Jacobsthal number identities using the generalized Brioschi formula

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In this paper, we consider some families of Hessenberg determinants the entries of which are Jacobsthal numbers. These determinant formulas may also be rewritten as identities that involve products of multinomial coefficients and powers of Jacobsthal numbers.

Keywords: Jacobsthal sequence, Jacobsthal number, Brioschi's formula, Hessenberg matrix, multinomial coefficient.

1. Introduction. The *Jacobsthal sequence* is considered as one of the important sequences among the well-known integer sequences. The *Jacobsthal sequence* $\{J_n\}_{n\geq 0}$ is defined by recurrence

$$J_n = J_{n-1} + 2J_{n-2},$$

with $J_0 = 0$, $J_1 = 1$, for $n \geq 2$. The number J_n is called the nth Jacobsthal number.

The list of the 12 terms of the Jacobsthal sequence is given in Table 1.

Table 1: Terms of J_n

n	0	1	2	3	4	5	6	7	8	9	10	11	12
J_n	0	1	1	3	5	11	43	85	171	341	683	1365	2731

The Jacobsthal numbers have many interesting properties and applications in many fields of mathematics, as geometry, number theory, combinatorics, and probability theory; see entry A001045 in the On-Line Encyclopedia of Integer Sequences (Sloane, 2019).

As examples of recent works involving the Jacobsthal numbers and its various generalizations, see Akbulak and Öteleş (2014), Aktaş and Köse (2015), Aydın (2018), Catarino et al. (2015), Cılasum (2016), Daşdemir (2019), Goy (2018a), Goy (2018b), Goy (2019b), Köken and Bozkurt (2008), Öteleş et al. (2018), Zatorsky and Goy (2016) and related references contained therein. For example, in Akbulak and Öteleş (2014) defined two n-square upper Hessenberg matrices one of which corresponds to the adjacency matrix a directed pseudo graph and investigated relations between determinants and permanents of these Hessenberg matrices and sum formulas of the Jacobsthal sequences. In Köken and Bozkurt (2008) defined the n-square Jacobsthal matrix and using this matrix derived some properties of Jacobsthal numbers. In Öteleş et al. (2018) investigated the relationships between the Hessenberg matrices and the Jacobsthal numbers. In Cılasun (2016) introduced recurrence relation for multiple-counting Jacobsthal sequences and showed their application with Fermat's little theorem. In Daşdemir (2019) extended the Jacobsthal numbers to the terms with negative subscripts and presented many identities for new forms of these numbers. In

Catarino et al. (2015) presented new families of sequences that generalize the Jacobsthal numbers and established some identities.

The purpose of the present paper is to study the Jacobsthal numbers. We investigate some families of Hessenberg determinants the entries of which are Jacobsthal numbers with successive, odd and even subscripts. Consequently, we obtain for these numbers new combinatorial identities involving multinomial coefficients.

2. Hessenberg matrices and determinants. Consider the *Hessenberg matrix* of order n having the form

$$H_n^{(k)}(a_1, a_2, \ldots, a_n) = \begin{pmatrix} k_1 a_1 & 1 & 0 & \cdots & 0 & 0 \\ k_2 a_2 & a_1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k_{n-1} a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & 1 \\ k_n a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{pmatrix},$$

where $a_0 \neq 0$ and $a_i \neq 0$ for at least one $i \geq 1$.

The following lemma gives the multinomial extension for $\det(H_n)$. This result directly follows from Theorem 2 in Zatorsky (2013).

Lemma 1. *Let n be a positive integer. Then*

$$\det(H_n^{(k)}) = \sum_{s_1 + 2s_2 + \cdots + ns_n = n} \frac{(-1)^{n-\sigma_n}}{\sigma_n} \left(\sum_{i=1}^{n} s_i k_i \right) m_n(s) a_1^{s_1} a_2^{s_2} \cdots a_n^{s_n}, \qquad (1)$$

where the summation is over integers $s_j \geq 0$ satisfying Diophantine equation $s_1 + 2s_2 + \cdots + ns_n = n$, $\sigma_n = s_1 + \cdots + s_n$, and $m_n(s) = \dfrac{(s_1 + \cdots + s_n)!}{s_1! \cdots s_n!}$ is the multinomial coefficient.

In the case $k_1 = k_2 = \ldots = k_n = 1$ we have *Brioschi's formula* (Muir, 1960).

Many combinatorial identities involving sums over integer partitions can be generated in this way. For example, similar results for Fibonacci, Lucas, Mersenne, Pell, Catalan, Oresme numbers have been recently discovered in Goy (2018c), Goy (2019b), Goy (2019c), Goy and Shattuck (2019a), Goy and Shattuck (2019b), Goy and Shattuck (2019c).

3. Determinant formulas for Jacobsthal numbers. In this section, we investigate a particular case of determinants $\det(H_n^{(k)})$, in which $k_i = i$. To simplify our notation, we write $\det(a_1, a_2, \ldots, a_n)$ in place of $\det(H_n^{(k)}(a_1, a_2, \ldots, a_n))$.

Recall that the *Fibonacci sequence* $\{F_n\}_{n \geq 0}$ is defined by the initial values $F_0 = 0$, $F_1 = 1$ and the recurrence

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2.$$

Theorem 2. *For* $n \geq 1$*, the following identities hold*

$$\det(J_0, J_1, \ldots, J_{n-1}) = (-1)^n (2F_{n+1} - F_n) - (-2)^n - 1,$$

$$\det(J_1, J_2, \ldots, J_n) = (-1)^{n-1} (2^{n/2} - 1)(2^{n/2} - (-1)^n),$$

$$\det(J_2, J_3, \ldots, J_{n+1}) = -(-2)^n - 1,$$

$$\det(J_3, J_4, \ldots, J_{n+2}) = 2^n - (-2)^n - 1,$$

$$\det(J_4, J_5, \ldots, J_{n+3}) = 2^{n+1} - (-2)^n - 1,$$

$$\det(J_3, J_5, \ldots, J_{2n+1}) = (-1)^n (4^n - 2^n + 1),$$

$$\det(J_2, J_4, \ldots, J_{2n}) = (-1)^{n-1} (2^n - 1)^2,$$

$$\det(J_4, J_6, \ldots, J_{2n+2}) = (-1)^{n-1} (4^n - 1),$$

where F_n *is the* n^{th} *Fibonacci number.*

Next, we focus on multinomial extension of Theorems 2. Formula (1), coupled with Theorem 2 above, yields the following combinatorial identities for Jacobsthal numbers.

Theorem 3. *Let* $n \geq 1$, $\sigma_n = s_1 + \cdots + s_n$, $s_i \geq 0$, *and* $m_n(s) = \dfrac{(s_1 + \cdots + s_n)!}{s_1! \cdots s_n!}$ *denotes the multinomial coefficient. Then*

$$\sum_{2s_1 + \cdots + s_{n-1} = n} \frac{(-1)^{\sigma_{n-1}}}{\sigma_{n-1}} m_{n-1}(s) J_1^{s_1} J_2^{s_2} \cdots J_{n-1}^{s_{n-1}} = \frac{2F_{n+1} - F_n - 2^n - (-1)^n}{n},$$

$$\sum_{s_1 + 2s_2 + \cdots + s_n = n} \frac{(-1)^{\sigma_n}}{\sigma_n} m_n(s) J_1^{s_1} J_2^{s_2} \cdots J_n^{s_n} = \frac{(1 - 2^{n/2})(2^{n/2} - (-1)^n)}{n},$$

$$\sum_{s_1 + 2s_2 + \cdots + ns_n = n} \frac{(-1)^{\sigma_n}}{\sigma_n} m_n(s) J_2^{s_1} J_3^{s_2} \cdots J_{n+1}^{s_n} = -\frac{2^n + (-1)^n}{n},$$

$$\sum_{s_1 + 2s_2 + \cdots + ns_n = n} \frac{(-1)^{\sigma_n}}{\sigma_n} m_n(s) J_3^{s_1} J_4^{s_2} \cdots J_{n+2}^{s_n} = \frac{(-2)^n - 2^n - (-1)^n}{n},$$

$$\sum_{s_1 + 2s_2 + \cdots + ns_n = n} \frac{(-1)^{\sigma_n}}{\sigma_n} m_n(s) J_4^{s_1} J_5^{s_2} \cdots J_{n+3}^{s_n} = \frac{2(-2)^n - 2^n - (-1)^n}{n},$$

$$\sum_{s_1 + 2s_2 + \cdots + ns_n = n} \frac{(-1)^{\sigma_n}}{\sigma_n} m_n(s) J_3^{s_1} J_5^{s_2} \cdots J_{2n+1}^{s_n} = \frac{4^n - 2^n + 1}{n},$$

$$\sum_{s_1 + 2s_2 + \cdots + ns_n = n} \frac{(-1)^{\sigma_n}}{\sigma_n} m_n(s) J_2^{s_1} J_4^{s_2} \cdots J_{2n}^{s_n} = -\frac{(2^n - 1)^2}{n},$$

$$\sum_{s_1+2s_2+\cdots+ns_n=n} \frac{(-1)^{\sigma_n}}{\sigma_n} m_n(s) J_4^{s_1} J_6^{s_2} \cdots J_{2n+2}^{s_n} = \frac{1-4^n}{n}.$$

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