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SEMIPOSITONE FRACTIONAL BOUNDARY VALUE PROBLEMS WITH N POINT FRACTIONAL INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this paper, we obtain the existence of positive solutions for the semipositone fractional boundary value problem with n point fractional integral boundary conditions. The existence of positive solutions is established using the five functionals fixed point theorem. An example is given to ratify that our main result is theoretically feasible.

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1. INTRODUCTION

Nowadays, fractional boundary value problems sprung up dramatically due to the wide applications in various research fields such as chemistry, biology, engineering, economy and other areas. The books of Kilbas et al. [8], Podlubny [10], Samko et al. [11] are mostly cited for the theory and applications of fractional calculus. Inspired by the extensive application area of fractional boundary value problems, many works focus on the existence of positive solutions for such boundary value problems. Some kinds of procedures are applied to establish positive solutions for fractional boundary value problems such as the Krasnoselskii fixed point theorem on cones, the Leggett Williams fixed point theorem, the Avery Henderson fixed point theorem, upper and lower solutions method [2–4, 9, 12–14, 16, 18]. To apply these theories and procedures, boundary value problems admit nonnegative and continuous nonlinear term. Besides, negative term can also appear in our problems dealing with real word problems. If boundary value problems involve both negative and nonnegative nonlinearity, we say that those problems are semipositone problems, which occur in astrophysics, chemical reactions, envisagement of suspension bridges. Investigations on existence results obtained for semipositone problems are more complicated than those for positive ones. Many authors study on semipositone boundary value problems using variational methods, fixed point theory, critical point theory [5–7, 15, 17, 19]. Generally, the existence results of solutions are obtained for fractional semipositone

Riemann-Liouville differential equations for zero boundary values or Caputo fractional differential equations with boundary conditions such as three-point, m-point, integral boundary conditions. For all we know, only a few papers concerned with the existence of solutions for semipositone fractional boundary value problems subject to the Riemann-Liouville fractional integral boundary conditions.

The aim of this paper is to establish multiple positive solutions for the fractional differential equation with Caputo derivative of order $\upsilon \in (1,2]$

$$D^{\upsilon}\phi(t) + f(t,\phi(t)) = 0, \quad t \in (0,1), \tag{1.1}$$

with fractional integral boundary conditions

$$\begin{aligned} \phi(0) - \sigma\phi'(0) &= \sum_{j=1}^{n-2} a_j I^q \phi(\eta_j), \\ \phi(1) + \rho\phi'(1) &= \sum_{j=1}^{n-2} b_j I^q \phi(\xi_j), \end{aligned} \tag{1.2}$$

in which $q > 0$ and $\sigma, \rho > 0$, $a_j, b_j, \eta_j, \xi_j > 0$, $a_j \leq b_j$ and $\eta_j \leq \xi_j$ for $j \in \{1, n-2\}$, $\sum_{j=1}^{n-2} \frac{a_j \eta_j^q}{\Gamma(q+1)} < 1$, $\sum_{j=1}^{n-2} \frac{b_j \xi_j^q}{\Gamma(q+1)} < 1$, $f \in C([0,1] \times [0,\infty), \mathbb{R})$ and $f(t,0) \not\equiv 0$.

In order to assert our main result, we will give some background materials and the five functionals fixed point theorem.

Let Φ, κ, Θ be nonnegative continuous convex functionals on the cone P and let χ, τ be nonnegative continuous concave functionals on P. Then for nonnegative real numbers l, k, r, h and g, we define the following convex sets:

$$\begin{aligned} P(\Phi, g) &= \{\vartheta \in P : \Phi(\vartheta) < g\}, \\ P(\Phi, \chi, k, g) &= \{\vartheta \in P : k \leq \chi(\vartheta), \Phi(\vartheta) \leq g\}, \\ Q(\Phi, \kappa, h, g) &= \{\vartheta \in P : \kappa(\vartheta) \leq h, \Phi(\vartheta) \leq g\}, \\ P(\Phi, \Theta, \chi, k, r, g) &= \{\vartheta \in P : k \leq \chi(\vartheta), \Theta(\vartheta) \leq r, \Phi(\vartheta) \leq g\}, \\ Q(\Phi, \kappa, \tau, l, h, g) &= \{\vartheta \in P : l \leq \tau(\vartheta), \kappa(\vartheta) \leq h, \Phi(\vartheta) \leq g\}. \end{aligned}$$

The five functionals fixed point theorem is very significant in proving our main theorem, which is given below.

Theorem 1 ([1]). *Let P be a cone in a real Banach space E. Assume there exist $g > 0$ and $m > 0$ satisfying*

$$\chi(\vartheta) \leq \kappa(\vartheta) \ \textit{and} \ \|\vartheta\| \leq m\Phi(\vartheta)$$

for all $\vartheta \in \overline{P(\Phi, g)}$. If

$$S : \overline{P(\Phi, g)} \to \overline{P(\Phi, g)}$$

is completely continuous and there exist nonnegative numbers k, r, h *and* l *with* $0 < h < k$ *such that*
(i) $\{\vartheta \in P(\Phi, \Theta, \chi, k, r, g) : \chi(\vartheta) > k\} \neq \varnothing$ *and* $\chi(S\vartheta) > k$ *for* $\vartheta \in P(\Phi, \Theta, \chi, k, r, g)$,
(ii) $\{\vartheta \in Q(\Phi, \kappa, \tau, l, h, g) : \kappa(\vartheta) < h\} \neq \varnothing$ *and* $\kappa(S\vartheta) < h$ *for* $\vartheta \in Q(\Phi, \kappa, \tau, l, h, g)$,
(iii) $\chi(S\vartheta) > k$ *for* $\vartheta \in P(\Phi, \chi, k, g)$ *with* $\Theta(S\vartheta) > r$,
(iv) $\kappa(S\vartheta) < h$ *for* $\vartheta \in Q(\Phi, \kappa, h, g)$ *with* $\tau(S\vartheta) < l$.
Then, S has at least three fixed points $\vartheta_1, \vartheta_2, \vartheta_3 \in \overline{P(\Phi, g)}$ *satisfying*

$$\kappa(\vartheta_1) < h,\ k < \chi(\vartheta_2) \text{ and } h < \kappa(\vartheta_3) \text{ with } \chi(\vartheta_3) < k.$$

2. Preliminaries

In order to assert our main result, we assemble some definitions and lemmas from the fractional calculus [8, 10, 11].

Definition 1. The Riemann-Liouville fractional integral of order υ for a function y is given as

$$I^{\upsilon} y(t) = \frac{1}{\Gamma(\upsilon)} \int_0^1 (t-s)^{\upsilon-1} y(s)ds, \ \ \upsilon > 0,$$

provided that such integral exists.

Definition 2. If $y \in \mathcal{C}^n[0,1]$, then the Caputo fractional derivative of order υ is defined by

$$D^{\upsilon} y(t) = \frac{1}{\Gamma(n-\upsilon)} \int_0^t (t-s)^{n-\upsilon-1} y^{(n)}(s)ds = I^{n-\upsilon} y^{(n)}(t),$$

where $n-1 < \upsilon < n, n = [\upsilon]+1$ and $[\upsilon]$ denotes the integer part of the real number υ.

Lemma 1. *Let* $\upsilon > 0$ *then the fractional differential equation* $D^{\upsilon} u(t) = 0$ *has a solution*

$$u(t) = d_0 + d_1 t + d_2 t^2 + \ldots + d_{n-1} t^{n-1},$$

in which $d_i \in \mathbb{R},\ \ i = 0, 1, 2, \ldots, n$*;* $n-1 < \upsilon < n,\ n = [\upsilon]+1$.

Next, we state some auxiliary lemmas for fractional BVP (1.1)-(1.2).

Lemma 2 ([4])**.** *If* $h \in \mathcal{C}[0,1]$, *then the fractional boundary value problem (fractional BVP for short)*

$$D^{\upsilon}\vartheta(t) + h(t) = 0, \quad t \in (0,1),$$
$$\vartheta(0) - \sigma\vartheta'(0) = 0,$$
$$\vartheta(1) + \rho\vartheta'(1) = 0$$

possesses the integral expression

$$\vartheta(t)=\int_0^1 G(t,s)h(s)ds,$$

where

$$G(t,s)=\frac{1}{\Gamma(\upsilon)}\begin{cases} -(t-s)^{\upsilon-1}+\dfrac{(t+\sigma)(1-s)^{\upsilon-1}}{1+\sigma+\rho}+\dfrac{(t+\sigma)(\upsilon-1)\rho(1-s)^{\upsilon-2}}{1+\sigma+\rho}, \\ 0\le s\le t\le 1, \\ \\ \dfrac{(t+\sigma)(1-s)^{\upsilon-1}}{1+\sigma+\rho}+\dfrac{(t+\sigma)(\upsilon-1)\rho(1-s)^{\upsilon-2}}{1+\sigma+\rho}, \\ 0\le t\le s\le 1. \end{cases}$$

Lemma 3 ([9, 18]). *$G(t,s)$ satisfies the following properties*

(i) *$G(t,s)$ is continuous on $[0,1]\times[0,1]$,*
(ii) *If $\sigma>\frac{2-\upsilon}{\upsilon-1}$, then $0\le G(t,s)\le G(s,s)$ for any $t,s\in[0,1]$,*
(iii) *If $\sigma>\frac{2-\upsilon}{\upsilon-1}$, then there exists $\omega>0$ such that $\omega G(s,s)\le G(t,s)\le G(s,s)$ for any $t,s\in[0,1]$,*

where $\omega=min\{\omega_1,\omega_2\}$ can be given by

$$\omega_1=\frac{4\rho[\sigma(\upsilon-1)+(\upsilon-2)]}{[\rho(\upsilon-1)+1-\sigma]^2+4\sigma[\rho(\upsilon-1)+1]}, \tag{2.1}$$

$$\omega_2=\frac{4\sigma\rho[\sigma(\upsilon-1)+(\upsilon-2)]}{[\rho(\upsilon-1)+1-\sigma]^2+4\sigma[\rho(\upsilon-1)+1]}. \tag{2.2}$$

Lemma 4. *For $h\in\mathcal{C}[0,1]$, the fractional BVP*

$$D^{\upsilon}\phi(t)+h(t)=0,\quad t\in(0,1), \tag{2.3}$$

$$\begin{aligned}\phi(0)-\sigma\phi'(0)&=\sum_{j=1}^{n-2}a_jI^q\phi(\eta_j),\\ \phi(1)+\rho\phi'(1)&=\sum_{j=1}^{n-2}b_jI^q\phi(\xi_j),\end{aligned} \tag{2.4}$$

possesses the integral expression

$$\phi(t)=\int_0^1 H(t,s)h(s)ds, \tag{2.5}$$

where

$$
\begin{aligned}
H(t,s) = G(t,s) &+ \frac{1}{\Lambda}\Big(\sum_{j=1}^{n-2} a_j I^q G(\eta_j,s)\Big)\Big(1+\rho-\sum_{j=1}^{n-2}\frac{b_j\xi_j^{q+1}}{\Gamma(q+2)}\Big)\\
&+\frac{1}{\Lambda}\Big(\sum_{j=1}^{n-2} b_j I^q G(\xi_j,s)\Big)\Big(\sigma+\sum_{j=1}^{n-2}\frac{a_j\eta_j^{q+1}}{\Gamma(q+2)}\Big)\\
&+\frac{t}{\Lambda}\Big[\Big(\sum_{j=1}^{n-2} b_j I^q G(\xi_j,s)\Big)\Big(1-\sum_{j=1}^{n-2}\frac{a_j\eta_j^{q}}{\Gamma(q+1)}\Big)\\
&-\Big(\sum_{j=1}^{n-2} a_j I^q G(\eta_j,s)\Big)\Big(1-\sum_{j=1}^{n-2}\frac{b_j\xi_j^{q}}{\Gamma(q+1)}\Big)\Big]
\end{aligned}
$$

and

$$
\Lambda = \Big(1-\sum_{j=1}^{n-2}\frac{a_j\eta_j^{q}}{\Gamma(q+1)}\Big)\Big(1+\rho-\sum_{j=1}^{n-2}\frac{b_j\xi_j^{q+1}}{\Gamma(q+2)}\Big)+\Big(1-\sum_{j=1}^{n-2}\frac{b_j\xi_j^{q}}{\Gamma(q+1)}\Big)\Big(\sigma+\sum_{j=1}^{n-2}\frac{a_j\eta_j^{q+1}}{\Gamma(q+2)}\Big).
$$

Here, $I^qG(\eta_j,s)$ *and* $I^qG(\xi_j,s)$ *are the Riemann-Liouville fractional integrals of* $G(t,s)$ *with respect to* $t=\eta_j$ *and* $t=\xi_j$ *respectively.*

Proof. Let

$$
\vartheta(t) = \int_0^1 G(t,s)h(s)ds. \tag{2.6}
$$

By employing Lemma 2, $\vartheta(t)$ holds

$$
\begin{aligned}
&D^{\upsilon}\vartheta(t)+h(t)=0, \quad t\in(0,1),\\
&\vartheta(0)-\sigma\vartheta'(0)=0,\\
&\vartheta(1)+\rho\vartheta'(1)=0.
\end{aligned}
$$

Suppose $\phi(t)$ is a solution of the BVP (2.3)-(2.4) and

$$
z(t)=\phi(t)-\vartheta(t), \quad t\in[0,1],
$$

then $z(t)$ holds the fractional BVP :

$$
\begin{aligned}
&D^{\upsilon}z(t)=0, \quad t\in(0,1),\\
&z(0)-\sigma z'(0)=\sum_{j=1}^{n-2} a_j I^q z(\eta_j)+\sum_{j=1}^{n-2} a_j I^q\vartheta(\eta_j),\\
&z(1)+\rho z'(1)=\sum_{j=1}^{n-2} b_j I^q z(\xi_j)+\sum_{j=1}^{n-2} b_j I^q\vartheta(\xi_j).
\end{aligned} \tag{2.7}
$$

Lemma 1 implies that

$$
z(t)=d_0+d_1t, \quad t\in[0,1], d_0,d_1\in\mathbb{R}, \tag{2.8}
$$

and replacing $z(t)$ into (2.7) leads to

$$
\begin{aligned}
d_0 = \frac{1}{\Lambda}[(\sum_{j=1}^{n-2} a_j I^q \vartheta(\eta_j))(1+\rho-\sum_{j=1}^{n-2}\frac{b_j\xi_j^{q+1}}{\Gamma(q+2)}) \\
+(\sum_{j=1}^{n-2} b_j I^q \vartheta(\xi_j))(\sigma+\sum_{j=1}^{n-2}\frac{a_j\eta_j^{q+1}}{\Gamma(q+2)})]
\end{aligned} \tag{2.9}
$$

and

$$
\begin{aligned}
d_1 = \frac{1}{\Lambda}[(1-\sum_{j=1}^{n-2}\frac{a_j\eta_j^{q}}{\Gamma(q+1)})(\sum_{j=1}^{n-2} b_j I^q \vartheta(\xi_j)) \\
-(1-\sum_{j=1}^{n-2}\frac{b_j\xi_j^{q}}{\Gamma(q+1)})(\sum_{j=1}^{n-2} a_j I^q \vartheta(\eta_j))],
\end{aligned} \tag{2.10}
$$

where

$$
\begin{aligned}
\Lambda = (1-\sum_{j=1}^{n-2}\frac{a_j\eta_j^{q}}{\Gamma(q+1)})(1+\rho-\sum_{j=1}^{n-2}\frac{b_j\xi_j^{q+1}}{\Gamma(q+2)}) \\
+(1-\sum_{j=1}^{n-2}\frac{b_j\xi_j^{q}}{\Gamma(q+1)})(\sigma+\sum_{j=1}^{n-2}\frac{a_j\eta_j^{q+1}}{\Gamma(q+2)}).
\end{aligned}
$$

Finally, rewriting (2.9) and (2.10) into (2.8), we have

$$
\begin{aligned}
z(t) = \frac{1}{\Lambda}(\sum_{j=1}^{n-2} a_j I^q \vartheta(\eta_j))[(1+\rho-\sum_{j=1}^{n-2}\frac{b_j\xi_j^{q+1}}{\Gamma(q+2)})-t(1-\sum_{j=1}^{n-2}\frac{b_j\xi_j^{q}}{\Gamma(q+1)})] \\
+\frac{1}{\Lambda}(\sum_{j=1}^{n-2} b_j I^q \vartheta(\xi_j))[(\sigma+\sum_{j=1}^{n-2}\frac{a_j\eta_j^{q+1}}{\Gamma(q+2)})+t(1-\sum_{j=1}^{n-2}\frac{a_j\eta_j^{q}}{\Gamma(q+1)})].
\end{aligned} \tag{2.11}
$$

Thus, we can conclude from (2.6) and (2.11) that (2.5) is satisfied. Therefore, the proof of Lemma 4 is accomplished. □

Lemma 5. *$H(t,s)$ holds the following properties*

(i) $H(t,s) \in C([0,1]\times[0,1])$, $H(t,s) \geq 0$ *for any* $t,s \in (0,1)$.

(ii) *There exist nonnegative numbers* φ *and* ω *such that*

$$H(t,s) \leq \varphi G(s,s), \quad t,s \in [0,1],$$

and

$$H(t,s) \geq \omega G(s,s), \quad t,s \in [0,1], \tag{2.12}$$

in which

$$\begin{aligned}\varphi = 1 &+ \frac{1}{\Lambda}\Big(\sum_{j=1}^{n-2}\frac{a_j\eta_j^q}{\Gamma(q+1)}\Big)\Big(1+\rho-\sum_{j=1}^{n-2}\frac{b_j\xi_j^{q+1}}{\Gamma(q+2)}\Big)\\ &+\frac{1}{\Lambda}\Big(\sum_{j=1}^{n-2}\frac{b_j\xi_j^q}{\Gamma(q+1)}\Big)\Big(\sigma+\sum_{j=1}^{n-2}\frac{a_j\eta_j^{q+1}}{\Gamma(q+2)}\Big)\\ &+\frac{1}{\Lambda}\Big(1-\sum_{j=1}^{n-2}\frac{a_j\eta_j^q}{\Gamma(q+1)}\Big)\Big(\sum_{j=1}^{n-2}\frac{b_j\xi_j^q}{\Gamma(q+1)}\Big),\end{aligned} \tag{2.13}$$

and ω *is defined by* (2.1)-(2.2).

Proof. Apparently, (i) is satisfied using the definition of $H(t,s)$. We will show property (ii). Lemma 3 implies that

$$\begin{aligned}H(t,s) \le G(s,s) &+ \frac{1}{\Lambda}\Big(\sum_{j=1}^{n-2}\frac{a_j\eta_j^q}{\Gamma(q+1)}G(s,s)\Big)\Big(1+\rho-\sum_{j=1}^{n-2}\frac{b_j\xi_j^{q+1}}{\Gamma(q+2)}\Big)\\ &+\frac{1}{\Lambda}\Big(\sum_{j=1}^{n-2}\frac{b_j\xi_j^q}{\Gamma(q+1)}G(s,s)\Big)\Big(\sigma+\sum_{j=1}^{n-2}\frac{a_j\eta_j^{q+1}}{\Gamma(q+2)}\Big)\\ &+\frac{t}{\Lambda}\Big(\sum_{j=1}^{n-2}\frac{b_j\xi_j^q}{\Gamma(q+1)}G(s,s)\Big)\Big(1-\sum_{j=1}^{n-2}\frac{a_j\eta_j^q}{\Gamma(q+1)}\Big)\\ &\le \Big[1+\frac{1}{\Lambda}\Big(\sum_{j=1}^{n-2}\frac{a_j\eta_j^q}{\Gamma(q+1)}\Big)\Big(1+\rho-\sum_{j=1}^{n-2}\frac{b_j\xi_j^{q+1}}{\Gamma(q+2)}\Big)\\ &+\frac{1}{\Lambda}\Big(\sum_{j=1}^{n-2}\frac{b_j\xi_j^q}{\Gamma(q+1)}\Big)\Big(\sigma+\sum_{j=1}^{n-2}\frac{a_j\eta_j^{q+1}}{\Gamma(q+2)}\Big)\\ &+\frac{1}{\Lambda}\Big(\sum_{j=1}^{n-2}\frac{b_j\xi_j^q}{\Gamma(q+1)}\Big)\Big(1-\sum_{j=1}^{n-2}\frac{a_j\eta_j^q}{\Gamma(q+1)}\Big)\Big]G(s,s)\\ &= \varphi G(s,s).\end{aligned}$$

On the other hand, one can see easily that (2.12) is satisfied. Therefore, the proof of Lemma 5 is accomplished. □

Theorem 1 will be utilized to prove the presence of positive solution. For this purpose, we are in position to introduce the Banach space $\mathcal{B} = \mathcal{C}[0,1]$ and a cone

$$P = \{\vartheta \in \mathcal{B} : \vartheta(t) \ge \Omega\|\vartheta\|, t\in[0,1]\},$$

where $\Omega = \frac{\omega}{\varphi}$, ω and φ are defined by (2.1)-(2.2) and (2.13).

To prove that fractional BVP (1.1)-(1.2) possesses multiple positive solutions, we choose $l=0$, $m=1$ and the following three functionals are defined by

$$\chi(\vartheta)=\min_{t\in[0,1]}\vartheta(t),\quad \tau(\vartheta)=0,\quad \Phi(\vartheta)=\kappa(\vartheta)=\Theta(\vartheta)=\|\vartheta\|.$$

Furthermore

$$\chi(\vartheta)\le\kappa(\vartheta)\ \text{ and }\ \|\vartheta\|\le m\Phi(\vartheta)\ \ for\ \ \vartheta\in P.$$

Let us denote

$$\mu=\omega\int_0^1 G(s,s)ds,$$
$$\zeta=\varphi\int_0^1 G(s,s)ds,$$
$$D_1=\frac{\varphi^2}{\omega}\int_0^1 G(s,s)ds.$$

Theorem 2. *Assume that there exist constants* $KD_1<h<h+KD_1\Omega<k<\dfrac{k}{\Omega^2}<g$ *such that* $\dfrac{1}{\Omega}<N<\dfrac{g\mu}{k\zeta}$ *holds.*
Furthermore f verifies the following conditions:

(C_0) *There exists $K>0$ such that $f(t,\vartheta)\ge -K$ for $(t,\vartheta)\in[0,1]\times\mathbb{R}^+$,*
(C_1) $f(t,\vartheta)\le\dfrac{g}{\zeta}-K$ *for* $t\in[0,1]$, $\vartheta\in[0,g]$,
(C_2) $f(t,\vartheta)\ge\dfrac{kN}{\mu}-K$ *for* $t\in[0,1]$, $\vartheta\in[k-KD_1\Omega,g]$,
(C_3) $f(t,\vartheta)<\dfrac{h}{\zeta}-K$ *for* $t\in[0,1]$, $\vartheta\in[0,h]$.

Then fractional BVP (1.1)-(1.2) has at least two positive solutions.

Proof. Assume w is a solution of

$$D^{\upsilon}\phi(t)+1=0,\quad t\in(0,1),$$
$$\phi(0)-\sigma\phi'(0)=\sum_{j=1}^{n-2}a_jI^q\phi(\eta_j),$$
$$\phi(1)+\rho\phi'(1)=\sum_{j=1}^{n-2}b_jI^q\phi(\xi_j),$$

and $z(t)=Kw(t)$ for $t\in[0,1]$. Then

$$z(t)=Kw(t)=K\int_0^1 H(t,s)ds\le K\varphi\int_0^1 G(s,s)ds\le KD_1\Omega.$$

We shall show that the fractional BVP

$$D^{\upsilon}\vartheta(t)+h(t,\overline{\vartheta}(t))=0,\quad t\in(0,1), \tag{2.14}$$

$$\begin{aligned}\vartheta(0)-\sigma\vartheta'(0)&=\sum_{j=1}^{n-2}a_jI^q\vartheta(\eta_j),\\ \vartheta(1)+\rho\vartheta'(1)&=\sum_{j=1}^{n-2}b_jI^q\vartheta(\xi_j),\end{aligned} \tag{2.15}$$

has at least three positive solutions in which

$$h(t,\overline{\vartheta}(t))=f(t,\overline{\vartheta}(t))+K \text{ and } \overline{\vartheta}(t)=max\{\vartheta(t)-z(t),0\}.$$

For $\vartheta\in P$, denote an operator S by

$$S\vartheta(t)=\int_0^1 H(t,s)h(s,\overline{\vartheta}(s))ds.$$

Clearly, the fractional BVP (2.14)-(2.15) has a solution provided that the operator S admits a fixed point.

Now, we check that $S(P)\subseteq P$. Indeed, for $\vartheta\in P$, Lemma 5 implies

$$\int_0^1 \omega G(s,s)h(s,\overline{\vartheta}(s))ds\ \le S\vartheta(t)\le\int_0^1 \varphi G(s,s)h(s,\overline{\vartheta}(s))ds.$$

Hence,

$$S\vartheta(t)\ge\int_0^1 \omega G(s,s)h(s,\overline{\vartheta}(s))ds\ge\Omega\|S\vartheta\|.$$

Furthermore by employing standard methods, the operator $S:P\to P$ is completely continuous. In what follows, we will show that all the conditions of Theorem 1 are satisfied.

We prove that $S(\overline{P(\Phi,g)})\subseteq\overline{P(\Phi,g)}$. Let $\vartheta\in\overline{P(\Phi,g)}$ then $0\le\overline{\vartheta}(t)\le\vartheta(t)\le g$. By C_1, we get

$$\Phi(S\vartheta)=\|S\vartheta\|=\max_{t\in[0,1]}S\vartheta(t)\le\varphi\int_0^1 G(s,s)h(s,\overline{\vartheta}(s))ds\le\varphi\frac{g}{\zeta}\int_0^1 G(s,s)ds\le g.$$

So $S:\overline{P(\Phi,g)}\to\overline{P(\Phi,g)}$. In the following, we now prove that the conditions of Theorem 1 is satisfied with $r=g$.

To verify condition (i) of Theorem 1, t let $\vartheta(t)=\frac{k}{\Omega^2}$, then one can see easily that

$$\{\vartheta\in P(\Phi,\Theta,\chi,k,r,g):\chi(\vartheta)>k\}=\left\{\vartheta\in P:\min_{t\in[0,1]}\vartheta(t)>k,\|\vartheta\|\le g\right\}\ne\varnothing.$$

Moreover, if $\vartheta \in P(\Phi,\Theta,\chi,k,r,g)$, then $\vartheta(t)-z(t) \leq \vartheta(t) \leq g$, that is $k-KD_1\Omega \leq \vartheta(t)-z(t) \leq g$. Applying (C_2), we get

$$\chi(S\vartheta) = \min_{t\in[0,1]} S\vartheta(t) \geq \Omega\|S\vartheta\| \geq \Omega\frac{kN}{\mu}\omega\int_0^1 G(s,s)ds = \Omega Nk > k. \tag{2.16}$$

Hence, condition (i) of Theorem 1 is satisfied.

Apparently,

$$\{\vartheta \in Q(\Phi,\kappa,\tau,l,h,g) : \kappa(\vartheta) < h\} = \{\vartheta \in P : \|\vartheta\| < h\} \neq \varnothing.$$

Using (C_3) leads that for $\vartheta \in Q(\Phi,\kappa,\tau,l,h,g)$

$$\kappa(S\vartheta) = \|S\vartheta\| = \max_{t\in[0,1]} S\vartheta(t) \leq \varphi\int_0^1 G(s,s)h(s,\overline{\vartheta}(s))ds < \varphi\frac{h}{\zeta}\int_0^1 G(s,s)ds = h.$$

Hence (ii) of Theorem 1 is satisfied. Let $\vartheta \in P(\Phi,\chi,k,g)$. Using the same method followed in (2.16) results in $\chi(S\vartheta) > k$ for $\vartheta \in P(\Phi,\chi,k,g)$. Hence, (iii) of Theorem 1 holds.

Finally, we omit (iv) because $\tau(S\vartheta) < l = 0$ is not possible. Theorem 1 implies that fractional BVP (2.14)-(2.15) has at least three positive solutions ϑ_1^*, ϑ_2^* and ϑ_3^* such that

$$\|\vartheta_1^*\| < h, \;\; k < \chi(\vartheta_2^*), \;\; \|\vartheta_3^*\| > h, \;\; \chi(\vartheta_3^*) < k.$$

Moreover,

$$\vartheta_2^*(t) \geq \Omega\|\vartheta_2^*\| > \Omega\chi(\vartheta_2^*) > \Omega k > \Omega KD_1 \geq z(t), \;\; t \in [0,1],$$
$$\vartheta_3^*(t) \geq \Omega\|\vartheta_3^*\| > \Omega h \geq \Omega KD_1 \geq z(t), \;\; t \in [0,1].$$

$\vartheta_2 = \vartheta_2^* - z$, $\vartheta_3 = \vartheta_3^* - z$ are two positive solutions of (1.1)-(1.2). This completes the proof. ☐

Example 1. Consider the fractional boundary value problem

$$\begin{cases} D^{3/2}\phi(t) + f(t,\phi(t)) = 0, \quad t \in (0,1), \\ \phi(0) - \frac{5}{2}\phi'(0) = \sum_{j=1}^{2} a_j I^{1/2}\phi(\eta_j), \\ \phi(1) + \frac{1}{2}\phi'(1) = \sum_{j=1}^{2} b_j I^{1/2}\phi(\xi_j), \end{cases} \tag{2.17}$$

in which $\upsilon = \frac{3}{2}, \sigma = \frac{5}{2}$, $\rho = \frac{1}{2}$, $q = \frac{1}{2}$, $n > 3, n = 4$, $a_1 = \frac{1}{4}$, $b_1 = \frac{1}{3}$, $\eta_1 = \frac{1}{4}$, $\xi_1 = \frac{1}{3}$, $a_2 = \frac{1}{8}$, $b_2 = \frac{1}{2}$, $\eta_2 = \frac{1}{8}$, $\xi_2 = \frac{1}{2}$.

$$f(t,\vartheta)=\begin{cases}-\frac{1}{2}cos\frac{\pi}{2}t+\frac{\vartheta}{20000}, & t\in[0,1],\vartheta\in[0,50],\\ -\frac{1}{2}cos\frac{\pi}{2}t+\frac{\vartheta}{20000}+\frac{100000}{29}(\vartheta-50), & t\in[0,1],\vartheta\in[50,52.9],\\ -\frac{1}{2}cos\frac{\pi}{2}t+\frac{\vartheta}{20000}+10000, & t\in[0,1],\vartheta\in[52.9,23000],\\ -\frac{1}{2}cos\frac{\pi}{2}t+\frac{23}{20}+10000, & t\in[0,1],\vartheta\in[23000,\infty),\end{cases}$$

Through calculation, we get $\omega_1=0.106$, $\omega_2=0.266$, $\Lambda=2.037$, $\varphi=2.132$, $D_1=42.53$, $\mu=0.105$, $\zeta=2.11$, $\Omega=0.049$. Let $K=1$, $N=19$, $h=50$, $k=55$, $g=23000$, then $f(t,\vartheta)$ satisfies

$$\begin{aligned}&f(t,\vartheta)\geq -K=-1,\text{for}t\in[0,1],\\ &f(t,\vartheta)\leq \frac{g}{\zeta}-K\approx 10898.47,\text{for}t\in[0,1],\vartheta\in[0,23000],\\ &f(t,\vartheta)\geq \frac{kN}{\mu}-K\approx 9951.38,\text{for}t\in[0,1],\vartheta\in[52.9,23000],\\ &f(t,\vartheta)\leq \frac{h}{\zeta}-K\approx 22,69,\text{for}t\in[0,1],\vartheta\in[0,50].\end{aligned}$$

We conclude that all the assumptions of Theorem 2 are verified, thus problem (2.17) has at least two positive solutions.

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