map, only three patterns are left as open enumeration problems.

The positions of the occurrences of the pattern π involve our equivalence relation. So, it would be interesting to study the weaker equivalence relation where two maps are equivalent when they have the same number of occurrences of a given pattern in their shuffle representations.

More generally, such a study could be made on maps using other representations and patterns.

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