# A MULTIDIMENSIONAL UNFOLDING LATENT TRAIT MODEL FOR BINARY DATA

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**Abstract**

We introduce a multidimensional latent trait model for binary data with non-monotone item response functions. We assume that the conditional probability of endorsing an item is a normal probability density function, and that the latent traits are normally distributed. The model yields closed form expressions for the moments of the multivariate Bernoulli (MVB) distribution. As a result, cell probabilities can be computed also in closed form, regardless of the dimensionality of the latent traits. The model is an ideal point model in the sense that a respondent –precisely at the ideal point (the mode of the item response function)- endorses the item with probability one.

**Keywords**

item response theory, categorical data analysis

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## 1. Introduction

Consider $n$ random variables $\mathbf{Y} = (Y_1, \cdots, Y_n)'$ each with two possible outcomes. Without loss of generality we may assign the values {0, 1} to their outcomes so that each $Y_i$ is Bernoulli and the joint distribution of $\mathbf{Y}$ is multivariate Bernoulli (MVB: Teugels, 1990; Maydeu-Olivares & Joe, in press[a]). Latent trait models are a popular class of MVB models in the Social Sciences. Any latent trait model for binary data can be written as (Bartholomew & Knott, 1999)

$$\Pr\left(\bigcap_{i=1}^{n}(Y_i = y_i)\right) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left\{\prod_{i=1}^{n}\left[\Pr\left(Y_i = 1|\boldsymbol{\eta}\right)\right]^{y_i}\left[1 - \Pr\left(Y_i = 1|\boldsymbol{\eta}\right)\right]^{1-y_i}\right\}\gamma_p(\boldsymbol{\eta})\,d\boldsymbol{\eta} \quad (1)$$

where we use $\Pr\left(\bigcap_{i=1}^{n}(Y_i = y_i)\right)$ to denote the probability of observing each of the possible $2^n$ binary patterns, and $\gamma_p(\boldsymbol{\eta})$ denotes the probability density function of a *p*-dimensional vector of continuous *unobserved* latent traits $\boldsymbol{\eta}$.

Latent trait models are used to reduce dimensionality. But the latent traits are also often used to represent unobserved psychological characteristics of the respondents of which the binary variables are indicators (proxies). In the psychological literature, latent trait models such as (1) are generally referred to as item response theory (IRT) models, and the function $\Pr(Y_i = 1|\boldsymbol{\eta})$ is denoted item response function. Many latent trait models for binary as well as for polytomous data have been proposed. For a good overview of these models see van der Linden and Hambleton (1997). However, due to the difficulty in evaluating the multidimensional integral in (1) most latent trait models proposed to date assume a single latent trait. Obviously, unidimensional latent trait models (i.e., $p = 1$) are less likely to be able to yield a good fit in applications than multidimensional models (McDonald, 1999). The most widely used multidimensional latent trait models is the normal ogive model (e.g., McDonald, 1997). This model assumes that the item response function is a standard normal distribution function evaluated at $\alpha_i + \boldsymbol{\beta}_i'\boldsymbol{\eta}$, that is $\Pr(Y_i = 1|\boldsymbol{\eta}) = \Phi_1\left(\alpha_i + \boldsymbol{\beta}_i'\boldsymbol{\eta}\right)$, and that the density of the latent traits is multivariate standard normal with correlation matrix $\boldsymbol{\Psi}$, i.e., $\gamma_p(\boldsymbol{\eta}) = \phi_p(\boldsymbol{\eta} : \mathbf{0}, \boldsymbol{\Psi})$.

Like the normal ogive model, most latent trait models for binary data use monotonically increasing item response functions. However, some social scientist (e.g. Andrich, 1996; Roberts, 1995; van Schuur & Kiers, 1994) have argued following Coombs (1964) that if the psychological mechanism by which individuals respond to attitude and preference items is a proximity mechanism (i.e., the closer the individual and the item are on the latent trait continua the higher the probability of the individual endorsing the item) non-monotonic item response functions should be used instead. Models based on this assumption of a proximity psychological response mechanism are generally referred to as unfolding models (see van der Linden and Hambleton, 1997: Part 5) and also as ideal point models. Yet, to our knowledge all unfolding IRT models proposed to date assume a unidimensional latent trait.

Here we propose a new multidimensional latent trait model with non-monotone item response functions. In this model the item response function is obtained by using a normal probability density function as link function instead of a normal cumulative function as in the normal ogive model. Accordingly, we use the term *normal PDF model* to denote our new model. We show that under the normal PDF model the joint moments of the MVB distribution have a closed form. Since in the MVB distribution there is a one-to-one linear map between the set of joint moments and the set of cell probabilities (Teugels, 1990), the estimation of this model need not involve multidimensional integration. This is a very attractive feature of the model. The strengths and limitations of the normal PDF model will be illustrated in a series of applications where we compare the fit of this model against the fit of a multidimensional normal ogive model.

## 2. The normal PDF model

Under the normal PDF model cell probabilities are given by (1) where as in the normal ogive model we shall assume that

$$\gamma_p(\boldsymbol{\eta}) = \phi_p(\boldsymbol{\eta} : \mathbf{0}, \boldsymbol{\Psi}). \tag{2}$$

where $\boldsymbol{\Psi}$ is a correlation matrix. However, in the normal PDF model we assume that

$$\Pr(Y_i = 1|\boldsymbol{\eta}) = \sqrt{2\pi}\phi_1(\alpha_i + \boldsymbol{\beta}_i'\boldsymbol{\eta} : 0,1), \tag{3}$$

where $\boldsymbol{\beta}_i' = (\beta_{i1}, \cdots, \beta_{ip})$. In (3), $\sqrt{2\pi}$ is a constant used to ensure that (3) takes the value of one for some value of the latent traits $\boldsymbol{\eta}$. Thus, according to the model if the respondent and item positions coincide in the space of the latent traits, the respondent will endorse the item with probability one.

When $p = 1$, the item response function (3) reaches its maximum at $\eta = -\frac{\alpha_i}{\beta_i}$ and has two inflexion points $\eta = \frac{\pm 1 - \alpha_i}{\beta_i}$. In general, $p \geq 1$, (3) satisfies

$$\begin{aligned} \sqrt{2\pi}\phi_1\left(\alpha_i + \boldsymbol{\beta}_i'\boldsymbol{\eta}\right) &= \sqrt{2\pi}\phi_1\left(-\alpha_i - \boldsymbol{\beta}_i'\boldsymbol{\eta}\right) \\ \sqrt{2\pi}\phi_1\left(-\alpha_i + \boldsymbol{\beta}_i'\boldsymbol{\eta}\right) &= \sqrt{2\pi}\phi_1\left(\alpha_i - \boldsymbol{\beta}_i'\boldsymbol{\eta}\right) \end{aligned}. \tag{4}$$

To resolve this indeterminacy we estimate the model parameters with the restriction that the intercepts $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)'$ be negative.

Let $\boldsymbol{B} = (\boldsymbol{\beta}_1, \cdots, \boldsymbol{\beta}_n)'$ be a $n \times p$ matrix of regression slopes, and $\boldsymbol{\Sigma} = \mathbf{I}_n + \boldsymbol{B\Psi B}'$. Furthermore, let $\mathbf{s}$ be any subset of $k$ Bernoulli variables. In the MVB distribution, the $k^{\text{th}}$ joint moment involving the variables in $\mathbf{s}$ is $\mu_s = E\left[\prod_{i\in\mathbf{s}} Y_i\right] = \Pr\left[\bigcap_{i\in\mathbf{s}} (Y_i = 1)\right]$ (Teugels, 1990). We show in the Appendix that $\mu_{\mathbf{s}}$ have the following closed form solution under the normal PDF model

$$\begin{aligned} \mu_{\mathbf{s}} &= \Pr\left[\bigcap_{i\in\mathbf{s}} (Y_i = 1)\right] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\prod_{i\in\mathbf{s}} \varphi\left(\alpha_i + \boldsymbol{\beta}_i'\boldsymbol{\eta}\right)\right] \phi_p\left(\boldsymbol{\eta} : \mathbf{0}, \boldsymbol{\Psi}\right) d\boldsymbol{\eta} = \\ &= (2\pi)^{\frac{k}{2}} \phi_k\left(\boldsymbol{\alpha}_{\mathbf{s}} : \mathbf{0}, \boldsymbol{\Sigma}_{\mathbf{s}}\right) \end{aligned} \tag{5}$$

where $\boldsymbol{\alpha}_{\mathbf{s}}$ and $\boldsymbol{\Sigma}_{\mathbf{s}}$ denote a $k \times 1$ vector and a $k \times k$ matrix, respectively, obtained by taking the appropriate rows and columns of $\boldsymbol{\alpha}$ and $\boldsymbol{\Sigma}$. Thus, the joint moments of the MVB distribution under this model are obtained by evaluating the $k$-dimensional normal density (5). Since there is a one to one linear correspondence between the set of joint moments of the MVB distribution and the set of cell probabilities (Teugels, 1990; Maydeu-Olivares & Joe, in press[a]) under this model, the cell probabilities (1) can be computed also in closed form. To our knowledge, this is the only multidimensional IRT model for binary variables besides the linear model (Maydeu-Olivares, in press[b]) with this attractive property.

To identify the model it suffices to consider univariate and bivariate MVB moments. Identification restrictions remain unchanged when higher order moments are considered. Therefore, the identification conditions for the normal PDF model are identical to those in the normal ogive model (see McDonald, 1999). Thus, when $p = 1$, all parameters of the normal PDF model are identified. When $p > 1$ a model with minimal identification restrictions (i.e., an unrestricted model) is obtained by setting $\boldsymbol{\Psi} = \mathbf{I}$ and to solve the rotational indeterminacy of $\mathbf{B}$, we let $\mathbf{B}$ be a low echelon matrix. That is, we let $\beta_{hl} = 0$, $l = 1, ..., p$; $h = 1, ..., l - 1$. In the multidimensional case, after $\hat{\boldsymbol{B}}$ is estimated from a sample it may be rotated orthogonally or obliquely to help interpreting the model. Alternatively, based on some a priori information about the data, a researcher may wish to fit a restricted model in which some elements of $\boldsymbol{\alpha}$, $\mathbf{B}$, and $\boldsymbol{\Psi}$ are subject to normalization, exclusion or equality constraints.

## 3. Limited information estimation and testing

Because MVB moments have a closed form solution under this model, estimation methods that minimize a discrepancy function between sample and expected moments are a natural choice. However, in most latent trait applications the number of binary variables is large and the observed contingency tables are very sparse. As a result, high order sample moments may be very poorly estimated. In contrast, univariate and bivariate sample moments can be reasonably estimated in very small samples regardless of $n$. Limited information procedures based on univariate and bivariate information are the most widely used approaches to estimate the multidimensional normal ogive model (see Christoffersson, 1975; Muthén, 1978, 1993; Maydeu-Olivares, 2001[b]). Limited information estimation methods yield as a side product limited information goodness-of-fit tests. Maydeu-Olivares and Joe (in press[a]) have recently provided a unified treatment of limited and full information estimation and goodness-of-fit testing methods for MVB models. They show that bivariate information methods have high efficiency. They also show that bivariate information tests have more precise Type I errors and are asymptotically more powerful in large and sparse binary tables than full information goodness-of-fit tests such as Pearson's $X^2$.

Limited information methods using only univariate and bivariate moments will be employed here to estimate and evaluate the goodness-of-fit of the normal PDF model. From (5), the univariate and bivariate moments of the MVB distribution under the normal PDF model are

$$\pi_i = E\left[Y_i\right] = \Pr\left(Y_i = 1\right) = \sqrt{2\pi}\phi_1\left(\alpha_i : 0, 1 + \boldsymbol{\beta}_i'\boldsymbol{\beta}_i\right), \tag{6}$$

$$\pi_{ij} = E\left[Y_i Y_j\right] = \Pr\left[\left(Y_i = 1\right) \cap \left(Y_j = 1\right)\right] = 2\pi\phi_2\left(\boldsymbol{\alpha}_{ij} : \mathbf{0}, \mathbf{I} + \boldsymbol{B}_{ij}'\boldsymbol{\Psi}\boldsymbol{B}_{ij}\right), \tag{7}$$

where $\boldsymbol{\alpha}_{ij} = \left(\alpha_i, \alpha_j\right)'$ and $\boldsymbol{B}_{ij} = \left(\boldsymbol{\beta}_i \mid \boldsymbol{\beta}_j\right)$ is a $p \times 2$ matrix. To estimate the model, we collect all $n$ univariate moments $\dot{\boldsymbol{\pi}}_1$ and all $\frac{n(n-1)}{2}$ bivariate moments $\dot{\boldsymbol{\pi}}_2$ in $\dot{\boldsymbol{\pi}}' = \left(\dot{\boldsymbol{\pi}}_1{}', \dot{\boldsymbol{\pi}}_2{}'\right)$, where $\dot{\boldsymbol{\pi}}_1{}' = \left(\pi_1, \cdots, \pi_n\right)$ and $\dot{\boldsymbol{\pi}}_2{}' = \left(\pi_{12}, \cdots, \pi_{1n}, \pi_{23}, \cdots, \pi_{2n}, \cdots \pi_{n-1,n}\right)$. We use $\dot{\boldsymbol{\pi}}\left(\boldsymbol{\theta}\right)$ to denote the restrictions (6) and (7) imposed by model on $\dot{\boldsymbol{\pi}}$, where $\boldsymbol{\theta}$ denotes a $q$-dimensional vector containing all mathematically independent elements in $\boldsymbol{\alpha}$, $\mathbf{B}$, and $\boldsymbol{\Psi}$. Thus, the degrees of freedom available for testing are $r = \frac{n(n+1)}{2} - q$. We assume that $\boldsymbol{\Delta} = \frac{\partial \dot{\boldsymbol{\pi}}}{\partial \boldsymbol{\theta}'}$ is of full rank so that the model is locally identified. Furthermore, let $N$ denote sample size and $\dot{\mathbf{p}}$ be the sample counterpart of $\dot{\boldsymbol{\pi}}$. Then, the model parameters can be estimated by minimizing

$$F = \mathbf{e}'\hat{\mathbf{W}}\mathbf{e}\,. \tag{8}$$

where $\mathbf{e} = \left(\dot{\mathbf{p}} - \dot{\boldsymbol{\pi}}\left(\boldsymbol{\theta}\right)\right)$, and $\hat{\mathbf{W}}$ is a matrix converging in probability to $\mathbf{W}$, a positive definite matrix. Letting $\boldsymbol{\Gamma}$ be the asymptotic covariance matrix of $\sqrt{N}\dot{\mathbf{p}}$, obvious choices of $\hat{\mathbf{W}}$ in (8) are $\hat{\mathbf{W}} = \hat{\boldsymbol{\Gamma}}^{-1}$ (weighted least squares, WLS), $\hat{\mathbf{W}} = \left(\mathrm{Diag}\left(\hat{\boldsymbol{\Gamma}}\right)\right)^{-1}$ (diagonally weighted least squares, DWLS), and $\hat{\mathbf{W}} = \mathbf{I}$ (unweighted least squares, ULS).

In the Social Sciences this general estimation framework is denoted as weighted least squares for moment structures (see Browne, 1984; Browne and Arminger, 1995; Satorra and Bentler, 1994). Maydeu-Olivares and Joe (in press[a]; see also Maydeu-Olivares, 200[b]), provide a unified framework of full and limited information weighted least squares estimation methods for MVB models. Large sample properties for the parameter estimates, standard errors and goodness of fit tests of the model can be readily obtained using standard theory for the estimation of moment structures. Letting $\mathbf{H} = \left(\boldsymbol{\Delta}'\mathbf{W}\boldsymbol{\Delta}\right)^{-1}\boldsymbol{\Delta}'\mathbf{W}$, the estimator $\hat{\boldsymbol{\theta}}$ obtained by minimizing (8) is consistent and

$$\sqrt{N}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right) \xrightarrow{d} N\left(\mathbf{0}, \mathbf{H}\boldsymbol{\Gamma}\mathbf{H}'\right) \tag{9}$$

$$\sqrt{N}\hat{\mathbf{e}} \xrightarrow{d} N\left(\mathbf{0}, \left(\mathbf{I} - \mathbf{H}\right)\boldsymbol{\Gamma}\left(\mathbf{I} - \mathbf{H}\right)\right). \tag{10}$$

where $\hat{\mathbf{e}} = \left(\dot{\mathbf{p}} - \dot{\boldsymbol{\pi}}\left(\hat{\boldsymbol{\theta}}\right)\right)$ denotes the univariate and bivariate residuals. These residuals can be divided by their standard error to obtain standardized residuals that are asymptotically standard normal. Also, in unrestricted multidimensional solutions where the columns of $\hat{\boldsymbol{B}}$ have been rotated, standard errors for the rotated loadings can be obtained using formulae given by Maydeu-Olivares (2001[b]) for rotated normal ogive model estimates.

Now, from standard theory, $T := N\hat{F} \overset{a}{=} \mathbf{e}'\mathbf{W}(\mathbf{I}-\mathbf{H})\mathbf{e}$, where $\overset{a}{=}$ denotes asymptotic equality. Thus, in general,

$$T \xrightarrow{d} \sum_{i=1}^{r} \lambda_i \chi^2 , \tag{11}$$

where the $\chi^2$'s are independent chi-square variables with one degree of freedom and the $\lambda_i$'s are the non-null eigenvalues of $\mathbf{M} = \mathbf{W}(\mathbf{I}-\boldsymbol{\Delta}\mathbf{H})\boldsymbol{\Gamma}$. In particular, when $\hat{\mathbf{W}} = \hat{\boldsymbol{\Gamma}}^{-1}$, we obtain an estimator with minimum asymptotic variance among the class of estimators (8). In this special case (9) and (11) simplify to $\sqrt{N}\left(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right) \xrightarrow{d} N\left(\mathbf{0},\left(\boldsymbol{\Delta}\boldsymbol{\Gamma}^{-1}\boldsymbol{\Delta}\right)^{-1}\right)$ and $T \xrightarrow{d} \chi^2_r$. respectively. However, the use of $\hat{\mathbf{W}} = \hat{\boldsymbol{\Gamma}}^{-1}$ requires inverting a very large matrix. Thus, WLS estimation is not suitable for large applications.

When $\hat{\mathbf{W}} \neq \hat{\boldsymbol{\Gamma}}^{-1}$, following Satorra and Bentler (1994; see also Rao & Scott, 1987; Maydeu-Olivares, 2001[a], 2001[b]) to assess the goodness of fit of the model we may scale $T$ by its asymptotic mean, or we may adjust $T$ by its asymptotic mean and variance using the following two test statistics

$$\bar{T} = \frac{T}{\mathrm{Tr}(\mathbf{M})/r} \qquad \bar{\bar{T}} = \frac{T}{\mathrm{Tr}\left(\mathbf{M}^2\right)/r}. \tag{12}$$

In (12), $\bar{T}$ and $\bar{\bar{T}}$ denote the scaled (for mean) and adjusted (for mean and variance) test statistics. The former is referred to a chi-square distribution with $r$ degrees of freedom, whereas the latter is referred to a chi-square distribution with $d = \dfrac{\mathrm{Tr}(\mathbf{M})^2}{\mathrm{Tr}\left(\mathbf{M}^2\right)/r}$ degrees of freedom.

Here we shall estimate the normal PDF model by simply using $\hat{\mathbf{W}} = \mathbf{I}$ (i.e., ULS) where standard errors, standardized residuals, and goodness of fit tests will be computed via (9), (10) and (12) by evaluating $\boldsymbol{\Delta}$ and $\boldsymbol{\Gamma}$ at the estimated parameter values. This approach is very similar to the one implemented in the computer program NOHARM (Fraser & McDonald, 1988) which estimates the multidimensional normal ogive model also using ULS from univariate and bivariate moments. However, there are two differences between the present approach and the approach used in NOHARM (see Maydeu-Olivares, 2001[b]). The first difference is that in NOHARM estimation is performed in two stages to improve computational efficiency exploiting a separability of parameters that exist in the normal ogive model but not in the normal PDF model. The second difference is that to obtain standard errors and goodness of fit tests for NOHARM $\mathbf{\Gamma}$ is consistently estimated using sample proportions, whereas in our estimation of the normal PDF model $\mathbf{\Gamma}$ is consistently estimated by evaluating it at the estimated parameter values. This is because $\mathbf{\Gamma}$ depends on fourth order joint moments. These can be computed in closed form under the normal PDF model, but they require multivariate integration in the normal ogive model.

## 4. Applications

We present four applications where we compare the fit of the normal PDF model estimated using the limited information methods described against the fit of the normal ogive model estimated using NOHARM. In the first application, we model attitudes towards censorship. This is a typical application where the item stems suggest that it is plausible to assume that individuals use a proximity mechanism in responding the items. In applications like this one, the normal PDF model should provide a better fit than the normal ogive model. In the second application, given the item stems, a proximity mechanism in responding the items does not seem plausible. Rather, a priori, a model with monotonically increasing item response functions seems more reasonable for these items. The third application is used to illustrate that if the sample size is small, we may not be able to empirically distinguish between the normal PDF and normal ogive model, although the estimated item response functions appear quite distinct. Finally, the fourth application is used to illustrate that in some situations we may not empirically distinguish between these two models, even with large sample sizes, because their item response functions are very similar in the region of high density of respondents.

### 4.1 Attitudes towards censorship

In this example, we model a set of 223 observations collected by Roberts (1995) on 20 statements reflecting attitudes toward censorship[1]. The statements were originally published in Rosander and Thurstone (1931). Roberts (1995) asked the respondents to rate each statement using a 6 point scale ranging from "Strongly disagree" to "Strongly agree". Their responses were dichotomized (0 = disagree, 1 = agree) for this analysis.

In Table 1 we provide goodness of fit results for the normal PDF model

[1] The data is available at http://www.education.umd.edu/EDMS/tutorials/data.html.

applied to these data with 1, 2, and 3 latent traits. In this table, we also provide the goodness of fit results obtained for the normal ogive model using NOHARM. As can be seen in this table, the normal PDF model reproduces the univariate and bivariate moments of the data better than the normal ogive model. Two latent trait dimensions seem to be necessary to reproduce these data using the normal PDF model, whereas three dimensions seem to be necessary for the normal ogive model. To verify that two latent trait dimensions suffice to reproduce these data using the normal PDF we performed nested tests using the mean scaled test statistic (see Satorra & Bentler, 2001). The results shown in Table 2 suggest that two dimensions are sufficient to model these data.

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Insert Tables 1 and 2 about here

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In Figure 1 we provide a plot of the slope parameters estimated in the two dimensional solution. The plot suggests the following restricted solution: We let statements {1, 2, 8, 9, 11, 14, 16, 17, 20} be a function of the first latent trait only, we let statements {3, 7, 10, 15, 18, 19} be a function of the second latent trait only, and we let the remaining statements {4, 5, 9, 13, 14} be a function of both latent traits. The latent traits are allowed to be correlated. As shown in Table 1, this model also fits reasonably the univariate and bivariate moments of the data. However, the estimated correlation among the latent traits was -0.93 with a standard error of 0.04. Clearly, this is not an appealing solution from a substantive viewpoint.

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Insert Figure 1 about here

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Figure 1 suggest an alternative approach to modeling these data. In this figure, most items fall roughly on a straight line close to the latent trait 1 axis. High scores on this axis indicate an anti-censorship attitude and low scores on this axis a pro-censorship attitude. We can not meaningfully interpret the second latent trait. This, along with the results obtained for the restricted model, suggests that the second latent trait is just "noise" induced by some items which are not appropriate indicators of the first latent trait. To identify which items are poor indicators of the latent trait "attitude towards censorship" we shall inspect the standardized univariate and bivariate residuals obtained from fitting the one dimensional model. The five largest standardized residuals for the one dimensional solution are {-6.33, -4.84, 3.64, -3.14, -2.96} which correspond to the following univariate and bivariate residuals {(4), (12), (19,4), (5), (4,1)}. These residuals suggest that a one dimensional solution may fit the 17 statements remaining after deleting statements {4, 5, 12}. The goodness of fit indices for this model are: $\overline{T} = 145.01$ on 119 df, $p = 0.05$, and $\overline{\overline{T}} = 105.30$ on 86.41 df, $p = 0.08$. Interestingly, the $\beta$ estimate for item 16 is very low, 0.01, with a standard error of 0.06. Under a unidimensional normal PDF model this statement "Education of the public taste is preferable to censorship" is a poor indicator of attitude towards censorship. After removing this item, we fitted a one dimensional model for the remaining 16 items obtaining finally $\overline{T} = 133.64$ on 104 df, $p = 0.03$, and $\overline{\overline{T}} = 95.92$ on 74.65 df, $p = 0.05$. Thus, we have been able to identify a set of 16 items from the original set that can be used to measure attitude towards censorship. The para-meter estimates and standard errors for this final model are given in Table 3.

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Insert Table 3 and Figure 2 about here

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We provide in Figure 2 plots of the item response function for selected items. The plots in Figure 2 illustrate the versatility of the model. The item response function for item 9 corresponds to an item that is endorsed only by respondents with a pro-censorship view. The probability of endorsing this item is maximum for extreme pro-censorship respondents and the item response function in the region of high density of respondents is monotonically increasing. The plot for item 3 (“We must have censorship to protect the morals of young people”) on the other hand is non-monotonic. According to the model, the probability of endorsing the item is maximum for respondents with a moderately positive attitude towards censorship. The more anti-censorship the attitude the less likely the item is endorsed. But respondents with an extreme pro-censorship view are also less likely to endorse the item than respondents with an moderate pro-censorship view. They may not endorse the item because they believe that the morals of all people should be protected not only the morals of young people. Finally, the model can handle well “I do not have a clear opinion on the topic” items such as item 14. The probability of endorsing this item is maximum for respondents that are neither pro nor against-censorship and minimum for respondents with extreme pro or against-censorship views.

Given this discussion, it is not surprising that, as shown in Table 1, a model with monotonically increasing item response functions, such as the normal ogive model, fails to fit adequately these 16 items.

**4.2 Satisfaction with Life**

Edward Diener kindly provided the responses of 7167 individuals from 42 countries to the Satisfaction with Life Scale (Diener, Emmons, Larsen & Griffin, 1985). The questionnaire consists of these 5 items

1. In most ways my life is close to my ideal.
2. The conditions of my life are excellent.
3. I am satisfied with my life.
4. So far I have gotten the important things I want in life.
5. If I could live my life over, I would change almost nothing.

which are to be rated on a 7 point scale ranging from "Strongly disagree" to "Strongly agree". For this analysis we discarded those individuals who chose the middle category “neither agree nor disagree” for any of the items, and dichotomized the responses of the remaining individuals (0 = disagree, 1 = agree). The resulting sample size was 4073.

Of these item stems only the first and third may be consistent with the notion of a proximity response mechanism. Thus, a priori, we expect a model with monotonic curves to fit better these data than the normal PDF model. In Table 4 we provide goodness of fit results for one and two dimensional normal PDF and normal ogive models fitted to these data. We also include in this table the results of a restricted two dimensional model suggested by McDonald (1999). In this model, the first three items are indicators of present satisfaction with life, the two last items measure past satisfaction with life, and both latent dimensions are correlated. As can be seen in this table, the two dimensional restricted normal ogive model provides a good fit to these data given the sample size. On the other hand, all the normal PDF models provide an extremely poor fit to these data.

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Insert Table 4 about here

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**4.3 Attitudes of morality and equality**

Jöreskog and Sörbom (1996) provide data on 200 Swedish school children in grade 9 who used a 4-point scale (“unimportant”, “not important”, “important”, and “very important”) to rate how important to them were

1. human rights
2. equal conditions for all people
3. racial problems
4. equal value of all people
5. euthanasia
6. crime and punishment
7. conscientious objectors
8. guilt and bad conscience

Their responses were dichotomized for this analysis (0 = not important, 1 = important)[2].

We fitted one and two dimensional normal PDF and normal ogive models to these data. We also fitted a restricted two dimensional model suggested in Jöreskog and Sörbom (1996). In this restricted model items {1,2,4,5} are taken as indicators of the latent trait "equality", and items {3,6,7,8} are taken as indicators of the latent trait "morality". Both latent traits are correlated. A priori, we do not believe that a model based on a proximity mechanism is suitable for these data. Rather, we expected the normal ogive model with its monotonic item response functions to fit better these data. This is because a priori we expected that the higher the sense of morality and equality of respondents the more likely they would endorse these items. Yet, as can be seen in Table 5 all models provide a good fit to the data. Nested tests using the scaled test statistic revealed that for both the normal PDF and normal ogive model, neither the two dimensional unrestricted nor the two dimensional restricted solutions provide significant improvement over the one dimensional model. Furthermore, as seen in Table 5, the difference in fit between the one dimensional normal PDF and normal ogive models is negligible. Yet, the estimated item response functions of these two models are markedly different for most items. This is shown in Figure 3 where we provide plots for selected items.

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Insert Table 5 and Figure 3 about here

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If we can not choose between these models based on their fit to the univariate and bivariate moments, could we choose between them using full information statistics? To answer this question, we estimated both unidimensional models using full information maximum likelihood. We obtained $X^2$ = 230.68 and $G^2$ = 152.30 for the normal PDF model and $X^2$ = 234.10 and $G^2$ = 148.28 for the normal ogive model. The number of degrees of freedom is 239. Thus, although a priori we consider that the normal PDF model is not an appropriate model for these data, the fit of the model is not outperformed by the normal ogive model. A larger dataset would be needed to distinguish between these two models in this application.

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[2] Five respondents had out of range values for some of the items. Therefore the actual sample size used in the analysis was 195.

**4.4 Political action survey**

Jöreskog and Moustaki (2001) modeled the USA sample of the Political Action Survey. 1719 individuals responded to the 6 items of this questionnaire using the following categories "Strongly agree", "Agree", "Disagree", "Strongly disagree", "Do not know" and "No answer". After eliminating those cases with "Do not know" and "No answer" responses the total sample size is 1554. The data was dichotomized (0 = disagree, 1 = agree) for the present analysis.

We estimated one dimensional normal PDF and normal ogive models to these data. We obtained the following goodness of fit statistics for the normal PDF model: $T = 0.04$, $\bar{T} = 15.39$ on 9 df, $p = 0.08$, and $\bar{\bar{T}} = 13.72$ on 8.03 df, $p = 0.09$. For the normal ogive model estimated using NOHARM we obtained $T = 0.04$, $\bar{T} = 16.19$ on 9 df, $p = 0.06$, and $\bar{\bar{T}} = 11.9$ on 6.62 df, $p = 0.09$. Thus, both models yield a similar fit to the univariate and bivariate moments of these data even though the sample size is large in this case.

This occurs because in this dataset the proportion of respondents that endorse each item is in all cases very low: {0.08, 0.05, 0.04, 0.02, 0.02}. As a result, the item response functions of the normal PDF model are monotonically increasing in the area of high density of respondents, $\eta \in (-3,3)$, and they are very hard to distinguish from the item response functions of the normal ogive model. This is shown in Figure 3, where we plot the item response functions for the normal PDF and normal ogive models for the first four items.

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Insert Figure 4 about here

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In fact, the predictions of these two models are hard to distinguish even when all the information available in these data is employed. We estimated by full information maximum likelihood the two competing models. For the normal PDF model we obtained $X^2 = 44.84$ and $G^2 = 52.14$ on 51 df. For the normal ogive model we obtained $X^2 = 44.89$ and $G^2 = 53.04$ also on 51 df.

## 5. Conclusions

In some item response applications it is reasonable to assume that individuals use a proximity mechanism in responding the items. Latent trait models based on this assumption are generally denoted as unfolding or ideal point models. In applications, models with several latent traits may often be needed to reproduce the data adequately. We have introduced here a multidimensional unfolding model in which the item response function is modeled using a normal probability density function and the latent trait density is assumed to be multivariate normal. The model has the very appealing property of yielding closed form expressions for the moments of the multivariate Bernoulli distribution. As a result, cell probabilities under this model can be computed without resorting to numerical integration regardless of the number of traits involved. The model proposed is indeed an ideal point model in the sense that a respondent –precisely at the ideal point (the mode of the item response function)- endorses the item with probability one. A more general model with an additional parameter controlling the probability of the modal point is easily conceived. Such a model may deserve future investigation, though problems can be anticipated related to bounds on the probabilities and possibly identifiability. Further research should consider the extension of this model to the polytomous case.

We have seen that in applications where a proximity response mechanism is plausible the model indeed fits better than the multidimensional normal ogive model. The model seems particularly suited to model items of the type "I don't have a clear opinion on the topic". However, it is important to consider a priori whether a proximity response mechanism is plausible for the application of interest. This is because, as we have seen, when only a small sample is available the normal PDF and normal ogive models may be hard to distinguish even though their item response functions are quite distinct. Furthermore, even when large samples are available, if the proportion of respondents endorsing the items is either very low or very large for all items it may be hard to distinguish these two models model as their item response functions will coincide in the region of high density of respondents,.

Here, we have estimated the model simply by minimizing the sum of squared errors between the observed and expected univariate and bivariate moments of the MVB distribution. Limited information methods are an attractive option to estimate latent trait models in sparse binary tables. Maydeu-Olivares (2001[a]) reports a simulation study where 100 observations sufficed to obtain parameter estimates and standard errors with a relative bias of less than 10% in estimating a 6-dimensional latent trait model for 21 binary variables. Limited information methods offer two important additional benefits. On the one hand, they yield residuals that can be readily used to detect the source of misfit in poorly fitting models. On the other hand, they provide a goodness of fit test with a reasonable behavior in sparse tables. Maydeu-Olivares (2001[b]) found that the mean and variance adjusted test statistic employed here closely matched its reference chi-square distribution even in the extremely sparse tables of his simulation study.

# Appendix

To prove (5) we apply the change of variable $\mathbf{z} = \boldsymbol{\Psi}^{-\frac{1}{2}}\boldsymbol{\eta}$ to obtain $\boldsymbol{\eta} = \boldsymbol{\Psi}^{\frac{1}{2}}\mathbf{z}$ and $\frac{d\boldsymbol{\eta}}{d\mathbf{z}} = |\boldsymbol{\Psi}|^{\frac{1}{2}}$. Thus,

$$\begin{aligned}\Pr\left[\bigcap_{i\in\mathbf{s}}(Y_i = 1)\right] &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\phi_p\left(\boldsymbol{\eta} : \mathbf{0}, \boldsymbol{\Psi}\right)\left[\prod_{i\in\mathbf{s}}\sqrt{2\pi}\phi_1\left(\alpha_i + \boldsymbol{\beta}_i'\boldsymbol{\eta}\right)\right]d\boldsymbol{\eta} = \\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\phi_p\left(\mathbf{z} : \mathbf{0}, \mathbf{I}\right)\left[\prod_{i\in\mathbf{s}}\sqrt{2\pi}\phi_1\left(\alpha_i + \boldsymbol{\beta}_i'\boldsymbol{\Psi}^{\frac{1}{2}}\mathbf{z}\right)\right]d\mathbf{z} = \\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(2\pi)^{-\frac{p}{2}}\exp\left\{-\frac{1}{2}\left[\mathbf{z}'\mathbf{z} + \sum_{i\in\mathbf{s}}\left(\alpha_i + \boldsymbol{\beta}_i'\boldsymbol{\Psi}^{\frac{1}{2}}\mathbf{z}\right)^2\right]\right\}d\mathbf{z}\end{aligned}.$$

Now, inside the curled brackets we have

$$\begin{aligned}\mathbf{z}'\mathbf{z} + \sum_{i\in\mathbf{s}}\left(\alpha_i + \boldsymbol{\beta}_i'\boldsymbol{\Psi}^{\frac{1}{2}}\mathbf{z}\right)^2 &= \mathbf{z}'\mathbf{z} + \sum_{i\in\mathbf{s}}\alpha_i^2 + 2\sum_{i\in\mathbf{s}}\alpha_i\boldsymbol{\beta}_i'\boldsymbol{\Psi}^{\frac{1}{2}}\mathbf{z} + \sum_{i\in\mathbf{s}}\mathbf{z}'\boldsymbol{\Psi}^{\frac{1}{2}}\boldsymbol{\beta}_i\boldsymbol{\beta}_i'\boldsymbol{\Psi}^{\frac{1}{2}}\mathbf{z} = \\ &= \mathbf{z}'\left(I + \sum_{i\in\mathbf{s}}\boldsymbol{\Psi}^{\frac{1}{2}}\boldsymbol{\beta}_i\boldsymbol{\beta}_i'\boldsymbol{\Psi}^{\frac{1}{2}}\right)\mathbf{z} + 2\sum_{i\in\mathbf{s}}\alpha_i\boldsymbol{\beta}_i'\boldsymbol{\Psi}^{\frac{1}{2}}\mathbf{z} + \sum_{i\in\mathbf{s}}\alpha_i^2 = \\ &= (\mathbf{z} - \boldsymbol{\mu})'\,\mathbf{A}\,(\mathbf{z} - \boldsymbol{\mu}) + \mathbf{C}\end{aligned}$$

where

$$\mathbf{A} = \mathbf{I}_p + \sum_{i\in\mathbf{s}}\boldsymbol{\Psi}^{\frac{1}{2}}\boldsymbol{\beta}_i\boldsymbol{\beta}_i'\boldsymbol{\Psi}^{\frac{1}{2}} = \mathbf{I} + \boldsymbol{\Psi}^{\frac{1}{2}}\boldsymbol{B}_s'\boldsymbol{B}_s\boldsymbol{\Psi}^{\frac{1}{2}},$$

$$\boldsymbol{\mu} = \mathbf{A}^{-1}\boldsymbol{\Psi}^{1/2}\left(\sum_{i\in\mathbf{s}}\alpha_i\boldsymbol{\beta}_i\right) = \mathbf{A}^{-1}\boldsymbol{\Psi}^{1/2}\boldsymbol{B}_s'\boldsymbol{\alpha}_s,$$

$$\mathbf{C} = \sum_{i\in\mathbf{s}}\alpha_i^2 - \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu} = \boldsymbol{\alpha}_s'\boldsymbol{\alpha}_s - \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}$$

and $\boldsymbol{\alpha}_s$ and $\boldsymbol{B}_s$ are a $k \times 1$ vector and a $k \times p$ matrix obtained by selecting rows in $\boldsymbol{\alpha}$ and $\mathbf{B}$ according to $\mathbf{s}$. Thus,

$$\Pr\left[\bigcap_{i\in\mathbf{s}}(Y_i = 1)\right] = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(2\pi)^{-\frac{p}{2}}\exp\left\{-\frac{1}{2}\left[(\mathbf{z} - \boldsymbol{\mu})'\,\mathbf{A}\,(\mathbf{z} - \boldsymbol{\mu}) + \mathbf{C}\right]\right\}d\mathbf{z} = \frac{1}{|\mathbf{A}|^{1/2}}\exp\left(-\frac{1}{2}\mathbf{C}\right)$$

since $\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(2\pi)^{-\frac{p}{2}}|\mathbf{A}|^{1/2}\exp\left\{-\frac{1}{2}\left[(\mathbf{z} - \boldsymbol{\mu})'\,\mathbf{A}\,(\mathbf{z} - \boldsymbol{\mu})\right]\right\}d\mathbf{z} = 1$.

Finally, to prove (5) it suffices to show that

$$|\mathbf{A}| = |\boldsymbol{\Sigma}_s| \tag{A.1}$$

$$\mathbf{C} = \boldsymbol{\alpha}_s{}'\boldsymbol{\Sigma}_s^{-1}\boldsymbol{\alpha}_s\,, \tag{A.2}$$

where $\boldsymbol{\Sigma}_s = \mathbf{I}_k + \boldsymbol{B}_s\boldsymbol{\Psi}\boldsymbol{B}_s{}'$.

Now, to prove (A.1) we write $\mathbf{A} = \mathbf{I}_p + \mathbf{H}\mathbf{H}'$ with $\mathbf{H} = \boldsymbol{\Psi}^{\frac{1}{2}}\boldsymbol{B}_s{}'$. Therefore, $|\mathbf{A}| = |\mathbf{I}_p + \mathbf{H}\mathbf{H}'| = |\mathbf{I}_k + \mathbf{H}'\mathbf{H}| = \left|\mathbf{I}_k + \boldsymbol{B}_s\boldsymbol{\Psi}\boldsymbol{B}_s{}'\right| = |\boldsymbol{\Sigma}_g|$.

To prove (A.2), since $\mathbf{A}^{-1} = \mathbf{I}_p - \mathbf{H}\left(\mathbf{I}_k + \mathbf{H}'\mathbf{H}\right)\mathbf{H}' = \mathbf{I}_p - \boldsymbol{\Psi}^{\frac{1}{2}}\boldsymbol{B}_s{}'\boldsymbol{\Sigma}_s^{-1}\boldsymbol{B}_s\boldsymbol{\Psi}^{\frac{1}{2}}$ and $\boldsymbol{B}_s\boldsymbol{\Psi}\boldsymbol{B}_s{}' = \boldsymbol{\Sigma}_s - \mathbf{I}_k$,

$$\begin{aligned}
\mathbf{C} &= \boldsymbol{\alpha}_s{}'\boldsymbol{\alpha}_s - \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu} = \boldsymbol{\alpha}_s{}'\boldsymbol{\alpha}_s - \boldsymbol{\alpha}_s{}'\boldsymbol{B}_s\boldsymbol{\Psi}^{\frac{1}{2}}\mathbf{A}^{-1}\mathbf{A}\mathbf{A}^{-1}\boldsymbol{\Psi}^{\frac{1}{2}}\boldsymbol{B}_s{}'\boldsymbol{\alpha}_s = \\
&= \boldsymbol{\alpha}_s{}'\left(\mathbf{I}_k - \boldsymbol{B}_s\boldsymbol{\Psi}^{\frac{1}{2}}\mathbf{A}^{-1}\boldsymbol{\Psi}^{\frac{1}{2}}\boldsymbol{B}_s{}'\right)\boldsymbol{\alpha}_s = \boldsymbol{\alpha}_s{}'\left(\mathbf{I}_k - \boldsymbol{B}_s\boldsymbol{\Psi}\boldsymbol{B}_s{}' + \boldsymbol{B}_s\boldsymbol{\Psi}\boldsymbol{B}_s{}'\boldsymbol{\Sigma}_s^{-1}\boldsymbol{B}_s\boldsymbol{\Psi}\boldsymbol{B}_s{}'\right)\boldsymbol{\alpha}_s = \\
&= \boldsymbol{\alpha}_s{}'\left[\mathbf{I}_k + \boldsymbol{B}_s\boldsymbol{\Psi}\boldsymbol{B}_s{}'\left(-\mathbf{I}_k + \boldsymbol{\Sigma}_s^{-1}\boldsymbol{B}_s\boldsymbol{\Psi}\boldsymbol{B}_s{}'\right)\right]\boldsymbol{\alpha}_s = \boldsymbol{\alpha}_s{}'\left[\mathbf{I}_k + \boldsymbol{B}_s\boldsymbol{\Psi}\boldsymbol{B}_s{}'\left(-\mathbf{I}_k + \mathbf{I}_k - \boldsymbol{\Sigma}_s^{-1}\right)\right]\boldsymbol{\alpha}_s = \\
&= \boldsymbol{\alpha}_s{}'\left(\mathbf{I}_s - \boldsymbol{B}_s\boldsymbol{\Psi}\boldsymbol{B}_s{}'\boldsymbol{\Sigma}_s^{-1}\right)\boldsymbol{\alpha}_s = \boldsymbol{\alpha}_s{}'\left(\mathbf{I}_k - \mathbf{I}_k + \boldsymbol{\Sigma}_s^{-1}\right)\boldsymbol{\alpha}_s = \boldsymbol{\alpha}_s{}'\boldsymbol{\Sigma}_s^{-1}\boldsymbol{\alpha}_s
\end{aligned}$$

which concludes the proof.

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Table 1

Goodness of fit tests for the censorship data

| | items | traits | $T$ | $\overline{T}$ | $df$ | $p$ | $\overline{\overline{T}}$ | $df$ | $p$ |
|---|---|---|---|---|---|---|---|---|---|
| Normal PDF | 20 | 1 | 5.40 | 244.38 | 170 | $< 0.01$ | 183.40 | 127.58 | $< 0.01$ |
| | 20 | 2 | 3.58 | 180.15 | 151 | 0.05 | 126.86 | 106.33 | 0.09 |
| | 20 | 3 | 2.66 | 145.42 | 133 | 0.22 | 104.00 | 95.12 | 0.25 |
| | 16 | 1 | 2.59 | 133.64 | 104 | 0.03 | 95.92 | 74.65 | 0.05 |
| Normal ogive | 20 | 1 | 9.73 | 408.50 | 170 | $< 0.01$ | 165.05 | 68.67 | $< 0.01$ |
| | 20 | 2 | 4.27 | 214.21 | 151 | $< 0.01$ | 99.30 | 70.00 | 0.01 |
| | 20 | 3 | 3.07 | 169.77 | 133 | 0.02 | 83.83 | 65.67 | 0.07 |
| | 16 | 1 | 3.63 | 171.01 | 104 | $< 0.01$ | 85.98 | 52.29 | $< 0.01$ |

Notes: $N = 223$; $T = N\hat{F}$ ; $\overline{T}$ denotes $T$ adjusted by its asymptotic mean; $\overline{\overline{T}}$ denotes $T$ adjusted by its asymptotic mean and variance

Table 2

Nested tests for comparing the fit of normal pdf models with increasing number of latent traits to the 20 censorship items

| model | $T$ | $\bar{T}$ | $df$ | $p$ |
|---|---|---|---|---|
| 1 vs. 2 | 1.82 | 45.76 | 19 | $< 0.01$ |
| 2 vs. 3 | 2.59 | 29.16 | 18 | 0.05 |

Notes: $N = 223$; $T = N\hat{F}$ ; $\bar{T}$ denotes $T$ adjusted by its asymptotic mean.

Table 3

Parameter estimates and standard errors for a one dimensional model applied to 16 statements on censorship

| Item | Stem | α | β |
|---|---|---|---|
| 1 | I doubt if censorship is wise. | -1.20<br>(0.10) | -0.87<br>(0.24) |
| 2 | A truly free people must be allowed to choose their own reading and entertainment. | -0.28<br>(0.10) | -0.33<br>(0.10) |
| 3 | We must have censorship to protect the morals of young people. | -1.05<br>(0.09) | 0.77<br>(0.16) |
| 6 | The whole theory of censorship is utterly unreasonable. | -1.54<br>(0.08) | -0.50<br>(0.10) |
| 7 | Until public taste has been educated, we must continue to have censorship. | -1.48<br>(0.11) | 0.98<br>(0.22) |
| 8 | Many of our greatest literary classics would be suppressed if the censors thought they could get away with it. | -0.63<br>(0.06) | -0.27<br>(0.07) |
| 9 | Everything that is printed for publication should first be examined by government censors. | -1.91<br>(0.11) | 0.63<br>(0.13) |
| 10 | Plays and movies should be censored but the press should be free. | -1.88<br>(0.09) | 0.30<br>(0.10) |
| 11 | Censorship has practically no effect on peoples morals. | -1.36<br>(0.07) | -0.33<br>(0.08) |
| 13 | Censorship protects those who lack judgment or experience to choose for themselves. | -1.02<br>(0.06) | 0.38<br>(0.08) |
| 14 | Censorship is a very difficult problem and I am not sure how far I think it should go. | -0.16<br>(0.10) | 0.76<br>(0.07) |
| 15 | Censorship is a good thing on the whole although it is often abused. | -0.75<br>(0.11) | 0.88<br>(0.14) |
| 17 | Human progress demands free speech and a free press. | -0.25<br>(0.09) | -0.43<br>(0.08) |
| 18 | Censorship is effective in raising moral and aesthetic standards. | -1.13<br>(0.10) | 0.89<br>(0.18) |
| 19 | Censorship might be warranted if we could get reasonable censors. | -0.47<br>(0.10) | 0.80<br>(0.10) |
| 20 | Morality is produced by self-control, not by censorship. | -0.30<br>(0.10) | -0.42<br>(0.09) |

Note: The statement numbering corresponds to the original 20 item set; standard errors in parentheses

Table 4

Goodness of fit tests for the Satisfaction with Life Scale

| | model | $T$ | $\bar{T}$ | $df$ | $p$ | $\bar{\bar{T}}$ | $df$ | $p$ |
|---|---|---|---|---|---|---|---|---|
| Normal PDF | 1 trait | 7.24 | 273.70 | 5 | $< 0.01$ | 258.53 | 4.72 | $< 0.01$ |
| | 2 traits, unrestricted | 7.18 | 198.79 | 1 | $< 0.01$ | 198.79 | 1.00 | $< 0.01$ |
| | 2 traits, restricted | 6.37 | 358.19 | 4 | $< 0.01$ | 178.16 | 1.79 | $< 0.01$ |
| Normal ogive | 1 trait | 0.93 | 42.84 | 5 | $< 0.01$ | 41.14 | 4.80 | $< 0.01$ |
| | 2 traits, unrestricted | 0.01 | 0.75 | 1 | 0.39 | 0.75 | 1.00 | 0.39 |
| | 2 traits, restricted | 0.21 | 10.20 | 4 | 0.04 | 10.04 | 3.94 | 0.04 |

Notes: $N = 4073$; $\bar{T}$ denotes $T$ adjusted by its asymptotic mean; $\bar{\bar{T}}$ denotes $T$ adjusted by its asymptotic mean and variance

Table 5

Goodness of fit tests for the morality and equality data

| | model | $T$ | $\bar{T}$ | $df$ | $p$ | $\bar{\bar{T}}$ | $df$ | $p$ |
|---|---|---|---|---|---|---|---|---|
| Normal PDF | 1 trait | 0.24 | 17.48 | 20 | 0.62 | 14.22 | 16.28 | 0.60 |
| | 2 traits, unrestricted | 0.13 | 12.16 | 13 | 0.51 | 10.69 | 11.43 | 0.51 |
| | 2 traits, restricted | 0.24 | 17.44 | 19 | 0.56 | 14.14 | 15.4 | 0.54 |
| Normal ogive | 1 trait | 0.30 | 16.93 | 20 | 0.66 | 12.34 | 14.57 | 0.62 |
| | 2 traits, unrestricted | 0.11 | 7.20 | 13 | 0.89 | 5.53 | 9.98 | 0.85 |
| | 2 traits, restricted | 0.26 | 14.60 | 19 | 0.75 | 10.64 | 13.84 | 0.70 |

Notes: $N = 223$; $T = N\hat{F}$ ; $\bar{T}$ denotes $T$ adjusted by its asymptotic mean; $\bar{\bar{T}}$ denotes $T$ adjusted by its asymptotic mean and variance

Figure 1

Censorship data. Plot of the regression slopes for the two dimensional solution

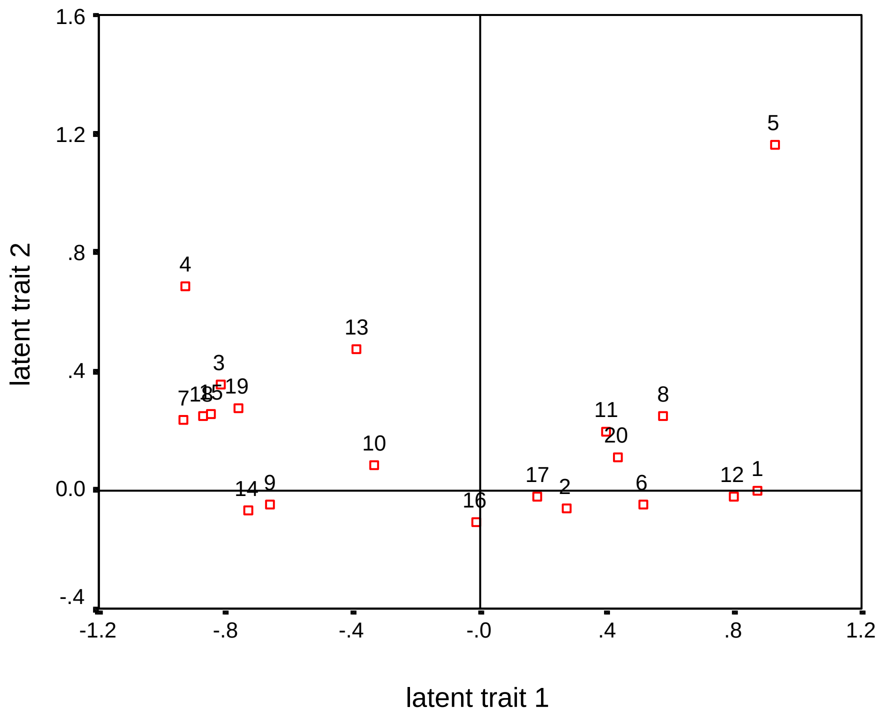
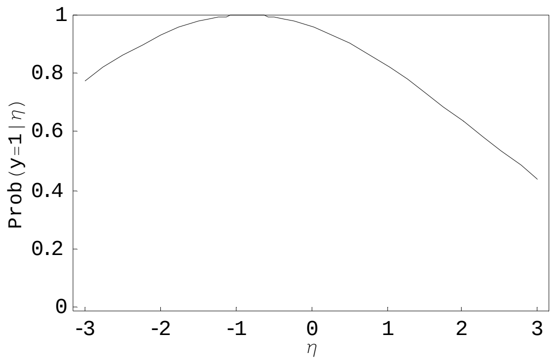


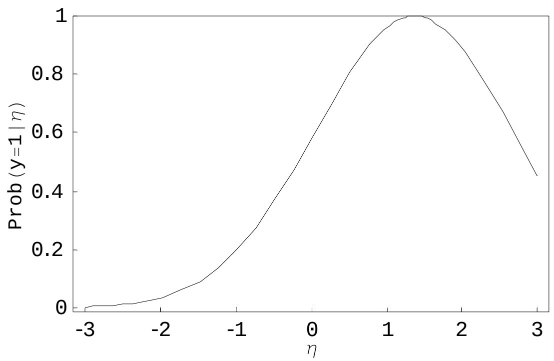
Figure 2

Censorship data. Plot of selected item response functions

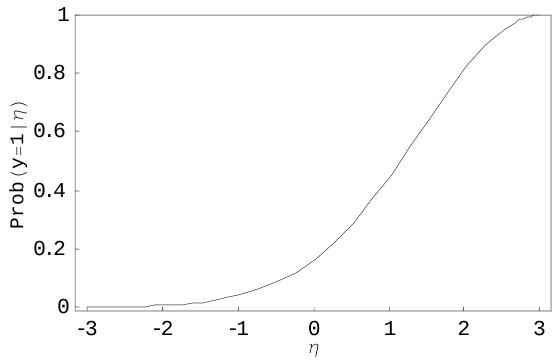
Item 2: A truly free people must be allowed to choose their own reading and entertainment




Item 3: We must have censorship to protect the morals of young people.




Item 9: Everything that is printed for publication should first be examined by government censors




Item 14: Censorship is a very difficult problem and I am not sure how far I think it should go.

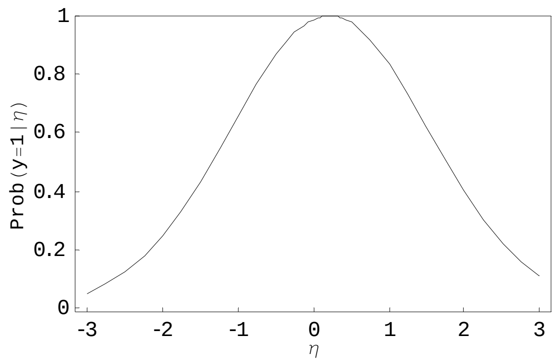
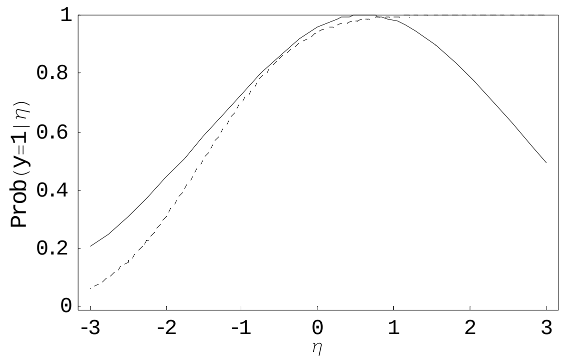


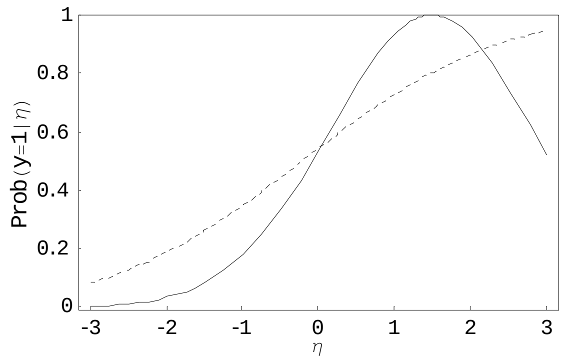
Figure 3

Morality and equality data. Plot of selected item response functions

Item 2: Are equal conditions for all people important to you?




Item 7: Are conscientious objectors important to you?

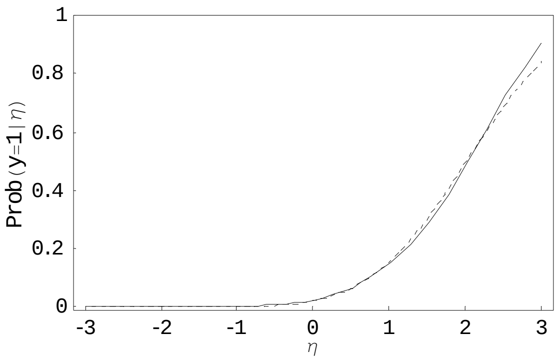



Note: the dashed line is the normal ogive model and the solid line is the normal PDF model

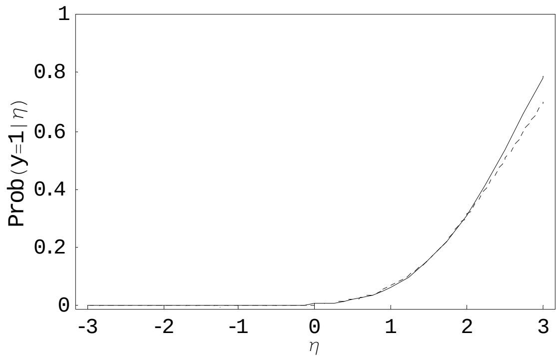
Figure 4

Political action data. Plot of selected item response functions

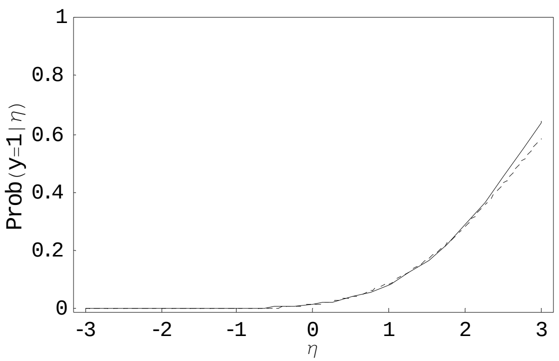
Item 1



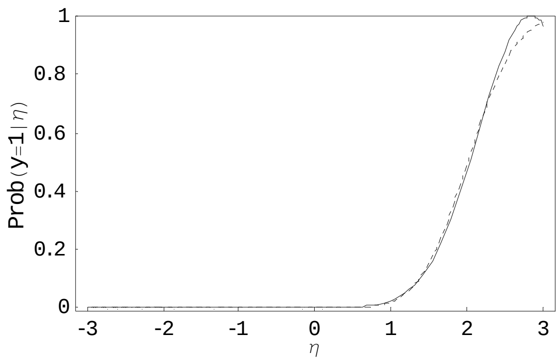

Item 3




Item 2




Item 4




Note: the dashed line is the normal ogive model and the solid line is the normal PDF model