Peter Scholze's lectures on p-adic geometry,* Fall 2014

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Contents

1 Introduction, 2 September

1.1 Motivation: Drinfeld, L. Lafforgue and V. Lafforgue

The starting point for this course is Drinfeld's work [Dri80] on the global Langlands correspondence over function fields. Fix $C/\mathbf{F}_q$ a smooth projective geometrically connected curve.

Definition 1.1.1. A *shtuka* of rank n over an $\mathbf{F}_q$-scheme S is a vector bundle $\mathcal{E}$ over $C \times_{\mathbf{F}_q} S$, together with a meromorphic isomorphism $\phi_\mathcal{E} \colon \mathrm{Frob}_S^* \mathcal{E} \to \mathcal{E}$. This means that $\phi_\mathcal{E}$ is defined on an open subset $U \subset C \times_{\mathbf{F}_q} S$ which is fiberwise dense in C. Here, $\mathrm{Frob}_S \colon S \to S$ always refers to the qth power Frobenius map.

The role of Frobenius is very important in this story. We remark that *geometric Langlands* studies the stack $\mathrm{Bun}_{\mathrm{GL}_n}$ of rank n vector bundles on C, even in circumstances where C is a complex curve. Whereas *arithmetic Langlands* studies $\mathrm{Bun}_{\mathrm{GL}_n}$ together with its Frobenius map, which gives you moduli spaces of shtukas (which roughly correspond to Frobenius fixed points).

Suppose we are given a shtuka $(\mathcal{E}, \phi_\mathcal{E})$ over $S = \operatorname{Spec} k$, where k is algebraically closed. We can attach to it the following data:

1. Points $x_1, \dots, x_m \in C(k)$, the points of indeterminacy of $\phi_\mathcal{E}$. We call these points the *legs* of the shtuka.

2. For each $i = 1, \dots, m$, a conjugacy class of cocharacters $\mu_i \colon \mathbf{G}_\mathrm{m} \to \mathrm{GL}_n$.

We consider the ordered set of points $x_1, \dots, x_m$ to be packaged along with the shtuka. The second item deserves some explanation: let $i \in \{1, \dots, m\}$, and let t_i be a uniformizing parameter at x_i. We have the completed stalks $(\mathrm{Frob}_S^* \mathcal{E})^\wedge_{x_i}$ and $\mathcal{E}^\wedge_{x_i}$. These are free rank n modules over $\mathcal{O}^\wedge_{C,x_i} \cong k[\![t_i]\!]$, and their generic fibres are identified using $\phi_\mathcal{E}$. In other words, we have two $k[\![t_i]\!]$-lattices in the same n-dimensional $k(\!(t_i)\!)$-vector space.

By the theory of elementary divisors, there exists a basis $e_1, \dots, e_n$ of $\mathcal{E}^\wedge_{x_i}$ such that $t_i^{k_1} e_1, \dots, t_i^{k_n} e_n$ is a basis of $\mathrm{Frob}_S^* \mathcal{E}_{x_i}$, where $k_1 \geq \cdots \geq k_n$. These integers are well-defined. Another way to package this data is as a conjugacy class of cocharacters $\mu_i \colon \mathbf{G}_\mathrm{m} \to \mathrm{GL}_n$, which in turn corresponds to a representation of the Langlands dual group, which is just GL_n again.

So now we fix discrete data: an integer m, and a collection of cocharacters $\{\mu_1, \dots, \mu_m\}$. Then there exists a moduli space $\mathrm{Sht}_{m,\{\mu_1,\dots,\mu_m\}}$ whose k-points classify the following data:

1. An m-tuple of points $(x_1, \dots, x_m)$ of $C(k)$, together with
2. A shtuka $(\mathcal{E}, \phi_{\mathcal{E}})$, for which $\phi_{\mathcal{E}}$ is regular outside of $\{x_1, \dots, x_m\}$ and for which the relative positions of $\mathcal{E}^{\wedge}_{x_i}$ and $(\mathrm{Frob}_S^* \mathcal{E})^{\wedge}_{x_i}$ is described by the cocharacter μ_i.

$\mathrm{Sht}_{m,\{\mu_1,\dots,\mu_m\}}$ is a Deligne-Mumford stack, which is unfortunately not quasi-compact. Let $f\colon \mathrm{Sht}_{m,\{\mu_1,\dots,\mu_m\}} \to C^m$ which maps a shtuka onto its m-tuple of legs. One can think of $\mathrm{Sht}_{m,\{\mu_1,\dots,\mu_m\}}$ as an equal-characteristic analogue of Shimura varieties. However, an important difference is that Shimura varieties live over $\mathrm{Spec}\,\mathbf{Z}$ (or some open subset thereof), and not anything like "$\mathrm{Spec}\,\mathbf{Z} \times \mathrm{Spec}\,\mathbf{Z}$".

In order to construct Galois representations, we consider the cohomology of $\mathrm{Sht}_{m,\{\mu_1,\dots,\mu_m\}}$. In this introduction we only consider the middle cohomology (middle being relative to the map f). Let $d = \dim \mathrm{Sht}_{m,\{\mu_1,\dots,\mu_m\}} - m$, and consider the cohomology $R^d f_! \overline{\mathbf{Q}}_\ell$, an étale sheaf on C^m. (This is not constructible, as f is not of finite type, but we can ignore this for now.)

Let $\eta = \mathrm{Spec}\, F$ be the generic point of C, and let η^m be the generic point of C^m. The generic fiber of this sheaf is $(R^d f_! \overline{\mathbf{Q}}_\ell)_{\overline{\eta}^m}$, which carries an action of $\mathrm{Gal}(\overline{\eta}^m/\eta^m)$. This last group the Galois group of the function field of the product C^m. This group is bigger than the product of m copies of $\mathrm{Gal}(\overline{\eta}/\eta)$, but we can control this by taking into account the partial Frobenii.

For $i = 1, \dots, m$, we have a morphism $F_i\colon C^m \to C^m$, which is Frob_C on the ith component, and the identity everywhere else. We will see that there are canonical commuting isomorphisms $F_i^*(R^d f_! \overline{\mathbf{Q}}_\ell) \cong R^d f_! \overline{\mathbf{Q}}_\ell$. We now apply an important lemma of Drinfeld:

Lemma 1.1.2. *For $U \subset C$ a dense open subset, let $\pi_1(U^m/\text{partial Frob.})$ classify finite étale covers of U^m equipped with partial Frobenii. Then*

$$\pi_1(U^m/\textit{partial Frob.}) \cong \pi_1(U) \times \cdots \times \pi_1(U) \ \textit{(m copies)}.$$

The lemma shows that if $\mathcal{F}$ is a local system on U^m, which comes equipped with isomorphisms $F_i^*\mathcal{F} \cong \mathcal{F}$, then the action of $\mathrm{Gal}(\overline{\eta}^m/\eta^m)$ on $\mathcal{F}_{\overline{\eta}^m}$ factors through $\mathrm{Gal}(\overline{\eta}/\eta)^m$.

Pedantic note: We can't literally apply this lemma to our sheaf $R^d f_! \overline{\mathbf{Q}}_\ell$ (because it isn't constructible), but once you take care of this, you really do get a space with an action of $\mathrm{Gal}(\overline{\eta}/\eta)^m$.

One can add level structures to these spaces of shtukas, parametrized by finite closed subschemes $N \subset C$ (that is, effective divisors). A level N structure on $(\mathcal{E}, \phi_{\mathcal{E}})$ is then a trivialization of the pullback of $\mathcal{E}$ to N in a way which is compatible with $\phi_{\mathcal{E}}$ (we will review the precise definition later).

As a result we get a family of stacks $\mathrm{Sht}_{m,\{\mu_1,\dots,\mu_m\},N}$ and morphisms

$$f_N\colon \mathrm{Sht}_{m,\{\mu_1,\dots,\mu_m\},N} \to (C\backslash N)^m.$$

The stack $\mathrm{Sht}_{m,\{\mu_1,\dots,\mu_m\},N}$ carries an action of $\mathrm{GL}_n(\mathcal{O}_N)$, by altering the trivialization of $\mathcal{E}$ on N.

Passing to the limit, you get a big representation of $\mathrm{GL}_n(\mathbf{A}_F) \times G_F \times \cdots \times G_F$ on $\varinjlim_N R^d(f_N)_!\overline{\mathbf{Q}}_\ell$, where G_F is the absolute Galois group of F. Roughly, the way one expects this space to decompose is as follows:

$$\varinjlim_N R^d(f_N)_!\overline{\mathbf{Q}}_\ell = \bigoplus_\pi \pi \otimes r_1 \circ \sigma(\pi) \otimes \cdots \otimes r_m \circ \sigma(\pi),$$

where

- π runs over cuspidal automorphic representations of $\mathrm{GL}_n(\mathbf{A}_F)$,
- $\sigma(\pi)\colon G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$ is the corresponding L-parameter, and
- $r_i\colon \mathrm{GL}_n \to \mathrm{GL}_{n_i}$ is the highest weight representation corresponding to μ_i.

Drinfeld ($n = 2$, [Dri80]) and L. Lafforgue (general n, [Laf02]) considered the case of $m = 2$, with μ_1 and μ_2 corresponding to the n-tuples $(1, 0, \cdots, 0)$ and $(0, \cdots, 0, -1)$ respectively. These cocharacters correspond to the tautological representation $r_1\colon \mathrm{GL}_n \to \mathrm{GL}_n$ and its dual r_2. Then they were able to prove the claimed decomposition, and in doing so constructed a bijection $\pi \mapsto \sigma(\pi)$ between cuspidal automorphic representations of $\mathrm{GL}_n(\mathbf{A}_F)$ and irreducible n-dimensional representations of G_F.

V. Lafforgue [Laf] considered general reductive groups G in place of GL_n. He didn't quite prove this conjecture, but he did construct L-parameters for all cuspidal automorphic representations of G, using all of the spaces $\mathrm{Sht}_{m,\{\mu_1,\dots,\mu_n\}}$ simultaneously in a crucial way. This overcomes a problem in defining L-parameters $\sigma(\pi)$ using Shimura varieties associated to the datum (G, μ): there you can only "see" $r \circ \sigma(\pi)$, where r is the heighest weight representation corresponding to a single minuscule cocharacter μ.

1.2 The possibility of shtukas in mixed characteristic

Of course, it would be great to do something similar for number fields. But the first immediate problem is that such a space of shtukas would live over a product like $\operatorname{Spec} \mathbf{Z} \times \operatorname{Spec} \mathbf{Z}$, where the product is over $\mathbf{F}_1$ somehow.

The first aim of this course is to describe the completion of $\operatorname{Spec}\mathbf{Z} \times \operatorname{Spec}\mathbf{Z}$ at (p,p), which we will give a rigorous definition for.

Open problem: Describe the completion of $\operatorname{Spec}\mathbf{Z} \times \operatorname{Spec}\mathbf{Z}$ along the diagonal. Alas, we may not be able to hope for more than this. As far as we know, $\operatorname{Spec}\mathbf{F}_p \times \operatorname{Spec}\mathbf{F}_\ell \subset \operatorname{Spec}\mathbf{Z} \times \operatorname{Spec}\mathbf{Z}$ does not make sense.

What we will actually do is construct something like $\operatorname{Spa}\mathbf{Q}_p \times \operatorname{Spa}\mathbf{Q}_p$. It lives in a world of nonarchimedean analytic geometry.

It may help to spell out the equal characteristic analogue of this, where nothing terribly esoteric happens. The product

$$\operatorname{Spa}\mathbf{F}_p((t)) \times_{\operatorname{Spa}\mathbf{F}_p} \operatorname{Spa}\mathbf{F}_p((t))$$

exists as an adic space, namely a punctured open unit disc. Indeed for all nonarchimedean fields $K/\mathbf{F}_p$, we have

$$\operatorname{Spa}\mathbf{F}_p((t)) \times_{\operatorname{Spa}\mathbf{F}_p} \operatorname{Spa} K = \mathbf{D}^*_K = \{x | 0 < |x| < 1\}.$$

In particular $\operatorname{Spa}\mathbf{F}_p((t)) \times_{\operatorname{Spa}\mathbf{F}_p} \operatorname{Spa}\mathbf{F}_p((u))$ is a punctured unit disc in two different ways: one where u is a parameter and t is in the field of constants, and one where the reverse happens.

There is a local version of Drinfeld's lemma, which we present for $m = 2$. We have the punctured disc $\mathbf{D}^*_{\mathbf{F}_p((t))}$ with parameter u, which comes with a Frobenius action ϕ, which takes t to t and u to u^p. This is a totally discontinuous action, so we can take the quotient $X = D^*_{\mathbf{F}_p((t))}/\phi^{\mathbf{Z}}$.

Lemma 1.2.1 ("local Drinfeld lemma", Fargues-Fontaine, Weinstein)**.** $\pi_1(X) \cong G_{\mathbf{F}_p((t))} \times G_{\mathbf{F}_p((t))}$.

It turns out there is an analogue of this story for $\mathbf{Q}_p$. Here is one model for the product $\operatorname{Spa}\mathbf{Q}_p \times \operatorname{Spa}\mathbf{Q}_p$, specified by picking one of the factors.

Definition 1.2.2. $\operatorname{Spa}\mathbf{Q}_p \times \operatorname{Spa}\mathbf{Q}_p = \widetilde{\mathbf{D}}^*_{\mathbf{Q}_p}/\mathbf{Z}_p^*$, the quotient being taken in a formal sense.

Here we see one of the $\mathbf{Q}_p$s appearing as the field of scalars of this object, but the other copy appears in a strange way.

Here, $\mathbf{D}_{\mathbf{Q}_p} = \{x | \, |x| < 1\} \hookrightarrow \mathbf{G}_{\mathrm{m}}$ is a group, and even a $\mathbf{Z}_p$-module , via $x \mapsto 1 + x$. Let

$$\widetilde{\mathbf{D}}_{\mathbf{Q}_p} = \varprojlim_{x \mapsto (1+x)^p - 1} \mathbf{D}_{\mathbf{Q}_p}$$

This is a pre-perfectoid space, which carries the structure of a $\mathbf{Q}_p$-vector space. Thus $\widetilde{\mathbf{D}}^*_{\mathbf{Q}_p}$ has an action of $\mathbf{Q}_p^\times$, and so we consider the quotient by $\mathbf{Z}_p^\times$. Note that this does not exist in the category of adic spaces!

On $\widetilde{\mathbf{D}}^*_{\mathbf{Q}_p}/\mathbf{Z}_p^\times$, we have an operator ϕ, corresponding to $p \in \mathbf{Q}_p^\times$. Let $X = (\widetilde{D}^*_{\mathbf{Q}_p}/\mathbf{Z}_p^*)/\phi^{\mathbf{Z}} = \widetilde{D}^*_{\mathbf{Q}_p}/\mathbf{Q}_p^\times$. Then

$$\pi_1(X) \cong G_{\mathbf{Q}_p} \times G_{\mathbf{Q}_p}.$$

The first aim of the course is to introduce the category of "diamonds", which contains these objects. In that category, the description of $\operatorname{Spa}\mathbf{Q}_p \times \operatorname{Spa}\mathbf{Q}_p$ becomes a proposition, rather than an *ad hoc* definition.

The second aim is to define spaces of local shtukas in this setup.

Let's go back to the analogy between function fields and number fields.

In the function field context, we have moduli spaces of shtukas. These are associated to data $(G, \{\mu_1, \dots, \mu_m\})$ where G is a reductive group and the μ_i are conjugacy classes of cocharacters of G (over an algebraic closure). These live over a product of m copies of the curve.

In the number field context, we have Shimura varieties associated with data (G, μ), where G is a reductive group and μ is a conjugacy class of *minuscule* cocharacters. This lives over one copy of the "curve" $\operatorname{Spec}\mathbf{Z}$. These are moduli spaces of abelian varieties with extra structure.

In this course we look at local analogues of these spaces. For function fields, these are moduli spaces of local shtukas, as studied by Pink, Hartl, Viehmann, and others. In the number field case, one has Rapoport-Zink spaces, where are moduli space of p-divisible groups. It was recently suggested by Rapoport-Viehmann that there should exist local Shuimura varieties which don't have anything to do with p-divisible groups, but which are still attached to data of the form (G, μ).

The goal of the entire course is to

1. Define a notion of local shtuka in mixed characteristic.

2. Construct moduli spaces of these for any $(G, \{\mu_1, \dots, \mu_m\})$, for any collection of cocharacters, living over m copies of $\operatorname{Spa}\mathbf{Q}_p$, considered as a diamond.

3. Show that these generalize RZ spaces and specialize to local Shimura varieties (which should be classical rigid spaces). For this we will have to relate local shtukas to p-divisible groups.

The hope is to carry out V. Lafforgue's program to define L-parameters for smooth representations of p-adic group.

We leave open the problem of defining shtukas over $\operatorname{Spec}\mathbf{Z}$, and moduli spaces of such.

2 Adic spaces, 4 September

2.1 Overview: formal schemes and their generic fibres

Today we will give a crash course in nonarchmidean geometry, since it plays such a vital role. The reference is Huber, [Hub93]. See also Wedhorn's notes on adic spaces.

Recall that a *formal scheme* is a topologically ringed space which is locally of the form $\operatorname{Spf} A$. We are only interested in the case that A is an *adic ring*, meaning a topological ring which is separated and complete for the topology induced by an ideal $I \subset A$, called an *ideal of definition*. For an adic ring A, $\operatorname{Spf} A$ is the set of open prime ideals of A. This is given a topology and a sheaf of topological rings much in the same way as is done for the usual spectrum of a ring. The category of formal schemes contains the category of schemes as a full subcategory, via a functor which carries $\operatorname{Spec} A$ onto $\operatorname{Spf} A$, where A is considered with its discrete topology.

The goal here is to construct a category which contains formal schemes and their generic fibers (which are often rigid spaces) as full subcategories. There is a fully faithful functor $X \mapsto X^{\text{ad}}$ from the category of formal schemes[1] to the category of adic spaces. Before giving precise definitions, let us explain a typical example.

The formal scheme $\operatorname{Spf} \mathbf{Z}_p$ is a one-point space (consisting of the open prime ideal $p\mathbf{Z}_p$). Its corresponding adic space $(\operatorname{Spf} \mathbf{Z}_p)^{\text{ad}}$ has two points: a generic point η and a special point s. Thus as a topological space it is the same as $\operatorname{Spec} \mathbf{Z}_p$.

If X is a formal scheme over $\operatorname{Spf} \mathbf{Z}_p$, then we can define the *generic fibre* of X by setting

$$X_\eta = X^{\text{ad}} \times_{(\operatorname{Spf} \mathbf{Z}_p)^{\text{ad}}} \{\eta\}.$$

For instance if $X = \operatorname{Spf} \mathbf{Z}_p[\![T]\!]$ is the formal open unit disc over $\mathbf{Z}_p$, then X_η is the "adic open disc" over $\mathbf{Q}_p$. It contains as points all $\operatorname{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$-orbits of elements $x \in \overline{\mathbf{Q}}_p$ with $|x| < 1$ (and many more exotic points besides these). The construction is in very much the same spirit as Berthelot's generic fibre functor, for which the reference is Berthelot's paper *Cohomologie rigide et cohomologie rigide à supports propres*.

Just as schemes are built out of affine schemes associated to rings, adic spaces are built out of affinoid adic spaces associated to pairs of rings

[1] At least, this functor is well-defined on formal schemes X whose topology is locally defined by a finitely generated ideal. Everything is fine for locally noetherian formal schemes, but many of the objects we will consider are not at all noetherian.

(A, A^+). The affinoid adic space associated to such a pair is written $\mathrm{Spa}(A, A^+)$, the *adic spectrum*.

Today we will define which pairs (A, A^+) are allowed, and define $\mathrm{Spa}(A, A^+)$ as a topological space.

2.2 Huber rings

Definition 2.2.1. A topological ring A is *Huber*[2] if A admits an open subring $A_0 \subset A$ which is adic with respect to a finitely generated ideal of definition. That is, there exists a finitely generated ideal $I \subset A_0$ such that $\{I^n | n \geq 0\}$ forms a basis of open neighborhoods of 0. Any such A_0 is called a *ring of definition* of A.

Note that A is not assumed to be I-adically complete. One can always take the I-adic completion $\widehat{A}$ and the theory remains essentially unchanged: $\widehat{A}$ is f-adic, and $\widehat{A}_0 \subset \widehat{A}$ is just the closure of A_0, which is $\varprojlim A_0/I^n$.

We now give three examples to indicate that adic spaces encompass schemes, formal schemes, and rigid spaces, respectively.

1. (Schemes) Any discrete ring A is Huber, with any $A_0 \subset A$ allowed (take $I = 0$ as an ideal of definition).

2. (Formal schemes) An adic ring A is Huber if it has a finitely generated ideal of definition. In that case, $A_0 = A$ is a ring of definition.

3. (Rigid spaces) Let A_0 be any ring, let $g \in A_0$ be a nonzero-divisor, and let $A = A_0[g^{-1}]$, equipped with the topology making $\{g^n A_0\}$ a basis of open neighborhoods of 0. This is Huber, with ring of definition A_0 and ideal of definition gA_0. For example, if A is a Banach algebra over a nonarchimedean field K, we can take $A_0 \subset A$ to be the unit ball, and $g \in K$ an nonzero element with $|g| < 1$. Then A is a Huber ring of this type.

The Banach algebras relevant to rigid analytic geometry arise as quotients of the Tate algebra $A = \mathbf{Q}_p\langle T_1, \dots, T_n\rangle$, consisting of power series in $T_1, \dots, T_n$ whose coefficients tend to 0. This is a Banach $\mathbf{Q}_p$-algebra with unit ball $A_0 = \mathbf{Z}_p\langle T_1, \dots, T_n\rangle$.

Definition 2.2.2. A subset S of a topological ring A is *bounded* if for all open neighborhoods U of 0 there exists an open neighborhood V of 0 such that $VS \subset U$.

[2] We propose to use the term *Huber ring* to replace Huber's terminology *f-adic ring*. The latter poses a slight threat of confusion since there is no variable f.

In verifying this condition for subsets of Huber rings, you are allowed to shrink U, and without loss of generality you may assume that U is closed under addition, because after all $\{I^n\}$ forms a basis of open neighborhoods of 0.

Lemma 2.2.3. *A subring A_0 of a Huber ring A is a ring of definition if and only if it is open and bounded.*

Proof. If A_0 is a ring of definition, it is open (by definition). Let U be an open neighborhood of 0 in A. Without loss of generality $U = I^n$, with $n \gg 0$. But then of course $V = I^n$ suffices.

The converse is left as an easy exercise. □

Definition 2.2.4. A Huber ring A is *Tate* if it contains a topologically nilpotent unit $g \in A$.

Proposition 2.2.5.

1. *If $A = A_0[g^{-1}]$ is as in Example 3, then A is Tate.*
2. *If A is Tate with topologically nilpotent unit g, and $A_0 \subset A$ is any ring of definition, then there exists n large enough so that $g^n \in A_0$, and then A_0 is g^n-adic. Furthermore $A = A_0[(g^n)^{-1}]$.*
3. *Suppose A is Tate with g as above and A_0 a ring of definition. A subset $S \subset A$ is bounded if and only if $S \subset g^{-n}A_0$ for some n.*

Proof.

1. Since $g \in A = A_0[g^{-1}]$ is a topologically nilpotent unit, A is Tate by definition.
2. Let $I \subset A_0$ be an ideal of definition. Since g is topologically nilpotent, we can replace g by g^n for n large enough to assume that $g \in I$. Since gA_0 is the preimage of A_0 under the continuous map $g^{-1}\colon A \to A$, we have that $gA_0 \subset A_0$ is open, and thus it contains I^m for some m. Thus we have $g^m A_0 \subset I^m \subset gA_0$, which shows that A_0 is g-adic.

   It remains to show that $A = A_0[g^{-1}]$. Clearly $A_0[g^{-1}] \hookrightarrow A$. If $x \in A$ then $g^n x \to 0$ as $n \to \infty$, since g is topologically nilpotent. Thus there exists n with $g^n x \in A_0$, and therefore $x \in A_0[g^{-1}]$.
3. Left as exercise.

□

We remark that if A is a complete Tate ring and $A_0 \subset A$ is a ring of definition, with $g \in A_0$ a topologically nilpotent unit in A, then one can define a norm $|\cdot| : A \to \mathbf{R}_{\geq 0}$ by

$$|a| = \inf_{n\colon g^n a \in A_0} 2^n$$

Thus $|g| = 1/2$ and $\left|g^{-1}\right| = 2$. Note that this really is a norm: if $|a| = 0$, then $a \in g^n A_0$ for all $n \geq 0$, and thus $a = 0$. Under this norm, A is a Banach ring with whose unit ball is A_0.

Note that this norm is not in general multiplicative. If $A = \mathbf{Q}_p(\sqrt{p})$ and $g = p$, then the above definition gives $\left|\sqrt{p}\right| = 1$, but $|p| = 1/2$. One can fix this by considering the modification

$$|a| = \inf_{n,m\colon g^n a^m \in A_0} 2^{n/m},$$

which gives the "correct" norm on $\mathbf{Q}_p(\sqrt{p})$. But also note that if $A = \mathbf{Q}_p\langle X, Y\rangle/(XY - p)$, with ring of definition $A_0 = \mathbf{Z}_p\langle X, Y\rangle/(XY - p)$, then the above definition gives $|X| = |Y| = 1$, but $|XY| < 1$. The most we can really say in generality is that $|g|\left|g^{-1}\right| = 1$.

This construction gives an equivalence of categories between the category of separated and complete Tate rings (with continuous homomorphisms), and the category of Banach rings A which admit an element $g \in A^\times$, $|g| < 1$ such that $|g|\left|g^{-1}\right| = 1$ (with bounded homomorphisms).

Definition 2.2.6. Let A be a Huber ring. An element $x \in A$ is *power-bounded* if $\{x^n | n \geq 0\}$ is bounded. Let $A^\circ \subset A$ be the subring of power-bounded elements (one checks easily that it really is a ring).

Example 2.2.7. If $A = \mathbf{Q}_p\langle T\rangle$, then $A^\circ = \mathbf{Z}_p\langle T\rangle$, which as we have seen is a ring of definition. However, if $A = \mathbf{Q}_p[T]/T^2$, then $A^\circ = \mathbf{Z}_p \oplus \mathbf{Q}_p T$. Since A° is not bounded, it cannot be a ring of definition.

Proposition 2.2.8. *1. Any ring of definition $A_0 \subset A$ is contained in A°.*

2. A° is the filtered direct limit of the rings of definition $A_0 \subset A$. (The word filtered *here means that any two subrings of definition are contained in a third.)*

Proof. 1. For any $x \in A_0$, $\{x^n\} \subset A_0$ is bounded, so $x \in A^\circ$.

2. The poset of rings of definition is filtered: if A_0, A_0' are rings of definition, let $A_0'' = A_0 A_0'$ be the ring they generate. We show directly that A_0'' is bounded. Let $U \subset A$ be an open neighborhood of 0; we have to find V such that $VA_0'' \subset U$. Without loss of generality, U is closed under addition. Pick U_1 such that $U_1 A_0 \subset U$, and pick V such that $VA_0' \subset U_1$.

   A typical element of A_0'' is $\sum_i x_i y_i$, with $x_i \in A_0$, $y_i \in A_0'$. We have

   $$\left(\sum_i x_i y_i\right) V \subset \sum_i (x_i y_i V) \subset \sum_i (x_i U_1) \subset \sum_i U = U.$$

   Thus $VA_0'' \subset U$ and A_0'' is bounded.

   For the claim that A° is the union of the rings of definition of A: Let $x \in A^\circ$, and let A_0 be any ring of definition. Then $A_0[x]$ is still a ring of definition, since it is still bounded.

□

Definition 2.2.9. A Huber ring A is *uniform* if A° is bounded, or equivalently A° is a ring of definition.

We remark that if A is Tate and uniform, then A is reduced.

Definition 2.2.10.

1. Let A be a Huber ring. A subring $A^+ \subset A$ is a *ring of integral elements* if it is open and integrally closed and $A^+ \subset A^\circ$.
2. An *affinoid ring* is a pair (A, A^+), where A is Huber and and $A^+ \subset A$ is a ring of integral elements.

We remark that it is often the case that $A^+ = A^\circ$. On first reading one may make this assumption.

2.3 Continuous valuations

Definition 2.3.1. A *continuous valuation* on a topological ring A is a map

$$|\cdot| : A \to \Gamma_{|\cdot|} \cup \{0\}$$

into a totally ordered abelian group such that

1. $|ab| = |a|\,|b|$
2. $|a+b| \leq \max(|a|, |b|)$

3. $|1| = 1$

4. $|0| = 0$

5. (Continuity) For all $\gamma \in \Gamma_{|\cdot|}$, $\{a \in A | \, |a| < \gamma\}$ is open in A.

(Our convention is that ordered abelian groups Γ are written multiplicatively, and $\Gamma \cup \{0\}$ means the ordered monoid with $\gamma > 0$ for all $\gamma \in \Gamma$. Of course, $\gamma 0 = 0$.)

Two continuous valuations are *equivalent* if $|a| \geq |b|$ if and only if $|a|' \geq |b|'$. In that case, after replacing $\Gamma_{|\cdot|}$ by the subgroup generated by the image of A, and similarly for $\Gamma_{|\cdot|'}$, there exists an isomorphism $\Gamma_{|\cdot|} \cong \Gamma_{|\cdot|'}$ such that

$$\begin{array}{ccc} & & \Gamma_{|\cdot|} \cup \{0\} \\ & \overset{|\cdot|}{\nearrow} & \\ A & & \downarrow \cong \\ & \underset{|\cdot|'}{\searrow} & \\ & & \Gamma_{|\cdot|'} \cup \{0\} \end{array}$$

commutes.

Note that the kernel of $|\cdot|$ is an ideal of A.

Thus continuous valuations are like the multiplicative seminorms of Berkovich's theory. At this point we must apologize that continuous valuations are not called "continuous norms", since after all they are written multiplicatively. On the other hand, we want to consider value groups of higher rank (and indeed this is the point of departure from Berkovich's theory), which makes the use of the word "norm" somewhat awkward.

Definition 2.3.2. $\mathrm{Spa}(A, A^+)$ is the set of equivalence classes of continuous valuations $|\cdot|$ on A such that $|A^+| \leq 1$.

For $x \in \mathrm{Spa}(A, A^+)$, write $g \mapsto |g(x)|$ for a choice of corresponding valuation.

The topology on $\mathrm{Spa}(A, A^+)$ is generated by open subsets of the form

$$\left\{ x \, \middle| \, |f(x)| \leq |g(x)| \neq 0 \right\},$$

with $f, g \in A$.

The shape of these open sets is dictated by the desired properties that both $\{x | |f(x)| \neq 0\}$ and $\{x | |f(x)| \leq 1\}$ be open. These desiderata combine features of classical algebraic geometry and rigid geometry, respectively.

Huber shows that the topological space $\mathrm{Spa}(A, A^+)$ is reasonable (at least from the point of view of an algebraic geometer!).

Definition 2.3.3 ([Hoc69])**.** A topological space T is *spectral* if the following equivalent conditions are satisfied.

1. $T \cong \mathrm{Spec}\, R$ for some ring R.

2. $T \cong \varprojlim T_i$ where T_i is a finite T_0-space. (Recall that T_0 means that given any two points, there exists an open set which contains one but not the other.)

3. There exists a basis of quasi-compact opens of T which is stable under finite intersection, and also T is *sober*: every irreducible closed subset has a unique generic point.

(Exercise: Let T_i be the topological space consisting of the first i primes (taken to be closed), together with a generic point whose closure is all of T_i. Let $\mathrm{Spec}\, \mathbf{Z} \to T_i$ be the map which sends the first i primes to their counterparts in T_i, and sends everything else to the generic point. Then there is a homeomorphism $\mathrm{Spec}\, \mathbf{Z} \cong \varprojlim T_i$.)

Example 2.3.4. Let R be a discrete ring. Then $\mathrm{Spa}(R, R)$ is the set of valuations on R bounded by 1. We list the points of $\mathrm{Spa}(\mathbf{Z}, \mathbf{Z})$:

1. A point η, which takes all nonzero integers to 1,

2. A special point s_p for each prime p, which is the composition $\mathbf{Z} \to \mathbf{F}_p \to \{0, 1\}$, where the second arrow sends all nonzero elements to 1,

3. A point η_p for each prime p, which is the composition $\mathbf{Z} \to \mathbf{Z}_p \to p^{\mathbf{Z}_{\leq 0}} \cup \{0\}$, where the second arrow is the usual p-adic absolute value.

Then $\{s_p\}$ is closed, whereas $\overline{\{\eta_p\}} = \{\eta_p, s_p\}$, and $\overline{\{\eta\}} = \mathrm{Spa}(\mathbf{Z}, \mathbf{Z})$.

In general we have a map $\mathrm{Spec}\, R \to \mathrm{Spa}(R, R)$, which sends $\mathfrak{p}$ to the valuation $R \to \mathrm{Frac}(R/\mathfrak{p}) \to \{0, 1\}$, where the second map is 0 on 0 and 1 everywhere else. There's also a map $\mathrm{Spa}(R, R) \to \mathrm{Spec}\, R$, which sends a valuation to its kernel. The composition of these two maps is the identity on $\mathrm{Spec}\, R$.

In fact we get a fully faithful functor from schemes to adic spaces in general.

One final remark: if K is a field, a subring $R \subset K$ is a *valuation ring* if whenever $f \in K^\times \backslash R$, we have $f^{-1} \in R$. If $A \subset K$ is any subring, the *Zariski-Riemann space* $\mathrm{Zar}(K, A)$ is the set of valuation rings $R \subset K$ containing A. This set is given a topology which makes $\mathrm{Zar}(K, A)$ a quasi-compact ringed space (see Matsumura, *Commutative Ring Theory*, Thm. 10.5). As an application, if K is a function field with field of constants k (meaning that K/k is purely transcendental of transcendence degree 1), then $\mathrm{Zar}(K, k)$ is the smooth projective curve over k whose function field is K.

Each valuation ring $R \in \mathrm{Zar}(K, A)$ induces a continuous valuation $|\ |_R \in \mathrm{Spa}(K, A)$, where K is given the discrete topology, here $\Gamma_{|\ |_R}$ is the value group of R (*i.e.*, the lattice of R-submodules of K of the form fR, $f \in K$, and then $|f|_R = fR$. Conversely, every continuous valuation $x \in \mathrm{Spa}(K, A)$ gives rise to a valuation ring $R = \{f \in K \mid |f(x)| \leq 1\}$ which contains A. We get a homeomorphism $\mathrm{Zar}(K, A) \cong \mathrm{Spa}(K, A)$.

3 Adic spaces II, 9 September

3.1 Complements on the topological space $\mathrm{Spa}(A, A^+)$

Arthur Ogus asked about the following tricky lemma, which helps show why the Huber condition is so important.

Lemma 3.1.1. *Let A be a ring, let M be an A-module, and let $I \subset A$ be a finitely generated ideal. Then for*

$$\widehat{M} = \varprojlim M/I^n M$$

the I-adic completion of M, one has $\widehat{M}/I\widehat{M} = M/IM$.

This implies that $\widehat{M}$ is I-adically complete.

Proof. Stacks project, Lemma 10.91.7. □

Today we define a structure (pre)sheaf $\mathcal{O}_X$ on $X = \mathrm{Spa}(A, A^+)$. The reference is [Hub94].

Recall from last time:

1. A Huber ring is a topological ring A which admits an open subring $A_0 \subset A$ which is adic with finitely generated ideal of definition.

2. A Huber pair is a pair (A, A^+), where A is a Huber ring and $A^+ \subset A$ is open and integrally closed.

Last time we constructed a topological space $X = \mathrm{Spa}(A, A^+)$ consisting of equivalence classes of continuous valuations $|\cdot|$ on A such that $|A^+| \leq 1$.

Theorem 3.1.2 (Huber)**.** *X is spectral (*i.e. *quasicompact, sober, and there is a basis for its topology consisting of quasicompact open subsets which is stable under intersection.*

Proposition 3.1.3. *Let $(\widehat{A}, \widehat{A}^+)$ be the completion of (A, A^+). Then there is a homeomorphism*

$$\mathrm{Spa}(\widehat{A}, \widehat{A}^+) \cong \mathrm{Spa}(A, A^+).$$

Today, all Huber pairs will be assumed complete. The next proposition shows that the adic spectrum $\mathrm{Spa}(A, A^+)$ is "large enough":

Proposition 3.1.4. *1. If $A \neq 0$ then $\mathrm{Spa}(A, A^+)$ is nonempty.*

2. $A^+ = \{f \in A | \, |f(x)| \leq 1, \text{ for all } x \in X\}$.

3. $f \in A$ is invertible if and only if for all $x \in X$, $|f(x)| \neq 0$.

3.2 Rational Subsets

Definition 3.2.1. Let $s_1, \ldots, s_n \in A$ and let $T_1, \ldots, T_n \subset A$ be finite subsets such that $T_i A \subset A$ is open for all i. We define a subset

$$U\left(\left\{\frac{T_i}{s_i}\right\}\right) = U\left(\frac{T_1}{s_1}, \ldots, \frac{T_n}{s_n}\right) = \{x \in X | \, |t_i(x)| \leq |s_i(x)| \neq 0, \text{ for all } t_i \in T_i\}$$

This is open because it is an intersection of a finite collection of the sort of opens which generate the topology on X. Subsets of this form are called *rational subsets.*

Note that a finite intersection of rational subsets is again rational, just by concatenating the data that define the individual rational subsets.

The following theorem shows that rational subsets are themselves adic spectra.

Theorem 3.2.2. *Let $U \subset \mathrm{Spa}(A, A^+)$ be a rational subset. Then there exists a complete Huber pair $(A, A^+) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ such that the map $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to \mathrm{Spa}(A, A^+)$ factors over U, and is universal for such maps. Moreover this map is a homeomorphism onto U. In particular, U is quasi-compact.*

Proof. (Sketch.) Choose s_i and T_i such that $U = U(\{T_i/s_i\})$. Choose $A_0 \subset A$ a ring of definition, $I \subset A_0$ a finitely generated ideal of definition. Take $(A, A^+) \to (B, B^+)$ such that $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ factors over U. Then

1. The s_i are invertible in B, so that we get a map $A[\{1/s_i\}] \to B$.

2. All t_i/s_i are of $|\cdot| \leq 1$ everywhere on $\mathrm{Spa}(B, B^+)$, so that $t_i/s_i \in B^+ \subset B^\circ$.

3. Since B° is the inductive limit of the rings of definition B_0, we can choose a B_0 which contains all t_i/s_i. We get a map

$$A_0[t_i/s_i | i = 1, \ldots, n, t_i \in T_i] \to B_0.$$

Endow $A_0[\{t_i/s_i\}]$ with the $IA_0[\{t_i/s_i\}]$-adic topology.

Lemma 3.2.3. *This defines a ring topology on $A[\{1/s_i\}]$ making $A_0[\{t_i/s_i\}]$ an open subring.*

The crucial point is to show that there exists n such that $\frac{1}{s_i} I^n \subset A_0[\{t_i/s_i\}]$, so that multiplication by $1/s_i$ can be continuous. It is enough to show that $I^n \subset T_i A_0$.

Lemma 3.2.4. *If $T \subset A$ is a subset such that $TA \subset A$ is open, then TA_0 is open.*

Proof. After replacing I with some power we may assume that $I \subset TA$. Write $I = (f_1, \ldots, f_k)$. There exists a finite set R such that $f_1, \ldots, f_k \in TR$.

Since I is topologically nilpotent, there exists n such that $RI^n \subset A_0$. Then for all $i = 1, \ldots, k$, $f_i I^n \subset TRI^n \subset TA_0$. Sum this over all i and conlude that $I^{n+1} \subset TA_0$. □

Back to the proof of the proposition. We have $A[\{1/s_i\}]$, a (non-complete) Huber ring. Let $A[\{1/s_i\}]^+$ be the integral closure of the image of $A^+[\{t/s_i\}]$ in $A[\{1/s_i\}]$.

Let $(A\langle\{T_i/s_i\}\rangle, A\langle\{T_i/s_i\}\rangle^+)$ be its completion, a Huber pair. This has the desired universal property.

For the claim that Spa of this pair is homeomorphic to U: Use that Spa doesn't change under completion. (Also that the operation of taking the integral closure doesn't change much, either.) □

Definition 3.2.5. Define a presheaf $\mathcal{O}_X$ of topological rings on $\mathrm{Spa}(A, A^+)$: If $U \subset X$ is rational, $\mathcal{O}_X(U)$ is as in the theorem. On a general open $W \subset X$, we put

$$\mathcal{O}_X(W) = \varprojlim_{U \subset W \text{ rational}} \mathcal{O}_X(U).$$

One defines $\mathcal{O}_X^+$ similarly.

Proposition 3.2.6. *For all $U \subset \mathrm{Spa}(A, A^+)$,*

$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) \,|\, |f(x)| \leq 1, \text{ all } x \in U\}.$$

In particular $\mathcal{O}_X^+$ is a sheaf if $\mathcal{O}_X$ is.

Theorem 3.2.7. *$\mathcal{O}_X$ is a sheaf in the following situations.*

1. *(Schemes) A is discrete.*

2. *(Formal schemes) A is finitely generated (as an algebra) over a noetherian ring of definition. This includes the case when X comes from a noetherian formal scheme.*

3. *(Rigid spaces) A is Tate and* strongly noetherian*: the rings*

$$A\langle X_1, \ldots, X_n \rangle = \left\{ \sum_{\underline{i}=(i_1,\ldots,i_n)\geq 0} a_{\underline{i}} T^{\underline{i}} \,\middle|\, a_{\underline{i}} \in A,\ a_{\underline{i}} \to 0 \right\}$$

   are noetherian for all $n \geq 0$.

Example 3.2.8. $A = \mathbf{C}_p$ is not covered by case 2, because $\mathcal{O}_{\mathbf{C}_p}$ is not noetherian. But $\mathbf{C}_p\langle T_1, \ldots, T_n \rangle$ is noetherian, so case 3 applies. The same goes for $A = \mathbf{C}_p\langle T_1, \ldots, T_n \rangle$.

3.3 Example: the adic open unit disc over $\mathbf{Z}_p$

Let us make the following convenient abbreviation: whenever A is a Huber ring, write $\mathrm{Spa}\, A$ for $\mathrm{Spa}(A, A^\circ)$.

The adic spectrum $\mathrm{Spa}\, \mathbf{Z}_p$ consists of two points, a special point and a generic point. The same is true for $\mathrm{Spa}(\mathbf{F}_p[\![T]\!], \mathbf{F}_p[\![T]\!])$ (or $\mathrm{Spa}\, A$ for any DVR A for that matter).

But now consider $A = \mathbf{Z}_p[\![T]\!]$ with the (p, T)-adic topology; this is a complete regular local ring of dimension 2. Then $\mathrm{Spa}\, A$ falls under case (2) of Thm. 3.2.7. Let us try to describe $X = \mathrm{Spa}\, A$.

There is a unique point $x_{\mathbf{F}_p} \in X$ whose kernel is open. It is the composition $\mathbf{Z}_p[\![T]\!] \to \mathbf{F}_p \to \{0,1\}$, where the second arrow is 1 on nonzero elements. Let $\mathcal{Y} = X \setminus \{x_{\mathbf{F}_p}\}$. All points in $\mathcal{Y}$ have non-open kernel, *i.e.* they are *analytic*.

Definition 3.3.1. Let (A, A^+) be a Huber pair. A point $x \in \operatorname{Spa}(A, A^+)$ is *non-analytic* if the kernel of $|\cdot|_x$ is open. That is, x comes from $\operatorname{Spec} A/I$ for an open ideal I. Otherwise we say x is *analytic*.

Suppose $A_0 \subset A$ is a ring of definition, and $I \subset A_0$ is an ideal of definition. If $x \in \operatorname{Spa}(A, A^+)$ is analytic, then the kernel of $|\cdot|_x$, not being open, cannot contain I. Thus there exists $f \in I$ such that $|f(x)| \neq 0$. Let $\gamma = |f(x)| \in \Gamma = \Gamma_x$. Since $f^n \to 0$ as $n \to \infty$, we must have $|f(x)|^n \to 0$. This means that for all $\gamma' \in \Gamma$ there exists $n \gg 0$ such that $\gamma^n < \gamma'$.

Lemma 3.3.2. *Let Γ be a totally ordered abelian group, and let $\gamma < 1$ in Γ. Suppose that for all $\gamma' \subset \Gamma$ there exists $n \gg 0$ such that $\gamma^n < \gamma'$. Then there exists a unique order-preserving map $\Gamma \to \mathbf{R}_{\geq 0}$ which sends $\gamma \mapsto 1/2$. (The kernel of this map consists of elements which are "infinitesimally close to 1".)*

Proof. Exercise. □

As an example, if x has value group $\Gamma_x = \mathbf{R}_{>0} \times \delta^{\mathbf{Z}}$ where $r < \delta < 1$ for all $r \in \mathbf{R}$, $r < 1$, then the map $\Gamma_x \to \mathbf{R}_{>0}$ of the lemma is just the projection.

Thus, any analytic point x gives rise to $\widetilde{x} \colon A \to \mathbf{R}_{\geq 0}$. (The equivalence class of $\widetilde{x}$ will be well-defined.)

Lemma 3.3.3. *The point $\widetilde{x}$ is the maximal generalization of x. Note that x and $\widetilde{x}$ define the same topology on A.*

Proof. If $U = \{|f(y)| \leq |g(y)| \neq 0\}$ contains x then it also contains $\widetilde{x}$, so that $\widetilde{x}$ is a generalization of x. The maximality of $\widetilde{x}$ is an exercise. □

In particular if x is an analytic point let $K(x)$ be the completion of $\operatorname{Frac}(A/\ker(x))$ with respect to $|\ |_x$ is a nonarchimedean field in the following sense.

Definition 3.3.4. A *nonarchimedean field* is a complete nondiscrete topological field K whose topology is inudced by a nonarchimedean norm $|\ | : K \to \mathbf{R}_{\geq 0}$.

For $x \in X$, let $K(x)$ be the completion of $\mathrm{Frac}(A/\ker|\cdot|_x)$ with respect to $|\ |_x$. The lemma shows that if x is analytic, then $K(x)$ is a nonarchimedean field. At non-analytic points of x, $K(x)$ is discrete. (In the situation of the special point s of our example, $K(s) = \mathbf{F}_p$.)

Let us return to our example $\mathcal{Y} = X \setminus \{x_{\mathbf{F}_p}\}$, with $X = \mathrm{Spa}(A, A)$ and $A = \mathbf{Z}_p[\![T]\!]$. For $x \in Y$, we have that $|T(x)|$ and $|p(x)|$ cannot both be zero. Both are elements of the value group which are topologically nilpotent. We can measure their relative position as an element of $[0, \infty]$.

Proposition 3.3.5. *There is a unique continuous surjection*

$$\kappa\colon \mathcal{Y} \to [0, \infty]$$

characterized by the following property: $\kappa(x) = r$ *if and only for all rational numbers* $m/n > r$, $|T(x)|^n \geq |p(x)|^m$, *and for all* $m/n < r$, $|T(x)|^n \leq |T(x)|^m$.

Proof. (Sketch.) Any $x \in \mathcal{Y}$ is analytic, so there exists a maximal generalization $\widetilde{x}$ which is real-valued. We define

$$\kappa(x) = \frac{\log|T(\widetilde{x})|}{\log|p(\widetilde{x})|} \in [0, \infty].$$

The numerator and denominator both lie in $[-\infty, 0)$, with at most one being equal to $-\infty$, so the quotient is indeed well-defined in $[0, \infty]$. We have $\kappa(x) = 0$ if and only if $|p(x)| = 0$, which is to say that $|\cdot|_x$ factors through $\mathbf{F}_p[\![T]\!]$. Similarly $\kappa(x) = \infty$ if and only if $|\cdot|_x$ factors through $\mathbf{Z}_p[\![T]\!] \to \mathbf{Z}_p$, $T \mapsto 0$.

To check continuity: look at the preimage of $(m/n, \infty]$. It is the set of $x \in \mathcal{Y}$ such that $|T^m(x)| \leq |p^n(x)| \neq 0$, which is open by definition. Similarly for $(0, m/n)$. □

For an interval $I \subset [0, \infty]$, let $\mathcal{Y}_I = \kappa^{-1}(I)$. Thus $\mathcal{Y}_{(0,\infty]}$ is the locus $p \neq 0$. This is the generic fibre of $\mathrm{Spa}\, A$ over $\mathrm{Spa}\, \mathbf{Z}_p$. It is not quasi-compact (otherwise its image under κ would lie in a compact interval), and in particular it is not affinoid.

One might think that $\mathrm{Spa}(\mathbf{Z}_p[\![T]\!][1/p], \mathbf{Z}_p[\![T]\!])$ should be the generic fibre of $\mathrm{Spa}\, A$. But $\mathbf{Z}_p[\![T]\!][1/p]$ isn't even a Huber ring! If one gives $\mathbf{Z}_p[\![T]\!][1/p]$ the topology induced from the (p, T)-adic topology of $\mathbf{Z}_p[\![T]\!]$, then $p^{-1}T^n \to 0$. Since this sequence never enters $\mathbf{Z}_p[\![T]\!]$, we can conclude that $\mathbf{Z}_p[\![T]\!]$ isn't even open.

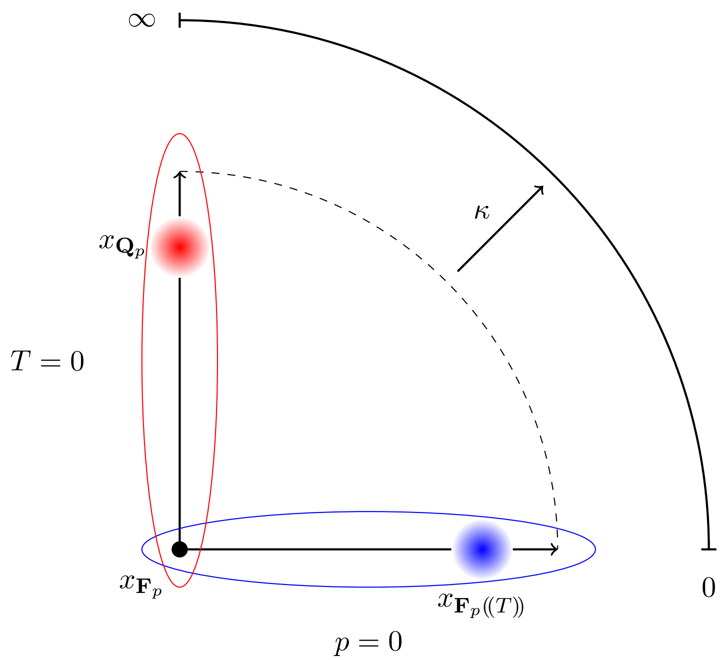


Figure 1: A depiction of $\operatorname{Spa} A$, where $A = \mathbf{Z}_p[\![T]\!]$. The two closed subspaces $\operatorname{Spa} \mathbf{F}_p[\![T]\!]$ and $\operatorname{Spa} \mathbf{Z}_p$ appear as the x-axis and y-axis, respectively. Their intersection is the unique non-analytic point $x_{\mathbf{F}_p}$ of $\operatorname{Spa} A$. The complement of $x_{\mathbf{F}_p}$ in $\operatorname{Spa} A$ is the adic space $\mathcal{Y}$, on which the continuous map $\kappa\colon \mathcal{Y} \to [0, \infty]$ is defined.

Also, $\mathbf{Z}_p[\![T]\!][1/p]$ would have to be Tate (as p is a topologically nilpotent unit), but we have seen that any ring of definition of a Tate ring has ideal of definition generated by one element, in this case p. But the topology on $\mathbf{Z}_p[\![T]\!]$ is not the p-adic one.

To get affinoid subsets of $\mathcal{Y}_{[0,\infty)}$, we need to impose some condition $|T^n| \leq |p|$. This is equivalent to exhausting $\mathcal{Y}_{(0,\infty]}$ by all

$$\mathcal{Y}_{[1/n,\infty]} = \operatorname{Spa}(\mathbf{Q}_p\langle T, T^n/p\rangle, \mathbf{Z}_p\langle T, T^n/p\rangle).$$

This is indeed a rational subset, because (T^n, p) is an open ideal.

This picture may seem rather esoteric, but it is really necessary to study it. As we progress in the course, we will encounter adic spaces similar to $\mathcal{Y}$ which are built out of much stranger rings, but for which the picture is essentially the same.

4 General adic spaces, 11 September

4.1 Honest adic spaces

Today[3] we will finally define what an adic space is. So far, we have defined the notion of a Huber pair (A, A^+), a topological space $X = \operatorname{Spa}(A, A^+)$, its presheaf of topological rings $\mathcal{O}_X$, and the subpresheaf $\mathcal{O}_X^+$. Moreover, for each $x \in X$, we have an equivalence class of continuous valuations $|\cdot(x)|$ on $\mathcal{O}_{X,x}$.

Remark 4.1.1. There are examples due to Rost (see [Hub94], end of §1) where $\mathcal{O}_X$ is not a sheaf. See [BV] and [Mih] for further examples.

Definition 4.1.2. A Huber pair (A, A^+) is *sheafy* if $\mathcal{O}_X$ is a sheaf of topological rings. (This implies that $\mathcal{O}_X^+$ is a sheaf.)

Recall that a scheme is a ringed space which locally looks like the spectrum of a ring. An adic space will be something similar. First we have to define the adic version of "ringed space". Briefly, it is a locally topologically ringed topological space equipped with valuations.

Definition 4.1.3. We define a category (V) as follows. The objects are triples $(X, \mathcal{O}_X, (|\cdot(x)|)_{x\in X})$, where X is a topological space, $\mathcal{O}_X$ is a sheaf

[3] Much of the content of this lecture was intended to provide a unified formalism for the definitions of *general adic space* and *diamond*. Since then the definitions of those concepts have evolved somewhat, to the point that the reader who wishes to learn these concepts may skip this lecture.

of topological rings, and for each $x \in X$, $|\cdot(x)|$ is an equivalence class of continuous valuations on $\mathcal{O}_{X,x}$. (Note that this data determines $\mathcal{O}_X^+$.) The morphisms are maps of locally topologically ringed topological spaces $f\colon X \to Y$ (so that the map $\mathcal{O}_Y(f^{-1}(U)) \to \mathcal{O}_X(U)$ is continuous for each open $U \subset X$), which make the following diagram commute up to equivalence:

$$\begin{array}{ccc} \mathcal{O}_{Y,f(x)} & \longrightarrow & \mathcal{O}_{X,x} \\ \downarrow & & \downarrow \\ \Gamma_{f(x)} \cup \{0\} & \longrightarrow & \Gamma_x \cup \{0\} \end{array}$$

An *honest adic space*[4] is an object $(X, \dots)$ of (V) which admits a covering by spaces U_i such that the triple $(U_i, \mathcal{O}_X|_{U_i}, (|\cdot(x)|)_{x \in U})$ is isomorphic to $\operatorname{Spa}(A_i, A_i^+)$ for a sheafy Huber pair (A_i, A_i^+).

For sheafy (A, A^+), the topological space $X = \operatorname{Spa}(A, A^+)$ together with its structure sheaf and continuous valuations is an *affinoid adic space*, which we continue to write as $\operatorname{Spa}(A, A^+)$.

Proposition 4.1.4. *The functor* $(A, A^+) \mapsto \operatorname{Spa}(A, A^+)$ *from sheafy complete Huber pairs to adic spaces is fully faithful.*

Proof. From the adic space $X = \operatorname{Spa}(A, A^+)$ one can recover A as $H^0(X, \mathcal{O}_X)$, and similarly A^+. Therefore if $Y = \operatorname{Spa}(B, B^+)$ is given, a map $X \to Y$ induces a map $(B, B^+) \to (A, A^+)$. So we get a map

$$\operatorname{Hom}(\operatorname{Spa}(A, A^+), \operatorname{Spa}(B, B^+)) \to \operatorname{Hom}((B, B^+), (A, A^+)).$$

For a proof that this is inverse to the natural map in the other direction, see Huber, *A generalization of formal schemes and rigid analytic varieties*, Prop. 2.1(i). □

4.2 General adic spaces: an overview

What can be done about non-sheafy Huber pairs (A, A^+)? It really is a problem that the structure presheaf on $\operatorname{Spa}(A, A^+)$ isn't generally a sheaf. It ruins any hope of defining a general adic space as what you get when you glue together spaces of the form $\operatorname{Spa}(A, A^+)$; indeed, without the sheaf property this gluing doesn't make any sense.

Here are some of our options for how to proceed:

[4] This is what Huber simply calles an *adic space*.

1. Ignore them. Maybe non-sheafy Huber pairs just don't appear in nature, so to speak.
2. It may be possible to redefine the structure sheaf on $X = \mathrm{Spa}(A, A^+)$ so that for a rational subset U, $\mathcal{O}_X(U)$ is formed using a henselization rather than a completion. Then it might be possible to show that $\mathcal{O}_X$ is always a sheaf. However, proceeding this way diverges quite a bit from the classical theory of rigid spaces.
3. Construct a larger category of adic spaces using a "functor of points" approach. This is analogous to the theory of algebraic spaces, which are functors on the (opposite) category of rings which may not be representable.

We take (what else?) the third approach. There is some abstract nonsense to grapple with, but the virtue of it is that it is conceptually unified with other constructions (such as algebraic spaces), and we will also use it to define the category of diamonds.

The problem is that the structure presheaf on $X = \mathrm{Spa}(A, A^+)$ can fail to be a sheaf. This suggests that the solution might be a kind of sheafification of X. It is tempting to simply keep the topological space X and use the sheafification of $\mathcal{O}_X$ to arrive at an object of the category (V), but then one runs into problems: for instance, the analogue of Prop. 4.1.4 will fail in general.

Instead we take a "Yoneda-style" point of view. Let CAff be the category of complete Huber pairs[5], where morphisms are continuous homomorphisms. Let $\mathrm{CAff}^{\mathrm{op}}$ be the opposite category. An object $X = (A, A^+)^{\mathrm{op}}$ of $\mathrm{CAff}^{\mathrm{op}}$ induces a set-valued covariant functor h_X on $\mathrm{CAff}^{\mathrm{op}}$, by $Y \mapsto \mathrm{Hom}_C(X, Y)$. Rather than sheafifying $\mathcal{O}_X$, *we sheafify this functor.* In order for this to make sense, we need to give $\mathrm{CAff}^{\mathrm{op}}$ a topology which turns it into a *site*. In this topology, "opens" in $X = (A, A^+)^{\mathrm{op}}$ come from rational subsets, and "covers" of X come from families of rational subsets which cover X. We can then define $\mathrm{Spa}(A, A^+)$ as the sheafification of the functor h_X.

Then we must define adic spaces in general, as a special class of sheaves $\mathcal{F}$ on the site $\mathrm{CAff}^{\mathrm{op}}$. The idea is that $\mathcal{F}$ should be considered an adic space if it is formed by gluing together "open subsets" which are each of the form $\mathrm{Spa}(A, A^+)$. However, one needs to make sense of what an open subset of $\mathcal{F}$ is, and also what gluing means, which is a little bit subtle. In the end

[5]This notation appears in [SW13], and anyway recall that Huber calls such objects *affinoid algebras*.

one arrives at a category Adic, together with a morphism $\mathrm{CAff}^{\mathrm{op}} \to \mathrm{Adic}$ which is fully faithful on the full subcategory consisting of objects $(A, A^+)^{\mathrm{op}}$, where (A, A^+) is sheafy.

In fact we will generalize this story to an arbitrary category C equipped with some extra structures, to arrive at a category of "C-spaces".

4.3 Direct limits

In order to glue basic objects together to get more general ones, we will make heavy use of the notion of a direct limit in a category.

Let I and C be categories. We will think of I as being a diagram, whose objects are meaningless indices, whereas C will be a category of geometric objects. In pratice, I is often a small category. A functor $D\colon I \to C$ is called a *diagram in C of type I*, and an object X of C is a *colimit* of D if there is a morphism $D(i) \to X$ for every object i of I making all possible diagrams commute, and if X is the initial such object.

Example: Let C be the category of schemes, and let I be the category containing two objects $0, 1$ and two morphisms $0 \rightrightarrows 1$. An instance of a diagram in C of type I is $\mathbf{G}_{\mathrm{m}} \rightrightarrows \mathbf{A}^1 \coprod \mathbf{A}^1$, where the two maps are $z \mapsto z^{\pm 1}$ into the respective copies of $\mathbf{A}^1$. The colimit of this diagram is $\mathbf{P}^1$. Colimits of this type are called *coequalizers*; we will use the notation $\mathrm{Coeq}(X \rightrightarrows Y)$.

We say that I is a *directed (or filtered) category* if every pair of objects admits a morphism into a common object, and if for every pair of morphisms $i \rightrightarrows j$ there exists a morphism $j \to k$ for which the two composite morphisms agree. A diagram in X whose index category I is directed is called a *directed system*. The colimit X of a directed system is a *direct limit*. If $X_i = D(i)$, we write $X = \varinjlim X_i$.

Examples: in the category of sets, every set is the direct limit of a diagram of finite sets, where the morphisms are inclusions. In the category of topological spaces, $\mathbf{R}$ is the direct limit of the system of intervals $[-n, n]$, where $n \geq 0$. Thus a direct limit captures the idea of a "increasing union".

4.4 Categories with étale morphisms

Definition 4.4.1. A *category with étale morphisms* is a category C endowed with a special class of morphisms, which we will call "C-étale". These will be assumed to satisfy:

1. Isomorphisms are C-étale.

2. Compositions of C-étale morphisms are C-étale.

3. If

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & Y & \end{array}$$

with g and h C-étale, then f is C-étale as well.

4. Suppose $X' \to X$ is C-étale, and $Y \to X$ is any morphism. Then the fibre product $X' \times_X Y$ is *ind-representable* in C in the following sense: there exists a directed system of C-étale morphisms $Y_i' \to Y$, which fit compatibly into diagrams

$$\begin{array}{ccc} Y_i' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X, \end{array}$$

such that the induced map of functors

$$\varinjlim h_{Y_i'} \to h_{X'} \times_{h_X} h_Y$$

is an isomorphism.

Condition 4 deserves some explanation. If it were the case that pullbacks of C-étale morphisms existed and were C-étale, then this condition would be satisfied: the directed system would consist of the single object $X' \times_X Y$. Examples of categories of étale morphisms where this happens include the class of open immersions in the category of topological spaces, and the class of étale morphims in the category of schemes.

However, we are interested in the category $C = \mathrm{CAff}^{\mathrm{op}}$. We define C-étale morphisms to be those of the form $\big((A, A^+) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))\big)^{\mathrm{op}}$ with $U \subset X = \mathrm{Spa}(A, A^+)$ a rational subset. It is not necessarily the case that the pullback of a C-étale morphism even exists! Indeed, let $X = (\mathbf{Z}_p, \mathbf{Z}_p)^{\mathrm{op}}$, and let $X' = (\mathbf{Q}_p, \mathbf{Z}_p)^{\mathrm{op}}$ be its generic point. This is a rational subset, so that $X' \to X$ is C-étale. Let $Y = \mathrm{Spa}(\mathbf{Z}_p[\![T]\!], \mathbf{Z}_p[\![T]\!])$. Presumably $X' \times_X Y$ is supposed to be the generic fibre of Y (though we have not defined it precisely yet). As we have seen, this is not quasicompact, and therefore cannot come from a Huber pair. However, this generic fibre is exhausted by an increasing sequence of rational subsets, namely those defined by the conditions $|T^n(x)| \leq |p(x)| \neq 0$ for $n = 1, 2, \ldots$. This is the phenomenon captured by condition 4.

Lemma 4.4.2. *Condition 4 is satisfied for* CAff^{op}.

Proof. Let (A, A^+) be a complete Huber pair, and let $X = (A, A^+)^{\mathrm{op}}$. Let $U(\{T_i/s_i\}) \subset X$ be a rational subset, where $T_1, \ldots, T_k \subset A$ are finite subsets such that $T_i A \subset A$ is open and $s_i \in A$ for $i = 1, \ldots, k$. Let $(A', A'^+) = (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$, so that $A' = A\langle\{T_i/s_i\}\rangle$. Let $X' = (A', A'^+)^{\mathrm{op}}$, so that $X' \to X$ is C-étale.

Now suppose we have a commutative diagram in $\mathrm{CAff}^{\mathrm{op}}$ of the form

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X, \end{array}$$

where $Y \to X$ corresponds to a morphism $\lambda\colon (A, A^+) \to (B, B^+)$ of complete Huber pairs, and $Y' = (B', B'^+)$. The image of $\mathrm{Spa}(B', B'^+)$ in $\mathrm{Spa}(B, B^+)$ is contained in the preimage of $\mathrm{Spa}(A', A'^+)$ in $\mathrm{Spa}(B, B^+)$, which is

$$V = \left\{x \in \mathrm{Spa}(B, B^+) \,|\, |\lambda(T_i)(x)| \leq |\lambda(s_i)(x)| \neq 0,\ i = 1, \ldots, k\right\}$$

This does not necessarily describe a rational subset of $\mathrm{Spa}(B, B^+)$, because $\lambda(T_i)B \subset B$ might not be open. Suppose $B_0 \subset B$ is a ring of definition, and $I \subset B_0$ is an ideal of definition. For $n \geq 1$ let $S_i^{(n)} = \lambda(T_i) \cup I^n$. Then clearly $S_i^{(n)} B \subset B$ is open. Let

$$V_n = U\left(\left\{S_i^{(n)}/s_i\right\}\right) = \left\{y \in \mathrm{Spa}(B, B^+) \,|\, |t_i(y)| \leq |s_i(y)| \neq 0,\ t_i \in S_i^{(n)}\right\},$$

a rational subset of $\mathrm{Spa}(B, B^+)$ contained in V. We claim that $V = \bigcup_n V_n$. Indeed, let $y \in V$. Since elements of I are topologically nilpotent, I is finitely generated, and $|\cdot(y)| : B \to \Gamma_y \cup \{0\}$ is continuous, we can say that for all $\gamma \in \Gamma$, there exists $n \gg 0$ such that $|f(y)| \leq \gamma$ for all $f \in I^n$. In particular there exists $n \gg 0$ such that $|f(y)| \leq |s_i(y)|$ for all $f \in I^n$, $i = 1, \ldots, k$. Then $y \in V_n$.

Since the V_n cover V, and the image of $\mathrm{Spa}(B', B'^+)$ in $\mathrm{Spa}(B, B^+)$ is a quasi-compact subset of V, we conclude that $\mathrm{Spa}(B', B'^+) \to \mathrm{Spa}(B, B^+)$ factors through V_n for some n. Let (B'_n, B'^+_n) be the Huber pair corresponding to V_n. By Thm. 3.2.2, $(B, B^+) \to (B', B'^+)$ factors through (B'_n, B'^+_n). Then $Y'_n = (B'_n, B'^+_n)^{\mathrm{op}}$ is the desired directed system (in fact it is just a sequence). □

Definition 4.4.3. An *abstract étale site* is a category with étale morphisms C together with a special class of collections of C-étale morphisms $\{X_i \to X\}$, called *covers*, which satisfy the following properties:

1. Isomorphisms are covers.

2. Pullbacks of covers are covers, in the following sense: If $\{X_i \to X\}$ is a cover, and $Y \to X$ is a morphism, then by Condition 4 in Defn. 4.4.1, $X_i \times_X Y$ is ind-representable by a directed system Y_{ij}, with $Y_{ij} \to Y$ being C-étale; then $\{Y_{ij} \to Y\}$ is a cover.

3. Compositions of covers are covers.

Examples include:

(A) Let C be opposite to the category of commutative rings, with C-étale morphisms given by maps $(A \to A[f^{-1}])^{\mathrm{op}}$, with $f \in A$, where covers are Zariski covers.

(B) Same C, but now C-étale means étale, and covers are étale covers.

(C) Let $C = \mathrm{CAff}^{\mathrm{op}}$, where covers are topological covers by rational subsets.

(D) Let C be (V), with C-étale morphisms given by open embeddings, and covers are open covers.

Recall the notion of a presheaf on a site C: it is simply a contravariant set-valued functor $\mathcal{F}$ on C. We also want to give a precise definition of a sheaf on C. Because the topology on C is a little strange (we did not require that pullbacks of covers are covers), the definition of a sheaf has to be adapted. Suppose $\mathcal{F}$ is a presheaf on C. If $X_1 \to X$ and $X_2 \to X$ are two C-étale morphisms, then $X_1 \times_X X_2$ is ind-representable by a directed system Y_i, and then we put[6] $\mathcal{F}(X_1 \times_X X_2) = \varprojlim \mathcal{F}(Y_i)$. (We remark, though, that in our applications, the fibre product of two C-étale morphisms will always exist as another C-étale morphism.)

[6] Brian Conrad points out that this definition raises some well-posedness issues because it is not clear that this definition is independent of the system Y_i. He points out, however, that there is a broader notion of site defined (as in SGA) using "covering sieves", which defines sheaves without reference to fibre products. I have his assurance that this definition of sheaf agrees with the ad hoc one I give here. Perhaps one day I will rewrite this lecture using sieves, but the need is not so great: when we define diamonds, starting from the category C of perfectoid spaces in characteristic p, there will be fibre products, so none of these issues will arise. (JW)

Definition 4.4.4. A presheaf $\mathcal{F}$ on an abstract étale site C is a *sheaf* if for every cover $\{X_i \to X\}$, the diagram

$$\mathcal{F}(X) \to \prod_i \mathcal{F}(X_i) \rightrightarrows \prod_{ij} \mathcal{F}(X_i \times_X X_j)$$

represents the first arrow as the equalizer of the other two.

Our focus will now shift from objects of C to (set-valued) sheaves on C.

Definition 4.4.5. Let C be an abstract étale site.

1. For an object X of C, let $\underline{X}$ be the sheafification of the presheaf $Y \mapsto \mathrm{Hom}_C(Y, X)$. (In examples (A),(B) and (D), this is already a sheaf. Those sites are *subcanonical*, meaning that every representable presheaf is a sheaf.) If A is an object of C^{op}, we write $\mathrm{Space}(A)$ for $\underline{A}^{\mathrm{op}}$.

2. A map $f\colon \mathcal{F} \to \mathcal{G}$ of sheaves on C is *C-étale* if for all $X \in C$ and all morphisms $\underline{X} \to \mathcal{G}$, the pullback $\mathcal{F} \times_{\mathcal{G}} \underline{X} \to \underline{X}$ can be written as the colimit of a directed system of morphisms of the form $\underline{Y} \to \underline{X}$, where $Y \to X$ is C-étale.

3. A sheaf $\mathcal{F}$ on C is called a *C-space* if $\mathcal{F}$ is the colimit of a system of C-étale morphisms of the form $\underline{X_i} \to \mathcal{F}$.

In our examples:

(A) A C-space is a scheme.

(B) A C-space is an algebraic space.

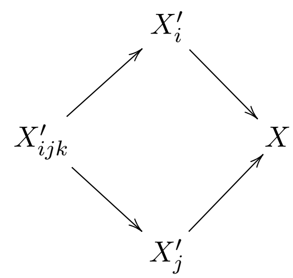
(C) A $\mathrm{CAff}^{\mathrm{op}}$-space is what we are now defining as a (general) *adic space*.

(D) A (V)-space is just an object of (V).

The category of sheaves is complete in the sense that limits always exist. In particular if $\mathcal{F} \to \mathcal{G}$ and $\mathcal{F}' \to \mathcal{G}$ are two morphisms of sheaves, then the fibre product $\mathcal{F} \times_{\mathcal{G}} \mathcal{F}'$ always exists: its value on an object X of C is simply $\mathcal{F}(X) \times_{\mathcal{G}(X)} \mathcal{F}'(X)$.

Lemma 4.4.6. *Let C be an abstract étale site. If $Y \to X$ is C-étale, then $\underline{Y} \to \underline{X}$ is C-étale.*

Proof. (Sketch: we are going to assume that fibre products of C-étale morphisms over a common base exist and are again C-étale, which is the case for our applications.) Given a morphism $\underline{X}' \to \underline{X}$, we have to check that the pullback $\underline{Y} \times_{\underline{X}} \underline{X}' \to \underline{X}'$ is C-étale. Such a morphism $\underline{X}' \to \underline{X}$ does not necessarily arise from a morphism $X' \to X$, because these objects are defined as sheafifications. Rather, $\underline{X}' \to \underline{X}$ is specified by a C-étale cover $\{X'_i \to X'\}$ and maps $X'_i \to X$, which agree locally on overlaps. This agreement means that each $X'_{ij} = X'_i \times_{X'} X'_j$ has a C-étale cover $\left\{X'_{ijk} \to X'_{ij}\right\}$ such that



commutes.

By the sheaf property, $\underline{X}'$ is the coequalizer of the diagram $\coprod_{ijk} \underline{X}'_{ijk} \rightrightarrows \coprod_i \underline{X}'_i$. (The two arrows are $X'_{ijk} \to X'_{ij}$ followed by the respective projections on to X'_i and X'_j.) Now we consider the fibre product with $\underline{Y}$. We are going to use the fact that in a topos, taking fibre products commutes with colimits[citation needed]. We find that

$$\begin{aligned}
\underline{Y} \times_{\underline{X}} \underline{X}' &= \underline{Y} \times_{\underline{X}} \operatorname{Coeq}\left(\coprod_{ijk} \underline{X}'_{ijk} \rightrightarrows \coprod_i \underline{X}'_i\right) \\
&= \operatorname{Coeq}\left(\underline{Y} \times_{\underline{X}} \coprod_{ijk} \underline{X}'_{ijk} \rightrightarrows \underline{Y} \times_{\underline{X}} \coprod_i \underline{X}'_i\right)
\end{aligned}$$

Recall by Condition 4 that the fibre product $Y \times_X X'_i$ is ind-representable by a directed system of objects which are C-étale over X'_i, and hence over X'. We conclude that $\underline{Y} \times_{\underline{X}} \underline{X}'$ is a direct limit of sheaves[7] attached to objects which are C-étale over X', as required. □

Let C and C' be abstract étale sites.

Lemma 4.4.7. *Let $F\colon C \to C'$ be a functor which carries C-étale morphisms onto C'-étale morphisms, and which carries covers onto covers. Then*

[7] This step requires further explanation, which I will add at some later date.

one gets a canonical functor

$$\widetilde{F}\colon \{C\text{-spaces}\} \to \{C'\text{-spaces}\}$$

such that $\widetilde{F}(\underline{X}) = \underline{F(X)}$ *for all objects* X *of* C.

Proof. (Sketch.) The functor F, being a morphism of sites, pulls back sheaves on C' to sheaves on C by the formula

$$F^*(\mathcal{F}) = \text{sheafification of } X \mapsto \varinjlim_{X' \to F(X)} \mathcal{F}(X').$$

To prove the lemma one needs to check that

1. If X is an object of C then $F^*(\underline{X}) = \underline{F(X)}$.
2. If $f\colon \mathcal{F} \to \mathcal{G}$ is a C-étale morphism of sheaves on C then $F^*(f)\colon F^*(\mathcal{F}) \to F^*(\mathcal{G})$ is C-étale.
3. F^* maps C-spaces to C'-spaces.

(Thus in examples A and B, the morphism of sites, which takes Zariski covers to étale covers, induces the functor which takes a scheme to its corresponding algebraic space.) □

Proposition 4.4.8. *There is a fully faithful functor from honest adic spaces to adic spaces.*

Example 4.4.9. Consider the fucntor

$$\begin{array}{rcl} \{\text{complete Huber pairs}\}^{\text{op}} & \to & (\text{V}) \\ (A, A^+) & \mapsto & \text{Spa}(A, A^+) \text{ with sheafified } \mathcal{O}_X \end{array}$$

This induces a functor from adic spaces to (V), which sends X to $(|X|, \dots)$. In particular any adic space has an associated topological space. However, this functor is not fully faithful.

From now on, if (A, A^+) is a Huber pair, we understand $\text{Spa}(A, A^+)$ to mean the adic space $\text{Space}((A, A^+))$, a sheaf on the category CAff^{op}. We hope this will not cause confusion. After all, the topological space associated to this adic space is $\text{Spa}(A, A^+)$ as we had previously defined it.

4.5 Analytic adic spaces

Definition 4.5.1. An adic space is *analytic* if all its points are analytic.

Proposition 4.5.2. *An adic space is analytic if and only if it is the colimit of a diagram of spaces* $\mathrm{Spa}(A, A^+)$*, with* A *Tate.*

Proof. Let $X = \mathrm{Spa}(A, A^+)$, with $x \in X$ an analytic point. We need to show that there exists a rational neighborhood U of x such that $\mathcal{O}_X(U)$ is Tate. Let $I \subset A_0$ be as usual. Take $f \in I$ be such that $|f(x)| \neq 0$. Then $\{g \in A | \, |g(x)| < |f(x)|\}$ is open (by the continuity of the valuations). This means that there exists n so that this set contains I^n. Write $I^n = (g_1, \dots, g_k)$. Then

$$U = \{y | \, |g_i(y)| \leq |f(y)| \neq 0\}$$

is a rational subset. On U, f is a unit (because it is everywhere nonzero), but it must also be topologically nilpotent, because it is containined in I. $\square$

Example 4.5.3. In $X = \mathrm{Spa}(\mathbf{Z}_p[\![T]\!], \mathbf{Z}_p[\![T]\!])$, we had a unique non-analytic point s.

Consider the point where $T = 0$ and $p \neq 0$. This has a rational neighborhood $U = \{|T(x)| \leq |p(x)| \neq 0\}$. After adding the trivial condition $|p(x)| \leq |p(x)|$, we see that this really is a rational subset. Then $\mathcal{O}_X(U)$ is the completion of $\mathbf{Z}_p[\![T]\!][1/p]$ with respect to the $(p, T) = (p, p(T/p)) = p$-adic topology on $\mathbf{Z}_p[\![T]\!][T/p]$. This is just $\mathbf{Q}_p\langle T/p \rangle$.

Now consider the point where $p = 0$ and $T \neq 0$. Let V be the rational subset $\{|p(x)| \leq |T(x)| \neq 0\}$. Then $\mathcal{O}_X(V)$ is the completion of $\mathbf{Z}_p[\![T]\!][1/T]$ with respect to the T-adic topology on $\mathbf{Z}_p[\![T]\!][p/T]$. One might call this $\mathbf{Z}_p[\![T]\!]\langle p/T \rangle[1/T]$. It is still a Tate ring, because T is topologically nilpotent. But it does not contain a nonarchimedean field! Thus you cannot make sense of it in the world of classical rigid spaces.

Complete Tate rings are "as good as" Banach algebras over nonarchimedean fields. For example:

Proposition 4.5.4 ([Hub93, Lemma 2.4(i)])**.** *Complete Tate rings satisfy Banach's open mapping theorem. That is, if* A *is a complete Tate ring, and* M *and* N *are complete Banach* A*-modules, then any continuous surjective map* $M \to N$ *is also open.*

5 Complements on adic spaces, 16 September

Today's lecture is a collection of complements in the theory of adic spaces.

5.1 Adic morphisms

Definition 5.1.1. A morphism $f\colon A \to B$ of Huber rings is *adic* if for one (or any) choice of rings of definition $A_0 \subset A$, $B_0 \subset B$ with $f(A_0) \subset B_0$, and $I \subset A_0$ an ideal of definition, $f(I)B_0$ is an ideal of definition.

A morphism $(A, A^+) \to (B, B^+)$ of Huber pairs is adic if $A \to B$ is.

Lemma 5.1.2. *If A is Tate, then any $f\colon A \to B$ is adic.*

Proof. If A contains a topologically nilpotent unit ϖ, then $f(\varpi) \in B$ is also a topologically nilpotent unit, and thus B is Tate as well. By Prop. 2.2.5(2), B contains a ring of definition B_0 admitting $f(\varpi)^n B_0$ as an ideal of definition for some $n \geq 1$. This shows that f is adic. □

The next proposition shows that fibre products $X \times_S Y$ exist in the category of adic spaces, so long as $X \to S$ and $Y \to S$ are adic in the appropriate sense.

Proposition 5.1.3.

1. *If $(A, A^+) \to (B, B^+)$ is adic, then pullback along the associated map of topological spaces $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ preserves rational subsets.*
2. *Let*

$$\begin{array}{ccc} & & (B, B^+) \\ & \nearrow & \\ (A, A^+) & & \\ & \searrow & \\ & & (C, C^+) \end{array}$$

be a diagram of Huber pairs where both morphisms are adic. Let A_0, B_0, C_0 be rings of definition compatible with the morphisms, and let $I \subset A_0$ be an ideal of definition. Let $D = B \otimes_A C$, and let D_0 be the image of $B_0 \otimes_{A_0} C_0$ in D. Make D into a Huber ring by declaring D_0 to be a ring of definition with ID_0 as its ideal of definition. Let D^+ be the integral closure of the image of $B^+ \otimes_{A^+} C^+$ in D. Then (D, D^+) is a Huber pair, and it is the pushout of the diagram in the category of Huber pairs.

Remark 5.1.4. If the objects in the diagram were complete Huber pairs, then after completing (D, D^+), one would obtain the pushout of the diagram in the category of complete Huber pairs.

Remark 5.1.5. An example of a non-adic morphism of Huber rings is $\mathbf{Z}_p \to \mathbf{Z}_p[\![T]\!]$. We claim that the diagram

$$\begin{array}{ccc} & & (\mathbf{Z}_p[\![T]\!], \mathbf{Z}_p[\![T]\!]) \\ & \nearrow & \\ (\mathbf{Z}_p, \mathbf{Z}_p) & & \\ & \searrow & \\ & & (\mathbf{Q}_p, \mathbf{Z}_p), \end{array}$$

has no pushout in the category of Huber pairs. (At this point this should come as no surprise: we have already indicated that the generic fiber of $\mathrm{Spa}(\mathbf{Z}_p[\![T]\!], \mathbf{Z}_p[\![T]\!])$ is not quasi-compact.) Suppose it did, say (D, D^+); then we have a morphism

$$(D, D^+) \to \varprojlim_n \left(\mathbf{Q}_p\langle \frac{T^n}{p} \rangle, \mathbf{Z}_p\langle \frac{T^n}{p} \rangle \right)$$

for each $n \geq 1$, since the diagram admits a morphism to the object on the right. Since (D, D^+) fits into the diagram, we have that $T \in D^+$ is topologically nilpotent, and $1/p \in D$. Therefore $T^n/p \to 0$ in D; now since $D^+ \subset D$ is open we have $T^n/p \in D^+$ for some n. But then D^+ cannot admit a morphism to $\mathbf{Z}_p\langle T^{n+1}/p \rangle$, contradiction.

5.2 A remark on the presentation of rational subsets

Proposition 5.2.1. *Let (A, A^+) be a Huber pair. Then any rational subset $U \subset \mathrm{Spa}(A, A^+)$ is of the form $U = \{x \mid |f_i(x)| \leq |g(x)| \neq 0\}$, for $f_1, \dots, f_n, g \in A$, where $(f_1, \dots, f_n)A \subset A$ is open.*

(Recall that Huber's definition of rational subset is an arbitrary finite intersection of sets of this type.)

Proof. Any such U is rational. Conversely, any rational U is a finite intersection of subsets of this form. Take two such: let $U_1 = \{x \mid |f_i(x)| \leq |g(x)| \neq 0\}$ and $U_2 = \left\{x \mid \left|f'_j(x)\right| \leq |g'(x)| \neq 0\right\}$. Their intersection is

$$\left\{x \mid \left|f_i f'_j(x)\right|, \left|f_i g'(x)\right|, \left|f'_j g(x)\right| \leq \left|g g'(x)\right| \neq 0\right\}$$

Now we just have to check that the $f_i f'_j$ generate an open ideal of A. By hypothesis there exists an ideal of definition I such that $I \subset (f_i)A$ and $I \subset (f'_j)A$. Then the ideal generated by the $f_i f'_j$ contains I^2. □

5.3 The role of A^+ in a Huber pair

The presence of the subring A^+ in a Huber pair (A, A^+) may seem unnecessary at first: why not just consider all continuous valuations on A? For a Huber ring A, let $\mathrm{Cont}(A)$ be the set of equivalence classes of continuous valuations on A, with topology generated by subsets of the form $\{|f(x)| \leq |g(x)| \neq 0\}$, with $f, g \in A$.

Proposition 5.3.1. *1.* $\mathrm{Cont}(A)$ *is a spectral space.*

2. The following sets are in bijection:

   (a) The set of subsets $F \subset \mathrm{Cont}(A)$ of the form $\cap_{f\in S}\{|f| \leq 1\}$, as S runs over arbitrary subsets of A°.

   (b) The set of open and integrally closed subrings $A^+ \subset A^\circ$.

   The map is

$$F \mapsto \{f \in A |\, |f(x)| \leq 1 \textit{ for all } x \in F\}$$

   with inverse

$$A^+ \mapsto \left\{x \in \mathrm{Cont}(A) |\, |f(x)| \leq 1 \textit{ for all } f \in A^+\right\}.$$

Thus specifying A^+ keeps track of which inequalities have been enforced among the continuous valuations in $\mathrm{Cont}(A)$.

As further explanation, suppose $A = K$ is a nonarchimedean field. This is an important case because points of an adic space X take the form $\mathrm{Spa}(K, K^+)$ (just as points of a scheme are spectra of fields). We cannot replace $\mathrm{Spa}(K, K^+)$ with $\mathrm{Cont}(K)$ because the topological space $\mathrm{Cont}(K)$ may have more than one point! In fact if k is the residue field of K (meaning power-bounded elements modulo topologically nilpotent elements), then $\mathrm{Cont}(K)$ is homeomorphic to the space of valuations on k.

For instance, let K be the completion of the fraction field of $\mathbf{Q}_p\langle T\rangle$ with respect to the supremum (Gauss) norm $|\ |_\eta$. The residue field of K is $\mathbf{F}_p(T)$, and so we have a homeomorphism $\mathrm{Cont}(K) \cong \mathbf{P}^1_{\mathbf{F}_p}$, where η corresponds to the generic point. The other points correspond to rank 2 valuations on K. For instance, if $x \in \mathrm{Cont}(K)$ corresponds to $0 \in \mathbf{P}^1(\mathbf{F}_p)$, and $\gamma = |T(x)|$, then $\gamma < 1$, but $\gamma^n > |p(x)|$ for all $n \geq 1$. If $K^+ \subset K$ is the valuation ring of x, then $K^+ \subsetneq K^\circ$, since $1/T \notin K^+$.

5.4 Important examples, and fibre products

We gather here some facts about the category of adic spaces.

- The final object is $\mathrm{Spa}(\mathbf{Z}, \mathbf{Z})$.

- (The adic closed unit disc) $\mathrm{Spa}(\mathbf{Z}[T], \mathbf{Z}[T])$ represents the functor $X \mapsto \mathcal{O}_X^+(X)$. This is true not just for honest adic spaces, but for general ones, where the sheafification of $\mathcal{O}_X^+$ must be used. Note that if K is a nonarchimedean field, then

$$\mathrm{Spa}(\mathbf{Z}[T], \mathbf{Z}[T]) \times \mathrm{Spa}(K, \mathcal{O}_K) = \mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$$

  A discussion of this appears in 2.20 of [Sch12].

- (The adic affine line) $X \mapsto \mathcal{O}_X(X)$ is also representable, by $\mathrm{Spa}(\mathbf{Z}[T], \mathbf{Z})$. Once again if K is any nonarchimedean field, then

$$\mathrm{Spa}(\mathbf{Z}[T], \mathbf{Z}) \times \mathrm{Spa}(K, \mathcal{O}_K) = \bigcup_{n \geq 1} \mathrm{Spa}\left(K\langle \frac{T^n}{\varpi}\rangle, \mathcal{O}_K\langle \frac{T^n}{\varpi}\rangle\right)$$

  is an increasing union of closed discs $|T| \leq |\varpi|^{-n}$. Here $\varpi \in \mathcal{O}_K$ is any *pseudo-uniformizer* (meaning a topologically nilpotent unit). For this you just have to check the universal property.

- (Fibre products do not exist in general) In the sense of hom-functors, the product

$$\mathrm{Spa}(\mathbf{Z}[T_1, T_2, \ldots], \mathbf{Z}) \times \mathrm{Spa}(K, \mathcal{O}_K)$$

  equals

$$\varinjlim_{(n_i) \to \infty} \mathrm{Spa}(K\langle \varpi^{n_1} T_1, \ldots\rangle, \mathcal{O}_K\langle \varpi^{n_i}, \ldots\rangle).$$

  But in this direct limit, the transition maps are not open; they are given by infinitely many inequalities $|T_i| \leq |\varpi|^{-n_i}$. So this direct limit is not representable as an adic space. This example suggests that we restrict the class of Huber pairs we work with a little. Let us call a Huber pair (A, A^+) *admissible* if A is finitely generated over a ring of definition $A_0 \subset A^+$. This is always the case for instance if A is Tate. If we build a category of adic spaces starting from admissible Huber pairs, then fibre products will always exist.

- (The open unit disc) Let $\mathbf{D} = \mathrm{Spa}(\mathbf{Z}[\![T]\!], \mathbf{Z}[\![T]\!])$.

$$\begin{aligned}\mathbf{D} \times \mathrm{Spa}(K, \mathcal{O}_K) &= [\mathbf{D} \times \mathrm{Spa}(\mathcal{O}_K, \mathcal{O}_K)] \times_{\mathrm{Spa}(\mathcal{O}_K, \mathcal{O}_K)} \mathrm{Spa}(K, \mathcal{O}_K) \\ &= \mathrm{Spa}(\mathcal{O}_K[\![T]\!], \mathcal{O}_K[\![T]\!]) \times_{\mathrm{Spa}(\mathcal{O}_K, \mathcal{O}_K)} \mathrm{Spa}(K, \mathcal{O}_K) \\ &= \bigcup_{n \geq 1} \mathrm{Spa}\left(K\langle \frac{T^n}{\varpi}\rangle, \mathcal{O}_K\langle \frac{T^n}{\varpi}\rangle\right)\end{aligned}$$

  Call this $\mathbf{D}_K$. This is another honest adic space, even though the intermediate space $\mathrm{Spa}(\mathcal{O}_K[\![T]\!], \mathcal{O}_K[\![T]\!])$ might not be. This shows the importance of allowing non-honest adic spaces.

- (The open punctured unit disc) Let $\mathbf{D}^* = \mathrm{Spa}(\mathbf{Z}(\!(T)\!), \mathbf{Z}[\![T]\!])$. Then $\mathbf{D}^*_K = \mathbf{D}_K \backslash \{T = 0\}$.

5.5 Analytic Adic Spaces

We discuss some very recent results from [BV], [Mih], and [KL]. Let (A, A^+) be a Tate-Huber pair (meaning a Huber pair with A Tate), and let $X = \mathrm{Spa}(A, A^+)$. When is $\mathcal{O}_X$ a sheaf?

Recall that A is *uniform* if $A^\circ \subset A$ is bounded.

Theorem 5.5.1 (Berkovich)**.** *For A uniform, the map*

$$A \to \prod_{x \in \mathrm{Spa}(A, A^+)} K(x)$$

is a homeomorphism of A onto its image. Here $K(x)$ is the completed residue field. Also we have

$$A^\circ = \{f \in A | f \in K(x)^\circ,\ x \in X\}.$$

(This also follws from $A^+ = \{f \in A |\, |f(x)| \leq 1,\ x \in X\}$.)

Corollary 5.5.2. *Let $\widetilde{\mathcal{O}}_X$ be the sheafification of $\mathcal{O}_X$. Then $A \to H^0(X, \widetilde{\mathcal{O}}_X)$ is injective.*

Proof. Indeed, the H^0 maps into $\prod_x K(x)$, into which A maps injectively. □

Definition 5.5.3. (A, A^+) is *stably uniform* if $\mathcal{O}_X(U)$ is uniform for all rational subsets $U \subset X = \mathrm{Spa}(A, A^+)$.

Theorem 5.5.4 ([BV, Thm. 7],[KL, Thm. 2.8.10],[Mih])**.** *If (A, A^+) is stably uniform, then it is sheafy.*

Theorem 5.5.5 ([KL, Thm. 2.4.23])**.** *If (A, A^+) is sheafy, then $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. (This is a version of Tate acyclicity.)*

Strategy of proof: Using combinatorial arguments going back to Tate, we can reduce to checking everything for a simple Laurent covering $X = U \cup V$, where $U = \{|f| \leq 1\}$ and $V = \{|f| \geq 1\}$. Then $\mathcal{O}_X(U) = A\langle T\rangle/\overline{(T-f)}$ and $\mathcal{O}_X(V) = A\langle S\rangle/\overline{(Sf-1)}$. We have $\mathcal{O}_X(U \cap V) = A\langle T, T^{-1}\rangle/\overline{(T-f)}$. We need to check that the Cech complex for this covering is exact. It is

$$A\langle T\rangle/\overline{(T-f)} \oplus A\langle S\rangle/\overline{(Sf-1)} \to A\langle T, T^{-1}\rangle/\overline{(T-f)}.$$

Lemma 5.5.6. *1. This map is surjective.*

2. If A is uniform, the kernel is A.

Proof. 1) is clear. For 2), the hard part is to show that the ideals were closed to begin with. By Berkovich, the norm on $A\langle T\rangle$ is the supremum norm: $\sup_{x \in \mathrm{Spa}(A,A^+)} |\ |_{K(x)} = \sup_x |\cdot|_{K(x)\langle T\rangle}$. This is the Gauss norm, and it is multiplicative.

We claim that for all $g \in A\langle T\rangle$, $|(T-f)g|_{A\langle T\rangle} \geq |g|_{A\langle T\rangle}$. Indeed, using the above, the LHS is the supremum of $|(T-f)g|$, but this is multiplicative, so get $\sup|T-f| \sup|g| \geq \sup|g|$. □

Example 5.5.7. In general, $\mathcal{O}_X$ is not a sheaf. In fact, uniform does not imply sheafy. (Therefore it does not imply stably uniform.) Both parts of the sheaf property fail: the injectivity and the surjectivity parts! (See the papers of Buzzard-Verberkmoes and Mihara.)

Theorem 5.5.8 ([KL, Thm. 2.7.7])**.** *Let (A, A^+) be a sheafy Tate-Huber pair. Let $X = \mathrm{Spa}(A, A^+) = \bigcup_i U_i$, with $U_i = \mathrm{Spa}(A_i, A_i^+)$, and also U_{ij} (the overlaps), etc. Then vector bundles "behave as expected." In particular, finitely generated projective A-modules M are in correspondence with data (M_i, β_{ij}), where M_i is a finitely generated projective A_i-modules, and*

$$\beta_{ij} \colon M_j \otimes_{A_i} A_{ij} \to M_i \otimes_{A_j} A_{ij}$$

is a system of isomorphisms satsifying the cocycle condition $\beta_{ij} \circ \beta_{jk} = \beta_{ik}$.

Corollary 5.5.9. *On honest analytic adic spaces X, there is a category $\mathrm{VB}(X)$ of vector bundles on X (consisting of certain sheaves of $\mathcal{O}_X$-modules) such that $\mathrm{VB}(\mathrm{Spa}(A, A^+))$ classifies finite projective A-modules.*

It is not immediately clear how to get a good theory of coherent sheaves on adic spaces, although there is some forthcoming work of Kedlaya-Liu that defines a category of $\mathcal{O}_X$-modules which are "pseudo-coherent", meaning that locally they come from modules which admit a (possibly infinite) resolution by free modules.

The strategy of proof of the theorem is to reduce to simple Laurent coverings, and then imitate the proof of Beauville-Laszlo [BL95], who prove the following lemma.

Lemma 5.5.10. *Let R be a (not necessarily noetherian!) ring, let $f \in R$ be a non-zero-divisor, and let $\widehat{R}$ be the f-adic completion of R. Then the category of R-modules M where f is not a zero-divisor is equivalent to the category of pairs $(M_{\widehat{R}}, M[f^{-1}], \beta)$, where $M_{\widehat{R}}$ is an $\widehat{R}$-module such that f is not a zero-divisor, $M_{R[f^{-1}]}$ is an $R[f^{-1}]$-module, and $\beta \colon M_{\widehat{R}}[f^{-1}] \to M_{R[f^{-1}]} \otimes_R \widehat{R}$ is an isomorphism.*

This does not follow from fpqc descent because of two subtle points: $R \to \widehat{R}$ might not be flat, and also we have not included a descent datum on $\widehat{R} \otimes_R \widehat{R}$.

6 Perfectoid rings, 18 September

Today we begin discussing perfectoid spaces. Let p be a fixed prime throughout.

6.1 Perfectoid Rings

Recall that a Huber ring R is Tate if it contains a topologically nilpotent unit; such elements are called pseudo-uniformizers. If $\varpi \in R$ is a pseudo-uniformizer, then necessarily $\varpi \in R^\circ$. Furthermore, if $R^+ \subset R$ is a ring of integral elements, then $\varpi \in R^+$. Indeed, since $\varpi^n \to 0$ as $n \to \infty$ and since R^+ is open, we have $\varpi^n \in R^+$ for some $n \geq 1$. Since R^+ is integrally closed, $\varpi \in R^+$.

The following definition is due to Fontaine [Fon13].

Definition 6.1.1. A complete Tate ring R is *perfectoid* if R is uniform (recall this means that $R^\circ \subset R$ is bounded) and there exists a pseudo-uniformizer $\varpi \in R$ such that $\varpi^p | p$ in R°, and such that the pth power Frobenius map

$$\Phi \colon R^\circ/\varpi \to R^\circ/\varpi^p$$

is an isomorphism.

Hereafter we use the following notational convention. If R is a ring, and $I, J \subset R$ are ideals containing p such that $I^p \subset J$, then $\Phi\colon R/I \to R/J$ will refer to the ring homomorphism $x \mapsto x^p$.

Remark 6.1.2. Let us explain why the isomorphism condition above is independent of ϖ. For any complete Tate ring R and pseudo-uniformizer ϖ satisfying $\varpi^p|p$ in R°, the Frobenius map $\Phi\colon R^\circ/\varpi \to R^\circ/\varpi^p$ is necessarily injective. Indeed, if $x \in R^\circ$ satisfies $x^p = \varpi^p y$ for some $y \in R^\circ$ then the element $x/\varpi \in R$ lies in R° since its pth power does. Thus, the isomorphism condition on Φ in Defn. 6.1.1 is really a *surjectivity* condition. In fact, this surjectivity condition is equivalent to the surjectivity of the (necessarily injective) Frobenius map

$$R^\circ/(p, \varpi^n) \to R^\circ/(p, \varpi^{np})$$

for *any* $n \geq 1$. (Proof: the surjectivity of $\Phi\colon R^\circ/\varpi \to R^\circ/\varpi^p$ is just the case $n = 1$, and if this map is surjective for n it is clearly surjective for all $1 \leq n' < n$. In the other direction, if this map is surjective for some n, and $x \in R^\circ/p$, we can write $x = y^p + \varpi^{np} z$ for some $y, z \in R^\circ/p$, yet likewise $z = t^p + \varpi^p u$ for some $t, u \in R^\circ/p$, so $x = (y + \varpi^n t)^p + \varpi^{(n+1)p} u$. Therefore this map is surjective for $n + 1$.)

Suppose ϖ' is another pseudo-uniformizer satisfying $\varpi'^p|p$ in R°. If we take n large enough then $\varpi^n \in \varpi' R^\circ$, then the surjectivity of $\Phi\colon R^\circ/(p, \varpi^n) \to R^\circ/(p, \varpi^{np})$ implies the surjectivity of $\Phi\colon R^\circ/\varpi' \to R^\circ/\varpi'^p$. Therefore the isomorphism condition on Φ in Defn. 6.1.1 holds for all ϖ satisfying $\varpi^p|p$ if it holds for one such choice.

Remark 6.1.3. $\mathbf{Q}_p$ is not perfectoid, even though the Frobenius map on $\mathbf{F}_p$ is an isomorphism. Certainly there is no element $\varpi \in \mathbf{Z}_p$ whose pth power divides p. But more to the point, a discretely valued non-archimedean field K of residue characteristic p cannot be perfectoid. Indeed, if ϖ is a pseudo-uniformizer as in Defn. 6.1.1, then ϖ is a non-zero element of the maximal ideal, so the quotients K°/ϖ and K°/ϖ^p are Artin local rings of different lengths and hence they cannot be isomorphic.

Example 6.1.4.

1. $\mathbf{Q}_p^{\text{cycl}}$, the completion of $\mathbf{Q}_p(\mu_{p^\infty})$.
2. The t-adic completion of $\mathbf{F}_p(\!(t)\!)(t^{1/p^\infty})$, which we will write as $\mathbf{F}_p(\!(t^{1/p^\infty})\!)$.
3. $\mathbf{Q}_p^{\text{cycl}}\langle T^{1/p^\infty}\rangle$. This is defined as $A[1/p]$, where A is the p-adic completion of $\mathbf{Z}_p^{\text{cycl}}[T^{1/p^\infty}]$.

4. (An example which does not live over a field). Recall from our discussion in 4.5.3 the ring $\mathbf{Z}_p[\![T]\!]\langle p/T\rangle[1/T]$, which is Tate with pseudo-uniformizer T, but which does not contain a nonarchimedean field. One can also construct a perfectoid version of it,

$$R = \mathbf{Z}_p^{\text{cycl}}\langle (p/T)^{1/p^\infty}\rangle[1/T].$$

Here we can take $\varpi = T^{1/p}$, because $\varpi^p = T$ divides p in R°.

Question: Is there a more general definition of perfectoid Huber rings, which do not have to be Tate? This should include $\mathbf{Z}_p^{\text{cycl}}[\![T^{1/p^\infty}]\!]$, the (p,T)-adic completion of $\mathbf{Z}_p^{\text{cycl}}[T^{1/p^\infty}]$.

Proposition 6.1.5. *Let R be a topological ring with $pR = 0$. The following are equivalent:*

1. *R is perfectoid.*
2. *R is a perfect uniform complete Tate ring.*

Of course, perfect means that $\Phi\colon R \to R$ is an isomorphism.

Proof. Let R be a uniform complete Tate ring. If R is perfect, then take ϖ any pseudo-uniformizer. The condition $\varpi^p | p = 0$ is vacuous. If $x \in R$ is powerbounded, then so is x^p, and vice versa, which means that $\Phi\colon R^\circ \to R^\circ$ is an isomorphism. This shows that $\Phi\colon R^\circ/\varpi \to R^\circ/\varpi^p$ is surjective. For injectivity: suppose that $x \in R^\circ$, $x^p = \varpi^p y$, with $y \in R^\circ$. Write $y = z^p$ with $z \in R^\circ$, and then $x = \varpi z$.

Conversely if R is perfectoid, then $\Phi\colon R^\circ/\varpi \to R^\circ/\varpi^p$ is an isomorphism, and therefore so is $R^\circ/\varpi^n \to R^\circ/\varpi^{np}$ by induction. Taking inverse limits and using completeness, we find that $\Phi\colon R^\circ \to R^\circ$ is an isomorphism. Then invert ϖ. □

Definition 6.1.6. A *perfectoid field* is a perfectoid ring R which is a nonarchimedean field.

Remark 6.1.7. It is not clear that a perfectoid ring which is a field is a perfectoid field.

Proposition 6.1.8. *Let K be a nonarchimedean field. K is a perfectoid field if and only if the following conditions hold:*

1. *K is not discretely valued,*

2. $|p| < 1$, *and*

3. $\Phi\colon \mathcal{O}_K/p \to \mathcal{O}_K/p$ *is surjective.*

Proposition 6.1.9. *Let R be a complete uniform Tate ring.*

1. *If there exists a pseudo-uniformizer $\varpi \in R$ such that $\varpi^p|p$ and $\Phi\colon R^\circ/p \to R^\circ/p$ is surjective, then R is a perfectoid ring.*

2. *Conversely, if R is a perfectoid ring, then $\Phi\colon R^\circ/p \to R^\circ/p$ is surjective* under the additional assumption that the ideal $pR^\circ \subset R^\circ$ is closed.

Proof. For (1), suppose that $\Phi\colon R^\circ/p \to R^\circ/p$ is surjective. Then $\Phi\colon R^\circ/\varpi \to R^\circ/\varpi^p$ is surjective as well. Injectivity is true automatically; see Rmk. 6.1.2.

For (2), assume that R is a perfectoid ring such that pR° is closed. Take $\varpi \in R$ a pseudo-uniformizer such that $\varpi^p|p$, and such that $\Phi\colon R^\circ/\varpi \to R^\circ/\varpi^p$ is an isomorphism. By the discussion in Rmk. 6.1.2, $\Phi\colon R^\circ/(p, \varpi^n) \to R^\circ/(p, \varpi^{np})$ is an isomorphism for all n. Now we take the inverse limit over n. The proposition will follow once we show that

$$R^\circ/p \to \varprojlim_n R^\circ/(p, \varpi^n)$$

is an isomorphism. Certainly it is surjective: R°/p is ϖ-adically complete, since R° is. Injectivity is the statement that R°/p is ϖ-adically separated. If x lies in the kernel, then there exist $y_n, z_n \in R^\circ$ with $x = \varpi^n y_n + pz_n$, $n \geq 1$. Since R is uniform, R° is bounded and thus $\varpi^n y_n \to 0$ as $n \to \infty$. We find that $x = \lim_{n\to\infty} pz_n$ lies in the closure of pR°. Since pR° is closed, we are done. □

Remark 6.1.10. It is highly likely that there exists a perfectoid ring R such that pR° is not closed, *e.g.* by adapting the example appearing in Section 7 of the Stacks Project – Examples. However, if p is invertible in a perfectoid ring R, then $\Phi\colon R^\circ/p \to R^\circ/p$ is surjective: see [KL, Prop. 3.6.2(e)].

Theorem 6.1.11 ([Sch12, Thm. 6.3],[KL, Thm. 3.6.14])**.** *Let (R, R^+) be a Huber pair such that R is perfectoid. Then for all rational subsets $U \subset X = \mathrm{Spa}(R, R^+)$, $\mathcal{O}_X(U)$ is again perfectoid. In particular, (R, R^+) is stably uniform, hence sheafy by Thm. 5.5.4.*

Question 6.1.12. Does Thm. 6.1.11 extend to a more general class of perfectoid Huber rings (without the Tate condition)?

The hard part of Thm. 6.1.11 is showing that $\mathcal{O}_X(U)$ is uniform. The proof of this fact makes essential use of the process of *tilting*.

6.2 Tilting

Definition 6.2.1. Let R be a perfectoid Tate ring. The *tilt* of R is

$$R^\flat = \varprojlim_{x \mapsto x^p} R,$$

given the inverse limit topology. A priori this is only a topological multiplicative monoid. We give it a ring structure where the addition law is

$$(x^{(0)}, x^{(1)}, \dots) + (y^{(0)}, y^{(1)}, \dots) = (z^{(0)}, z^{(1)}, \dots)$$

where

$$z^{(i)} = \lim_{n \to \infty} (x^{(i+n)} + y^{(i+n)})^{p^n} \in R.$$

Lemma 6.2.2. *The limit $z^{(i)}$ above exists and defines a ring structure making $R^\flat$ a topological $\mathbf{F}_p$-algebra that is a perfect uniform Tate ring. The subset $R^{\flat\circ}$ of power-bounded elements is given by the topological ring isomorphism*

$$R^{\flat\circ} = \varprojlim_{\Phi} R^\circ \cong \varprojlim_{\Phi} R^\circ/p \cong \varprojlim_{\Phi} R^\circ/\varpi,$$

where $\varpi \in R$ is a pseudo-uniformizer which divides p in R°. Furthermore there exists a pseudo-uniformizer $\varpi \in R$ with $\varpi^p | p$ in R° which admits a sequence of pth power roots ϖ^{1/p^n}, giving rise to an element $\varpi^\flat = (\varpi, \varpi^{1/p}, \dots) \in R^{\flat\circ}$, which is a pseudo-uniformizer of $R^\flat$. Then $R^\flat = R^{\flat\circ}[1/\varpi^\flat]$.

Proof. Certainly $R^\flat$ is perfect by design. Let ϖ_0 be a pseudo-uniformizer of R. Let us check that the maps

$$\varprojlim_{\Phi} R^\circ \to \varprojlim_{\Phi} R^\circ/p \to \varprojlim_{\Phi} R^\circ/\varpi_0$$

are isomorphisms. The essential point is that any sequence $(\overline{x}_0, \overline{x}_1, \dots) \in \varprojlim_\Phi R^\circ/\varpi_0$ lifts uniquely to a sequence $(x_0, x_1, \dots) \in \varprojlim_\Phi R^\circ$. Here $x^{(i)} = \lim_{n\to\infty} x_{n+i}^{p^n}$, where $x_j \in R^\circ$ is any lift of $\overline{x}_j$. (For the convergence of that limit, note that if $x \equiv y \pmod{\varpi_0^n}$, then $x^p \equiv y^p \pmod{\varpi_0^{n+1}}$.) This shows that we get a well-defined ring $R^{\flat\circ}$.

Now assume that $\varpi_0^p | p$ in R°. We construct the element $\varpi^\flat$. The preimage of ϖ_0 under $R^{\flat\circ} = \varprojlim_\Phi R^\circ/\varpi_0^p \to R^\circ/\varpi_0^p$ is an element $\varpi^\flat$ with the right properties. It is congruent to ϖ_0 modulo ϖ_0^p, and therefore it is also a pseudo-uniformizer. Then $\varpi = \varpi^{\flat\sharp}$ is the desired pseudo-uniformizer of R°. $\square$

Remark 6.2.3. In the special case that $R = K$ is a perfectoid field, the construction of $K^\flat$ is due to Fontaine, [Fon82], as an intermediate step towards his construction of p-adic period rings. Here, $K^\flat$ is a complete algebraically closed valued field with absolute value defined by $f \mapsto \left|f^\sharp\right|$, where $|\ |$ is the absolute value on K. It is a nontrivial theorem that if K is algebraically closed, then so is $K^\flat$.

We have a continuous, multiplicative (but not additive) map $R^\flat \to \varprojlim R \to R$ by projecting onto the zeroth coordinate; call this $f \mapsto f^\sharp$. This projection defines a ring isomorphism $R^{\flat\circ}/\varpi^\flat \cong R^\circ/\varpi$. By topological nilpotence conditions, the open subrings of $R^{\flat\circ}$ and R° correspond exactly to the subrings of their common quotients modulo $\varpi^\flat$ and ϖ. Moreover, the open subring is integrally closed if and only if its image in this quotient is integrally closed. This defines an inclusion-preserving bijection between the sets of open integrally closed subrings of $R^{\flat\circ}$ and R°. This correspondence can be made more explicit:

Lemma 6.2.4. *The set of rings of integral elements $R^+ \subset R^\circ$ is in bijection with the set of rings of integral elements $R^{\flat+} \subset R^{\flat\circ}$, via $R^{\flat+} = \varprojlim_{x \mapsto x^p} R^+$. Also, $R^{\flat+}/\varpi^\flat = R^+/\varpi$.*

The following two theorems belong to a pattern of "tilting equivalence".

Theorem 6.2.5. *(Kedlaya, Scholze) Let (R, R^+) be a perfectoid Huber pair, with tilt $(R^\flat, R^{\flat+})$. There is a homeomorphism $\mathrm{Spa}(R, R^+) \cong \mathrm{Spa}(R^\flat, R^{\flat+})$ sending x to $x^\flat$, where $\left|f(x^\flat)\right| = \left|f^\sharp(x)\right|$. This homeomorphism preserves rational subsets.*

Theorem 6.2.6 ([Sch12])**.** *Let R be a perfectoid ring with tilt $R^\flat$. Then there is an equivalence of categories between perfectoid R-algebras and perfectoid $R^\flat$-algebras, via $S \mapsto S^\flat$.*

Let us at least describe the inverse functor, along the lines of Fontaine's Bourbaki talk. In fact we will answer a more general question. Given a perfectoid algebra R in characteristic p, what are all the untilts $R^\sharp$ of R? Let's start with a pair (R, R^+).

Lemma 6.2.7. *Let $(R^\sharp, R^{\sharp+})$ be an untilt of R;* i.e. *a perfectoid ring $R^\sharp$ together with an isomorphism $R^{\sharp\flat} \to R$, such that $R^{\sharp+}$ and R^+ are identified under Lemma 6.2.4.*

1. *There is a canonical surjective ring homomorphism*

$$\begin{array}{rcl} \theta\colon W(R^+) & \to & R^{\sharp+} \\ \sum_{n\geq 0}[r_n]p^n & \mapsto & \sum_{n\geq 0} r_n^\sharp p^n \end{array}$$

2. *The kernel of θ is generated by a nonzero-divisor ξ of the form $\xi = p + [\varpi]\alpha$, where $\varpi \in R^+$ is a pseudo-uniformizer, and $\alpha \in W(R^+)$.*

See [Fon13], [FF11] and [KL, Thm. 3.6.5]. We remark that there is no assumption that an untilt of R should have characteristic p. In particular R itself is an untilt of R, corresponding to $\alpha = 0$.

Definition 6.2.8. An ideal $I \subset W(R^+)$ is *primitive of degree 1* if I is generated by an element of the form $\xi = p + [\varpi]\alpha$, with $\varpi \in R^+$ a pseudo-uniformizer and $\alpha \in W(R^+)$. (This ξ is necessarily a non-zero-divisor.)

Proof. (Of the lemma)

1. This follows from the Deninger's universal property of Witt vectors: $W(R^+)$ is the universal p-adically complete ring A with a continuous multiplicative map $R^+ \to W(R^+)$. Thus there exists a map $\theta\colon W(R^+) \to R^{\sharp+}$. For surjectivity: we have that $R^+ \to R^{\sharp+}/p$ is surjective, which shows that θ mod p is surjective, which shows (since everything is p-adically complete) θ is surjective.

2. Fix $\varpi \in R^+$ a pseudo-uniformizer such that $\varpi^\sharp \in R^{\sharp+}$ satisfies $(\varpi^\sharp)^p | p$. We claim that there exists $f \in \varpi R^+$ such that $f^\sharp \equiv p \pmod{p\varpi^\sharp R^{\sharp+}}$. Indeed, consider $\alpha = p/\varpi^\sharp \in R^{\sharp+}$. There exists $\beta \in R^+$ such that $\beta^\sharp \equiv \alpha \pmod{pR^{\sharp+}}$. Then $(\varpi\beta)^\sharp = \varpi^\sharp\alpha \equiv p \pmod{p\varpi^\sharp R^{\sharp+}}$. Take $f = \varpi\beta$.

   Thus we can write $p = f^\sharp + p\varpi^\sharp \sum_{n\geq 0} r_n^\sharp p^n$, with $r_n \in R^+$. We can now define $\xi = p - [f] - [\varpi]\sum_{n\geq 0}[r_n]p^{n+1}$, which is of the desired form, and which lies in the kernel of θ.

Lemma 6.2.9. *Any ξ of this form, namely $p + [\varpi]\alpha$, is a non-zero-divisor.*

Proof. Assume $\xi \sum_{n\geq 0}[c_n]p^n = 0$. Mod $[\varpi]$, this reads $\sum_{n\geq 0}[c_n]p^{n+1} \equiv 0 \pmod{[\varpi]}$, meaning that all $c_n \equiv 0 \pmod{\varpi}$. Divide by ϖ, and induct. □

Finally we need to show that ξ generates $\ker(\theta)$. For this, note that θ induces is a surjective map $W(R^+)/(\xi) \to R^{\sharp+}$. It is enough to show that

f is an isomorphism modulo $[\varpi]$, because $W(R^+)/(\xi)$ is $[\varpi]$-torsion free and $[\varpi]$-adically complete. We have $W(R^+)/(\xi, [\varpi]) = W(R^+)/(p, [\varpi]) = R^+/[\varpi] = R^+/\varpi \cong R^{\sharp +}/\varpi^\sharp$. □

Theorem 6.2.10. *(Kedlaya-Liu, Fontaine) There is an equivalence of categories between:*

1. *Perfectoid Tate-Huber pairs* (S, S^+)
2. *Triples* $(R, R^+, \mathcal{J})$*, where* (R, R^+) *is a perfectoid Tate-Huber pair of characteristic* p *and* $\mathcal{J} \subset W(R^+)$ *is primitive of degree 1.*

In one direction the map is $(S, S^+) \mapsto (S^\flat, S^{\flat +}, \ker \theta)$*, and in the other, it is* $(R, R^+, \mathcal{J}) \mapsto (W(R^+)[[\varpi]^{-1}]/\mathcal{J}, W(R^+)/\mathcal{J})$.

6.3 Preperfectoid rings

Thm. 6.1.11 states that if (R, R^+) is a Huber pair with R perfectoid, then (R, R^+) is sheafy. It will be useful to extend this theorem to a slightly broader class of Huber pairs. The following definition slightly generalizes the one appearing in [SW13], Defn. 2.3.9.

Definition 6.3.1. Let R be a $\mathbf{Z}_p$-algebra which is Tate. R is *preperfectoid* if there exists a perfectoid field K of characteristic 0 such that $R \widehat{\otimes}_{\mathbf{Z}_p} \mathcal{O}_K$ is perfectoid.

Example 6.3.2.

1. $R = \mathbf{Q}_p\langle T^{1/p^\infty}\rangle$ is preperfectoid, since $R \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{Z}_p^{\mathrm{cycl}} = \mathbf{Q}_p^{\mathrm{cycl}}\langle T^{1/p^\infty}\rangle$ is perfectoid.
2. Let $R = \mathbf{Z}_p\langle (p/T)^{1/p^\infty}\rangle[1/T]$. Then R is Tate with pseudo-uniformizer T, and $R \widehat{\otimes} \mathbf{Z}_p^{\mathrm{cycl}}$ is the perfectoid ring from Example 6.1.4(4).
3. Unfortunately, perfectoid rings aren't necessarily preperfectoid. For instance, the perfectoid field $\mathbf{Q}_p^{\mathrm{cycl}}$ isn't preperfectoid, because one can show that for any perfectoid field K, $\mathbf{Q}_p^{\mathrm{cycl}} \widehat{\otimes}_{\mathbf{Z}_p} \mathcal{O}_K$ isn't uniform.

Proposition 6.3.3. *Let* (R, R^+) *be a Tate Huber pair such that* R *is uniform and preperfectoid. Then* (R, R^+) *is sheafy.*

Proof. Let K be a perfectoid field such that $\widetilde{R} = R \widehat{\otimes}_{\mathbf{Z}_p} \mathcal{O}_K$ is perfectoid. We claim that (R, R^+) is stably uniform, which by Thm. 5.5.4 imples that it is sheafy. Let $U = \mathrm{Spa}(R, R^+)$ and let $\widetilde{U} = \mathrm{Spa}(\widetilde{R}, \widetilde{R}^+)$, where $\widetilde{R}^+$ is the integral closure of $R^+ \widehat{\otimes}_{\mathbf{Z}_p} \mathcal{O}_K$ in $\widetilde{R}$. Let $V \subset U$ be a rational subset, and let $\widetilde{V}$ be its preimage under $\widetilde{U} \to U$, which is also a rational subset. Then $\mathcal{O}_U(V)$ is a topological subring of $\mathcal{O}_{\widetilde{U}}(\widetilde{V}) = \mathcal{O}_U(V)\widehat{\otimes}_{\mathbf{Z}_p}\mathcal{O}_K$.

Since $\widetilde{R}$ is perfectoid, Thm. 6.1.11 implies that $\mathcal{O}_{\widetilde{U}}(\widetilde{V})$ is perfectoid, hence uniform. Thus $\mathcal{O}_{\widetilde{U}}(\widetilde{V})^\circ$ serves as a ring of definition. Let $\varpi \in R$ be a pseudo-uniformizer; then ϖ serves as a pseudo-uniformizer for both $\mathcal{O}_U(V)$ and $\mathcal{O}_{\widetilde{U}}(\widetilde{V})$. Therefore $\varpi^n\mathcal{O}_{\widetilde{U}}(\widetilde{V})^\circ \cap \mathcal{O}_U(V)$ ($n \geq 0$) is a basis of neighborhoods of 0 in $\mathcal{O}_U(V)$. It follows that $\mathcal{O}_{\widetilde{U}}(\widetilde{V})^\circ \cap \mathcal{O}_U(V)$ is open and bounded in $\mathcal{O}_U(V)$, hence a ring of definition. With these facts in place, it is easy to see that $\mathcal{O}_U(V)^\circ = \mathcal{O}_U(V) \cap \mathcal{O}_{\widetilde{U}}(\widetilde{V})^\circ$, and that this is bounded in $\mathcal{O}_U(V)$. Therefore $\mathcal{O}_U(V)$ is uniform as required. □

7 Perfectoid spaces, 23 September

This will be the second lecture on perfectoid spaces. Recall that a *perfectoid Tate ring* R is a complete, uniform Tate ring containing a pseudo-uniformizer ϖ such that $\varpi^p | p$ in R° and such that $\Phi\colon R^\circ/\varpi \to R^\circ/\varpi^p$ is an isomorphism.

In fact there is a recent generalization of this notion due to Gabber-Ramero. They define (at least) a *perfectoid Huber ring* (and even show that these are sheafy). This would include $\mathbf{Z}_p^{\mathrm{cycl}}[\![T^{1/p^\infty}]\!]$.

7.1 Perfectoid spaces: definition and tilting equivalence

We also talked about tilting. Suppose (R, R^+) is a Huber pair, with R perfectoid. Let $R^\flat = \varprojlim_{x \mapsto x^p} R$, a perfectoid ring of characteristic p, together with a map $R^\flat \to R$ of multiplicative monoids $f \mapsto f^\sharp$.

Theorem 7.1.1. *A Huber pair (R, R^+) with R perfectoid is sheafy. Let $X = \mathrm{Spa}(R, R^+)$, $X^\flat = \mathrm{Spa}(R^\flat, R^{\flat,+})$, and then we have a homeomorphism $X \to X^\flat$, $x \mapsto x^\flat$, which preserves rational subsets. It is characterized by $\left|f(x^\flat)\right| = \left|f^\sharp(x)\right|$. Moreover for $U \subset X$ perfectoid, $\mathcal{O}_X(U)$ is perfectoid with tilt $\mathcal{O}_{X^\flat}(U)$.*

Definition 7.1.2. A *perfectoid space* is an adic space covered by $\mathrm{Spa}(R, R^+)$ with R perfectoid.

Remark 7.1.3. Perfectoid spaces are honest because such $\mathrm{Spa}(R, R^+)$ are sheafy. However if (R, R^+) is some Huber pair and $\mathrm{Spa}(R, R^+)$ is a perfectoid space, it is not clear whether R has to be perfectoid (although it is fine if we are in characteristic p). See Buzzard-Verberkmoes.

The tilting process glues to give a functor $X \mapsto X^\flat$.

Proposition 7.1.4. *Tilting induces an equivalence between the following categories:*

1. *Pairs (S, S^+), where S is perfectoid, and*
2. *Triples $(R, R^+, \mathcal{J})$, where (R, R^+) is perfectoid of characteristic p, and $\mathcal{J} \subset W(R^+)$ is a primitive ideal of degree 1.*

7.2 Why do we study perfectoid spaces?

1. Any adic space over $\mathbf{Q}_p$ is pro-étale locally perfectoid (Colmez). That is, if A is a Banach $\mathbf{Q}_p$-algebra, there exists a filtered directed system of finite étale A-algebras A_i such that $A_\infty = \widehat{\varinjlim A_i}$ is perfectoid. For example, if $X = \mathrm{Spa}(\mathbf{Q}_p\langle T^{\pm 1}\rangle, \mathbf{Z}_p\langle T^{\pm 1}\rangle)$ is the "unit circle", this has a pro-étale covering by $\widetilde{X} = \mathrm{Spa}(\mathbf{Q}_p^{\mathrm{cycl}}\langle T^{\pm 1/p^\infty}, \mathbf{Z}_p^{\mathrm{cycl}}\langle T^{\pm 1/p^\infty}\rangle\rangle)$, this being the inverse limit of the appropriate system X_i.
2. If X is a perfectoid space, all topological information (*e.g.* $|X|$, and even $X_{\text{ét}}$) can be recovered from $X^\flat$. However $X^\flat$ forgets the structure morphism $X \to \mathrm{Spa}(\mathbf{Z}_p, \mathbf{Z}_p)$. The following can be made precise (next lecture): The category of perfectoid spaces over $\mathbf{Q}_p$ is equivalent to the category of perefectoid spaces X of characteristic p, together with a "structure morphism $X \to \mathbf{Q}_p$".

7.3 The equivalence of étale sites

The tilting equivalence extends to the étale site of a perfectoid space. That is, if X is a perfectoid space then there is an equivalence $X_{\text{ét}} \cong X^\flat_{\text{ét}}$. First we discuss the case where X is a single point.

Theorem 7.3.1 ([FW79], [KL, Thm. 3.5.6], [Sch12])**.** *Let K be a perfectoid field with tilt $K^\flat$.*

1. *If L/K is finite, then L is perfectoid.*
2. *$L \mapsto L^\flat$ is an equivalence of categories between finite extensions of K and finite extensions of $K^\flat$ which preserves degrees. Thus, the absolute Galois groups of K and $K^\flat$ are isomorphic.*

Theorem 7.3.2 ([Tat67], [GR03])**.** *Let K be a perfectoid field, L/K a finite extension. Then $\mathcal{O}_L/\mathcal{O}_K$ is* almost finite étale*.*

For the precise meaning of almost finite étale, which is somewhat technical, we refer to [Sch12], Section 4. What Tate actually proved is that (for certain perfectoid fields K) if $\mathrm{tr}\colon L \to K$ is the trace map, then $\mathrm{tr}(\mathcal{O}_L)$ contains $\mathfrak{m}_L$, the maximal ideal of L.

Example 7.3.3. Say $K = \mathbf{Q}_p(p^{1/p^\infty})^\wedge$, a perfectoid field. Let $L = K(\sqrt{p})$ (and assume $p \neq 2$). Let $K_n = \mathbf{Q}_p(p^{1/p^n})$ and $L_n = K_n(\sqrt{p})$. Note that $p^{1/2p^n} \in L_n$, because $p^{1/2p^n} = (p^{1/p^n})^{(p^n+1)/2}p^{-1/2}$, and that

$$\mathcal{O}_{L_n} = \mathcal{O}_{K_n}[p^{1/2p^n}] = \mathcal{O}_{K_n}[x]/(x^2 - p^{1/p^n}).$$

Let $f(x) = x^2 - p^{1/p^n}$. The different ideal δ_{L_n/K_n} is the ideal of $\mathcal{O}_{L_n}$ generated by $f'(p^{1/2p^n})$, which is $p^{1/2p^n}$. The p-adic valuation of δ_{L_n/K_n} is $1/2p^n$, which tends to 0 as $n \to \infty$. Inasmuch as the different measures ramification, this means that the extensions L_n/K_n are getting less ramified as $n \to \infty$.

In other words, one can almost get rid of ramification along the special fibre by passing to a tower whose limit is perfectoid. This is what Tate does (using the cyclotomic tower) to do computations in Galois cohomology, which is an essential part of p-adic Hodge theory.

In fact Theorem 7.3.2 implies Theorem 7.3.1. The equivalence between finite étale algebras over K and $K^\flat$ goes according to the diagram (which uses the notations of [Sch12]):

$$\begin{array}{ccccc}
\left\{\begin{array}{l}\text{finite étale}\\ K\text{-algs.}\end{array}\right\} & \xrightarrow{\text{Thm. 7.3.2}} & \left\{\begin{array}{l}\text{almost}\\ \text{finite étale}\\ \mathcal{O}_K^a\text{-algs.}\end{array}\right\} & \longrightarrow & \left\{\begin{array}{l}\text{almost}\\ \text{finite étale}\\ (\mathcal{O}_K/\varpi)^a\text{-}\\ \text{algs.}\end{array}\right\} \\
 & & & & \downarrow \\
\left\{\begin{array}{l}\text{finite étale}\\ K^\flat\text{-algs.}\end{array}\right\} & \xleftarrow{\text{Thm. 7.3.2}} & \left\{\begin{array}{l}\text{almost}\\ \text{finite étale}\\ \mathcal{O}_{K^\flat}^a\text{-algs.}\end{array}\right\} & \longleftarrow & \left\{\begin{array}{l}\text{almost}\\ \text{finite étale}\\ (\mathcal{O}_{K^\flat}/\varpi^\flat)^{\flat a}\text{-}\\ \text{algs.}\end{array}\right\}
\end{array}$$

Philosophically, properties of K extend "almost integrally" to $\mathcal{O}_K$, which one can then pass to $\mathcal{O}_K/\varpi$.

7.4 Almost mathematics, after Faltings

Let R be a perfectoid Tate ring.

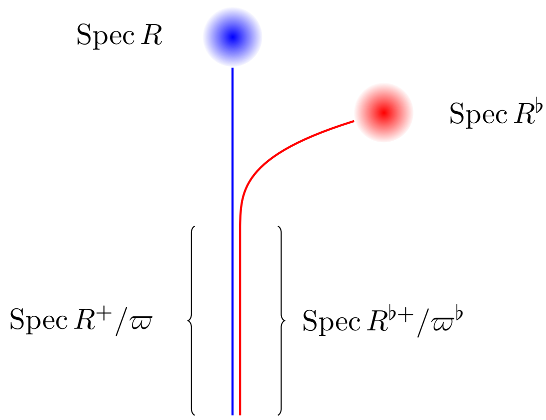


Figure 2: A (not to be taken too seriously) depiction of the tilting process for a perfectoid ring R. The blue figure represents Spec R^+ and the red figure represents Spec $R^{\flat+}$. Objects associated with R can "almost" be extended to R^+ and then reduced modulo ϖ. But then $R^+/\varpi = R^{\flat+}/\varpi^\flat$, so one gets an object defined over $R^{\flat+}/\varpi^\flat$. The process can be reversed on the $R^\flat$ side, so that one gets a tilted object defined over $R^\flat$.

Definition 7.4.1. An R°-module M is *almost zero* if $\varpi M = 0$ for all pseudo-uniformizers ϖ. Equivalently, if ϖ is a fixed pseudo-uniformizer admitting pth power roots, and $\varpi^{1/p^n}M = 0$ for all n. (Similarly for R^+-modules)

Example 7.4.2.

1. If K is a perfectoid field, $\mathcal{O}_K/\mathfrak{m}_K$ is almost zero. (A general almost zero module is a direct sum of such modules.)

2. If R is perfectoid and $R^+ \subset R^\circ$ is any ring of integral elements, then R°/R^+ is almost zero. Indeed, if ϖ is a pseudo-uniformizer, and $x \in R^\circ$, then ϖx is topologically nilpotent. Since R^+ is open, there exists n with $(\varpi x)^n \in R^+$, so that $\varpi x \in R^+$ by integral closedness.

Note: extensions of almost zero modules are almost zero. Thus the category of almost zero modules is a thick Serre subcategory of the category of all modules, and one can take the quotient.

Definition 7.4.3. The category of *almost R°-modules*, written $R^{\circ a}$-mod, is the quotient of the category of R°-modules by the subcategory of almost zero modules.

One can also define R^{+a}-mod, but the natural map $R^{\circ a}$-mod $\to R^{+a}$-mod is an equivalence.

Theorem 7.4.4. *Let (R, R^+) be a perfectoid Tate-Huber pair, and let $X = \mathrm{Spa}(R, R^+)$. (Thus by Kedlaya-Liu, $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.) Then $H^i(X, \mathcal{O}_X^+)$ is almost zero for $i > 0$, and $H^0(X, \mathcal{O}_X^+) = R$.*

Proof. By tilting one can reduce to the case of $pR = 0$. We will show that for any finite rational covering $X = \bigcup_i U_i$, all cohomology groups of the Cech complex

$$C^\bullet : \ 0 \to R^+ \to \prod_i \mathcal{O}_X^+(U_i) \to \prod_{i,j} \mathcal{O}_X^+(U_i \cap U_j) \to \cdots$$

are almost zero. We know that $C^\bullet[1/\varpi]$ (replace $\mathcal{O}_X^+$ with $\mathcal{O}_X$ everywhere) is exact. Now we use Banach's Open Mapping Theorem: each cohomology group of $C^\bullet$ is killed by a power of ϖ. (One knows each cohomology group is ϖ-power-torsion, but we assert that there is a single power of ϖ which kills everything.)

Since R is perfect, Frobenius induces isomorphisms on all cohomology groups of $C^\bullet$. So if these are killed by ϖ^n, they are also killed by all the ϖ^{n/p^k}, so they are almost zero. □

This is a typical strategy: bound the problem up to a power of ϖ, and then use Frobenius to shrink the power to zero.

Theorem 7.4.5 ([Fal02],[KL],[Sch12])**.** *Let R be perfectoid with tilt $R^\flat$.*

1. *For any finite étale R-algebra S, S is perfectoid.*
2. *Tilting induces an equivalence*

$$\begin{aligned} \{\textit{Finite étale } R\textit{-algebras}\} &\to \left\{\textit{Finite étale } R^\flat\textit{-algebras}\right\} \\ S &\mapsto S^\flat \end{aligned}$$

3. *(Almost purity) For any finite étale R-algebras S, then S° is almost finite étale over R°. (This means that S° is almost self-dual under the trace pairing.)*

See [**?**], Thm. 7.4.5 for a proof in characteristic 0, but the argument carries over to this context. See also Thm. 3.6.21 for parts (1) and (2) and Thm. 5.5.9 for (3).

The line of argument is to prove (2) and deduce (1) and (3) (by proving them in characteristic p). Let us sketch the proof of (2). We reduce to the case of perfectoid fields via the following argument. Let $x \in X = \mathrm{Spa}(R, R^+)$ with residue field $K(x)$, and similarly define $K(x^\flat)$. Then we have $K(x)^\flat = K(x^\flat)$.

Lemma 7.4.6.

$$2\text{-}\varinjlim_{U\ni x}\{\textit{finite étale } \mathcal{O}_X(U)\textit{-algs.}\} \to \{\textit{finite étale } K(x)\textit{-algs.}\}$$

is an equivalence.

Remark 7.4.7. One has here a directed system of categories $\mathcal{C}_i$, indexed by a filtered category I, with functors $F_{ij}\colon \mathcal{C}_i \to \mathcal{C}_j$ for each morphism $i \to j$ in I. The *2-limit* $\mathcal{C} = 2\text{-}\varinjlim \mathcal{C}_i$ is a category whose objects are objects of any $\mathcal{C}_i$. If X_i and X_j belong to $\mathcal{C}_i$ and $\mathcal{C}_j$, respectively, then

$$\mathrm{Hom}_{\mathcal{C}}(X_i, X_j) = \varinjlim_{i,j\to k} \mathrm{Hom}_{\mathcal{C}_k}(F_{ik}(X_i), F_{jk}(X_j)),$$

the limit being taken over pairs of morphisms from i and j into a common third object k.

Admitting the lemma for the moment, we can complete this to a diagram

$$\begin{array}{ccc} 2\text{-}\varinjlim_{U\ni x}\{\text{finite étale } \mathcal{O}_X(U)\text{-algs.}\} & \longrightarrow & \{\text{finite étale } K(x)\text{-algs}\} \\ \downarrow & & \downarrow \\ 2\text{-}\varinjlim_{U\ni x}\{\text{finite étale } \mathcal{O}_{X^\flat}(U)\text{-algs.}\} & \longrightarrow & \left\{\text{finite étale } K(x)^\flat\text{-algs}\right\} \end{array}$$

Thus we get equivalences locally at every point, which we can glue together to deduce (2). It remains to address Lemma 7.4.6. This rests on the following theorem.

Theorem 7.4.8 ([Elk73], [GR03])**.** *Let A be a Tate ring such that A is "topologically henselian". That is, for a ring of definition $A_0 \subset A$, $\varpi \in A_0$ a pseudo-uniformizer, then A_0 is henselian along ϖA_0. Then the functor*

$$\begin{array}{rcl} \{\textit{finite étale } A\textit{-algs.}\} & \to & \left\{\textit{finite étale } \widehat{A}\textit{-algs.}\right\} \\ B & \mapsto & \widehat{B} = B \otimes_A \widehat{A}. \end{array}$$

Remark 7.4.9. Also, the K-groups are the same: $K(A, \mathbf{Z}/\ell) \cong K(\widehat{A}, \mathbf{Z}/\ell)$ if $\ell \in A^\times$.

As a corollary, let A_i be a filtered directed system of complete Tate rings, $A_\infty = \widehat{\varinjlim A_i}$. Then

$$2\text{-}\varinjlim\{\text{finite étale } A_i\text{-algs}\} = \left\{\text{finite étale } \varinjlim A_i\text{-algs.}\right\} = \{\text{finite étale } A_\infty\text{-algs.}\}$$

To deduce Lemma 7.4.6 from Theorem 7.4.8, it remains to show that $K(x) = \widehat{\varinjlim \mathcal{O}_X(U)}$, for the topology making $\varinjlim \mathcal{O}_X^+(U)$ open and bounded. We have

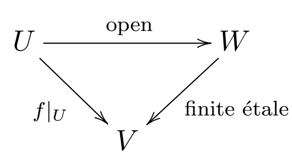
$$0 \to I \to \varinjlim_{x \in U} \mathcal{O}_X^+(U) \to K(x)^+$$

where the last arrow has dense image. We claim that ϖ is invertible in I. If $f \in \mathcal{O}_X^+(U)$ is such that $|f(x)| = 0$, then $V = |f(x)| \le |\varpi|$ is an open neighborhood of x, and then $f \in \varpi\mathcal{O}_X^+(V)$, and so $f \in \varpi I$. Thus the ϖ-adic completion of $\varinjlim \mathcal{O}_X^+(U)$ is $K(x)^+$.

7.5 The étale site

Definition 7.5.1. 1. A morphism $f\colon X \to Y$ of perfectoid spaces is *finite étale* if for all $\mathrm{Spa}(B, B^+) \subset Y$ open, the pullback $X \times_Y \mathrm{Spa}(B, B^+)$ is $\mathrm{Spa}(A, A^+)$, where A is a finite étale B-algebra, and A^+ is the integral closure of the image of B^+ in A.

2. A morphism $f\colon X \to Y$ is *étale* if for all $x \in X$ there exists an open $U \ni x$ and $V \supset f(U)$ such that there is a diagram



3. An *étale cover* is a jointly surjective family of étale maps.

Proposition 7.5.2. *1. Composition of étale morphisms are étale.*

2. Pullbacks of étale morphisms are étale. (Same with finite)

3. If g and gf are étale, then so is f.

4. f étale if and only if $f^\flat$ is étale.

Corollary 7.5.3. *There exists an étale site $X_{\acute{e}t}$, such that naturally $X_{\acute{e}t} \cong X^\flat_{\acute{e}t}$, and $H^i(X_{\acute{e}t}, \mathcal{O}_X^+)$ is almost zero for $i > 0$ for affinoids X.*

8 Diamonds, 25 September

8.1 Diamonds: motivation

Today we discuss the notion of a diamond. The idea is that there should be a (not necessarily fully faithful) functor

$$\begin{array}{rcl} \{\text{adic spaces over } \mathbf{Q}_p\} & \to & \{\text{diamonds}\} \\ X & \mapsto & X^\diamondsuit \end{array}$$

which "forgets the structure morphism to $\mathbf{Q}_p$". For a perfectoid space $X/\mathbf{Q}_p$, the functor $X \mapsto X^\flat$ has this property, so the desired functor should factor through $X \mapsto X^\flat$ on such objects. In general, if $X/\mathbf{Q}_p$ is an adic space, then X is pro-étale locally perfectoid:

$$X = \mathrm{Coeq}(\widetilde{\widetilde{X}} \rightrightarrows \widetilde{X}),$$

where $\widetilde{X} \to X$ is a pro-étale (in the sense defined below) perfectoid cover, and so is $\widetilde{\widetilde{X}} \to \widetilde{X} \times_X \widetilde{X}$. Then the functor should send X to $\mathrm{Coeq}(\widetilde{\widetilde{X}}^\flat \rightrightarrows \widetilde{X}^\flat))$. The only question now is, what category does this live in? (There is also the question of whether this construction depends on the choices made.) Whatever this object is, it is pro-étale *under* a perfectoid space in characteristic p, namely $\widetilde{X}^\flat$.

Example 8.1.1. If $X = \mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, then a pro-étale perfectoid cover of X is $\widetilde{X} = \mathrm{Spa}(\mathbf{Q}_p^{\mathrm{cycl}}, \mathbf{Z}_p^{\mathrm{cycl}})$. Then $\widetilde{\widetilde{X}} = \widetilde{X} \times_X \widetilde{X} = \widetilde{X} \times \mathbf{Z}_p^\times$ is again a perfectoid space, and so $X^\flat$ should be the coequalizer of $\widetilde{\widetilde{X}}^\flat \rightrightarrows \widetilde{X}$, which comes out to be the quotient $\mathrm{Spa}(\mathbf{Q}_p^{\mathrm{cycl},\flat}, \mathbf{Z}_p^{\mathrm{cycl},\flat})/\mathbf{Z}_p^\times$, whatever this means.

8.2 Pro-étale morphisms

Definition 8.2.1. A morphism $f\colon X \to Y$ of perfectoid spaces is *pro-étale* if it is locally (on the source) of the form $\mathrm{Spa}(A_\infty, A_\infty^+) \to \mathrm{Spa}(A, A^+)$, where A, A_∞ are perfectoid, where

$$(A_\infty, A_\infty^+) = \widehat{\varinjlim (A_i, A_i^+)}$$

is a filtered colimit of pairs (A_i, A_i^+) with A_i perfectoid, such that

$$\mathrm{Spa}(A_i, A_i^+) \to \mathrm{Spa}(A, A^+)$$

is étale.

Remark 8.2.2. Pro-étale morphisms are not necessarily open. For instance, Y could be a pro-finite set[8], and $f\colon X \to Y$ could be the inclusion of a point $X = \{x\}$. Indeed, this is the completed inverse limit of morphisms $U_i \to Y$, where $U_i \subset Y$ is an open subset and $\cap_i U_i = \{x\}$.

Lemma 8.2.3. *Let*

$$\begin{array}{ccc} \mathrm{Spa}(A_\infty, A_\infty^+) & & \mathrm{Spa}(B_\infty, B_\infty^+) \\ & \searrow \qquad \swarrow & \\ & \mathrm{Spa}(A, A^+) & \end{array}$$

be a diagram of pro-étale morphisms of perfectoid affinoids, where $(A_\infty, A_\infty^+) = \widehat{\varinjlim_{i\in I}(A_i, A_i^+)}$ *is as in the definition, and similarly with the B objects. Then*

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{Spa}(A,A^+)}(\mathrm{Spa}(A_\infty, A_\infty^+), \mathrm{Spa}(B_\infty, B_\infty^+)) \\ = \ &\varprojlim_J \varinjlim_I \mathrm{Hom}_{\mathrm{Spa}(A,A^+)}\left(\mathrm{Spa}(A_i, A_i^+), \mathrm{Spa}(B_j, B_j^+)\right). \end{aligned}$$

Proof. Without loss of generality J is a singleton, and we can write $(B, B^+) = (B_\infty, B_\infty^+)$. Now have to check that

$$\mathrm{Hom}(\mathrm{Spa}(A_\infty, A_\infty^+), \mathrm{Spa}(B, B^+)) = \varinjlim_I \mathrm{Hom}(\mathrm{Spa}(A_i, A_i^+), \mathrm{Spa}(B, B^+))$$

(where all Homs are over (A, A^+)). This can be checked locally on $\mathrm{Spa}(B, B^+)$. An étale morphism is a composition of rational embeddings and finite étale morphisms. So WLOG $f\colon \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ is one of these.

1. If f is a rational embedding: let $U = \mathrm{Spa}(B, B^+) \hookrightarrow \mathrm{Spa}(A, A^+)$, then the fact that $\mathrm{Spa}(A_\infty, A_\infty^+) \to \mathrm{Spa}(A, A^+)$ factors over U implies that there exists i such that $\mathrm{Spa}(A_i, A_i^+) \to \mathrm{Spa}(A, A^+)$ factors over U. Indeed, we can apply the following quasi-compactness argument, which applies whenever one wants to show that a "constructible" algebro-geometric property applies to a limit of spaces if and only if it applies to some stage of the limit.

[8] Probably I should have included this as an example earlier, but if Y is any profinite set, then Y can be realized as an adic space over a nonarchimedean field K as $\mathrm{Spa}(R, R^+)$, where R (resp., R^+) is the ring of continuous functions from Y to K (resp., $\mathcal{O}_K$). The resulting adic space is homeomorphic to Y. If K is a perfectoid field, then this construction produces a perfectoid space.

Topologically we have $\mathrm{Spa}(A_\infty, A_\infty^+) = \varprojlim_i \mathrm{Spa}(A_i, A_i^+)$.

$$\mathrm{Spa}(A_\infty, A_\infty^+) \setminus \{\text{preimage of } U\} = \varprojlim \left(\mathrm{Spa}(A_i, A_i^+) \setminus \{\text{preimage of } U\}\right)$$

The RHS is an inverse limit of spaces which are closed in a spectral space, thus spectral, and so they are compact and Hausdorff for the constructible topology. If the inverse limit is empty, one of the terms had to be empty: this is a version of Tychonoff's theorem, see [RZ10]. We get that $\mathrm{Spa}(A_i, A_i^+)$ equals the preimage of U for some i.

2. Suppose that f is finite étale. Recall from [GR03] that

$$\{\text{finite étale } A_\infty\text{-algs.}\} = 2\text{-}\varinjlim \{\text{finite étale } A_i\text{-algs.}\}$$

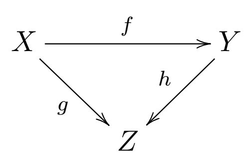
This shows that

$$\begin{aligned} \mathrm{Hom}_A(B, A_\infty) &= \mathrm{Hom}_{A_\infty}(B \otimes A_\infty, A_\infty) \\ &= \varinjlim_i \mathrm{Hom}_{A_i}(B \otimes A_i, A_i) \\ &= \varinjlim_i \mathrm{Hom}_A(B, A_i). \end{aligned}$$

□

Proposition 8.2.4. *1. Compositions of pro-étale maps are pro-étale.*

2. If



is a diagram of perfectoid spaces where g and h are pro-étale, then f is pro-étale.

3. Pulbacks of pro-étale morphisms are pro-étale. (Note: the category of affinoid perfectoid spaces has all limits. In particular fibre products of perfectoid spaces exist.)

Definition 8.2.5 (The big pro-étale site)**.** Consider the category Perf of perfectoid spaces of characteristic p. We endow this with the structure of a site by saying that a collection of morphisms $\{f_i\colon X_i \to X\}$ is a covering (a *pro-étale cover*) if the f_i are pro-étale, and if for all quasi-compact open $U \subset X$, there exists a finite subset $I_U \subset I$, and a quasicompact open $U_i \subset X_i$ for $i \in I_U$, such that $U = \cup_{i \in I_U} f_i(U_i)$.

Remark 8.2.6. It is not good enough to demand that the f_i are a topological cover. For instance, let X be a pro-finite set considered as a perfectoid space over some perfectoid field in characteristic p, and let $X_i \to X$ be a pro-étale morphism whose image is the point $i \in X$, as in Rmk. 8.2.2. The finiteness criterion in Defn. 8.2.5 prevents the $X_i \to X$ from constituting a cover in Perf. The same issue arises for the fpqc topology on the category of schemes: if X is an affine scheme, the collection of flat quasi-compact maps $\operatorname{Spec} \mathcal{O}_{X,x} \to X$ for $x \in X$ is generally not an fpqc cover.

Proposition 8.2.7. *For a perfectoid space X of characteristic p, let h_X be the presheaf on* Perf *defined by $h_X(Y) = \operatorname{Hom}(Y, X)$. Then h_X is a sheaf on the pro-étale site. (That is, all representable presheaves are sheaves.)*

Proof. The essential point is to show that $X \mapsto \mathcal{O}_X(X)$ is a sheaf for the pro-étale topology. Without loss of generality $X = \operatorname{Spa}(R, R^+)$ is a perfectoid affinoid. Fix a pseudo-uniformizer $\varpi \in R$. Say X has a finite cover $X_i = \operatorname{Spa}(R_{\infty i}, R^+_{\infty i})$, with each $(R_{\infty i}, R^+_{\infty i}) = \widehat{\varinjlim_{j \in J} (R_{ji}}, R^+_{ji})$. We claim that the complex

$$0 \to R^+/\varpi \to \prod_i R^+_{\infty i}/\varpi \to \cdots$$

is almost exact. For all j, the complex

$$0 \to R^+/\varpi \to \prod_i R^+_{ji}/\varpi \to \cdots$$

is almost exact, because $H^i(X_{\text{ét}}, \mathcal{O}^+_X/\varpi)$ is almost zero for $i > 0$ (and R^+/ϖ for $i = 0$). Now we can take a direct limit over j. A filtered direct limit of almost exact sequences is almost exact. Thus,

$$0 \to R^+ \to \prod_i R^+_{\infty i} \to \cdots$$

is almost exact (all terms are ϖ-torsion free and ϖ-adically complete). Now invert ϖ to get the "essential point".

For the general statement, can reduce to the case that $X = \operatorname{Spa}(R, R^+)$ and $Y = \operatorname{Spa}(S, S^+)$ is affinoid. Let $\{f_i : Y_i \to Y\}$ be a cover. We can further reduce to the case that this a finite cover. But then each Y_i admits a cover by affinoids of the standard form: $Y_i = \operatorname{Spa}(S_{\infty i}, S^+_{\infty i}) \to \operatorname{Spa}(S, S^+)$ as in the definition of pro-étale. We get a map $R \to H^0(Y_{\text{proét}}, \mathcal{O}_Y) = S$ (the latter because $\mathcal{O}_Y$ is a sheaf), as desired. □

Definition 8.2.8. 1. A map $f\colon \mathcal{F} \to \mathcal{G}$ of sheaves on Perf is *pro-étale* if for all perfectoid spaces X and maps $h_X \to \mathcal{G}$ (which correspond to sections in $\mathcal{G}(X)$), the pullback $h_X \times_{\mathcal{G}} \mathcal{F}$ is representable by a perfectoid space Y, and $Y \to X$ (corresponding to $h_Y = h_X \times_{\mathcal{G}} \mathcal{F} \to h_X$) is pro-étale[9].

2. A *diamond* is a sheaf $\mathcal{F}$ on Perf such that there exists a collection of pro-étale morphisms $h_{X_i} \to \mathcal{F}$ such that $\coprod_i h_{X_i} \to \mathcal{F}$ is surjective.

Remark 8.2.9. The underlying formalism used to define diamonds is the same as the one used to define algebraic spaces. An algebraic space is a sheaf on the category of schemes (with its étale topology) admitting a surjective étale morphism from a representable sheaf.

Let us introduce some notation. If (R, R^+) is pefectoid, write $\mathrm{Spd}(R, R^+)$ for $h_{\mathrm{Spa}(R^\flat, R^{\flat +})}$, considered as a diamond. We have a fully faithful functor $X \mapsto X^\diamondsuit$ from perfectoid spaces in characteristic p to diamonds; here $X^\diamondsuit = h_X$.

9 Diamonds II, 7 October

9.1 Complements on the pro-étale topology

Two issues came up last time which we would like to address.

The first issue was *pro-étale descent for perfectoid spaces.*

Question 9.1.1. Is the fibred category

$$X \mapsto \{\text{morphisms } Y \to X \text{ with } Y \text{ perfectoid}\}$$

on the category of perfectoid spaces a stack for the pro-étale topology? That is, if $X' \to X$ is a pro-étale cover, and we are given a morphism $Y' \to X'$ together with a descent datum over $X' \times_X X'$, etc, then does $Y' \to X'$ descend to $Y \to X$? (Such a descent is unique up to unique X-isomorphism if it exists, since by Prop. 8.2.7 h_Y is a sheaf.)

Remark 9.1.2. 1. It is enough to show that for affinoid perfectoid X, the fibred category of morphisms $Y \to X$ with Y perfectoid affinoid is a stack.

[9] It may be necessary to loosen this definition so that $h_X \times_{\mathcal{G}} \mathcal{F}$ has these properties pro-étale locally on X. This is because we are not sure if the property of being pro-étale is a pro-étale local property on the target.

2. The statement is true if X is "w-local" in the sense of [BS].

The other issue was that the property of being a pro-étale morphism is not local for the pro-étale topology on the source. That is, suppose $f\colon X \to Y$ is a morphism between perfectoid spaces, and suppose there exists a pro-étale surjection $X' \to X$, such that the composite $X' \to X \to Y$ is pro-étale. We cannot conclude that f is pro-étale, cf. Example 9.1.5. How can we characterize such f? It turns out there is a convenient "punctual" criterion.

Definition 9.1.3. Say $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ is *affinoid pro-étale* if

$$(B, B^+) = \widehat{\varinjlim_{\text{filtered}} (A_i, A_i^+)},$$

where A_i is perfectoid, and $\mathrm{Spa}(A_i, A_i^+) \to \mathrm{Spa}(A, A^+)$ is étale.

Proposition 9.1.4. *Let $f\colon X \to Y$ be a morphism of affinoid perfectoid spaces. The following are equivalent:*

1. *There exists $X' \to X$ which is affinoid pro-étale surjective, such that the composite $X' \to Y$ is affinoid pro-étale.*

2. *For all points $y \in Y$, corresponding to $\mathrm{Spa}(K(y), K(y)^+) \to Y$, the pullback $X \times_Y \mathrm{Spa}(K(y), K(y)^+) \to \mathrm{Spa}(K(y), K(y)^+)$ is pro-étale.*

3. *Same as (2), but for geometric points $\mathrm{Spa}(C, \mathcal{O}_C)$ of rank 1.*

The idea for the proof is to follow the argument in [BS], Thm. 2.3.4, which concerns the relationship between *weakly étale* and pro-étale morphisms between schemes.

Example 9.1.5 (A non-pro-étale morphism which is locally pro-étale)**.** Let

$$Y = \mathrm{Spa}(K\langle T^{1/p^\infty}\rangle, \mathcal{O}_K\langle T^{1/p^\infty}\rangle),$$

and let

$$X = \mathrm{Spa}(K\langle T^{1/2p^\infty}\rangle, \mathcal{O}_K\langle T^{1/2p^\infty}\rangle).$$

(Assume $p \neq 2$.) Then $X \to Y$ appears to be ramified at 0, and indeed it is not pro-étale. However, consider the following pro-étale cover of Y: let

$$Y' = \varprojlim Y_n',\ Y_n' = \{x \in Y \,|\, |x| \leq |\varpi^n|\} \sqcup \coprod_{i=1}^{n} \left\{x \in Y \,|\, |\varpi|^i \leq |x| \leq |\varpi|^{i-1}\right\}.$$

Let $X' = X \times_Y Y'$. We claim that the pullback $X' \to Y'$ is pro-étale, and that the composite $X' \to Y$ is affinoid pro-étale. Thus $X \to Y$ may not be pro-étale, but it is so pro-étale locally on X, and even on Y!

As a topological space, $\pi_0(Y') = \{1, 1/2, 1/3, \ldots, 0\} \subset \mathbf{R}$. The fibre of Y' over $1/i$ is $\left\{x \in Y \,|\, |\varpi|^i \leq |x| \leq |\varpi|^{i-1}\right\}$, and the fibre over 0 is just 0. Let

$$X'_n = \{x \,|\, |x| \leq |\varpi|^n\} \sqcup \coprod_{i=1}^{n} \left\{|\varpi|^i \leq |x| \leq |\varpi|^{i-1}\right\} \times_Y X$$

so that $X'_n \to Y'_n$ is finite étale. Then $X' \cong \varprojlim X'_n = \varprojlim (X'_n \times_{Y'_n} Y') \to Y'$ is pro-étale.

9.2 Quasi-profinite morphisms

Recall that the pro-étale topology on the category (Perf) is defined by declaring a family of morphisms $\{X_i \to X\}$ to be a cover when each morphism is pro-étale (together with a certain quasicompactness condition). Suppose we relax this condition so that each $X_i \to X$ is only locally pro-étale on X_i, in the sense that there are pro-étale coverings $\{X_{ij} \to X_i\}_{j \in J_i}$ such that the composite morphisms $X_{ij} \to X$ are pro-étale. Then we have changed the topology on (Perf), but not the topos: checking the sheaf condition for $\{X_i \to X\}$ is equivalent to checking it for the pro-étale covers $\{X_{ij} \to X_i\}_{j \in J_i}$ and $\{X_{ij} \to X\}_{i,j}$.

Prop. 9.1.4 suggests a class of morphisms which are pro-étale locally on the source.

Definition 9.2.1. A morphism $f \colon X \to Y$ is *quasi-profinite (qpf)* if locally on X it is of the form $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ as in Prop. 9.1.4.

Definition 9.2.2. Consider the site (Perf) of perfectoid spaces of characteristic p with covers generated by open covers and affinoid pro-étale maps (or affinoid qpf maps), subject to the same quasi-compactness condition on covers appearing in Defn. 8.2.5. If $f \colon \mathcal{F} \to \mathcal{G}$ is a map of sheaves on (Perf), say f is *quasi-profinite* if for all $h_X \to \mathcal{G}$, $\mathcal{F} \times_{\mathcal{G}} h_X$ is representable, say it is h_Y, with $Y \to X$ qpf.

A *diamond* is a sheaf $\mathcal{D}$ on (Perf) such that there exists a surjective qpf map $h_X \to \mathcal{D}$ from a representable sheaf.

Remark 9.2.3. Replacing pro-étale maps with qpf maps changes the topology on (Perf), but not the topos: Prop. 9.1.4 says that any pro-étale covering can be refined to a qpf covering. Thus we have not changed the definition of

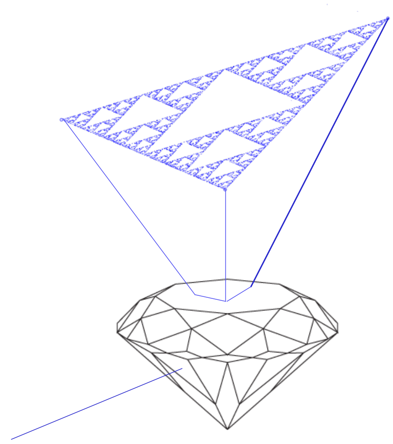


Figure 3: An illustration of a diamond $\mathcal{D}$. Suppose $h_X \to \mathcal{D}$ is a surjective qpf map. The incoming light beam represents a morphism $h_Y \to \mathcal{D}$ from a representable object. The outgoing light beam represents the pullback $h_Y \times_{\mathcal{D}} h_X$, which is representable by a perfectoid space Y' which is qpf over Y. If Y is a geometric point, then Y' is a profinite union of copies of Y; we have attempted to depict this scenario.

diamond. Meanwhile it is somewhat easier to check that a morphism is qpf than it is to check that it is pro-étale, since we have the punctual criterion of Prop. 9.1.4.

Remark 9.2.4. One would like to be able to check that $f\colon \mathcal{F} \to \mathcal{G}$ is qpf by checking it after pullback to a cover of $\mathcal{G}$. However, we run into the problem of Question 9.1.1. That is, suppose $\mathcal{G}' \to \mathcal{G}$ is a cover for which $\mathcal{F}' = \mathcal{F} \times_{\mathcal{G}} \mathcal{G}' \to \mathcal{G}$ is qpf. Then if $h_X \to \mathcal{G}$ is a morphism from a representable object, we want to check whether $\mathcal{F} \times_{\mathcal{G}} h_X$ is representable by some qpf $Y \to X$. We have the cover $h'_X = h_X \times_{\mathcal{G}} \mathcal{G}' \to \mathcal{G}'$. Since $\mathcal{F}' \to \mathcal{G}'$ is qpf, its pullback $h'_X \times_{\mathcal{G}'} \mathcal{F}' = h_{Y'}$ for some $Y' \to X'$ qpf. Then $Y' \to X'$ comes equipped with descent data to X. This is where we encounter Question 9.1.1: it is not clear that the descent $Y \to X$ exists.

Remark 9.2.5. Suppose $\mathcal{F}$ is a diamond, and $h_X \to \mathcal{F}$ is a qpf surjection from a representable sheaf as in Defn. 9.2.2. If $h_Y \to \mathcal{F}$ is any morphism from a representable sheaf, then $h_X \times_{\mathcal{F}} h_Y = h_Z$ for some qpf morphism $Z \to Y$. In particular the product $h_X \times_{\mathcal{F}} h_X \subset h_X \times h_X$ is representable by a space $R \subset X \times X$, where each of the maps $R \rightrightarrows X$ is qpf. This R is a *qpf equivalence relation*. (The morphism $R \to X \times X$ is functorially injective, but one cannot expect it to be Zariski closed.)

Thus diamonds are quotients of perfectoid spaces X by certain kinds of equivalence relations $R \subset X \times X$. A special case is when R comes from the action of a profinite group on X. We will encounter this in the next section when we introduce the diamond $\mathrm{Spd}\, \mathbf{Q}_p$.

9.3 G-torsors

If G is a finite group, we have the notion of *G-torsor* on any topos. This is a map $f\colon \mathcal{F}' \to \mathcal{F}$ with an action $G \times \mathcal{F}' \to \mathcal{F}'$ such that locally on $\mathcal{G}$, $\mathcal{F}' \cong \mathcal{F} \times G$.

We also have G-torsors when G is not an abstract group but rather a group object in the category of sheaves on (Perf). If G is a finite group, let $\underline{G}$ be the constant sheaf on (Perf). That is, if X is an affinoid in (Perf), then $\underline{G}(X) = \mathrm{Cont}(|\mathrm{X}|, \mathrm{G})$, the group of continuous maps $|X| \to G$ (since G is discrete, continuous here implies locally constant). Extend this to profinite groups G by setting $\underline{G} = \varprojlim G/H$ as a sheaf on (Perf), where H runs through the open normal subgroups. Then once again $\underline{G}(X) = \mathrm{Cont}(|\mathrm{X}|, \mathrm{G})$.

Note that $\underline{G}$ is not representable, even if G is finite. The problem is that (Perf) lacks an initial object X (in other words, a base). If it had one, then $\underline{G}$ would be representable by "G copies of X". And indeed, $\underline{G}$

becomes representable once we supply the base. If X is a perfectoid space, then $\underline{G} \times h_X$ is representable by a perfectoid space, namely

$$\underline{G} \times X := \varprojlim G/H \times X,$$

where $G/H \times X$ is just a finite disjoint union of copies of X. The inverse limit really does exist in (Perf): If $X = \mathrm{Spa}(R, R^+)$, then $\underline{G} \times X = \mathrm{Spa}(S, S^+)$, where

$$S = \left(\varinjlim_H \mathrm{Cont}(G/H, R) \right)^{\wedge} = \mathrm{Cont}(G, R),$$

and similarly $S^+ = \mathrm{Cont}(G, R^+)$. Finally, note that everything in this paragraph applies when G is a profinite set rather than a group.

Now if G is a profinite group, a $\underline{G}$-torsor is a morphism $f \colon \mathcal{F}' \to \mathcal{F}$ with an action $\underline{G} \times \mathcal{F}' \to \mathcal{F}'$ such that locally on $\mathcal{F}$ we have $\mathcal{F}' \cong \mathcal{F} \times G$.

Proposition 9.3.1. *Let $f \colon \mathcal{F}' \to \mathcal{F}$ be a $\underline{G}$-torsor, with G profinite. Then for any affinoid $Y = \mathrm{Spa}(B, B^+)$ and any morphism $h_Y \to \mathcal{F}$, the pullback $\mathcal{F}' \times_{\mathcal{F}} h_Y$ is representable by a perfectoid affinoid $X = \mathrm{Spa}(A, A^+)$. Furthermore, A is the completion of $\varinjlim_H A^H$, where for each open normal subgroup $H \subset G$, A_H/B is a finite étale G/H-torsor in the algebraic sense.*

Remark 9.3.2. In fact one can take $A_H = A^H$ to be the ring of elements of A fixed by H.

Proof. (Sketch.) We reduce to the case of G finite this way: If $H \subset G$ is open normal, then $\mathcal{F}'/\underline{H} \to \mathcal{F}$ is a G/H-torsor, and $\mathcal{F}' = \varprojlim \mathcal{F}'/H$.

The key point here is that the fibred category

$$Y \mapsto \{\text{finite étale } X/Y\}$$

over the category of affinoid perfectoid spaces is a stack for the pro-étale topology. If $Y' = \varprojlim Y_i'$ is an affinoid pro-étale cover, then

$$\left\{\text{finite étale descent data for } Y' \to Y\right\} = 2\text{-}\varinjlim \left\{\text{finite étale descent data for } Y_i'/Y\right\}$$

([Elk73], [GR03]). This in turn is the category of finite étale covers X/Y, and so we are reduced to showing that étale descent works. Thus, we are reduced to showing that 'etale descent is effective for finite étale morphisms of affinoid perfectoids. For this we can either use a descent to noetherian adic spaces together with a result of Huber. Or we can use de Jong-van der Put: étale descent follows from analytic descent (Kedlaya-Liu) plus classical finite étale descent. ☐

9.4 The diamond $\operatorname{Spd}\mathbf{Q}_p$

For a Huber ring R, let us abbreviate $\operatorname{Spa} R = \operatorname{Spa}(R, R^\circ)$. If R is perfectoid (of whatever characteristic), let us write $\operatorname{Spd} R$ for the diamond $h_{\operatorname{Spa}(R^\flat)}$. Thus we have the diamond $\operatorname{Spd}\mathbf{Q}_p^{\mathrm{cycl}}$: it is the sheaf on (Perf) whose (R, R^+)-valued sections are the set of continuous homorphisms $\mathbf{Q}_p^{\mathrm{cycl},\flat} \to R$. Since $\mathbf{Q}_p^{\mathrm{cycl},\flat} \cong \mathbf{F}_p(\!(t^{1/p^\infty})\!)$, this is nothing more than the set of topologically nilpotent elements of R (these automatically lie in R^+).

We now give an *ad hoc* definition of the diamond $\operatorname{Spd}\mathbf{Q}_p$.

Definition 9.4.1. $\operatorname{Spd}\mathbf{Q}_p = \operatorname{Spd}\mathbf{Q}_p^{\mathrm{cycl},\flat}/\underline{\mathbf{Z}}_p^\times$. That is, $\operatorname{Spd}\mathbf{Q}_p$ is the coequalizer of

$$\underline{\mathbf{Z}}_p^\times \times h_{\operatorname{Spa}\mathbf{Q}_p^{\mathrm{cycl},\flat}} \rightrightarrows h_{\operatorname{Spa}\mathbf{Q}_p^{\mathrm{cycl},\flat}},$$

where one map is the projection and the other is the action.

We would like to know that $\operatorname{Spd}\mathbf{Q}_p^{\mathrm{cycl}} \to \operatorname{Spa}\mathbf{Q}_p$ is a $\underline{\mathbf{Z}}_p^\times$-torsor. This is something like showing that the $\mathbf{Z}_p^\times$ action on $\mathbf{Q}_p^{\mathrm{cycl},\flat} = \mathbf{F}_p(\!(t^{1/p^\infty})\!)$ is sufficiently nontrivial.

Lemma 9.4.2. *Let* $g\colon \underline{\mathbf{Z}}_p^\times \times \operatorname{Spd}\mathbf{Q}_p^{\mathrm{cycl}} \to \operatorname{Spd}\mathbf{Q}_p^{\mathrm{cycl}} \times_{\operatorname{Spd}\mathbf{Q}_p} \operatorname{Spd}\mathbf{Q}_p^{\mathrm{cycl}}$ *be the product of the projection onto the second factor by the group action. Then* g *is an isomorphism.*

Proof. The crucial point is that $\mathbf{Z}_p^\times \to \operatorname{Aut}\mathbf{F}_p(\!(t^{1/p^\infty})\!)$ is injective. This implies that $\mathbf{Z}_p^\times$ acts freely on geometric points of $\operatorname{Spa}\mathbf{F}_p(\!(t^{1/p^\infty})\!)$. Let us construct the inverse of g. We need a map

$$\operatorname{Spa}\mathbf{Q}_p^{\mathrm{cycl}} \times_{\operatorname{Spa}\mathbf{Q}_p} \operatorname{Spd}\mathbf{Q}_p^{\mathrm{cycl}} \to \underline{\mathbf{Z}}_p^\times$$

(whereas the map to $\operatorname{Spd}\mathbf{Q}_p^{\mathrm{cycl}}$ is just the projection onto the first factor). A section of the fibre product over (R, R^+) is a pair of maps $f_1, f_2\colon \mathbf{Q}_p^{\mathrm{cycl},\flat} \to R$, such that there exists an affinoid pro-étale cover $\operatorname{Spa}(\widetilde{R}, \widetilde{R}^+) \to \operatorname{Spa}(R, R^+)$, and a continuous map $\widetilde{\gamma}\colon \operatorname{Spa}(\widetilde{R}, \widetilde{R}^+) \to \mathbf{Z}_p^\times$ such that

$$f_1(t) = (1 + f_2(t))^{\widetilde{\gamma}} - 1 \in \widetilde{R}.$$

We want to show that $\widetilde{\gamma}$ factors through a continuous map $\widetilde{\gamma}\colon \operatorname{Spa}(R, R^+) \to \mathbf{Z}_p^\times$. It is enough to show that $\widetilde{\gamma}$ is constant on fibres of $\operatorname{Spa}(\widetilde{R}, \widetilde{R}^+) \to \operatorname{Spa}(R, R^+)$, so without loss of generality $R = C$ is an algebraically closed nonarchimedean field. If $\widetilde{\gamma}$ is not constant, we have $\gamma_0 \neq \gamma_1 \in \mathbf{Z}_p^\times$ such that

$$f_1(t) = (1 + f_2(t))^{\gamma_i} - 1 \in C$$

for $i = 0, 1$. That is, the two composites $f_2 \circ \widetilde{\gamma}_1, f_1 \circ \widetilde{\gamma}_2 \colon \mathbf{F}_p((t^{1/p^\infty})) \to C$ are the same. But this can't be, since the action of $\mathbf{Z}_p^\times$ on geometric points of $\operatorname{Spa} \mathbf{F}_p((t^{1/p^\infty}))$ is free. □

Corollary 9.4.3. $\operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl}} \to \operatorname{Spd} \mathbf{Q}_p$ *is a* $\underline{\mathbf{Z}}_p^\times$*-torsor.*

Corollary 9.4.4. $(\operatorname{Spd} \mathbf{Q}_p)(R, R^+)$ *is the set of isomorphism classes of data of the form:*

1. $R \to \widetilde{R}$*, a* $\mathbf{Z}_p^\times$*-torsor,*

2. *A* $\mathbf{Z}_p^\times$*-equivariant map* $\mathbf{Q}_p^{\mathrm{cycl},\flat} \to \widetilde{R}$.

Theorem 9.4.5. *There is an equivalence of categories between perfectoid spaces over* $\mathbf{Q}_p$*, and the category of perfectoid spaces* X *of characteristic* p *together with a "structure morphism"* $X^\diamondsuit \to \operatorname{Spd} \mathbf{Q}_p$.

Proof. We will show below that for any affinoid perfectoid (R, R^+), specifying a morphism $\operatorname{Spa}(R, R^+) \to \operatorname{Spd}(\mathbf{Q}_p)$ determines a *characteristic 0 untilt* of (R, R^+), by which we mean a pair $(R^\sharp, \iota)$, where $R^\sharp$ is a perfectoid $\mathbf{Q}_p$-algebra $R^\sharp$ and an isomorphism $\iota \colon R^{\sharp\flat} \to R$ of perfectoid algebras. This pair is *uniquely functorial* in the map to $\operatorname{Spd}(\mathbf{Q}_p)$. This gives a precise sense in which specifying an untilt of (R, R^+) is functorially the same as specifying a morphism $\operatorname{Spa}(R, R^+) \to \operatorname{Spd}(\mathbf{Q}_p)$, so we may then globalize to obtain the assertion in the theorem.

Suppose $(R^\sharp, \iota)$ is an untilt of (R, R^+). Let

$$\widetilde{R}^\sharp = R^\sharp \widehat{\otimes}_{\mathbf{Q}_p} \mathbf{Q}_p^{\mathrm{cycl}} = \left(\varinjlim R^\sharp \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(\mu_{p^n}) \right)^\wedge,$$

so that $\widetilde{R}^\sharp$ is a $\mathbf{Z}_p^\times$-torsor over $R^\sharp$. Tilting this gives $\widetilde{R}$, a $\underline{\mathbf{Z}}_p^\times$-torsor over R, together with a $\mathbf{Z}_p^\times$-equivariant map $\mathbf{Q}_p^{\mathrm{cycl},\flat} \to \widetilde{R}$.

In the other direction, let $R \to \widetilde{R}$ be a $\underline{\mathbf{Z}}_p^\times$-torsor, and let $\mathbf{Q}_p^{\mathrm{cycl},\flat} \to \widetilde{R}$ be a $\mathbf{Z}_p^\times$-equivariant map. Recall the tilting equivalence between perfectoid algebras over $\mathbf{Q}_p^{\mathrm{cycl},\flat}$ and $\mathbf{Q}_p^{\mathrm{cycl}}$. Thus we get a canonical untilt $\mathbf{Q}_p^{\mathrm{cycl}} \to \widetilde{R}^\sharp$ which is $\mathbf{Z}_p^\times$-equivariant. We let

$$R^\sharp = \left(\widetilde{R}^\sharp \right)^{\mathbf{Z}_p^\times},$$

and similarly for $R^{\sharp+}$. We would like to know that $(R^\sharp, R^{\sharp+})$ is "big enough" in the sense that $R^{\sharp\flat} \cong R$.

First we work modulo a pseudo-uniformizer. By an argument with cocycles, we can build a pseudo-uniformizer $\varpi \in \widetilde{R}$ such that $\varpi^\sharp \in R^\sharp$ (*i.e.*, it is $\mathbf{Z}_p^\times$-invariant), and such that $(\varpi^\sharp)^p | p$ in $\widetilde{R}^{\sharp,+}$.

Consider the short exact sequence

$$0 \longrightarrow \widetilde{R}^{\sharp+} \xrightarrow{\varpi^\sharp} \widetilde{R}^{\sharp+} \longrightarrow \widetilde{R}^{\sharp+}/\varpi^\sharp = \widetilde{R}^+/\varpi \longrightarrow 0$$

Taking $\mathbf{Z}_p^\times$ invariants, we see there is an inclusion $R^{\sharp+}/\varpi^\sharp \to H^0(\mathbf{Z}_p^\times, \widetilde{R}^+/\varpi)$. The latter H^0 is just R^+/ϖ, because $R^+ \to \widetilde{R}^+$ is a $\mathbf{Z}_p^\times$-torsor. The obstruction to $R^{\sharp+}/\varpi^\sharp \to R^+/\varpi$ being surjective lies in $H^1_{\mathrm{cont}}(\mathbf{Z}_p^\times, \widetilde{R}^{\sharp+})$. By successive approximation, this H^1 vanishes provided that $H^1_{\mathrm{cont}}(\mathbf{Z}_p^\times, \widetilde{R}/\varpi^\sharp)$ vanishes.

Lemma 9.4.6. *We have an almost isomorphism:*

$$H^i_{cont}(\mathbf{Z}_p^\times, \widetilde{R}^{\sharp+}/\varpi^\sharp) \cong \begin{cases} R^+/\varpi, & i = 0 \\ 0, & i > 0. \end{cases}$$

Proof. Use pro-étale descent for $(\mathcal{O}^+/\varpi)^a$ along $R \to \widetilde{R}$. □

Thus $R^{\sharp+}/\varpi^\sharp \cong R^+/\varpi$. We can also apply the lemma using ϖ^p instead of ϖ (the lemma only required that $\varpi^\sharp | p$) to obtain an isomorphism $R^{\sharp+}/(\varpi^\sharp)^p \cong R^+/\varpi^p$. Since R is perfectoid, $\Phi\colon R^+/\varpi \to R^+/\varpi^p$ is an isomorphism, and so is $\Phi\colon R^{\sharp+}/\varpi^\sharp \to R^{\sharp+}/(\varpi^\sharp)^p$. Therefore $R^\sharp$ is perfectoid as well.

The $\mathbf{Z}_p^\times$-invariant inclusion $R^\sharp \to \widetilde{R}^\sharp$ is a map between perfectoid rings, so the functoriality of tilting gives a $\mathbf{Z}_p^\times$-invariant map $R^{\sharp\flat} \to \widetilde{R}^{\sharp\flat} = \widetilde{R}$, which factors through the subalgebra R of $\mathbf{Z}_p^\times$-invariants. As a result we have a map $R^{\sharp\flat} \to R$ which induces a map $R^{\sharp\flat+} \to R^+$. We have already seen that the reduction of the latter map modulo ϖ is an isomorphism, and so by successive approximation, $R^{\sharp\flat+} \to R^+$ is surjective. But likewise we see from being an isomorphism modulo ϖ that it is an isomorphism modulo ϖ^n for $n = 1, 2, \ldots$. Taking the inverse limit, we deduce an isomorphism $R^{\sharp\flat+} \to R^+$, which becomes an isomorphism $R^{\sharp\flat} \to R$ upon inverting ϖ. □

10 The cohomology of diamonds, 9 October

10.1 The functor $X \mapsto X^\diamondsuit$

Our goal today is to construct a functor

$$\begin{array}{rcl} \{\text{analytic adic spaces}/\operatorname{Spa} \mathbf{Z}_p\} & \to & \{\text{diamonds}\} \\ X & \mapsto & X^\diamondsuit \end{array}$$

This is roughly analogous to the forgetful functor from complex-analytic spaces to topological spaces.

Definition 10.1.1. Let X be an analytic adic space over $\operatorname{Spa} \mathbf{Z}_p$. Define a pre-sheaf $X^\diamondsuit$ on (Perf) as follows. For a perfectoid space Y in characteristic p, let $X^\diamondsuit(Y)$ be the set of isomorphism classes of pairs $(Y^\sharp, \iota)$, with $Y^\sharp \to X$ a perfectoid space, and $\iota\colon Y^{\sharp\flat} \cong Y$ an isomorphism.

If $X = \operatorname{Spa}(R, R^+)$, we write $\operatorname{Spd}(R, R^+) = \operatorname{Spa}(R, R^+)^\diamondsuit$.

Remark 10.1.2. Note that if (R, R^+) is already perfectoid in characteristic p, then $\operatorname{Spd}(R, R^+) = h_{\operatorname{Spa}(R,R^+)}$ agrees with our prior definition. Also note that the pairs $(Y^\sharp, \iota)$ don't have nontrivial automorphisms (cf. Thm. 6.2.6), so $X^\diamondsuit$ has some hope of being a sheaf.

Theorem 10.1.3. *$X^\diamondsuit$ is a diamond.*

Proof. By Prop. 4.5.2 we may assume $X = \operatorname{Spa}(R, R^+)$ is affinoid, with R a Tate ring. Since $\operatorname{Spa} \mathbf{Z}_p$ is the base, $p \in R$ is topologically nilpotent (though not necessarily a unit).

Lemma 10.1.4 ([Fal02], [Col02])**.** *Let R be a Tate ring such that $p \in R$ is topologically nilpotent. Let $\varinjlim R_i$ be a filtered direct limit of algebras R_i finite étale over R_i, which admits no nonsplit finite étale covers. Endow $\varinjlim R_i$ with the topology making $\varinjlim R_i^\circ$ open and bounded. Let $\widetilde{R}$ be the completion. Then $\widetilde{R}$ is perfectoid.*

Proof. First, find $\varpi \in \widetilde{R}$ a pseudo-uniformizer such that $\varpi^p | p$ in $\widetilde{R}^\circ$. To do this, let $\varpi_0 \in R$ be any pseudo-uniformizer. Let N be large enough so that $\varpi_0 | p^{p^N}$. Now look at the equation $x^p - \varpi_0 x = \varpi_0$. This determines a finite étale $\widetilde{R}$-algebra, and so it admits a solution $x = \varpi_1 \in \widetilde{R}$. Note that $\varpi_1^p | \varpi_0$ in $\widetilde{R}^\circ$, and ϖ_1 is a unit in $\widetilde{R}$. Repeat to obtain a pseudo-uniformizer $\varpi = \varpi_{N+1}$ with $\varpi^p | p$.

Now we must check that $\Phi\colon \widetilde{R}^\circ/\varpi \to \widetilde{R}^\circ/\varpi$ is surjective. Let $f \in \widetilde{R}^\circ$, and consider the equation $x^p - \varpi x - f$. We claim this determines an étale

$\widetilde{R}$-algebra (which consequently has a section). This can be checked at each geometric point $\operatorname{Spa}(C, C^+)$ of $\operatorname{Spa}(\widetilde{R}, \widetilde{R}^+)$. We claim that the image of the polynomial $x^p - \varpi x - f$ in $C[x]$ is separable. If $x \in C$ is a repeated root, then $px^{p-1} = \varpi$; since p/ϖ is topologically nilpotent in R, we have that $x \in C$ is a unit with x^{-1} topologically nilpotent. Thus $x^{-1}C^\circ$ is an ideal of definition for $C^\circ \subset C$. On the other hand, x is integral over R°, and so it must lie in C°. By definition of power-bounded, there exists $n \geq 1$ such that $x^{-n}\left\{1, x, x^2, \dots\right\} \subset \varpi C^\circ$. But this implies $1 \in \varpi C^\circ$, contradiction. Thus $x^p - \varpi x - f$ has a root $g \in \widetilde{R}$, which automatically lies in $\widetilde{R}^\circ$. Then $g^p \equiv f \pmod{\varpi \widetilde{R}^\circ}$.

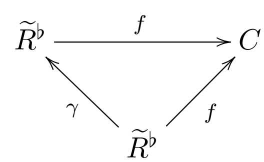
□

We can also assume that each R_i is a G_i-torsor over R, compatibly with change in i for an inverse system $\{G_i\}$ of finite groups. Let $G = \varprojlim_i G_i$.

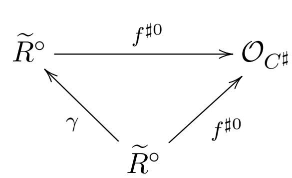
Lemma 10.1.5. $\operatorname{Spd}(R, R^+) = \operatorname{Spd}(\widetilde{R}, \widetilde{R}^+)/G$. *Also,* $\operatorname{Spd}(\widetilde{R}, \widetilde{R}^+) \to \operatorname{Spd}(R, R^+)$ *is a* $\underline{G}$*-torsor.*

Proof. (Sketch.) The proof is similar to the case of $\operatorname{Spd} \mathbf{Q}_p$. We need the fact that for any algebraically closed nonarchimedean field C of characteristic p, the group G acts freely on $\operatorname{Hom}(\widetilde{R}^\flat, C)$. Fix $f \colon \widetilde{R}^\flat \to C$. By the tilting equivalence, this corresponds to a map $f^\sharp \colon \widetilde{R} \to C^\sharp$. More precisely, $\widetilde{R}^\circ = W(\widetilde{R}^{\flat\circ})/I$, where I is G-stable. We get $W(f^\circ) \colon W(\widetilde{R}^{\flat\circ}) \to W(\mathcal{O}_C)$, and then $W(f^\circ) \mod I \colon \widetilde{R}^\circ \to \mathcal{O}_{C^\sharp}$.

Assume there exists $\gamma \in G$ such that



commutes. Apply W and reduce modulo I to obtain



Now invert p. By G-equivariance, we get that

$$
\begin{array}{ccc}
R_i & \xrightarrow{f^\sharp} & C^\sharp \\
{\scriptstyle\gamma}\nwarrow & & \nearrow{\scriptstyle f^\sharp} \\
& R_i &
\end{array}
$$

commutes, which shows that $\gamma = 1$. □

□

Example 10.1.6. Consider the product $\operatorname{Spd} \mathbf{Q}_p \times \operatorname{Spd} \mathbf{Q}_p$. Let $\widetilde{\mathbf{D}}^*_{\mathbf{Q}_p}$ be the punctured unit disc, considered as an adic space over $\mathbf{Q}_p$. We can consider $\widetilde{\mathbf{D}}^*_{\mathbf{Q}_p}$ as a subspace of $\mathbf{G}_{\mathrm{m}}$ via $x \mapsto 1 + x$. Let $\widetilde{\mathbf{D}}^*_{\mathbf{Q}_p} = \varprojlim \mathbf{D}^*_{\mathbf{Q}_p}$, where the inverse limit is taken with respect to the pth power maps on $\mathbf{G}_{\mathrm{m}}$.

We claim that there is an isomorphism of diamonds $(\widetilde{\mathbf{D}}^*_{\mathbf{Q}_p})^\diamondsuit/\mathbf{Z}_p^\times \cong \operatorname{Spd} \mathbf{Q}_p \times \operatorname{Spd} \mathbf{Q}_p$ which makes the following diagram commute:

$$
\begin{array}{ccc}
(\widetilde{\mathbf{D}}^*_{\mathbf{Q}_p})^\diamondsuit/\mathbf{Z}_p^\times & \longrightarrow & \operatorname{Spd} \mathbf{Q}_p \times \operatorname{Spd} \mathbf{Q}_p \\
\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\
\operatorname{Spd} \mathbf{Q}_p & \xrightarrow{=} & \operatorname{Spd} \mathbf{Q}_p.
\end{array}
$$

For this it is enough to know that there is an isomorphism

$$\left(\widetilde{\mathbf{D}}^*_{\mathbf{Q}_p^{\mathrm{cycl}}}\right)^\diamondsuit \cong \operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl}} \times \operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl}}$$

which is $\underline{\mathbf{Z}}_p^\times \times \underline{\mathbf{Z}}_p^\times$-equivariant. It is not hard to see that $\widetilde{\mathbf{D}}^*_{\mathbf{Q}_p^{\mathrm{cycl}}}$ is a perfectoid space whose tilt is $\widetilde{\mathbf{D}}^*_{\mathbf{Q}_p^{\mathrm{cycl},\flat}}$, the corresponding object in characteristic p.

Meanwhile, $\operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl}} \times \operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl}}$ is representable by the perfectoid space $\operatorname{Spa} \mathbf{Q}_p^{\mathrm{cycl},\flat} \times \operatorname{Spa} \mathbf{Q}_p^{\mathrm{cycl},\flat}$. We have an isomorphism $\mathbf{Q}_p^{\mathrm{cycl},\flat} \cong \mathbf{F}_p(\!(t^{1/p^\infty})\!)$, which when applied to one of the factors in the product gives

$$\operatorname{Spa} \mathbf{Q}_p^{\mathrm{cycl},\flat} \times \operatorname{Spa} \mathbf{Q}_p^{\mathrm{cycl},\flat} \cong \operatorname{Spa} \mathbf{Q}_p^{\mathrm{cycl},\flat} \times \operatorname{Spa} \mathbf{F}_p(\!(t^{1/p^\infty})\!) = \mathbf{D}^*_{\mathbf{Q}_p^{\mathrm{cycl},\flat}}$$

once again. We leave it as an exercise to check that the group actions are compatible.

10.2 The cohomology of diamonds

Definition 10.2.1. Let $\mathcal{D}$ be a diamond. Define

$$\mathcal{D}_{\text{proét}} = \{\text{qpf morphisms } \mathcal{E} \to \mathcal{D}\}$$

with covers defined as jointly surjective maps.

Remark 10.2.2. If $\mathcal{E}$ is a sheaf on (Perf) and $\mathcal{E} \to \mathcal{D}$ is quasiprofinite, then $\mathcal{E}$ is a diamond. Indeed if $h_X \to \mathcal{D}$ is qpf, then $h_X \times_{\mathcal{D}} \mathcal{E} = h_Y$ is representable, and then $Y \to \mathcal{E}$ is qpf and surjective.

Remark 10.2.3. If $\mathcal{D} = X^\diamondsuit$ for a perfectoid space X, then $\mathcal{D}_{proet}$ is the category of $Y \to X$ qpf, with Y a perfectoid space, with covers as in (Perf). Thus the pro-étale site on diamonds recovers that on perfectoid spaces.

Proposition 10.2.4. *Let $\mathcal{F}$ be a sheaf on $\mathcal{D}_{proét}$, and let $X^\diamondsuit \to \mathcal{D}$ be a surjective qpf morphism from a perfectoid space X. For $i \geq 0$, let X_i be the perfectoid space $X^\diamondsuit \times_{\mathcal{D}} \cdots \times_{\mathcal{D}} X^\diamondsuit$, with $i+1$ factors. Then there is a spectral sequence*

$$E_1^{ij} = H^j(X_i^\diamondsuit, \mathcal{F}) = H^j(X_{i,proét}, \mathcal{F}) \implies H^{i+j}(\mathcal{D}_{proét}, \mathcal{F})$$

In this way the cohomology of diamonds can be computed from the cohomology of perfectoid spaces.

Proposition 10.2.5. *Let X be a rigid space over $\mathbf{Q}_p$ considered as an adic space. (Or more generally, suppose X is locally of the form $X = \mathrm{Spa}(A, A^+)$, where A is strongly noetherian.) Huber defines an étale site $X_{ét}$ (can be defined as for perfectoid spaces). There is a morphism of sites $\nu\colon X^\diamondsuit_{proét} \to X_{ét}$, such that for all sheaves $\mathcal{F}$ on $X_{ét}$, $\mathcal{F} \to R\nu_*\nu^*\mathcal{F}$ is an isomorphism. As a result $H^i(X_{ét}, \mathcal{F}) = H^i(X^\diamondsuit_{proét}, \nu^*\mathcal{F})$.*

Proof. (Sketch.) The morphism ν sends an étale morphism $Y \to X$ to $Y^\diamondsuit \to X^\diamondsuit$.

We have the site $X^{\text{old}}_{\text{proét}}$ as defined in [Sch13a], where it was proved that $\mathcal{F} \cong R\nu^{\text{old}}_* \nu^{\text{old}*}\mathcal{F}$. Then ν is the composite

$$X^\diamondsuit \xrightarrow{\lambda} X^{\text{old}}_{\text{proét}} \xrightarrow{\nu^{\text{old}}} X_{\text{ét}}$$

It just remains to show that $\nu^{\text{old}*}\mathcal{F} \cong R\lambda_*\nu^*\mathcal{F}$. That is, we want the sheafification of $U \in X^{\text{old}}_{\text{proét}} \mapsto H^i(U^\diamondsuit_{\text{proét}}, \nu^*\mathcal{F})$ to vanish for $i > 0$, and $\nu^{\text{old}}\mathcal{F}$ for $i = 0$.

For the vanishing: let $i > 0$. WLOG U is perfectoid. Let $\alpha \in H^i(U^\diamondsuit_{\text{proét}}, \nu^*\mathcal{F})$. This vanishes after pullback $V^\diamondsuit \to U^\diamondsuit$, where $V = \varprojlim V_i$, with $V_j \in U$ étale (and thus they live in the old pro-étale site). Now in the cohomology

$$H^i(V^\diamondsuit_{\text{proét}}, \nu^*\mathcal{F}) = \varinjlim_j H^i(V^\diamondsuit_{j,\text{proét}}, \nu^*\mathcal{F})$$

we must have that α already vanishes in $H^i(V^\diamondsuit_{j,\text{proét}}, \nu^*\mathcal{F})$ for $j \gg 0$. □

Proposition 10.2.6. *Let $f\colon X \to X'$ be a finite universal homeomorphism of locally strongly noetherian adic spaces over $\operatorname{Spa} \mathbf{Q}_p$. (This means that any pullback is also a homeomorphism.) Then $f^\diamondsuit\colon X^\diamondsuit \to (X')^\diamondsuit$ is an isomorphism.*

Proof. Let $Y = \operatorname{Spa}(S, S^+)$ be an affinoid pefectoid space, and let $Y \to X'$ be a morphism. We claim there exists a unique factorization

$$\begin{array}{ccc} Y & \dashrightarrow & X \\ & \searrow \quad \swarrow & \\ & X' & \end{array}$$

We have $|f| : |Y| \to |X| = |X'|$. We need also $|f|^* \mathcal{O}_X \to \mathcal{O}_Y$. We have $|f|^* \mathcal{O}^+_{X'}/\varpi^n \to \mathcal{O}^+_Y/\varpi^n$.

Lemma 10.2.7. *$\mathcal{O}^+_{X'}/\varpi^n \to \mathcal{O}^+_X/\varpi^n$ is an isomorphism of sheaves on $|X| = |X'|$.*

Proof. Check on stalks. They are $K(x')^+/\varpi^n \to K(x)^+/\varpi^n$, where x' corresponds to x. But the residue fields are the same. □

So we have a map $|f|^* \mathcal{O}^+_X/\varpi^n \to \mathcal{O}^+_Y/\varpi^n$, which in the limit becomes $|f|^* \mathcal{O}^+_X \to \varprojlim_n \mathcal{O}^+_Y/\varpi^n = \mathcal{O}^+_Y$. The last equality is because Y is perefectoid: $H^i(Y, \mathcal{O}^+_Y/\varpi^n)$ is almost S^+/ϖ^n if $i = 0$ and 0 if $i > 0$. □

In [Sch13b], we had the perfectoid space $\mathcal{A}^*_{g,\infty} = \varprojlim \mathcal{A}^*_{g,K_p}$. Within this there is a smaller Shimura variety $\mathrm{Sh}^*_\infty = \varprojlim \mathrm{Sh}^*_{K_p}$. But then there is a universal homeomorphism $\mathrm{Sh}^*_{K_p} \to \mathrm{Sh}^*_{K_p}$ from the compatification of interest. Nonetheless the two Shs have the same diamond, and so they have the same cohomology.

Proposition 10.2.8. *The functor*

$$\begin{array}{rcl} \{\textit{normal rigid-analytic spaces}/\mathbf{Q}_p\} & \to & \{\textit{diamonds}/\operatorname{Spd}\mathbf{Q}_p\} \\ X & \mapsto & X^\diamondsuit \end{array}$$

is fully faithful.

Proof. The crucial statement is that if $R \to \widetilde{R}$ is as above, with group G, and R is normal, then $R = \widetilde{R}^G$. (This is an analogue of the Ax-Sen-Tate theorem.) One can use resolution of singularities to reduce to the smooth case, and then use an explicit computation. $\square$

11 Mixed-characteristic shtukas, 14 October

Today we begin talking about shtukas.

11.1 Review of the equal characteristic story

Let $X/\mathbf{F}_q$ be a smooth projective geometrically connected curve.

Definition 11.1.1. Let $S/\mathbf{F}_q$ be a scheme. A *shtuka of rank n over S with paws*[10] $x_1, \dots, x_m \in X(S)$ is a rank n vector bundle $\mathcal{E}$ over $X \times_{\mathbf{F}_q} S$ together with an isomorphism $\phi_{\mathcal{E}} \colon (1 \times \operatorname{Frob}_S)^*\mathcal{E} \to \mathcal{E}$ on $(X \times_{\mathbf{F}_q} S) \backslash \bigcup_i \Gamma_{x_i}$, where $\Gamma_{x_i} \subset X \times_{\mathbf{F}_q} S$ is the graph of $x_i \colon S \to X$.

We currently have no hope of replacing X with the spectrum of the ring of integers in a number field, but rather only a p-adic field. So let us discuss the local analogue of Defn. 11.1.1. Fix a point $x \in X(\mathbf{F}_q)$, and let $\widehat{X}_x$ be the formal completion of X at x, so that $\widehat{X} \cong \operatorname{Spf}\mathbf{F}_q[\![T]\!]$. The paws will be elements of $\widehat{X}_x(S)$, which is to say morphisms $S \to \operatorname{Spf}\mathbf{F}_q[\![T]\!]$, where $S/\mathbf{F}_q$ is an adic formal scheme, or more generally an adic space.

Definition 11.1.2. A *local shtuka* of rank n over an adic space $S/\operatorname{Spa}\mathbf{F}_q$ with paws $x_1, \dots, x_n \in \widehat{X}_x(S)$ is a rank n vector bundle $\mathcal{E}$ over $\widehat{X}_x \times_{\mathbf{F}_q} S$ together with an isomorphism $\phi_{\mathcal{E}} \colon (1 \times \operatorname{Frob}_S)^*\mathcal{E} \to \mathcal{E}$ over $(\widehat{X}_x \times S) \backslash \bigcup_i \Gamma_{x_i}$.

Remark 11.1.3. The space S should be sufficiently nice that $\widehat{X}_x \times_{\mathbf{F}_q} S$ exists and has a good theory of vector bundles. This is the case for instance if S is locally of the form $\operatorname{Spa}(R, R^+)$, with R strongly noetherian. One should also impose the condition that $\phi_{\mathcal{E}}$ is meromorphic along Γ_{x_i}.

[10] "Paws" is a translation of *pattes*, which appears in the French version of [Laf].

Example 11.1.4 (One paw)**.** Let $C/\mathbf{F}_q$ be an algebraically closed nonarchimedean field with pseudo-uniformizer ϖ and residue field k, and let $S = \operatorname{Spa} C$. Then the product $\operatorname{Spa} \mathbf{F}_q[\![T]\!] \times_{\mathbf{F}_q} S = \mathbf{D}_C$ is the open unit disc over C. (Recall that $\mathbf{D}_C$ is the increasing union $\bigcup_{n\geq 1} \operatorname{Spa} C\langle T/\varpi^{1/n}\rangle$.) The paw will be a continuous map $\mathbf{F}_q[\![T]\!] \to C$, which is to say an element $\zeta \in C$ (the image of T) which is topologically nilpotent. We can think of ζ as a C-point of $\mathbf{D}_C$.

Then a shtuka is a rank n vector bundle $\mathcal{E}$ over $\mathbf{D}_C$ together with an isomorphism $\phi_{\mathcal{E}} \colon \phi^*\mathcal{E} \to \mathcal{E}$ on $\mathbf{D}_C \backslash \{\zeta\}$. Here, $\phi \colon \mathbf{D}_C \to \mathbf{D}_C$ sends the parameter T to T, but is Frobenius on C. Note that $\mathbf{D}_C$ is a classical rigid space over C, but ϕ is not a morphism of rigid spaces because it is not C-linear. In the case $\zeta = 0$, such pairs $(\mathcal{E}, \phi_{\mathcal{E}})$ are studied in [HP04], where they are called *σ-bundles*.

We could also have taken $S = \operatorname{Spa} \mathcal{O}_C$, in which case the product $\widehat{X}_x \times \operatorname{Spa} \mathcal{O}_C$ is $\operatorname{Spa} \mathcal{O}_C[\![T]\!]$. This is similar to the space $\operatorname{Spa} \mathbf{Z}_p[\![T]\!]$, which we analyzed in §3.3. It contains a unique non-analytic point x_k. Let $\mathcal{Y}$ be the complement of x_k in $\operatorname{Spa} \mathcal{O}_C[\![T]\!]$. Once again there is a continuous surjective map $\kappa \colon \mathcal{Y} \to [0, \infty]$, defined by

$$\kappa(x) = \frac{\log |\varpi(\widetilde{x})|}{\log |T(\widetilde{x})|},$$

where $\widetilde{x}$ is the maximal generalization of x. The Frobenius map ϕ is a new feature of this picture. It satisfies $\kappa \circ \phi = p\kappa$. See Figure 11.1.

11.2 The adic space "$S \times \operatorname{Spa} \mathbf{Z}_p$"

In the mixed characteristic setting, $\widehat{X}_x$ will be replaced with $\operatorname{Spa} \mathbf{Z}_p$. Our test objects S will be drawn from (Perf), the category of perfectoid spaces in characteristic p. For an object S of (Perf), a shtuka over S should be a vector bundle over an adic space "$S \times \operatorname{Spa} \mathbf{Z}_p$" together with a Frobenius structure. The product is not meant to be taken literally (if so, one would just recover S), but rather it is to be interpreted as a fiber product over a deeper base.

The main idea is that if R is a $\mathbf{F}_p$-algebra, then "$R \otimes \mathbf{Z}_p$" ought to be $W(R)$. As justification for this, note $W(R)$ is a ring admitting a ring homomorphism $\mathbf{Z}_p \to W(R)$ and also a map $R \to W(R)$ which is not quite a ring homomorphism (it is only multiplicative). Motivated by this, we will define an analytic adic space "$S \times \operatorname{Spa} \mathbf{Z}_p$" and then show that its associated diamond is the appropriate product of sheaves on (Perf).

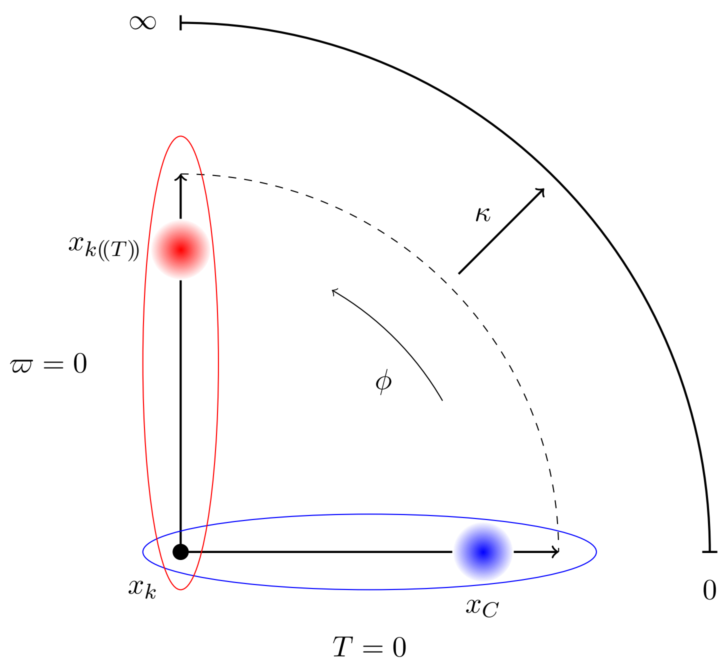


Figure 4: A depiction of $\operatorname{Spa} A$, where $A = \mathcal{O}_C[\![T]\!]$. The two closed subspaces $\operatorname{Spa} \mathcal{O}_C$ and $\operatorname{Spa} k[\![T]\!]$ appear as the x-axis and y-axis, respectively. Their intersection is the unique non-analytic point x_k of $\operatorname{Spa} A$. The complement of x_k in $\operatorname{Spa} A$ is the adic space $\mathcal{Y}$, on which the continuous map $\kappa\colon \mathcal{Y} \to [0, \infty]$ is defined. The automorphism ϕ of $\operatorname{Spa} A$ tends to rotate points towards the y-axis (though it fixes both axes).

Definition 11.2.1. Let $\operatorname{Spd} \mathbf{Z}_p$ be the presheaf on (Perf) which sends S to the set of isomorphism classes of pairs $(S^\sharp, \iota)$, where $S^\sharp$ is a perfectoid space (in any characteristic) and $\iota \colon S^{\sharp\flat} \to S$ is an isomorphism. Such a pair $(S^\sharp, \iota)$ will be called an *untilt* of S (and often we will drop the ι from the notation).

Recall that $\operatorname{Spd} \mathbf{Q}_p$ is the sheaf on (Perf) which sends S to the set of isomorphism classes of untilts $S^\sharp$ which are $\mathbf{Q}_p$-algebras. This is an open subfunctor of $\operatorname{Spd} \mathbf{Z}_p$. The complement is represented by the single point $\operatorname{Spa} \mathbf{F}_p$, which sends S to $\{(S, \mathrm{id}_S)\}$.

We warn the reader that $\operatorname{Spd} \mathbf{Z}_p$ is not a diamond. (All diamonds are "analytic", but the special point of $\operatorname{Spd} \mathbf{Z}_p$ is not analytic.) However, the next proposition shows that $\operatorname{Spd} \mathbf{Z}_p$ becomes a diamond once one supplies a base.

Proposition 11.2.2. *For any perfectoid space S of characteristic p, $S^\diamondsuit \times \operatorname{Spd} \mathbf{Z}_p$ is a diamond. More precisely, there is an analytic adic space "$S \times \operatorname{Spa} \mathbf{Z}_p$" over* $\operatorname{Spa} \mathbf{Z}_p$ *such that* $(\text{"}S \times \operatorname{Spa} \mathbf{Z}_p\text{"})^\diamondsuit \cong S^\diamondsuit \times \operatorname{Spd} \mathbf{Z}_p$ *as diamonds over* $\operatorname{Spd} \mathbf{Z}_p$.

Remark 11.2.3. It follows that $\operatorname{Spd} \mathbf{Z}_p$ is a sheaf. Indeed, to check the sheaf property for $\operatorname{Spd} \mathbf{Z}_p$ on a cover of an object S is the same as checking it for the product $S^\diamondsuit \times \operatorname{Spd} \mathbf{Z}_p$, which is a diamond and therefore a sheaf.

Proof. We will treat the case that $S = \operatorname{Spa}(R, R^+)$ is affinoid; the reader may globalize the result by computing with rational subsets. Let $\varpi \in R$ be a pseudo-uniformizer. Give the ring of Witt vectors $W(R^+)$ the $(p, [\varpi])$-adic topology. Define

$$\text{"}S \times \operatorname{Spa} \mathbf{Z}_p\text{"} = \{[\varpi] \neq 0\} \subset \operatorname{Spa} W(R^+).$$

(The displayed condition is shorthand for $\{x \,|\, |[\varpi](x)| \neq 0\}$; note that it does not depend on the choice of ϖ.)

The element $[\varpi]$ is everywhere a topologically nilpotent unit on "$S \times \operatorname{Spa} \mathbf{Z}_p$". Consequently "$S \times \operatorname{Spa} \mathbf{Z}_p$" is an analytic adic space over $\operatorname{Spa} \mathbf{Z}_p$.

Now we check the identification of associated diamonds. For an object Y of (Perf), say $Y = \operatorname{Spa}(T, T^+)$, a Y-valued point of $(\text{"}S \times \operatorname{Spa} \mathbf{Z}_p\text{"})^\diamondsuit$ is an untilt $Y^\sharp$ lying over "$S \times \operatorname{Spa} \mathbf{Z}_p$". To give a morphism $Y^\sharp \to$ "$S \times \operatorname{Spa} \mathbf{Z}_p$" is to give a continuous homomorphism $W(R^+) \to T^{\sharp +}$ such that the image of $[\varpi]$ is invertible in T.

On the other hand, a Y-valued point of $S^\diamondsuit \times \operatorname{Spd} \mathbf{Z}_p$ is a pair consisting of a morphism $Y \to S$ together with an untilt $Y^\sharp$.

Define a map $(\text{“}S \times \operatorname{Spa} \mathbf{Z}_p\text{”})^\diamondsuit(Y) \to (S^\diamondsuit \times \operatorname{Spd} \mathbf{Z}_p)(Y)$ as follows. Suppose we are given an untilt $Y^\sharp$ and a map $W(R^+) \to T^{\sharp+}$. Reduce the map modulo the ideal $(p, [\varpi])$ to obtain a map $R^+/\varpi \to T^+/p$. Now apply $\varprojlim_\Phi$ to obtain a map $R^+ \to T^+$, in which the image of ϖ is invertible in T; this extends to $R \to T$. We have constructed a map $Y \to S$. The untilt $Y^\sharp$ was part of the given data, so we have constructed an element of $(S^\diamondsuit \times \operatorname{Spd} \mathbf{Z}_p)(Y)$.

In the other direction, suppose we are given a morphism $Y \to S$ and an untilt $Y^\sharp = \operatorname{Spa}(T^\sharp, T^{\sharp+})$. The morphism $Y \to S$ corresponds to a map $R^+ \to T$, which induces a map $W(R^+) \to W(T^+)$. Composing this with the map $\theta\colon W(T^+) \to T^{\sharp+}$ gives the desired map $W(R^+) \to T^{\sharp+}$. □

Example 11.2.4. Let $S = \operatorname{Spa} C$, where $C/\mathbf{F}_p$ is an algebraically closed nonarchimedean field with residue field k. The ring of Witt vectors $W(\mathcal{O}_C)$ is called A_{inf} in Fontaine's theory, [Fon94]. Its spectrum $\operatorname{Spa} W(\mathcal{O}_C)$ is rather like the formal unit disk $\operatorname{Spa} \mathcal{O}_C[\![T]\!]$. Note that there is no map "$S \times \operatorname{Spa} \mathbf{Z}_p$" $\to S$ (because there is no ring homomorphism $\mathcal{O}_C \to W(\mathcal{O}_C)$), but there is one on the level of topological spaces.

11.3 "Sections of "$S \times \operatorname{Spa} \mathbf{Z}_p$" $\to S$"

Even though there is no morphism "$S \times \operatorname{Spa} \mathbf{Z}_p$" $\to S$, Prop. 11.2.2 shows there is a morphism $(\text{“}S \times \operatorname{Spa} \mathbf{Z}_p\text{”})^\diamondsuit \to S^\diamondsuit$. The following proposition shows that sections of this morphism behave as expected.

Proposition 11.3.1. *Let S be an object of (Perf). The following sets are naturally identified:*

1. *Sections of* $(\text{“}S \times \operatorname{Spa} \mathbf{Z}_p\text{”})^\diamondsuit \to S^\diamondsuit$,
2. *Morphisms* $S^\diamondsuit \to \operatorname{Spd} \mathbf{Z}_p$, *and*
3. *Untilts $S^\sharp$ of S.*

Proof. By Prop. 11.2.2, $(\text{“}S \times \operatorname{Spa} \mathbf{Z}_p\text{”})^\diamondsuit = S^\diamondsuit \times \operatorname{Spd} \mathbf{Z}_p$. Thus a section as in (1) is a section of $S^\diamondsuit \times \operatorname{Spd} \mathbf{Z}_p \to S^\diamondsuit$, which is nothing but a morphism $S^\diamondsuit \to \operatorname{Spd} \mathbf{Z}_p$. Thus (1) and (2) are identified.

Let us show that (3) is identified with (1) and (2). Once again only treat the case that $S = \operatorname{Spa}(R, R^+)$ is affinoid. Let $\varpi \in R$ be a pseudo-uniformizer.

Given an untilt $S^\sharp = \operatorname{Spa}(R^\sharp, R^{\sharp+})$ as in (3), we have a map $\theta\colon W(R^+) \to R^{\sharp+}$ which sends $[\varpi]$ to a unit in $R^\sharp$. This means that the composite map

$S \to \operatorname{Spa} R^{\sharp +} \to \operatorname{Spa} W(R^+)$ factors through "$S \times \operatorname{Spa} \mathbf{Z}_p$" $\subset \operatorname{Spa} W(R^+)$, so we have a map $S \to$ "$S \times \operatorname{Spa} \mathbf{Z}_p$". Passing to diamonds gives an element of (1).

Conversely if an element of (2) is given, we a have a morphism of diamonds $S^\diamondsuit \to \operatorname{Spd} \mathbf{Z}_p$. The image of the identity morphism $S \to S$ under $S^\diamondsuit(S) \to (\operatorname{Spd} \mathbf{Z}_p)(S)$ is an S-point of $\operatorname{Spd} \mathbf{Z}_p$, which is by definition an untilt of S. We leave it to the reader to see that these maps are mutual inverses. □

Proposition 11.3.2. "$S \times \operatorname{Spa} \mathbf{Z}_p$" *is an honest adic space.*

Proof. Once again we assume $S = \operatorname{Spa}(R, R^+)$ is affinoid, with $\varpi \in R$ a pseudo-uniformizer. We claim that the adic space

$$\text{“}S \times \operatorname{Spa} \mathbf{Z}_p\text{”} = \operatorname{Spa} W(R^+) \backslash \{[\varpi] = 0\}$$

has a covering by affinoid subsets of $\operatorname{Spa} W(R^+)$ of the form $|p| \leq \left|[\varpi^{1/p^n}]\right| \neq 0$, for $n = 1, 2, \ldots$. Indeed, if $|\cdot(x)|$ is a continuous valuation on $W(R^+)$ with $|[\varpi](x)| \neq 0$, then (since p is topologically nilpotent) there exists $n \gg 0$ such that $|p(x)|^n \leq |[\varpi]|$.

Let $\operatorname{Spa}(R_n, R_n^+)$ be the rational subset $|p| \leq \left|[\varpi^{1/p^n}]\right| \neq 0$. Thus for instance R_n is the ring obtained by $[\varpi]$-adically completing $W(R^+)[p/[\varpi^{1/p^n}]$ and then inverting $[\varpi]$. One has the following presentation for R_n:

$$R_n = \left\{ \sum_{i \geq 0} [r_i] \left(\frac{p}{[\varpi^{1/p^n}]} \right)^i \;\middle|\; r_i \in R,\ r_n \to 0 \right\}.$$

(One might call this ring $W(R)\langle p/[\varpi^{1/p^n}] \rangle$.)

Note that R_n is Tate ($[\varpi]$ serves as a pseudo-uniformizer). We leave it to the reader to show that if K is a perfectoid $\mathbf{Q}_p$-algebra, then $R_n \widehat{\otimes}_{\mathbf{Z}_p} \mathcal{O}_K$ is a perfectoid ring, and therefore R_n is preperfectoid. (We just note that $\varpi' = [\varpi^{1/p}]$ serves as a pseudo-uniformizer satisfying the condition of Defn. 6.1.1. Note that R_n does not contain a field, so it is rather like Example 6.3.2(2).) By Prop. 6.3.3, (R_n, R_n^+) is sheafy. □

11.4 Definition of mixed-characteristic shtukas

By Thm. 5.5.8 one has a good notion of vector bundle for the space "$S \times \operatorname{Spa} \mathbf{Z}_p$".

Definition 11.4.1. Let S be a perfectoid space in characteristic p. Let $x_1, \dots, x_m : S^{\diamondsuit} \to \operatorname{Spd} \mathbf{Z}_p$ be a collection of morphisms; for $i = 1, \dots, m$ let $x_i^\sharp : S_i^\sharp \to$ "$S \times \operatorname{Spa} \mathbf{Z}_p$" be the corresponding morphism of adic spaces over $\operatorname{Spa} \mathbf{Z}_p$. Write Γ_{x_i} for the image of $x_i^\sharp$. A *(mixed-characteristic) shtuka of rank n over S with paws $x_1, \dots, x_m$* is a rank n vector bundle $\mathcal{E}$ over "$S \times \operatorname{Spa} \mathbf{Z}_p$" together with an isomorphism $\phi_{\mathcal{E}} : \operatorname{Frob}_S^*(\mathcal{E}) \to \mathcal{E}$ over ("$S \times \operatorname{Spa} \mathbf{Z}_p$")$\backslash \bigcup_i \Gamma_{x_i}$. This is required to be meromorphic along each Γ_{x_i}.

Let us discuss the case of shtukas with one paw over $S = \operatorname{Spa} C$, where $C/\mathbf{F}_p$ is an algebraically closed nonarchimedean field. The paw is a map $S^{\diamondsuit} \to \operatorname{Spd} \mathbf{Z}_p$. Let us assume that this factors over $\operatorname{Spd} \mathbf{Q}_p$ and thus corresponds to a characteristic 0 untilt $C^\sharp$, an algebraically closed field. We have a surjective homomorphism $W(\mathcal{O}_C) \to \mathcal{O}_{C^\sharp}$ whose kernel is generated by an element $\xi \in W(\mathcal{O}_C)$. Let $\phi = W(\operatorname{Frob}_{\mathcal{O}_C})$, an automorphism of $W(\mathcal{O}_C)$.

It turns out that such shtukas are in correspondence with linear-algebra objects which are essentially shtukas over all of $\operatorname{Spa} W(\mathcal{O}_C)$, rather than over just the locus $[\varpi] \neq 0$.

Definition 11.4.2 (Fargues)**.** A *Breuil-Kisin module over $W(\mathcal{O}_C)$* is a pair (M, ϕ_M), where M is a finite free $W(\mathcal{O}_C)$-module and $\phi_M : (\phi^* M)[\xi^{-1}] \xrightarrow{\sim} M[\xi^{-1}]$ is an isomorphism.

Remark 11.4.3. Note the analogy to Kisin's work [Kis06], which takes place in the context of a finite totally ramified extension $K/W(k)[1/p]$ (now k is any perfect field of characteristic p). Let ξ generate the kernel of a continuous surjective homomorphism $W(k)[\![u]\!] \to \mathcal{O}_K$. Kisin's ϕ-modules are pairs (M, ϕ_M), where M is a finite free $W(k)[\![u]\!]$-module and $\phi_M : (\phi^* M)[\xi^{-1}] \xrightarrow{\sim} M[\xi^{-1}]$ is an isomorphism. Kisin constructs a fully faithful functor from the category ϕ-modules up to isogeny into the category of crystalline representations of $\operatorname{Gal}(\overline{K}/K)$ and identifies the essential image.

Let (M, ϕ_M) be a Breuil-Kisin module over $W(\mathcal{O}_C)$. After inverting $[\varpi]$, M gives rise to a vector bundle on "$S \times \operatorname{Spa} \mathbf{Z}_p$" $\subset W(\mathcal{O}_C)$ and therefore a shtuka $\mathcal{E}$. In fact one can go in the other direction:

Theorem 11.4.4 ([Far13])**.** *The functor $M \mapsto \mathcal{E}$ is an equivalence between the category of Breuil-Kisin modules over $W(\mathcal{O}_C)$ and the category of shtukas over* $\operatorname{Spa} C$ *with one paw at $C^\sharp$.*

12 Shtukas with one paw, 16 October

12.1 p-divisible groups over $\mathcal{O}_C$

Today we connect mixed-characteristic shtukas to p-adic Hodge theory over an algebraically closed field. Let $C/\mathbf{Q}_p$ be an algebraically closed nonarchimedean field, with ring of integers $\mathcal{O}_C$ and residue field k. Let G be a p-divisible group over $\mathcal{O}_C$, and let $T_pG = \varprojlim G[p^n](C)$ be its Tate module, a free $\mathbf{Z}_p$-module of finite rank.

Theorem 12.1.1 (The Hodge-Tate exact sequence, [Far08])**.** *There is a natural short exact sequence*

$$0 \longrightarrow \operatorname{Lie} G \otimes_{\mathcal{O}_C} C(1) \xrightarrow{\alpha^*_{G^*}} T_pG \otimes_{\mathbf{Z}_p} C \xrightarrow{\alpha_G} (\operatorname{Lie} G^*)^* \otimes_{\mathcal{O}_C} C \longrightarrow 0$$

Remark 12.1.2. Tate [Tat67] treated the case where G comes from a p-divisible group over a DVR with perfect residue field.

The map α_G is defined as follows. An element of T_pG is really just a morphism of p-divisible groups $f\colon \mathbf{Q}_p/\mathbf{Z}_p \to G$, whose dual is morphism $f^*\colon G^* \to \mu_{p^\infty}$. The derivative of f^* is an $\mathcal{O}_C$-linear map $\operatorname{Lie} f^*\colon \operatorname{Lie} G^* \to \operatorname{Lie} \mu_{p^\infty}$. The dual of $\operatorname{Lie} f^*$ is a map $(\operatorname{Lie} \mu_{p^\infty})^* \to (\operatorname{Lie} G^*)^*$; let $\alpha_G(f)$ be the image of $dt/t \in (\operatorname{Lie} \mu_{p^\infty})^*$. (See for instance Ch. II of Messing's thesis for a discussion of this map for a p-divisible group over a p-adically separated and complete ring.)

Remark 12.1.3. It isn't quite formal that $\alpha_G \circ \alpha^*_{G^*} = 0$.

Definition 12.1.4. Let $\{(T, W)\}$ be the category of pairs consisting of a free $\mathbf{Z}_p$-module T of finite rank and a C-vector subspace $W \subset T \otimes_{\mathbf{Z}_p} C$.

Theorem 12.1.1 gives us a functor

$$\begin{array}{rcl} \{p\text{-div. gps.}/\mathcal{O}_C\} & \to & \{(T, W)\} \\ G & \mapsto & (T_pG, \operatorname{Lie} G \otimes_{\mathcal{O}_C} C(1)) \end{array}$$

Theorem 12.1.5 ([SW13])**.** *This is an equivalence of categories.*

Theorem 12.1.5 gives a classification of p-divisible groups over $\mathcal{O}_C$ in terms of linear algebra data, analogous to Riemann's classification of complex abelian varieties. Recall also the classification of p-divisible groups over a general perfect field k of characteristic p by *Dieudonné modules*: free $W(k)$-modules M of finite rank equipped with a σ-linear map $F\colon M \to M$

and a σ^{-1}-linear map $V\colon M \to M$ which satisfy $FV = VF = p$. Here $\sigma\colon W(k) \to W(k)$ is induced from the pth power Frobenius map on k. There is the interesting question of how these classifications interact. That is, we have a diagram

We get a diagram

$$\begin{array}{ccc} \{p\text{-div. gps.}/\mathcal{O}_C\} & \xrightarrow{\sim} & \{(T, W)\} \\ \downarrow & & \downarrow ? \\ \{p\text{-div. gps.}/k\} & \xrightarrow{\sim} & \{\text{Dieudonné modules}\} \end{array}$$

It is not at all clear how to give an explicit description of the arrow labeled "?". If we think of "?" only as a map between sets of isomorphism classes of objects, we get the following interpretation. Let (h, d) be a pair of nonnegative integers with $d \leq h$. The set of isomorphism classes of objects (T, W) with $\operatorname{rank} T = h$ and $\dim W = d$ together with a trivialization $T \cong \mathbf{Z}_p^n$ is $\operatorname{Grass}(h, d)$, the set of d-planes in C^h. The set of isogeny classes of Dieudonné modules is identified with the finite set $\mathcal{NP}_{h,d}$ of Newton polygons running between $(0, 0)$ and (d, h) whose slopes lie in $[0, 1]$. Thus we have a *canonical* $\mathrm{GL}_h(\mathbf{Q}_p)$-equivariant map $\operatorname{Grass}(d, h) \to \mathcal{NP}_{h,d}$. What are its fibers?

Example 12.1.6. In the cases $d = 0$ and $d = 1$ the map $\operatorname{Grass}(d, h) \to \mathcal{NP}_{h,d}$ can be calculated explicitly. The case $d = 0$ is trivial since both sets are singletons. So consider the case $d = 1$. Let $W \subset C^h$ be a line, so that $W \in \mathbf{P}^{h-1}(C)$. The p-divisible group corresponding to $(\mathbf{Z}_p^h, W)$ is determined by "how rational" W is. To wit, let $H \subset \mathbf{P}^{h-1}(C)$ be the smallest $\mathbf{Q}_p$-linear C-subspace containing W, and let $i = \dim_C H - 1$. Then the p-divisible group associated to $(\mathbf{Z}_p^h, W)$ is isogenous to $(\mathbf{Q}_p/\mathbf{Z}_p)^{\oplus(h-i)} \oplus G_i$, where G_i is a p-divisible group of height i and dimension 1. (Since k is algebraically closed, G_i is unique up to isomorphism.) Thus for instance the set of $W \in \mathbf{P}^{h-1}(C)$ which correspond to a p-divisible group with special fiber G_h is Drinfeld's upper half-space $\Omega^h(C)$.

12.2 Shtukas with one paw and p-divisible groups: an overview

In general the functor "?" is harder to describe, but there is a hint for how to proceed: If G is defined over $\mathcal{O}_K$ with $K/\mathbf{Q}_p$ finite, then we have the following Fontaine-style comparison isomorphism, valid after inverting p:

$$M = (T_p G \otimes_{\mathbf{Z}_p} A_{\mathrm{cris}})^{G_K}$$

up to p-torsion. This indicates that we will have to work with period rings. It turns out that the necessary period rings show up naturally in the geometry of $\operatorname{Spa} W(\mathcal{O}_C)$, and that there is an intimate link to shtukas over $\operatorname{Spa} C^\flat$ with one paw at C.

Remark 12.2.1. A notational remark: Fontaine gives the name $\mathcal{R}$ to $\mathcal{O}_{C^\flat}$, and Berger, Colmez call it $\widetilde{\mathbb{E}}^+$, reserving $\widetilde{\mathbb{E}}$ for what we call $C^\flat$. $W(\mathcal{O}_{C^\flat})$ is variously called $W(\mathcal{R})$ and $A_{\inf}$.

Recall that we have a surjective map $\theta\colon W(\mathcal{O}_{C^\flat}) \to \mathcal{O}_C$. The kernel of θ is generated by a non-zero divisor $\xi = p - [p^\flat]$, where $p^\flat = (p, p^{1/p}, p^{1/p^2}, \dots) \in \mathcal{O}_{C^\flat}$. Note that $p^\flat$ is a pseudo-uniformizer of $C^\flat$.

Consider the adic space $\operatorname{Spa} W(\mathcal{O}_{C^\flat})$. (It turns out that this is an honest adic space, but at present we won't be needing this fact.) We give names to four special points of $\operatorname{Spa} W(\mathcal{O}_{C^\flat})$, labeled by their residue fields:

1. x_k, the unique non-analytic point (recall that k is the residue field of C).
2. $x_{C^\flat}$, which corresponds to $W(\mathcal{O}_{C^\flat}) \to \mathcal{O}_{C^\flat} \to C^\flat$.
3. x_C, which corresponds to $W(\mathcal{O}_{C^\flat}) \to \mathcal{O}_C \to C$ (the first map is θ).
4. x_L, which corresponds to $W(\mathcal{O}_{C^\flat}) \to W(k) \to W(k)[1/p] = L$,

Let $\mathcal{Y} = \operatorname{Spa} W(\mathcal{O}_{C^\flat}) \backslash \{x_k\}$, an analytic adic space. Then as usual there exists a surjective continuous map $\kappa\colon \mathcal{Y} \to \mathbf{R}_{\geq 0} \cup \{\infty\}$, defined by

$$\kappa(x) = \frac{\log |[\varpi](\widetilde{x})|}{\log |p(\widetilde{x})|},$$

where $\widetilde{x}$ is the maximal generalization of x, cf. the discussion in §3.3. See Figure 12.2 for a depiction of the various structures associated with $\operatorname{Spa} W(\mathcal{O}_{C^\flat})$. We have:

$$\begin{aligned} \kappa(x_{C^\flat}) &= 0, \\ \kappa(x_C) &= 1, \\ \kappa(x_L) &= \infty. \end{aligned}$$

For an interval $I \subset \mathbf{R}_{\geq 0} \cup \{\infty\}$, let $\mathcal{Y}_I$ be the interior of the preimage of $\mathcal{Y}$ under κ. Thus $\mathcal{Y}_{[0,\infty)}$ is the complement in $\mathcal{Y}$ of the point x_L with residue field $L = W(k)[1/p]$. Also note that $\mathcal{Y}_{[0,\infty)} =$ "$\operatorname{Spa} C \times \operatorname{Spa} \mathbf{Z}_p$".

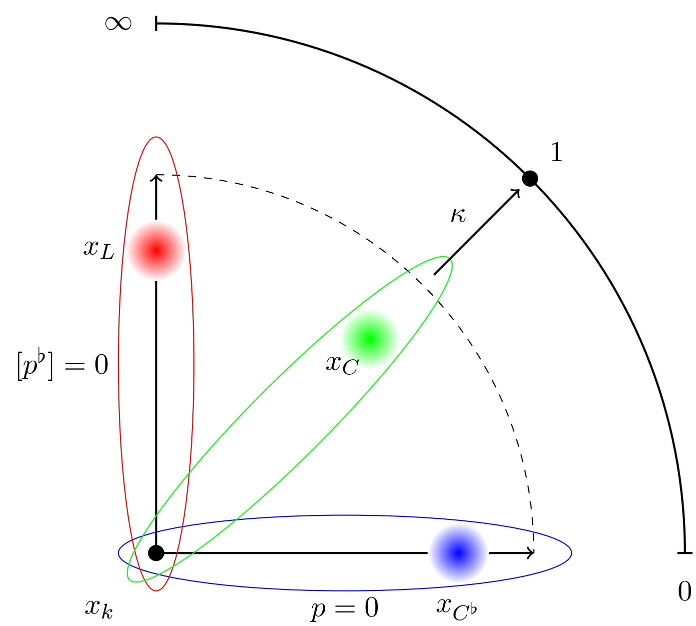


Figure 5: A depiction of $\operatorname{Spa} A_{\mathrm{inf}}$, where $A_{\mathrm{inf}} = W(\mathcal{O}_{C^\flat})$. The two closed subspaces $p = 0$ and $[p^\flat] = 0$ appear as the x-axis and y-axis, respectively. We have also depicted the closed subspace $p = [p^\flat]$, which cuts out $\operatorname{Spa} \mathcal{O}_C$, as a green ellipse. The unique non-analytic point x_k of $\operatorname{Spa} A_{\mathrm{inf}}$ appears at the origin. Its complement in $\operatorname{Spa} A_{\mathrm{inf}}$ is the adic space $\mathcal{Y}_{[0,\infty]}$, on which the continuous map $\kappa \colon \mathcal{Y}_{[0,\infty]} \to [0, \infty]$ is defined. The automorphism ϕ of $\operatorname{Spa} A_{\mathrm{inf}}$ (not shown) tends to rotate points towards the y-axis (though it fixes both axes).

The Frobenius automorphism of $\mathcal{O}_{C^\flat}$ induces an automorphism ϕ of $\operatorname{Spa} W(\mathcal{O}_{C^\flat})$, which preserves $\mathcal{Y}$ and which satisfies $\kappa \circ \phi = p\kappa$. Therefore ϕ sends $\mathcal{Y}_{[a,b]}$ to $\mathcal{Y}_{[ap,bp]}$.

We will outline the construction of "?" in three steps:

1. Give a linear algebra description of the category of shtukas over $\operatorname{Spa} C^\flat$ with one paw at C, in terms of modules over the de Rham period ring B_{dR}.
2. Show that any shtuka over $\mathcal{Y}_{[0,\infty)}$ extends to all of $\mathcal{Y}$ (*i.e.*, it extends over x_L).
3. Show that any shtuka over $\mathcal{Y}$ extends to all of $\operatorname{Spa} W(\mathcal{O}_{C^\flat})$ (*i.e.*, it extends over x_k).

The remainder of the lecture concerns Step (1).

12.3 Shtukas with no paws, and ϕ-modules over the integral Robba ring

Definition 12.3.1 (The integral Robba rings)**.** Let $\widetilde{\mathcal{R}}^{\mathrm{int}}$ be the local ring $\mathcal{O}_{\mathcal{Y},x_{C^\flat}}$. For a rational number $r > 0$, we define $\widetilde{\mathcal{R}}^{\mathrm{int},r}$ to be the ring of global sections of $\mathcal{O}_{\mathcal{Y}_{[0,r]}}$.

Unwinding the definition of κ, we see that $\mathcal{Y}_{[0,r]}$ is the rational subset of $\operatorname{Spa} W(\mathcal{O}_{C^\flat})$ cut out by the conditions $|p(x)|^r \leq |[\varpi](x)| \neq 0$. Thus $\widetilde{\mathcal{R}}^{\mathrm{int},r} = W(\mathcal{O}_{C^\flat})\langle p/[p^\flat]^{1/r}\rangle$ is the p-adic completion of $W(\mathcal{O}_{C^\flat})[p/[(p^\flat)^{1/r}]]$, where $(p^\flat)^{1/r} \in \mathcal{O}_{C^\flat}$ is any rth root of $p^\flat$. Thus $\widetilde{\mathcal{R}}^{\mathrm{int},r}$ can be described as

$$\widetilde{R}^{\mathrm{int},r} = \left\{ \sum_{n\geq 0} [c_n]p^n \;\middle|\; c_n \in C^\flat,\; c_n (p^\flat)^{n/r} \to 0 \right\}.$$

For $r' < r$, the inclusion of rational subsets $\mathcal{Y}_{[0,r']} \to \mathcal{Y}_{[0,r]}$ allows us to view $\widetilde{R}^{\mathrm{int},r}$ as a subring of $\widetilde{R}^{\mathrm{int},r'}$. Finally,

$$\widetilde{R}^{\mathrm{int}} = \varinjlim \widetilde{R}^{\mathrm{int},r} \text{ as } r \to 0.$$

Remark 12.3.2.

1. $\widetilde{\mathcal{R}}^{\mathrm{int}}$ is a henselian discrete valuation ring with uniformizer p, residue field $C^\flat$, and completion equal to $W(C^\flat)$.
2. The Frobenius automorphism of $\mathcal{O}_{C^\flat}$ induces isomorphisms $\phi\colon \widetilde{\mathcal{R}}^{\mathrm{int},r} \to \widetilde{\mathcal{R}}^{\mathrm{int},r/p}$ for $r > 0$.

Definition 12.3.3. Let R be a ring together with an automorphism $\phi\colon R \to R$. A *ϕ-module over R* is a finite projective R-module M with a ϕ-semilinear isomorphism $\phi_M\colon M \to M$.

The next theorem states that ϕ-modules over the rings $\widetilde{\mathcal{R}}^{\mathrm{int}}$ and $W(C^\flat)$ are trivial in the sense that one can always find a ϕ-invariant basis.

Theorem 12.3.4 ([KL, Thm. 8.5.3])**.** *The following categories are equivalent:*

1. $\left\{\phi\text{-modules over } \widetilde{\mathcal{R}}^{\mathrm{int}}\right\}$
2. $\left\{\phi\text{-modules over } W(C^\flat)\right\}$
3. $\{\text{finite free } \mathbf{Z}_p\text{-modules}\}$

The functor from (1) to (2) is base extension, and the functor from (2) to (3) is the operation of taking ϕ-invariants.

Proof. The equivalence between (1) and (2) is equivalent to the statement that if M is a ϕ-module over $\widetilde{\mathcal{R}}^{\mathrm{int}}$, then the map

$$M^{\phi=1} \to (M \otimes_{\widetilde{\mathcal{R}}^{\mathrm{int}}} W(C^\flat))^{\phi=1}$$

is an isomorphism. This can be checked by consideration of Newton polygons.

The equivalence between (2) and (3) is a special case of the following fact. Let R be a perfect ring. Then ϕ-modules over $W(R)$ are equivalent to $\mathbf{Z}_p$-local systems on $\operatorname{Spec} R$. This is sometimes called Artin-Shreier-Witt theory (the case of ϕ-modules over R and $\mathbf{F}_p$-local systems being due to Artin-Shreier). □

Proposition 12.3.5. *The following categories are equivalent:*

1. *Shtukas over* $\operatorname{Spa} C^\flat$ *with no paws, and*
2. *ϕ-modules over $\widetilde{\mathcal{R}}^{\mathrm{int}}$ (in turn equivalent to finite free $\mathbf{Z}_p$-modules by Thm. 12.3.4).*

Proof. A shtuka over $\operatorname{Spa} C^\flat$ with no paws is a vector bundle $\mathcal{E}$ on $\mathcal{Y}_{[0,\infty)}$ together with an isomorphism $\phi_{\mathcal{E}}\colon \phi^*\mathcal{E} \to \mathcal{E}$. The localization of $\mathcal{E}$ at $x_{C^\flat}$ is a ϕ-module over $\mathcal{O}_{\mathcal{Y},x_{C^\flat}} = \widetilde{\mathcal{R}}^{\mathrm{int}}$.

Going in the other direction, suppose (M, ϕ_M) is a ϕ-module over $\widetilde{\mathcal{R}}^{\mathrm{int}}$. Since $\widetilde{\mathcal{R}}^{\mathrm{int}} = \varinjlim \widetilde{\mathcal{R}}^{\mathrm{int},r}$, the category of finitely presented modules over $\widetilde{\mathcal{R}}^{\mathrm{int}}$

is the colimit of the directed system of categories of finitely presented modules over the $\widetilde{\mathcal{R}}^{\mathrm{int},r}$. This means that we can descend (M, ϕ_M) to a pair (M_r, ϕ_{M_r}), where M_r is a finitely presented module over $\widetilde{\mathcal{R}}^{\mathrm{int},r}$, together with an isomorphism $\phi_{M_r}\colon \phi^* M_r \to M_r \otimes_{\widetilde{\mathcal{R}}^{\mathrm{int},r}} \widetilde{\mathcal{R}}^{\mathrm{int},r/p}$.

In terms of vector bundles, we have a pair $(\mathcal{E}_r, \phi_{\mathcal{E}_r})$, where $\mathcal{E}_r$ is a vector bundle over $\mathcal{Y}_{[0,r]}$ together with an isomorphism $\phi_{\mathcal{E}_r}\colon \phi^*\mathcal{E}_r \to \mathcal{E}_r|_{\mathcal{Y}_{[0,r/p]}}$. We can now pull back through powers of ϕ^{-1} to construct a shtuka $(\mathcal{E}, \phi_{\mathcal{E}})$ over all of $\mathcal{Y}_{[0,\infty)}$. For $n \geq 1$, the pullback $(\phi^{-n})^*\mathcal{E}_r$ is a vector bundle on $\mathcal{Y}_{[0,p^n r]}$. These can be glued together using ϕ_{E_r}. To wit, $(\phi^{-n})^*\phi_{E_r}$ is an isomorphism $(\phi^{1-n})^*\mathcal{E}_r \to (\phi^{-n})^*\mathcal{E}_r|_{\mathcal{Y}_{[0,p^{n-1}r]}}$. Let $\mathcal{E}$ be the result of this gluing, a vector bundle on all of $\mathcal{Y}_{[0,\infty)}$.

We still need to construct the required isomorphism $\phi_{\mathcal{E}}\colon \phi^*\mathcal{E} \to \mathcal{E}$. Let us construct its restriction to $\mathcal{Y}_{[0,p^n r]}$: this is the composition

$$(\phi^*\mathcal{E})|_{\mathcal{Y}_{[0,p^n r]}} = \phi^*(\mathcal{E}|_{\mathcal{Y}_{[0,p^{n+1}r]}}) \cong \phi^*(\phi^{-n-1})^*\mathcal{E}_r = (\phi^{-n})^*\mathcal{E}_r \cong \mathcal{E}|_{\mathcal{Y}_{[0,p^n r]}}.$$

We leave it to the reader to check that these isomorphisms are compatible as n changes. □

12.4 Shtukas with one paw, and B_{dR}-modules

We now turn to the category of shtukas over $\operatorname{Spa} C^\flat$ with one paw at C. These are vector bundles $\mathcal{E}$ on $\mathcal{Y}_{[0,\infty)}$ together with an isomorphism $\phi_{\mathcal{E}}\colon \phi^*\mathcal{E} \to \mathcal{E}$ away from x_C. Our analysis will involve the completed local ring of $\mathcal{Y}$ at x_C, which is none other than the de Rham period ring B_{dR}^+.

Definition 12.4.1 (The de Rham period ring)**.** Let $B_{\mathrm{dR}}^+ = \widehat{\mathcal{O}}_{\mathcal{Y},x_C}$. That is, B_{dR} is the ξ-adic completion of $W(\mathcal{O}_{C^\flat})[1/p]$. It is a complete discrete valuation ring with residue field C and uniformizer ξ. The map $\theta\colon W(\mathcal{O}_{C^\flat}) \to \mathcal{O}_C$ extends to a map $B_{\mathrm{dR}} \to C$ which we continue to call θ. Let $B_{\mathrm{dR}} = B_{\mathrm{dR}}^+[\xi^{-1}]$ be the fraction field of B_{dR}^+.

Remark 12.4.2. The automorphism ϕ of $\mathcal{Y}$ allows us to identify $\widehat{\mathcal{O}}_{\mathcal{Y},\phi^n(x_C)}$ with B_{dR}^+ for any $n \in \mathbf{Z}$, and it will be convenient for us to do so. Thus if $\mathcal{E}$ is a vector bundle on an open subset of $\mathcal{Y}$ containing $\phi^n(x_C)$, its completed stalk $\widehat{\mathcal{E}}_{\phi^n(x_C)}$ is a B_{dR}^+-module.

Remark 12.4.3. The assumption that $\phi_{\mathcal{E}}$ is meromorphic at x_C means that it induces an isomorphism on stalks at C after inverting the parameter ξ. That is, we have an isomorphism $\phi_{\mathcal{E},x_C}\colon \widehat{\mathcal{E}}_{\phi(x_C)} \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}} \to \widehat{\mathcal{E}}_{x_C} \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}}$. Thus $\phi_{\mathcal{E},x_C}$ is an isomorphism between two B_{dR}-vector spaces which each contain a distinguished B_{dR}^+-lattice. The following theorem shows that shtukas can be classified in terms of the relative position of these two lattices.

Theorem 12.4.4 (Fargues)**.** *There is an equivalence between the category of shtukas over* $\mathrm{Spa}\, C^\flat$ *with one paw at* C *and the category of pairs* (T, Ξ)*, where* T *is a finite free* $\mathbf{Z}_p$*-module and* $\Xi \subset T \otimes_{\mathbf{Z}_p} B_{\mathrm{dR}}$ *is a* B_{dR}^+*-lattice.*

Remark 12.4.5. There is a fully faithful functor $\{(T, W)\} \to \{(T, \Xi)\}$ where $\Xi = \left\{x \in T \otimes_{\mathbf{Z}_p} B_{\mathrm{dR}}^+ | x \pmod{\xi} \in W\right\}$.

Proof. Let $(\mathcal{E}, \phi_{\mathcal{E}})$ be a shtuka over $\mathrm{Spa}\, C^\flat$ with one paw at C. The stalk $\mathcal{E}_{x_{C^\flat}}$ is a ϕ-module over $\widetilde{\mathcal{R}}^{\mathrm{int}}$. Applying Prop. 12.3.5, this stalk corresponds to a shtuka $(\mathcal{E}_0, \phi_{\mathcal{E}_0})$ with no paw. With this correspondence comes an isomorphism of ϕ-modules $\mathcal{E}_{0,x_{C^\flat}} \cong \mathcal{E}_{x_{C^\flat}}$ over $\widetilde{\mathcal{R}}^{\mathrm{int}} = \mathcal{O}_{\mathcal{Y}, x_{C^\flat}}$. This isomorphism descends to an isomorphism of vector bundles $\iota\colon \mathcal{E}_0 \cong \mathcal{E}$ over $\mathcal{Y}_{[0,r]}$ for some $0 < r < 1$, which is compatible with the ϕ-module structures in the sense that the diagram of vector bundles

$$\begin{array}{ccc} (\phi^*\mathcal{E}_0) & \xrightarrow{\phi^*(\iota)} & (\phi^* E) \\ \downarrow{\scriptstyle \phi_{\mathcal{E}_0}} & & \downarrow{\scriptstyle \phi_{\mathcal{E}}} \\ \mathcal{E}_0 & \xrightarrow[\iota]{} & \mathcal{E} \end{array}$$

commutes; here all morphisms are only defined over $\mathcal{Y}_{[0,r]}$. (The condition $r < 1$ is necessary so that $\mathcal{Y}_{[0,r]}$ avoids the paw x_C.) We now proceed as in the proof of Thm. 12.3.5 to extend ι over a larger domain. Since $(\phi^*\mathcal{E}_0)|_{\mathcal{Y}_{[0,r]}} = \phi^*(\mathcal{E}_0|_{\mathcal{Y}_{[0,pr]}})$, and similarly for $\mathcal{E}$, we can apply $(\phi^{-1})^*$ to obtain an isomorphism $\mathcal{E}_0 \cong \mathcal{E}$ over $\mathcal{Y}_{[0,pr]} \backslash \{x_C\}$ which extends ι. This process can be repeated to extend ι over $\mathcal{Y}_{(0,\infty)} \backslash \{x_C, \phi(x_C), \dots\}$, so that the above diagram commutes for any $r > 0$. At x_C itself, one only has an isomorphism $\widehat{\mathcal{E}}_{0,x}[1/\xi] \tilde{\to} \widehat{\mathcal{E}}_x[1/\xi]$.

As discussed in Rmk. 12.4.3 we have an isomorphism of B_{dR}-vector spaces $\phi_{\mathcal{E},x_C}\colon \widehat{\mathcal{E}}_{\phi(x_C)}[1/\xi] \to \widehat{\mathcal{E}}_{x_C}[1/\xi]$.

Now we apply Thm. 12.3.4 to trivialize the shtuka $\mathcal{E}_0$ with no paw. The completion $\widehat{\mathcal{E}}_{0,x_{C^\flat}}$ is a ϕ-module over $W(\mathcal{O}_{C^\flat})$. Let $T = (\widehat{\mathcal{E}}_{0,x_{C^\flat}})^{\phi=1}$. Then $\mathcal{E}_0 = T \otimes_{\mathbf{Z}_p} \mathcal{O}_{\mathcal{Y}_{[0,\infty)}}$.

Finally we can define the lattice $\Xi \subset T \otimes_{\mathbf{Z}_p} B_{\mathrm{dR}}$: it is the image of $\widehat{\mathcal{E}}_{x_C}$ under the isomorphism

$$\widehat{\mathcal{E}}_{x_C}[1/\xi] \xrightarrow{\phi_{\mathcal{E},x_C}} \widehat{\mathcal{E}}_{0,x_C}[1/\xi] \longrightarrow T \otimes_{\mathbf{Z}_p} B_{\mathrm{dR}}.$$

Thus we have defined a functor from shtukas with one paw to $\{(T, \Xi)\}$. To go in the other direction, suppose we are given (T, Ξ). Define $(\mathcal{E}_0, \phi_{\mathcal{E}_0}) = T \otimes_{\mathbf{Z}_p} \mathcal{O}_{\mathcal{Y}_{[0,\infty)}}$. The lattice Ξ lies in $\widehat{\mathcal{E}}_{0,x_C} \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}}$, and so

$$\phi^{-n}(\Xi) \subset \widehat{\mathcal{E}}_{0,\phi^n(x_C)} \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}}$$

is a B_{dR}^+-lattice, $n \geq 1$.

Using the lattices $\phi^{-n}(\Xi)$, we modify $\mathcal{E}_0$ at all $\phi^n(x_C)$, $n \geq 1$. We get a new vector bundle $\mathcal{E}$ over $\mathcal{Y}_{(0,\infty]}$ together with an isomorphism $\phi_{\mathcal{E}} \colon \phi^*\mathcal{E} \to \mathcal{E}$ away from x_C, as required. □

12.5 Description of the functor "?"

This conludes Step (1) of the program outlined in §12.2. The goal of the remaining steps is to show that shtukas over $\mathcal{Y}_{[0,\infty)}$ can be extended to all of $\operatorname{Spa} W(\mathcal{O}_C)$, where they become what we have called Breuil-Kisin modules. Then we will have the following theorem:

Theorem 12.5.1. *There is an equivalence of categories between Breuil-Kisin modules over $W(\mathcal{O}_{C^\flat})$ and pairs (T, Ξ).*

Recall that an *isocrystal over k* is a finite-dimensional vector space over $L = W(k)[1/p]$ equipped with a σ-semilinear automorphism F. Note that the isogeny category of Dieudonné modules is a full subcategory of the category of isocrystals, consisting of those objects where the slopes of F lie in $[0, 1]$. We have a functor

$$\{\text{Breuil-Kisin modules over } W(\mathcal{O}_{C^\flat})\} \to \{\text{isocrystals over } k\}$$

which sends M to $M \otimes_{W(\mathcal{O}_{C^\flat})} L$. This corresponds to the operation of taking the fiber of the shtuka over x_L. Note that since the image of $\xi = p - [p^\flat]$ under $W(\mathcal{O}_{C^\flat}) \to W(k)[1/p]$ is invertible, ϕ_M really does induce an automorphism of $M \otimes_{W(\mathcal{O}_{C^\flat})} L$.

Finally we can give a description of the functor "?" from $\{(T, W)\}$ to

Dieudonné modules, as in the following diagram:

$$\begin{array}{ccc}
\{p\text{-div. gps. over } \mathcal{O}_C\} & \longrightarrow & \{(T,W)\} \\
\Big\downarrow & & \Big\downarrow \\
 & & \{(T,\Xi)\} \\
 & & \Big\downarrow \wr \\
 & & \{\text{BK modules over } W(\mathcal{O}_{C^\flat})\} \\
\Big\downarrow & & \Big\downarrow \\
\{p\text{-div. gps. over } k\} & \longrightarrow & \{\text{isocrystals over } k\}
\end{array}$$

Proposition 12.5.2 (Fargues)**.** *This diagram commutes*[11].

13 Shtukas with one paw II, 21 October

Today we discuss Step 2 of the plan laid out in Lecture 11. We will show that a shtuka over $\operatorname{Spa} C^\flat$, which is a ϕ-module over $\mathcal{Y}_{[0,\infty)}$, actually extends to $\mathcal{Y}_{[0,\infty]}$. In doing so we will encounter the theory of ϕ-modules over the Robba ring, due to Kedlaya. These are in correspondence with vector bundles over the *Fargues-Fontaine curve*, [FF11].

13.1 $\mathcal{Y}$ is honest

As in the previous lecture, let $C/\mathbf{Q}_p$ be an algebraically closed nonarchimedean field, with tilt $C^\flat/\mathbf{F}_p$ and residue field k. Let $A_{\text{inf}} = W(\mathcal{O}_{C^\flat})$, with its $(p, [p^\flat])$-adic topology. We had set $\mathcal{Y} = \mathcal{Y}_{[0,\infty]} = (\operatorname{Spa} A_{\text{inf}}) \backslash \{x_k\}$, an analytic adic space.

Proposition 13.1.1. *$\mathcal{Y}$ is an honest adic space.*

Proof. From Prop. 11.3.2 applied to $S = \operatorname{Spa} C^\flat$ we know that "$S \times \operatorname{Spa} \mathbf{Z}_p$" $= \mathcal{Y}_{[0,\infty)}$ is honest, by exhibiting a covering by rational subsets $\operatorname{Spa}(R, R^+)$, where R is preperfectoid. We apply a similar strategy to the rational subsets $\mathcal{Y}_{[r,\infty]}$ for $r > 0$.

For $r > 0$ rational we have $\mathcal{Y}_{[r,\infty]} = \operatorname{Spa}(R_r, R_r^+)$, where R_r is the ring $W(\mathcal{O}_{C^\flat})\langle (p^\flat)^r/p \rangle[1/p]$. Here $W(\mathcal{O}_{C^\flat})\langle (p^\flat)^r/p \rangle$ is the completion of $W(\mathcal{O}_{C^\flat})[(p^\flat)^r]/p]$ with respect to the $([p^\flat], p)$-adic topology, but since p divides $[(p^\flat)^r]$ in this ring, the topology is just p-adic.

[11] Fargues denies responsibility for this if $p = 2$.

Let A'_{inf} be the ring $W(\mathcal{O}_{C^\flat})$ equipped with the p-adic topology (rather than the $(p,[p^\flat])$-adic topology, as we have defined A_{inf}). By the observation above, the morphism of adic spaces $\operatorname{Spa} A_{\text{inf}} \to \operatorname{Spa} A'_{\text{inf}}$ induces an isomorphism between $\mathcal{Y}_{[r,\infty]}$ and the rational subset of $\operatorname{Spa} A'_{\text{inf}}$ defined by $\left|[p^\flat](x)\right|^r \leq |p(x)| \neq 0$. Therefore to prove the proposition it is enough to show that the rational subset $\{|p(x)| \neq 0\}$ of $\operatorname{Spa} A'_{\text{inf}}$ is honest. This rational subset is $\operatorname{Spa}(A'_{\text{inf}}[1/p], A'_{\text{inf}})$.

We claim $A'_{\text{inf}}[1/p]$ is preperfectoid. Indeed, if $R = A'_{\text{inf}}[1/p]\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{Z}_p[p^{1/p^\infty}]$, then R is a Tate ring with pseudo-uniformizer $p^{1/p}$. Its subring of power-bounded elements is $R^\circ = A'_{\text{inf}}\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{Z}_p[p^{1/p^\infty}]$. Observe that Φ is surjective on $R^\circ/p = \mathcal{O}_{C^\flat} \otimes_{\mathbf{F}_p} \mathbf{F}_p[x^{1/p^\infty}]/x = \mathcal{O}_{C^\flat}[x^{1/p^\infty}]/x$. Thus R is perfectoid. By Prop. 6.3.3, $(A'_{\text{inf}}, A'_{\text{inf}}[1/p])$ is sheafy.

□

Remark 13.1.2. The same proof shows that $\operatorname{Spa} W(R)[1/p]$ is an honest adic space, where R is any (discrete) perfect ring and $W(R)$ has the p-adic topology.

13.2 The extension of shtukas over x_L

The main theorem of this lecture concerns the extension of ϕ-modules from $\mathcal{Y}_{[r,\infty)}$ to $\mathcal{Y}_{[r,\infty]}$, where $0 \leq r < \infty$.

Definition 13.2.1. Let $I \subset [0,\infty]$ be an interval of the form $[r,\infty)$ or $[r,\infty]$, so that $I \subset p^{-1}I$. A *ϕ-module over $\mathcal{Y}_I$* is a pair $(\mathcal{E},\phi_{\mathcal{E}})$, where $\mathcal{E}$ is a vector bundle over $\mathcal{Y}_I$, and $\phi_{\mathcal{E}} \colon \phi^*\mathcal{E}|_{\mathcal{Y}_I} \to \mathcal{Y}_I$ is an isomorphism. (Note that $\phi^*\mathcal{E}$ is a vector bundle over $\mathcal{Y}_{p^{-1}I} \supset \mathcal{Y}_I$.)

Theorem 13.2.2. *For $0 \leq r < \infty$, the restriction functor from ϕ-modules over $\mathcal{Y}_{[r,\infty]}$ to ϕ-modules over $\mathcal{Y}_{[r,\infty)}$ is an equivalence.*

Remark 13.2.3. In particular, suppose $(\mathcal{E},\phi_{\mathcal{E}})$ is a shtuka over $\operatorname{Spa} C^\flat$ with paws $x_1,\dots,x_n$. Thus $\mathcal{E}$ is a vector bundle over $\mathcal{Y}_{[0,\infty)}$ and $\phi_E \colon \phi^*\mathcal{E} \to \mathcal{E}$ is an isomorphism away from the Γ_{x_i}. Suppose $r > 0$ be greater than $\kappa(\Gamma_{x_i})$ for $i = 1,\dots,n$, so that $\mathcal{E}|_{\mathcal{Y}_{[r,\infty)}}$ is a ϕ-module over $\mathcal{Y}_{[r,\infty)}$. By the theorem, $\mathcal{E}|_{\mathcal{Y}_{[r,\infty)}}$ extends uniquely to a ϕ module over $\mathcal{Y}_{[r,\infty]}$. This can be glued to the given shtuka to obtain a vector bundle $\widehat{\mathcal{E}}$ together with an isomorphism $\phi_{\widehat{\mathcal{E}}} \colon \phi^*\widehat{\mathcal{E}} \to \widehat{\mathcal{E}}$ on $\mathcal{Y}\backslash \bigcup_i \Gamma_{x_i}$.

We only offer some ideas of the proof below.

13.3 Full faithfullness

We now sketch a proof that the functor described in Thm. 13.2.2 is fully faithful.

Suppose I is an interval of the form $[r,\infty)$ or $[r,\infty]$ with $r > 0$. If $(\mathcal{E},\phi_{\mathcal{E}})$ is a ϕ-module over $\mathcal{Y}_I$, let $H^0(\mathcal{Y}_I,\mathcal{E})^{\phi=1}$ denote the space of sections $s \in H^0(\mathcal{Y}_I,\mathcal{E})$ for which $\phi_{\mathcal{E}}(\phi^*(s)|_{\mathcal{Y}_I}) = s$.

Lemma 13.3.1. *Let $r' > r$, and let $I' = I \cap [r',\infty]$. The restriction map $H^0(\mathcal{Y}_I,\mathcal{E})^{\phi=1} \to H^0(\mathcal{Y}_{I'},\mathcal{E})^{\phi=1}$ is an isomorphism.*

Proof. The inverse map is as follows: if $s \in H^0(\mathcal{Y}_{I'},\mathcal{E})^{\phi=1}$, then $s_1 = \phi_{\mathcal{E}}(\phi^*(s)|_{\mathcal{Y}_I})$ is a section of $\mathcal{E}$ over $\mathcal{Y}_{p^{-1}I'\cap I}$ which agrees with s on $\mathcal{Y}_{I'}$. Inductively define $s_n \in H^0(\mathcal{Y}_{p^{-n}I'\cap I},\mathcal{E})$ by $s_n = \phi_{\mathcal{E}}(\phi^*(s_{n-1}))$. Then for n large enough so that $I \subset p^{-n}I'$, $s_n \in H^0(\mathcal{Y}_I,\mathcal{E})^{\phi=1}$ extends s. □

For full faithfulness of the restriction functor in Thm. 13.2.2, the key point is the following proposition.

Proposition 13.3.2. *Let $r > 0$, and let $\mathcal{E}$ be a ϕ-module over $\mathcal{Y}_{[r,\infty]}$. Restriction induces an isomorphism*

$$H^0(\mathcal{Y}_{[r,\infty]},\mathcal{E})^{\phi=1} \xrightarrow{\sim} H^0(\mathcal{Y}_{[r,\infty)},\mathcal{E})^{\phi=1}.$$

Proof. We may reduce to the case that $\mathcal{E} = \mathcal{O}^n_{\mathcal{Y}_{[r,\infty]}}$ is free. Indeed, $\mathcal{E}$ is free over the local ring $\mathcal{O}_{\mathcal{Y},x_L}$, and so it is free over $\mathcal{Y}_{[r',\infty]}$ for r' large enough; by Lemma 13.3.1 we may replace r by r' in the statement of the proposition.

Assume for simplicity $n = 1$, so that $\mathcal{E} = H^0(\mathcal{Y}_{[r,\infty]},\mathcal{O}_{\mathcal{Y}})$. The image of $1 \in H^0(\mathcal{Y}_{[r,\infty]},\mathcal{O}_{\mathcal{Y}})$ under $\phi_{\mathcal{E}}$ is an element of $H^0(\mathcal{Y}_{[r,\infty]},\mathcal{O}_Y)^\times$, call it A^{-1}. An element of $H^0(\mathcal{Y}_{[r,\infty)},\mathcal{E})^{\phi=1}$ corresponds to an element $f \in H^0(\mathcal{Y}_{[r,\infty)},\mathcal{O}_Y)$ such that $Af = \phi(f)$; our task is to show that any such f extends over $\mathcal{Y}_{[r,\infty]}$.

Consider the *Newton polygon* of f: if

$$f = \sum_{i\in\mathbf{Z}} [a_i]p^i,\ a_i \in C^\flat,$$

let $\mathrm{Newt}(f)$ be the convex hull of the polygon in $\mathbf{R}^2$ joining the points $(\mathrm{val}(a_i), i)$ for $i \in \mathbf{Z}$. Here val is a valuation on $C^\flat$, written additively. Then $\mathrm{Newt}(f)$ is independent of the expression of f as a series (which may not be unique). We have that f extends to $\mathcal{Y}_{[r,\infty]}$ if and only if $\mathrm{Newt}(f)$ lies on the right of the y-axis. (This takes some argument, see [FF11].) We have

$$\mathrm{Newt}(\phi(f)) = \mathrm{Newt}(Af) \geq \mathrm{Newt}(A) + \mathrm{Newt}(f).$$

Since A is a section over $\mathcal{Y}_{[r,\infty]}$, $\mathrm{Newt}(A)$ lies on the right of the vertical axis. Also $\mathrm{Newt}(\phi(f))$ is $\mathrm{Newt}(f)$ but scaled by p in the val-axis (which we are drawing as the x-axis). If $\mathrm{Newt}(f)$ goes to the left of the y-axis, then $\mathrm{Newt}(\phi(f))$ goes further to the left, contradiction. □

We can now show that the restriction functor in Thm. 13.2.2 is fully faithful. Suppose $\mathcal{E}$ and $\mathcal{E}'$ are two ϕ-modules over $\mathcal{Y}_{[r,\infty]}$ and $f\colon \mathcal{E} \to \mathcal{E}'$ is a morphism of ϕ-modules defined over $\mathcal{Y}_{[r,\infty)}$. Then f determines a ϕ-invariant global section s of the vector bundle $\mathcal{E} = \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)$ ("internal hom"), which is also a ϕ-module over $\mathcal{Y}_{[r,\infty]}$. Now we apply Prop. 13.3.2 to show that s (and therefore f) extends uniquely over $\mathcal{Y}_{[r,\infty]}$.

13.4 Essential surjectivity

For essential surjectivity in Thm. 13.2.2, the strategy is to classify all ϕ-modules over $\mathcal{Y}_{[r,\infty)}$ and show by inspection that each one extends over $\mathcal{Y}_{[r,\infty]}$.

Recall that $L = W(k)[1/p]$. If we choose an embedding $k \hookrightarrow \mathcal{O}_{C^{flat}}$ which reduces to the identity modulo the maximal ideal of $\mathcal{O}_{C^\flat}$, we obtain an embedding $L \hookrightarrow A_{\mathrm{inf}}[1/p]$.

Theorem 13.4.1 ([Ked04]). *Let $(\mathcal{E}, \phi_{\mathcal{E}})$ be a ϕ-module over $\mathcal{Y}_{[r,\infty)}$. Then there exists a ϕ-module (M, ϕ_M) over L such that $(\mathcal{E}, \phi_E) \cong (M, \phi_M) \otimes_L \mathcal{O}_{\mathcal{Y}_{[r,\infty)}}$. (Here we have fixed an embedding $L \to A_{\mathrm{inf}}[1/p]$.)*

Remark 13.4.2.

1. As a result, the ϕ-module $\widehat{\mathcal{E}} = M \otimes_L \mathcal{O}_{\mathcal{Y}_{[r,\infty]}}$ extends $\mathcal{E}$ to $\mathcal{Y}_{[r,\infty]}$.
2. ϕ-modules over L are by definition the same as isocrystals over k. The category of isocrystals over k admits a *Dieudonné-Manin classification*: it is semisimple, with simple objects M_λ classified by $\lambda \in \mathbf{Q}$. For a rational number $\lambda = d/h$ written in lowest terms with $h > 0$, the rank of M_λ is h, and ϕ_{M_λ} can be expressed in matrix form as

$$\phi_{M_\lambda} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ p^d & & & 0 & 1 \end{pmatrix}.$$

The usual "Frobenius pullback" trick shows that ϕ-modules over $\mathcal{Y}_{[r,\infty)}$ for $r > 0$ can be extended arbitrarily close to 0 in the sense that the restriction map

$$\{\phi\text{-modules over } \mathcal{Y}_{(0,\infty)}\} \xrightarrow{\sim} \{\phi\text{-modules over } \mathcal{Y}_{[r,\infty)}\}$$

is an equivalence. The category on the left is equivalent to the colimit as $r \to 0$ of the category of ϕ-modules over $\mathcal{Y}_{(0,r]}$.

Definition 13.4.3 (The extended Robba rings, [KL, Defn. 4.2.2])**.** Let $\widetilde{\mathcal{R}}^r = H^0(\mathcal{Y}_{(0,r]}, \mathcal{O}_{\mathcal{Y}})$, and let $\widetilde{\mathcal{R}} = \varinjlim \widetilde{\mathcal{R}}^r$.

Thus $\widetilde{\mathcal{R}}$ is the ring of functions defined on some punctured disc of small (and unspecified) radius around $x_{C^\flat}$. Note that ϕ induces an automorphism of $\widetilde{\mathcal{R}}$ (but not of any $\widetilde{\mathcal{R}}^r$. A similar Frobenius pullback trick shows that the category of ϕ-modules over $\mathcal{Y}_{(0,\infty)}$ is equivalent to the category of ϕ-modules over $\widetilde{\mathcal{R}}$. It is shown in [Ked05, §6.3,6.4] that this latter category admits a Dieudonné-Manin classification.

13.5 The Fargues-Fontaine curve

As ϕ acts properly discontinuously on $\mathcal{Y}_{(0,\infty)}$, it makes sense to form the quotient.

Definition 13.5.1. The *adic Fargues-Fontaine curve* is the quotient $\mathcal{X}_{\mathrm{FF}} = \mathcal{Y}_{(0,\infty)}/\phi^{\mathbf{Z}}$.

Theorem 13.5.2 ([Ked, Thm. 4.10])**.** *$\mathcal{Y}_{(0,\infty)}$ is strongly noetherian, and thus so is $\mathcal{X}_{FF}$.*

Now ϕ-modules over $\mathcal{Y}_{(0,\infty)}$ are visibly the same as vector bundles over $\mathcal{X}_{FF}$. $\mathcal{X}_{FF}$ comes equipped with a natural line bundle $\mathcal{O}(1)$, corresponding to the ϕ-module on $\mathcal{Y}_{(0,\infty)}$ whose underlying line bundle is trivial and for which $\phi_{\mathcal{O}(1)}$ is $p^{-1}\phi$. Let $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$, and let

$$P = \bigoplus_{n \geq 0} H^0(\mathcal{X}_{FF}, \mathcal{O}(n)),$$

a graded ring. The nth graded piece is $H^0(\mathcal{Y}_{(0,\infty)}, \mathcal{O}_{\mathcal{Y}})^{\phi = p^n}$. Note that by Lemma 13.3.1, this is the same as $H^0(\mathcal{Y}_{[r,\infty)})^{\phi=p^n}$ for any $r > 0$, and by Prop. 13.2.2 this is in turn the same as $H^0(\mathcal{Y}_{[r,\infty]}, \mathcal{O}_{\mathcal{Y}})^{\phi=p^n}$.

These spaces can be reformulated in terms of the crystalline period rings of Fontaine. Let A_{cris} be the p-adic completion of the divided power envelope of the surjection $A_{\mathrm{inf}} \to \mathcal{O}_C$. By a general fact about divided power envelopes for principal ideals in flat $\mathbf{Z}$-algebras, this divided power envelope is the same as the A_{inf}-subalgebra of $A_{\mathrm{inf}}[1/p]$ generated by $\xi^n/n!$ for $n \geq 1$. Let $B^+_{\mathrm{cris}} = A_{\mathrm{cris}}[1/p]$. An element of B^+_{cris} may be written

$$\sum_{n \geq 1} a_n \frac{\xi^n}{n!},\ a_n \in W(\mathcal{O}_{C^\flat})[1/p],\ a_n \to 0\ p\text{-adically}.$$

Using the estimate $n/(p-1)$ for the p-adic valuation of $n!$, one can show that such series converge in $\widetilde{B}^{[r,\infty]} = H^0(\mathcal{Y}_{[r,\infty]}, \mathcal{O}_{\mathcal{Y}})$. Thus we have an embedding $B^+_{\mathrm{cris}} \subset \widetilde{B}^{[r,\infty]}$; it can be shown that $B^{+,\phi=p^n}_{\mathrm{cris}} = (\widetilde{B}^{[r,\infty]})^{\phi=p^n}$.

The full crystalline period ring B_{cris} is defined by inverting the element $t \in B^+_{\mathrm{cris}}$, where $t = \log[\varepsilon]$, $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots)$. We have $\phi(t) = \log[\varepsilon^p] = \log[\varepsilon]^p = pt$, so that t is a section of $\mathcal{O}(1)$. Also, $\theta(t) = 0$, so that t has a zero on $x_C \in \mathcal{X}_{\mathrm{FF}}$; in fact this is the only zero.

Definition 13.5.3 (The algebraic Fargues-Fontaine curve)**.** Let $X_{\mathrm{FF}} = \operatorname{Proj} P$. The map $\theta\colon B^+_{\mathrm{cris}} \to C$ determines a distinguished point $x_C \in X_{\mathrm{FF}}(C)$ with residue field C.

Theorem 13.5.4 ([FF11])**.**

1. *X_{FF} is a regular noetherian scheme of Krull dimension 1 which is locally the spectrum of a Dedekind domain*
2. *In fact, $X_{FF} \backslash \{x_C\}$ is an affine scheme* $\operatorname{Spec} B_e$*, where $B_e = B^{\phi=1}_{\mathrm{cris}}$ is a principal ideal domain!*
3. *We have $\widehat{\mathcal{O}}_{X_{FF},x_C} = B^+_{\mathrm{dR}}$. Thus vector bundles over X_{FF} correspond to "B-pairs" (M_e, M^+_{dR}) consisting of modules over B_e and B^+_{dR} respectively, with an isomorphism over B_{dR}.*
4. *The set $|X_{\mathrm{FF}}|$ of closed points of X_{FF} is identified with the set of characteristic 0 untilts of $C^\flat$ modulo Frobenius. This identification sends $x \in |X_{\mathrm{FF}}|$ to its residue field.*

Question 13.5.5. Let C' be a characteristic 0 untilt of $C^\flat$. Is $C' \cong C$? This is true if C is *spherically complete*, but open if *e.g.* $C = \mathbf{C}_p$.

Moreover, [FF11] shows that there is a Dieudonné-Manin classification for vector bundles over X_{FF}, just as in Kedlaya's theory. That is, there is a faithful and essentially surjective functor from isocrystals over k to vector bundles over X_{FF}, which sends M to the vector bundle associated to the graded P-module $\bigoplus_{n\geq 0}(B^+_{\mathrm{cris}} \otimes M^*)^{\phi=p^n}$ (here M^* is the dual isocrystal). This functor induces a bijection on the level of isomorphism classes. For $\lambda \in \mathbf{Q}$, let $\mathcal{O}(\lambda)$ be the vector bundle corresponding to the simple isocrystal M_λ; if $\lambda = d/h$ is in lowest terms with $h > 0$, then $\mathcal{O}(\lambda)$ has rank h. If $\lambda = n \in \mathbf{Z}$, this definition is consistent with how we have previously defined the line bundle $\mathcal{O}(n)$. In general, if $\mathcal{E}$ is a vector bundle corresponding to $\bigoplus_i M_{\lambda_i}$, the rational numbers λ_i are the *slopes* of $\mathcal{E}$.

The global sections of $\mathcal{O}(\lambda)$ are

$$H^0(X_{FF}, \mathcal{O}(\lambda)) = \begin{cases} \text{big}, & \lambda > 0, \\ \mathbf{Q}_p, & \lambda = 0, \\ 0, & \lambda < 0 \end{cases}$$

In the "big" case, the space of global sections is a "finite-dimensional Banach Space" (with a capital S) in the sense of Colmez, [Col02]. For example if $\lambda = 1$, we have an exact sequence

$$0 \to \mathbf{Q}_p t \to H^0(X_{\mathrm{FF}}, \mathcal{O}(1)) \to C \to 0.$$

Furthermore, $H^1(X_{FF}, \mathcal{O}(\lambda)) = 0$ if $\lambda \geq 0$, and $H^1(X_{FF}, \mathcal{O}(-1))$ is isomorphic to the quotient $C/\mathbf{Q}_p$.

Theorem 13.5.6 ([KL, Thm. 8.7.7], [Far13], "GAGA for the curve")**.** *Vector bundles over X_{FF} and vector bundles over $\mathcal{X}_{\mathrm{FF}}$ are equivalent.*

There is a map of locally ringed spaces $\mathcal{X}_{FF} \to X_{FF}$ which induces the identification $\widehat{\mathcal{O}}_{X_{\mathrm{FF}}, x_C} = B_{\mathrm{dR}}$ of Thm. 13.5.4(3), so one really does have a functor from vector bundles over X_{FF} to vector bundles over $\mathcal{X}_{FF}$. Kedlaya even proves Thm. 13.5.6 for coherent sheaves.

Theorem 13.5.7 ([FF11])**.** *$\mathcal{X}_{FF,\overline{\mathbf{Q}}_p}$ is simply connected;* i.e. *any finite étale cover is split.*

Proof. The following argument also gives a proof that $\mathbf{P}^1$ is simply connected over an algebraically closed field, which avoids using the Riemann-Hurwitz formula.

We need to show that $A \mapsto \mathcal{O}_{X_{\mathrm{FF}}} \otimes_{\mathbf{Q}_p} A$ is an equivalence from finite étale $\mathbf{Q}_p$-algebras to finite étale $\mathcal{O}_{X_{\mathrm{FF}}}$-algebras. Suppose $\mathcal{E}$ is a finite étale $\mathcal{O}_{X_{\mathrm{FF}}}$-algebra. By Thm. 13.5.4, the underlying vector bundle of $\mathcal{E}$ is isomorphic to $\bigoplus_i \mathcal{O}(\lambda_i)$ for some $\lambda_1, \ldots, \lambda_s \in \mathbf{Q}$. The étaleness provides a perfect trace pairing on $\mathcal{E}$, hence a self-duality of the underlying vector bundle, which implies that $\sum_i \lambda_i = 0$. Let $\lambda = \max \lambda_i$, so that $\lambda \geq 0$.

Assume $\lambda > 0$. The multiplication map $\mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$ restricts to a map $\mathcal{O}(\lambda) \otimes \mathcal{O}(\lambda) \to \mathcal{E}$, which gives a global section of $\mathcal{E} \otimes \mathcal{O}(-\lambda)^{\otimes 2}$. But the latter has negative slopes, implying that $H^0(X_{\mathrm{FF}}, \mathcal{E} \otimes \mathcal{O}(-2\lambda)) = 0$. It follows that $f^2 = 0$ for every $f \in H^0(X_{\mathrm{FF}}, \mathcal{O}(\lambda)) \subset H^0(X_{\mathrm{FF}}, \mathcal{E})$. But since $\mathcal{E}$ is étale over $\mathcal{O}_{X_{\mathrm{FF}}}$, its ring of global sections is reduced, so in fact $H^0(X_{\mathrm{FF}}, \mathcal{O}(\lambda)) = 0$ and $\lambda < 0$, contradiction. Therefore $\lambda = 0$, and so $\lambda_i = 0$ for all i, meaning that $\mathcal{E}$ is trivial. But the category of trivial vector bundles on X_{FF} is the equivalent to the category of finite dimensional vector spaces, given by $\mathcal{E} \to H^0(X_{\mathrm{FF}}, \mathcal{E})$. Thus $H^0(X_{\mathrm{FF}}, \mathcal{E})$ is a finite étale $\mathbf{Q}_p$-algebra, which gives us the functor in the other direction. □

14 Shtukas with one paw III, 23 October

14.1 Extending vector bundles over the closed point of $\operatorname{Spec} A_{\mathrm{inf}}$

We continue in the usual setup: Let $C/\mathbf{Q}_p$ be an algebraically closed nonarchimedean field, $C^\flat$ its tilt, k its residue field, $L = W(k)[1/p]$, and $A_{\mathrm{inf}} = W(\mathcal{O}_{C^\flat})$.

Our goal is to complete the proof of the following theorem of Fargues.

Theorem 14.1.1. *The following categories are equivalent:*

1. *Breuil-Kisin modules over A_{inf}, and*
2. *Pairs (T, Ξ), where T is a finite free $\mathbf{Z}_p$-module, and $\Xi \subset T \otimes B_{\mathrm{dR}}$ is a B_{dR}^+-lattice.*

Let $\mathcal{Y} = \mathrm{Spa}(A_{\mathrm{inf}}) \setminus \{x_k\}$. What remains to be done is to "extend" ϕ-modules on $\mathcal{Y}$ over all of $\mathrm{Spa}\, A_{\mathrm{inf}}$ in the sense of the following theorem.

Theorem 14.1.2 (Kedlaya)**.** *There is an equivalence of categories between:*

1. *Finite free A_{inf}-modules, and*
2. *Vector bundles on $\mathcal{Y}$.*

One should think of this as being an analogue of a classical result: If $(R, \mathfrak{m})$ is a 2-dimensional regular local ring, then finite free R-modules are equivalent to vector bundles on $(\operatorname{Spec} R) \setminus \{\mathfrak{m}\}$. In fact the proof we give for Thm. 14.1.2 works in that setup as well.

14.2 Vector bundles on $\operatorname{Spec}(A_{\mathrm{inf}}) \setminus \{\mathfrak{m}\}$

First we prove the algebraic version of Thm. 14.1.2.

Lemma 14.2.1. *Let $\mathfrak{m} \in \operatorname{Spec} A_{\mathrm{inf}}$ be the closed point, and let $Y = \operatorname{Spec}(A_{\mathrm{inf}}) \setminus \{\mathfrak{m}\}$. Then $\mathcal{E} \mapsto \mathcal{E}_Y$ is an equivalence between vector bundles on $\operatorname{Spec} A_{\mathrm{inf}}$ (that is, finite free A_{inf}-modules) and the category of vector bundles on Y.*

Proof. Let $R = A_{\text{inf}}$, and let

$$\begin{aligned} R_1 &= R[1/p] \\ R_2 &= R[1/[p^\flat]] \\ R_{12} &= R[1/p[p^\flat]]. \end{aligned}$$

Then Y is covered by $\operatorname{Spec} R_1$ and $\operatorname{Spec} R_2$, with overlap $\operatorname{Spec} R_{12}$. Thus the category of vector bundles on Y is equivalent to the category of triples (M_1, M_2, h), where M_i is a finite projective R_i-module for $i = 1, 2$, and $h\colon M_1 \otimes_{R_1} R_{12} \xrightarrow{\sim} M_2 \otimes_{R_2} R_{12}$ is an isomorphism of R_{12}-modules. We wish to show that the obvious functor $M \mapsto (M \otimes_R R_1, M \otimes_R R_2, h_M)$ from finite free R-modules to such triples is an equivalence.

For full faithfulness, suppose we are given finite free R-modules M and M' and a morphism of triples $(M \otimes_R R_1, M \otimes_R R_2, h_M) \to (M' \otimes_R R_1, M' \otimes_R R_2, h_{M'})$. The matrix coefficients of such a morphism lie in $R_1 \cap R_2 = R$, and thus the morphism extends uniquely to a morphism $M \to M'$.

For essential surjectivity, suppose we are given a triple (M_1, M_2, h). Using h we may identify both $M_1 \otimes_{R_1} R_{12}$ and $M_2 \otimes_{R_2} R_{12}$ with a common R_{12}-module M_{12}. Consider the map $M_1 \oplus M_2 \to M_{12}$ defined by $(x, y) \mapsto x - y$. Let M be the kernel, an R-module. For $i = 1, 2$, the projection map $\operatorname{pr}_i\colon M_1 \oplus M_2 \to M_i$ induces a map $\operatorname{pr}_i\colon M \otimes_R R_i \to M_i$, which we claim is an isomorphism. We indicate the inverse map for $i = 1$. The image of an element $m_1 \in M_1$ under $h\colon M_1 \otimes_{R_1} R_{12} \xrightarrow{\sim} M_2 \otimes_{R_2} R_{12}$ takes the form $p^{-n} m_2$, with $m_2 \in M_2$ and $n \geq 0$. Here m_2 is uniquely determined by n because M_2 is p-torsion free. Then $m = (p^n m_1, m_2)$ lies in M and is clearly independent of n. The inverse map $M_1 \to M \otimes_R R_1$ sends m_1 to $p^{-n} m$. The case $i = 2$ is similar.

We now must show that M is a finite free R-module, given that its localizations to R_1 and R_2 are locally free of finite rank. First we present some generalizations concerning projective modules over Tate rings. (Even though R is not Tate, its localizations R_1 and R_2 are both Tate, with pseudo-uniformizers p and $[p^\flat]$, respectively.)

Let A be a Tate ring, let $f \in A$ be a topologically nilpotent unit, and let $A_0 \subset A$ be a ring of definition containing f. Then A_0 has the f-adic topology and $A = A_0[f^{-1}]$ (Prop. 2.2.5(2)). If M is a finite projective A-module, it comes with a canonical topology. This may be defined by writing M as a direct summand of A^n and giving M the induced subspace (or equivalently, quotient space) topology. We gather a few facts:

1. An A_0-submodule $N \subset M$ is open if and only if $N[f^{-1}] = M$.

2. An A_0-submodule $N \subset M$ is bounded if and only if it is contained in a finitely generated A_0-submodule.

3. If A is f-adically separated and complete, then an open and bounded A_0-submodule $N \subset M$ is also f-adically separated and complete.

4. Let A' be a Tate ring contaning A as a topological subring. If $X \subset M \otimes_A A'$ is a bounded subset, then $X \cap M \subset M$ is also bounded.

(For the last point: suppose that f is a pseudo-uniformizer of A, and thus also of A'. Let $A'_0 \subset A'$ be a ring of definition containing f; then $A_0 := A \cap A'_0$ is a ring of definition for A. Use a presentation of M as a direct summand of a free module to reduce to the case that M is free, and then to the case that $M = A$. By Prop. 2.2.5(3), boundedness of $X \subset A'$ means that $X \subset f^{-n}A'_0$ for some n, and therefore $X \cap A \subset f^{-n}A_0$ is bounded.)

Now we return to the situation of the lemma. Endow R_1 with the p-adic topology making R a ring of definition. Then R_1 is Tate and $p \in R_1$ is a topologically nilpotent unit. We claim that $M \subset M_1$ is an open and bounded R_1-submodule. Since $M \otimes_R R_1 = M_1$, point (1) above (applied to $A_0 = R$ and $A = R_1$) shows that M is open. For boundedness, endow R_{12} with the p-adic topology making R_2 a ring of definition. Then $R_1 \subset R_{12}$ is a topological subring. Since $M_2 \subset M_{12} = M_2 \otimes_{R_2} R_{12}$ is bounded, we can apply (4) to conclude that $M = M_2 \cap M_1 \subset M_1$ is bounded as well. (A similar statement holds when the roles of R_1 and R_2 are reversed.)

Thus $M \subset M_1$ is open and bounded. Since R_1 is p-adically separated and complete, point (3) shows that M is p-adically complete. It is also p-torsion free, since M_1 is. An approximation argument now shows that in order to prove that M is finite free, it is enough to prove that M/p is finite free over $R/p = \mathcal{O}_{C^\flat}$.

We claim that the inclusion $M \hookrightarrow M_2$ induces an injection $M/p \hookrightarrow M_2/p = M_2 \otimes_{R_2} C^\flat$. Assume $m \in M$ maps to $0 \in M_2/p$. Write $m = pm_2$, with $m_2 \in M_2$. Then $m' := (m/p, m_2) \in \ker(M_1 \oplus M_2 \to M_{12}) = M$, so that $m = pm'$, giving the claim.

Thus M/p is an open and bounded $\mathcal{O}_{C^\flat}$-submodule of a $C^\flat$-vector space of finite dimension d, and we want to show that it is actually free of rank d over $\mathcal{O}_{C^\flat}$. Note that if K is a discretely valued nonarchimedean field, then any open and bounded $\mathcal{O}_K$-submodule of $K^{\oplus d}$ is necessarily finite free of rank d. However, the same statement is false when K is not discretely valued: the maximal ideal $\mathfrak{m}_K$ of $\mathcal{O}_K$ is open and bounded in K, but it isn't even finitely generated.

Lemma 14.2.2. *Let $\Lambda \subset (C^\flat)^{\oplus d}$ be any $\mathcal{O}_{C^\flat}$-submodule. Then*

$$\dim_k \Lambda \otimes_{\mathcal{O}_{C^\flat}} k \leq d,$$

with equality if and only if $\Lambda \cong \mathcal{O}_{C^\flat}^{\oplus d}$.

Proof. We use induction to reduce to the case $d = 1$. If $d > 1$, choose a subspace $0 \neq W \subsetneq V = (C^\flat)^{\oplus d}$, let $\Lambda_W = \Lambda \cap W \subset W$, and let $\Lambda_{V/W}$ be the image of Λ in V/W. We have an exact sequence

$$0 \to \Lambda_W \to \Lambda \to \Lambda_{V/W} \to 0.$$

Since $\Lambda_{V/W}$ is flat over $\mathcal{O}_{C^\flat}$, the sequence remains exact after tensoring over $\mathcal{O}_{C^\flat}$ with k:

$$0 \to \Lambda_W \otimes_{\mathcal{O}_{C^\flat}} k \to \Lambda \otimes_{\mathcal{O}_{C^\flat}} k \to \Lambda_{V/W} \otimes_{\mathcal{O}_{C^\flat}} k \to 0.$$

Applying the induction hypothesis, $\dim \Lambda \otimes_{\mathcal{O}_{C^\flat}} k \leq d$, with equality if and only if Λ is an extension of free modules of respective ranks $\dim W$ and $\dim(V/W)$, which implies that Λ is free of rank d. Conversely, if Λ is free of rank d then equality obviously holds.

So assume $d = 1$. Then either $\Lambda = C^\flat$, in which case $\Lambda \otimes_{\mathcal{O}_{C^\flat}} k = 0$, or else (after rescaling) $\Lambda \subset \mathcal{O}_{C^\flat}$ is an ideal. There are two cases. Either Λ is principal, in which case Λ is a free $\mathcal{O}_{C^\flat}$-module and $\dim \Lambda \otimes_{\mathcal{O}_{C^\flat}} k = 1$, or else there exists $r \in \mathbf{R}_{>0}$ such that $\Lambda = \{x \in \mathcal{O}_{C^\flat} \,|\, |x| < r\}$. In the latter case, any $x \in \Lambda$ can be written εy, for some $y \in \mathcal{O}_{C^\flat}$ with $|y| < r$ and $\varepsilon \in \mathfrak{m}_{C^\flat}$. This shows that $\Lambda \otimes k = 0$. □

Our goal was to show that M/p is a finite free $\mathcal{O}_{C^\flat}$-module. By Lemma 14.2.2 it suffices to show that $\dim_k(M/p \otimes_{\mathcal{O}_{C^\flat}} k) = \dim_k(M \otimes_R k)$ is at least d. Let T be the image of $M \otimes_R W(k)$ in $M_1 \otimes_{R_1} W(k)[1/p] \cong L^{\oplus d}$. Since M is open and bounded in M_1, T is open and bounded in $L^{\oplus d}$, so $T \cong W(k)^{\oplus d}$, which implies that $M \otimes_R k$ surjects onto $T \otimes_R k \cong k^{\oplus d}$, and we conclude. □

Remark 14.2.3. At the end of the proof we really needed the fact that L is discretely valued. It is not clear if Lemma 14.2.1 applies to *e.g.* $A_{\mathrm{inf}} \widehat{\otimes}_{\mathbf{Z}_p} \mathcal{O}_C$.

14.3 Proof of Thm. 14.1.2

We can now prove Thm. 14.1.2, which is the statement that finite free A_{inf}-modules are in equivalence with vector bundles on $\mathcal{Y}$.

For full faithfulness: We have $\mathcal{Y} = \mathcal{Y}_{[0,1]} \cup \mathcal{Y}_{[1,\infty]}$, with intersection $\mathcal{Y}_{\{1\}}$. Let $\operatorname{Spa} S_1 = \mathcal{Y}_{[0,1]}$ and $\operatorname{Spa} S_2 = \mathcal{Y}_{[1,\infty]}$, with $Y_{\{1\}} = \operatorname{Spa} S_{12}$. Then S_1, S_2 and S_{12} are complete Tate rings:

1. $S_1 = A_{\mathrm{inf}}\langle [p^\flat]/p\rangle[1/p]$ has ring of definition $A_{\mathrm{inf}}\langle [p^\flat]/p\rangle$ and pseudo-uniformizer p,

2. $S_2 = A_{\mathrm{inf}}\langle p/[p^\flat]\rangle[1/p]$ has ring of definition $A_{\mathrm{inf}}\langle p/[p^\flat]\rangle$ and pseudo-uniformizer $[p^\flat]$,

3. $S_{12} = A_{\mathrm{inf}}\langle p/[p^\flat], [p^\flat]/p\rangle[1/p[p^\flat]]$ has ring of definition $A_{\mathrm{inf}}\langle p/[p^\flat], [p^\flat]/p\rangle$; both p and $[p^\flat]$ are pseudo-uniformizers.

The ring S_{12} contains S_1 and S_2 as topological subrings. The intersection $S_1 \cap S_2$ inside S_{12} is A_{inf}: this is [KL], Lemma 5.2.11(c). (This is not trivial, because the expansion of elements in S_1 is not unique.) This gives full faithfulness.

We turn to essential surjectivity. A vector bundle $\mathcal{E}$ on $\mathcal{Y}$ is the same (by [KL] again) as data $\mathcal{E}_i$ over S_i, for $i = 1, 2, 12$, which glue over S_{12}. We want to produce an algebraic gluing, which is to say we want to construct isomorphisms of modules over rings. Consider again $R_1 = W(\mathcal{O}_{C^\flat})[1/p]$, endowed with the p-adic topology on $W(\mathcal{O}_{C^\flat})$. Then $\operatorname{Spa}(R_1, W(\mathcal{O}_{C^\flat}))$ is covered by open subsets $\left\{\left|[p^\flat]\right| \leq |p| \neq 0\right\} = \operatorname{Spa} S_1$ and $\left\{|p| \leq \left|[p^\flat]\right| \neq 0\right\} =: \operatorname{Spa} S_2'$, where

$$S_2' = W(\mathcal{O}_{C^\flat})[p/[p^\flat]]_p^\wedge[1/p].$$

Lemma 14.3.1. *The identity map $W(\mathcal{O}_{C^\flat}) \to A_{\mathrm{inf}}$ induces an isomorphism of topological rings*

$$W(\mathcal{O}_{C^\flat})[p/[p^\flat]]_p^\wedge \cong A_{\mathrm{inf}}[p/[p^\flat]]_{(p,[p^\flat])}^\wedge.$$

Proof. The crucial point is that $W(\mathcal{O}_{C^\flat})[p/[p^\flat]]_p^\wedge$ is already $[p^\flat]$-adically complete. A devissage argument reduces this to checking that $W(\mathcal{O}_{C^\flat})[p/[p^\flat]]/p$ is $[p^\flat]$-adically complete. We have

$$\begin{aligned} W(\mathcal{O}_{C^\flat})[p/[p^\flat]]/p &= (W(\mathcal{O}_{C^\flat})/p)[T]/(p-[p^\flat]T) \\ &= \mathcal{O}_{C^\flat}[T]/p^\flat T = \mathcal{O}_{C^\flat} \oplus \bigoplus_{i\geq 1} (\mathcal{O}_{C^\flat}/p^\flat)T^i, \end{aligned}$$

which is indeed $[p^\flat]$-adically complete. □

After inverting p in Lemma 14.3.1 we find an isomorphism $S_2' \cong S_2[1/p]$. Under this isomorphism, $\mathcal{E}_2[1/p]$ can be considered as a vector bundle on S_2'. Then $\mathcal{E}_1$, $\mathcal{E}_2[1/p]$ and $\mathcal{E}_{12}$ define a gluing datum for a vector bundle on $\operatorname{Spa} R_1$. Since R_1 is sheafy (Prop. 11.3.2), they glue to give a finite projective R_1-module M_1.

Similarly, $\mathcal{E}_1[1/[p^\flat]]$, $\mathcal{E}_2$, and $\mathcal{E}_{12}$ glue to give a finite projective R_2-module M_2. These glue over R_{12} to give a vector bundle on $\operatorname{Spec}(R)\backslash\{\mathfrak{m}\}$, and we conclude by Lemma 14.2.1.

Question 14.3.2. One can ask whether Thm. 14.1.2 extends to vector bundles for groups other than GL_n. If $G/\mathbf{Q}_p$ is a connected reductive group with parahoric model $\mathcal{G}/\mathbf{Z}_p$, is it true that $\mathcal{G}$-torsors over A_{inf} are the same as $\mathcal{G}$-torsors on $\operatorname{Spec} A_{\mathrm{inf}}\backslash\{\mathfrak{m}\}$, and on $\mathcal{Y}$? Cf. work of Kisin-Pappas. The analogue for 2-dimensional regular local rings is treated in [CTS79, Thm. 6.13].

14.4 Some work in progress with Bhargav Bhatt and Matthew Morrow on integral p-adic Hodge theory

The equivalence between shtukas with one paw and Breuil-Kisin modules has a geometric interpretation in terms of integral p-adic Hodge theory for a proper smooth rigid-analytic variety X/C.

The results of [Sch13a], combined with a result of [CG14], imply:

Theorem 14.4.1. *There is a natural B_{dR}^+-lattice $\Xi \subset H^i_{\acute{e}t}(X, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} B_{\mathrm{dR}}$. If $X = Y_C$ for a smooth proper rigid-analytic variety Y over a discretely valued field $K \subset C$ with perfect residue field, then $\Xi = H^i_{\mathrm{dR}}(Y/K) \otimes_K B_{\mathrm{dR}}^+$ under the comparison isomorphims.*

Thus after getting rid of torsion in étale cohomology, one gets a pair (T, Ξ) as in Thm. 12.4.4, which corresponds by Thm. 12.5.1 to a Breuil-Kisin module which we denote $H^i_{A_{\mathrm{inf}}}(X)$.

Theorem 14.4.2. *Let $\mathcal{X}/\mathcal{O}_C$ be a proper smooth formal scheme, with generic fiber X.*

1. *If $H^i_{\mathrm{cris}}(\mathcal{X}_k/W(k))$ is p-torsion free for all i, then so are all $H^i_{\acute{e}t}(X, \mathbf{Z}_p)$.*
2. *In this case, there is an isomorphism*

$$H^i_{A_{\mathrm{inf}}}(X) \otimes_{A_{\mathrm{inf}}} W(k) \cong H^i_{\mathrm{cris}}(\mathcal{X}_k/W(k))$$

of ϕ-modules over $W(k)$.

15 The faithful topology, 28 October

The goals for the remaining lectures are:

1. Prove Drinfeld's lemma.
2. Discuss the affine Grassmannian.
3. Construct moduli spaces of local shtukas.

All of this will involve working quite extensively with diamonds. So today's lecture (and maybe Thursday's too) will cover foundational material on diamonds.

15.1 There is no good notion of affinoid morphism in rigid geometry

Recall that a morphism of schemes $f\colon X \to Y$ is *affine* if for all open affine subsets $U \subset Y$, $f^{-1}(U)$ is affine. It is a basic result that this is equivalent to the condition that there exists an open affine cover $\{U_i\}$ of X such that $f^{-1}(U_i)$ is affine for all i. As a result, the functor $X \mapsto \{Y/X \text{ affine}\}$ is a stack on the category of schemes in the Zariski (and even the fpqc) topology.

One might guess that there is a notion of "affinoid morphism" between adic spaces which works the same way. However, we run into the following counterexample. Let K be a nonarchimedean field, and let $X = \operatorname{Spa} K\langle x, y\rangle$. Let $V \subset X$ be $\{(x,y) | \, |x| = 1 \text{ or } |y| = 1\}$. This is covered by two affinoids, but certainly is not affinoid itself: $H^1(V, \mathcal{O}_V) = \widehat{\bigoplus}_{m,n\geq 0} K x^{-n} y^{-m}$ (analogous to the situation of the punctured plane in classical geometry).

We claim there is a cover $X = \bigcup_i U_i$ by rational subsets U_i such that $U_i \times_X V \subset U_i$ is rational, thus affinoid. Let $\varpi \in K$ be a pseudo-uniformizer, and take

$$\begin{aligned} U_0 &= \{|x|, |y| \leq |\varpi|\} \\ U_1 &= \{|y|, |\varpi| \leq |x|\} \\ U_2 &= \{|x|, |\varpi| \leq |y|\}. \end{aligned}$$

Then $X = U_0 \cup U_2 \cup U_3$, and

$$\begin{aligned} U_0 \times_X V &= \emptyset \\ U_1 \times_X V &= \{|x| = 1\} \subset V \\ U_2 \times_X V &= \{|y| = 1\} \subset V \end{aligned}$$

are all rational subsets of V. One gets a similar example in the perfectoid setting. Thus the functor $X \mapsto \{Y/X \text{ affinoid perfectoid}\}$ is *not* a stack on the category of affinoid perfectoid spaces for the analytic topology, let alone for a finer topology such as the pro-étale topology.

We will fix this by restricting the class of affinoids somewhat. We will also pass to an even finer topology, the *faithful topology*, which is reminiscent of the fpqc topology on schemes but which is more "topological" in nature.

15.2 w-local spaces, after Bhatt-Scholze

The paper [BS] introduces a *pro-étale topology* $X_{\text{proét}}$ for a scheme X, which is finer than the étale topology. Recall that the étale cohomology $H^i(X_{\text{ét}}, \mathbf{Q}_\ell)$ is *not* equal to the cohomology of $R\Gamma(X_{\text{ét}}, \mathbf{Q}_\ell)$, but rather is *defined* as $\varprojlim H^i(X_{\text{ét}}, \mathbf{Z}/\ell^n\mathbf{Z}) \otimes \mathbf{Q}_\ell$. This *ad hoc* definition is not required in the proétale topology: $H^i(X_{\text{proét}}, \mathbf{Q}_\ell)$ really is "correct as written".

The objects in $X_{\text{proét}}$ are *weakly étale morphisms* $f: Y \to X$, see [BS, Defn. 12]. This means that both f and the diagonal map $\Delta_f: Y \to Y \times_X Y$ are flat. Covers in $X_{\text{proét}}$ are fpqc covers. It turns out [BS, Thm. 13] the site of weakly étale morphisms is "generated by" morphisms which Zariski locally on X look like $\operatorname{Spec} B \to \operatorname{Spec} A$, where $A \to B$ is *ind-étale*, meaning that B is a filtered direct limit of étale A-algebras; this justifies the term "pro-étale topology".

The local nature of schemes in the pro-étale topology is surprisingly simple. Every affine scheme $\operatorname{Spec} A$ admits a pro-étale cover by another affine scheme $\operatorname{Spec} A^Z$ which is in a sense *as disconnected as possible*. To a first approximation, A^Z is the product $A' = \prod_{x\in X} A_x$ of the local rings of A. Note that the set of connected components of $\operatorname{Spec} A'$ is in bijection with X itself.

However, the homomorphism $A \to \prod_{x\in X} A_x$ is not flat in general (though it is if A is noetherian), which implies that $\operatorname{Spec} A' \to \operatorname{Spec} A$ is not in general weakly étale.

Theorem 15.2.1 ([BS, Thm. 1.7])**.** *There is an intermediate ring*

$$A \to A^Z \to \prod_{x\in X} A_x$$

(Z for "Zariski") such that

1. $\operatorname{Spec} A^Z \to \operatorname{Spec} A$ *is a cover in* $(\operatorname{Spec} A)_{\text{proét}}$.
2. *If* $(\operatorname{Spec} A^Z)^c$ *is the set of closed points of* $\operatorname{Spec} A^Z$*, then* $\operatorname{Spec} A^Z \to \operatorname{Spec} A$ *induces a homeomorphism* $(\operatorname{Spec} A^Z)^c \to \operatorname{Spec} A$.

3. *The set* $\pi_0 \operatorname{Spec} A^Z$ *of connected components (with its quotient topology) is homeomorphic to* $\operatorname{Spec} A$ *with its constructible topology.*

4. *The (scheme-theoretic) fibre of* $\operatorname{Spec} A^Z \to \operatorname{Spec} A$ *over* $x \in \operatorname{Spec} A$ *is* $\operatorname{Spec} A_x$.

The ring A^Z is a filtered colimit of algebras of the form $\prod_{i \in I} A[f_i^{-1}]$, where I is finite and the f_i generate the unit ideal. Thus $\operatorname{Spec} A^Z$ is an inverse limit of disjoint unions of opens in $\operatorname{Spec} A$. The effect of the functor $\operatorname{Spec} A \mapsto \operatorname{Spec} A^Z$ is a special case of a general and purely topological construction involving spectral spaces.

We recall some definitions and basic facts about spectral spaces. For further details and proofs we refer the reader to [Sta14, §5.22].

Definition 15.2.2. The category of *spectral spaces with spectral maps* is the pro-category of finite T_0-spaces. If X is a spectral space, the *constructible topology* on X is the topology generated by quasicompact opens and their complements.

Proposition 15.2.3. *Let* $f\colon X \to Y$ *be a continuous map between spectral spaces. The following are equivalent:*

1. f *is spectral.*

2. *The preimage of any quasicompact open is also quasicompact.*

3. *The preimage of any constructible set is constructible; that is, the induced map* $f_{\mathrm{cons}}\colon X_{\mathrm{cons}} \to Y_{\mathrm{cons}}$ *is continuous.*

Example 15.2.4. Let $X = \varprojlim X_i$, with X_i finite and T_0. One thinks of a finite T_0 space as finitely many points together with some specialization relations that define a poset. Recall that the constructible topolgy is generated by usual opens together with complements of quasicompact opens. Then

$$(X_i)_{\mathrm{cons}} = \varprojlim (X_i)_{\mathrm{cons}}.$$

Also, $\pi_0 X = \varprojlim \pi_0 X_i$ is profinite.

Definition 15.2.5. A topological space X is *w-local* if the following conditions hold:

1. X is spectral,

2. every connected component of X has a unique closed point,

3. $X^c := \{\text{closed points of } X\}$ is a closed subset of X.

A map of w-local spaces is *w-local* if it maps closed points to closed points. (The w stands for "weakly"; recall that a topological space is *local* if it is a connected space containing a unique closed point.)

Remark 15.2.6. 1. Any open cover of a w-local space splits. That is, if X is w-local and $\{U_i\}$ is an open cover of X, then $\coprod_i U_i \to X$ has a continuous section, or equivalently $\{U_i\}$ admits a refinement $\{V_j\}$ by pairwise disjoint opens V_j.

2. If X is w-local, the composition $X^c \hookrightarrow X \to \pi_0 X$ is a homeomorphism.

Proposition 15.2.7. *The inclusion of the category of w-local spaces with w-local maps into the category of spectral spaces and spectral maps admits a right adjoint $X \mapsto X^Z$. That is, for any spectral space X there exists w-local space X^Z and a spectral map $X^Z \to X$ which is final among all spectral maps from w-local spaces into X. The map $X^Z \to X$ is a filtered inverse limit of finite open covers $\coprod U_i \to X$. Moreover, $(X^Z)^c \cong \pi_0 X^Z \to X_{\mathrm{cons}}$ is a homeomorphism. The fibre of $X^Z \to X$ over $x \in X$ is the set of generalizations of x in X.*

Proof. The idea of the construction is as follows. Say $X = \varprojlim X_i$, with X_i finite and T_0. Let $X_i^Z = \coprod_{x \in X_i} X_{i,x}$, where $X_{i,x}$ is the set of generalizations of x in X_i. Then $X^Z = \varprojlim X_i^Z$. □

Definition 15.2.8. A ring A is *w-local* if $\operatorname{Spec} A$ is. A homomorphism $A \to B$ between w-local rings is *w-local* if $\operatorname{Spec} B \to \operatorname{Spec} A$ is w-local.

Suppose A is a ring and $X = \operatorname{Spec} A$. By keeping track of the structure sheaves in the construction of Prop. 15.2.7 one obtains:

Corollary 15.2.9. *The inclusion of the category of w-local rings and w-local homomorphisms into the category of all rings and homomorphisms admits a left adjoint $A \mapsto A^Z$. Moreover, $A \mapsto A^Z$ is faithfully flat.*

In a sense A^Z is the "total localization of A".

Example 15.2.10. If R is a DVR with fraction field K then $R^Z = R \times K$.

15.3 The faithful topology on (Perf)

All of the preceding constructions for the category of schemes carry over into the category of adic spaces. If X is an analytic adic space, then there exists a profinite étale cover of X by an adic space X^T (T for Tate) which is w-local as a topological space. In fact the situation is rather nicer than the situation for schemes: the connected components of X^T are all of the form $\mathrm{Spa}(K, K^+)$, where K is a nonarchimedean field and $K^+ \subset \mathcal{O}_K$ is an open valuation subring, and these are rather well-behaved. (In contrast, the connected components of X^Z for a scheme X are spectra of arbitrary local rings.) This fact will allow us to shift focus to the topological space $|X^Z|$; this is consistent with our philosophy that diamonds are purely topological entities.

If (R, R^+) is a perfectoid Huber pair, there exists a natural homomorphism $(R, R^+) \to (R^T, R^{T+})$ into another perfectoid Huber pair which induces a homeomorphism

$$\mathrm{Spa}(R^T, R^{T+}) \cong \mathrm{Spa}(R, R^+)^Z.$$

By construction, the morphism $\mathrm{Spa}(R^T, R^{T+}) \to \mathrm{Spa}(R, R^+)$ is affinoid pro-étale (Defn. 9.1.3). Say that a perfectoid Huber pair (R, R^+) is *w-local* if $\mathrm{Spa}(R, R^+)$ is.

Proposition 15.3.1. *Let (R, R^+) be w-local perfectoid, let $(R, R^+) \to (S, S^+)$ be a map to any Huber pair, and let $\varpi \in R$ a pseudo-uniformizer. Then S^+/ϖ is flat over R^+/ϖ, and even faithfully flat if $\mathrm{Spa}(S, S^+) \to \mathrm{Spa}(R, R^+)$ is surjective.*

Proof. The idea is to check flatness on stalks of $\mathrm{Spec}\, R^+/\varpi$. For this reduction step we need a lemma, which says that flatness can be checked on stalks over profinite bases.

Lemma 15.3.2. *Let T be a profinite set, let $\mathcal{A}$ be a sheaf of rings on T, and let $\mathcal{M}$ be an $\mathcal{A}$-module. Then $H^0(T, \mathcal{M})$ is flat over $H^0(T, \mathcal{A})$ if and only if for all $y \in T$, $\mathcal{M}_y$ is flat over $\mathcal{A}_y$.*

Proof. Write $T = \varprojlim T_i$, with T_i finite and discrete. Write $\mathrm{pr}_i \colon T \to T_i$ for the projection. For each i, the decomposition $T = \coprod_{j \in T_i} \mathrm{pr}_i^{-1}(j)$ into open and closed subsets induces a decomposition $H^0(T, \mathcal{A}) = \prod_{j \in T_i} H^0(\mathrm{pr}_i^{-1}(j), \mathcal{A})$. Let $f_i \colon \mathrm{Spec}\, H^0(T, \mathcal{A}) \to T_i$ be the continuous map which sends $\mathrm{Spec}\, H^0(\mathrm{pr}_i^{-1}(j), \mathcal{A})$ to j. Then $f_{i*}\mathcal{O}_{\mathrm{Spec}\, H^0(T,\mathcal{A})} = \mathrm{pr}_{i*}\mathcal{A}$. Passing to the limit, we obtain a continuous map $f \colon \mathrm{Spec}\, H^0(T, \mathcal{A}) \to T$ which satisfies $f_*\mathcal{O}_{\mathrm{Spec}\, H^0(T,\mathcal{A})} = \mathcal{A}$.

Note that $y \in T$ lies in the image of f if and only if $\mathcal{A}_y \neq 0$. Similarly, if $\widetilde{H}^0(T, \mathcal{M})$ is the $\mathcal{O}_{\operatorname{Spec} H^0(T,\mathcal{A})}$-module corresponding to $H^0(T, \mathcal{M})$, then $f_* \widetilde{H}^0(T, \mathcal{M}) = \mathcal{M}$.

Now for every $x \in \operatorname{Spec} H^0(T, \mathcal{A})$ the natural map $H^0(T, \mathcal{A}) \to H^0(T, \mathcal{A})_x$ factors as

$$H^0(T, \mathcal{A}) \to \mathcal{A}_{f(x)} \to H^0(T, \mathcal{A})_x.$$

Lying over these rings are the modules $H^0(T, \mathcal{M})$, $\mathcal{M}_{f(x)}$, and $H^0(T, \mathcal{M})_x$, respectively; each of the latter two modules is the base change of the first through the appropriate ring map. If $H^0(T, \mathcal{M})$ is flat over $H^0(T, \mathcal{A})$, then (by preservation of flatness under base change) $\mathcal{M}_y$ is flat over $\mathcal{A}_y$ for all y in the image of f, and thus for all $y \in T$ because $\mathcal{A}_y = 0$ if y is not in the image of f. Conversely, if $\mathcal{M}_y$ is flat over $\mathcal{A}_y$ for all $y \in T$, then $H^0(T, \mathcal{M})_y$ is flat over $H^0(T, \mathcal{A})_y$, and thus $H^0(T, \mathcal{M})_x$ is $H^0(T, \mathcal{A})_x$-flat for all $x \in \operatorname{Spec} H^0(T, \mathcal{A})$. Hence, $H^0(T, \mathcal{M})$ is flat over $H^0(T, \mathcal{A})$. □

Returning to the situation of the proposition, we have a w-local perfectoid Huber pair (R, R^+), and a map $(R, R^+) \to (S, S^+)$ into any Huber pair. Apply Lemma 15.3.2 with $T = \pi_0 \operatorname{Spa}(R, R^+)$, $\mathcal{A}$ the sheaf of rings pushed forward from $\mathcal{O}^+_{\operatorname{Spa}(R,R^+)}/\varpi$, and $\mathcal{M}$ the $\mathcal{A}$-module pushed forward from $\mathcal{O}^+_{\operatorname{Spa}(S,S^+)}/\varpi$. The lemma implies that the flatness of S^+/ϖ over R^+/ϖ can be checked on stalks over $\pi_0 \operatorname{Spa}(R, R^+)$.

Since $\operatorname{Spa}(R, R^+)$ is w-local, its connected components are of the form $\operatorname{Spa}(K, K^+)$, with K a perfectoid field and $K^+ \subset \mathcal{O}_K$ is an open valuation subring. If $y \in \pi_0 \operatorname{Spa}(R, R^+)$ represents one such connected component, it is easy to check that $\mathcal{A}_y = K^+/\varpi$. Similarly $\mathcal{M}_y = S_y^+/\varpi$, where (S_y, S_y^+) is a Huber pair lying over (K, K^+). Flatness over K^+ is equivalent to torsion-freeness; since $S_y^+ \subset S_y$ we have that S_y^+ is flat over K^+, and hence S_y^+/ϖ is flat over K^+/ϖ. □

Definition 15.3.3. The *faithful topology* on (Perf) is the topology generated by open covers and *all* surjective maps of affinoids.

Remark 15.3.4. A map $f\colon X \to Y$ is a faithful cover if and only if any quasicompact open $V \subset Y$ is contained in the image of some quasicompact open $U \subset X$.

It may appear at first sight that the faithful topology admits far too many covers to be a workable notion. However we have the following surprising theorem, which shows that the structure sheaf is a sheaf for the faithful topology on (Perf), just as it is for the fpqc topology on schemes.

Theorem 15.3.5. *The functors $X \mapsto H^0(X, \mathcal{O}_X)$ and $X \mapsto H^0(X, \mathcal{O}_X^+)$ are sheaves on the faithful site. Moreover if X is affinoid then $H^i_{\text{faithful}}(X, \mathcal{O}_X) = 0$ for $i > 0$.*

The proof will follow the pattern of the following lemma.

Lemma 15.3.6. *Let $\mathcal{F}$ be any abelian presheaf on the faithful site of perfectoid affinoid spaces. For any cover $X' \to X$ of affinoids, let $C(X'/X, \mathcal{F})$ be the associated Čech complex*

$$0 \to \mathcal{F}(X) \to \mathcal{F}(X') \to \mathcal{F}(X' \times_X X') \to \cdots .$$

Assume that

1. *If X is w-local, then $C(X'/X, F\mathcal{F})$ is acyclic.*
2. *For the w-localization $X^T \to X$, $C(X^T/X, \mathcal{F})$ is acyclic.*

Then $\mathcal{F}$ is a sheaf, and $H^i(X, \mathcal{F}) = 0$ for all X and $i > 0$.

Proof. First we show that $\mathcal{F}$ is separated. That is, suppose $X' \to X$ is a faithful cover; we show that $\mathcal{F}(X) \to \mathcal{F}(X')$ is injective. For this, let $X'' = X' \times_X X^Z$, so that we have morphisms $X'' \to X' \to X$. It is enough to check injectivity of $\mathcal{F}(X) \to \mathcal{F}(X'')$. This follows from the injectivity of $\mathcal{F}(X) \to \mathcal{F}(X^T)$ (assumption (2)) and the injectivity of $\mathcal{F}(X^T) \to \mathcal{F}(X'')$ (assumption (1)).

Next we check that $\mathcal{F}$ satisfies the sheaf property with respect to a cover $X' \to X$. Let $s' \in \mathcal{F}(X')$ be a section whose two pullbacks to $X' \times_X X'$ agree. With X'' as before, this gives $s'' \in \mathcal{F}(X'')$ whose two pullbacks to $X'' x_X X''$ agree. In particular, the two pullbacks to $X'' \times_{X^T} X''$ agree, which shows that (by assumption (1)) s'' descends to $s^T \in \mathcal{F}(X^T)$. Now the pullbacks of s^T to $X^T x_X X^T$ agree after further pullback to $X'' x_X X''$, so by separatedness, they agree on $X^T x_X X^T$ already. Thus, s^T descends to $s \in \mathcal{F}(X)$. The pullback of s to X'' agrees with the pullback of s' to X'', so by separatedness again, s pulls back to s' on X' as required.

The same line of argument can be continued to show that $H^i(X, \mathcal{F}) = 0$ for $i > 0$. □

Suppose X is an affinoid in (Perf) with pseudo-uniformizer ϖ. we apply Lemma 15.3.6 to the presheaf $(\mathcal{O}^+/\varpi)^a$ on the category of perfectoid spaces over X defined by $Y \mapsto H^0(Y, \mathcal{O}_Y^+/\varpi)$. The condition (1) is satisfied by Prop. 15.3.1 and almost faithfully flat descent. The condition (2) is satisfied by pro-étale descent (recall that $X^T \to X$ is pro-étale).

Thus $(\mathcal{O}^+/\varpi)^a$ is a sheaf on the faithful site, and $H^i(X, (\mathcal{O}^+/\varpi)^a) = 0$ for $i > 0$. A devissage argument can be used to transfer the same statements to $\mathcal{O}^+$ and $\mathcal{O}$, thus completing the proof of Thm. 15.3.5.

Corollary 15.3.7. *Representable presheaves are sheaves on the faithful site.*

Proof. The proof follows the pro-étale case, as in Prop. 8.2.7. □

We now return to the question of "affinoid morphisms". On w-local objects, the problem in §15.1 does not arise, even for the faithful site:

Corollary 15.3.8. *The functor which assigns to a w-local affinoid perfectoid X the category $\{Y/X$ affinoid perfectoid$\}$ is a stack for the faithful topology.*

Proof. This follows from Prop. 15.3.1 and faithfully flat descent. □

15.4 Faithful diamonds

One can also introduce diamonds in this setup.

Definition 15.4.1.
1. A morphism $f \colon \mathcal{F} \to \mathcal{G}$ of sheaves on the faithful site is *relatively representable* if for all $Y^\diamondsuit \to \mathcal{G}$ (with Y an object of (Perf)), the fibre product $\mathcal{F} \times_{\mathcal{G}} Y^\diamondsuit$ is representable by an object of (Perf).
2. A *faithful diamond* is a sheaf $\mathcal{F}$ on the faithful site such that there exists a relatively representable surjection $X^\diamondsuit \to \mathcal{F}$ for some object X of (Perf).

It is true but not obvious that any diamond is sheaf on the faithful site, and thus is a faithful diamond.

15.5 The underlying topological space of a (faithful) diamond

Proposition 15.5.1. *If $f \colon X \to Y$ is a faithful cover in (Perf) then $|f| \colon |X| \to |Y|$ is a quotient map. (That is, $|f|$ is surjective, and $V \subset |Y|$ is open if and only if $|f|^{-1}(V) \subset |X|$ is open.)*

Proof. (Sketch.) We may assume that X and Y are affinoid. Then $|f| \colon |X| \to |Y|$ is a surjective spectral map of spectral spaces which is also *generalizing*, meaning that if $x \in X$, and y' specializes to $f(x) \in Y$, then there exists $x' \in X$ specializing to x such that $f(x') = y'$.

(We remark that any map of *analytic* adic spaces is generalizing. This fact is akin to the "going up" theorem for flat maps. It can be proved using [Wed, Prop. 4.21], see also [HK94, Lemma 1.2.4]. These results describe *all* specializations in $\mathrm{Spv}(A)$ as a composition of two concrete cases, called "horizontal" and "vertical". For maps between analytic adic spaces, all specializations are vertical; see [Hub93, p. 468].)

This reduces us to the following lemma:

Lemma 15.5.2. *Any surjective generalizing spectral map of spectral spaces is a quotient map.*

Proof. Let as exercise. The key point is that a constructible set in a spectral space is open if and only if it is stable under generalization. □

□

Lemma **??** ensures the well-posedness and functoriality of the following definition.

Definition 15.5.3. Let $\mathcal{D}$ be a (faithful) diamond, and let $X^\diamondsuit \to \mathcal{D}$ be a relatively representable surjection (which we can take to be qpf if $\mathcal{D}$ is a diamond). The *underlying topological space* $|\mathcal{D}|$ is the coequalizer of $|R| \rightrightarrows |X|$, where $R^\diamondsuit = X^\diamondsuit \times_{\mathcal{D}} X^\diamondsuit$.

In other words, $|\mathcal{D}|$ is the quotient of $|X|$ by the image of the equivalence relation $|R| \to |X| \times |X|$. We will need $|\mathcal{D}|$ later without any quasi-separatedness conditions on $\mathcal{D}$. As is the case for non-qs algebraic spaces, $|\mathcal{D}|$ can be quite pathological.

16 Fake diamonds, 30 October

16.1 Fake diamonds: motivation

The motivation for defining the faithful site is that we want to define a *mixed characteristic affine Grassmannian*

$$\mathrm{Gr}_G = LG/L^+G$$

whose C-points are $G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+)$. It will easy to see that Gr_G is a faithful diamond, because LG is covered by opens that look like L^+G, which is representable. However it will be much more difficult to show that Gr_G is a diamond.

The aim of this lecture is to present a criterion for when a faithful diamond is a diamond.[12]

Recall that if $\mathcal{D}$ is a fake diamond, and $R^\diamond \rightrightarrows X^\diamond \to \mathcal{D}$ is a faithful presentation of $\mathcal{D}$, then we had defined $|\mathcal{D}|$ to be the coequalizer of $|R| \rightrightarrows |X|$ as a topological space.

Remark 16.1.1. In general $|\mathcal{D}|$ is ill-behaved. It might not even be T_0, for instance.

Proposition 16.1.2. *Let X be an analytic adic space over* $\operatorname{Spa} \mathbf{Z}_p$. *There is a natural homeomorphism* $\left|X^\diamond\right| \tilde{\to} |X|$.

Proof. Since X is analytic, it is covered by Tate affinoids; we can therefore reduce to the case that $X = \operatorname{Spa}(R, R^+)$ is affinoid and Tate over $\operatorname{Spa} \mathbf{Z}_p$. Let $(R, R^+) \to (\widetilde{R}, \widetilde{R}^+)$ be a G-torsor, with $(\widetilde{R}, \widetilde{R}^+) = (\varinjlim (R_i, R_i^+))^\wedge$ perfectoid, such that each (R_i, R_i^+) is a finite étale G_i-torsor, and $G = \varprojlim G_i$. Let $\widetilde{X} = \operatorname{Spa}(\widetilde{R}, \widetilde{R}^+)$. Then $X^\diamond = \widetilde{X}^\diamond / \underline{G}$, and $\left|X^\diamond\right| = \left|\widetilde{X}\right| / G = \varprojlim |X_i| / G_i = \varprojlim |X| = |X|$. □

Definition 16.1.3. A map $\mathcal{E} \to \mathcal{D}$ of fake diamonds is an *open immersion* if for any (equivalently, for one) surjection $X^\diamond \to \mathcal{D}$ (for the faithful topology), the fibre product $\mathcal{E} \times_{\mathcal{D}} X^\diamond \to X^\diamond$ is representable by an open subspace of X. In this case we say $\mathcal{E}$ is an *open subdiamond* of $\mathcal{D}$.

Proposition 16.1.4. *The category of open subdiamonds of $\mathcal{D}$ is equivalent to the category of open immersions into $|\mathcal{D}|$, via $\mathcal{E} \mapsto |\mathcal{E}|$.*

We want to single out those diamonds for which $|\mathcal{D}|$ is well-behaved. For starters, we want $\mathcal{D}$ to be quasiseparated. A scheme X is quasiseparated if the diagonal map $X \to X \times X$ is a closed immerson. But for a diamond $\mathcal{D}$, the diagonal map $\mathcal{D} \to \mathcal{D} \times \mathcal{D}$ might not be relatively representable, so this definition cannot be used. To handle this, we make a topos-theoretic digression.

Recall from SGA4 the following notions: In any topos, we have a notion of a quasicompact (qc) object: this means that any covering family has a finite subcover. An object Z is quasiseparated (qs) if for any qc $X, Y \to Z$, $X \times_Z Y$ is qc.

If there is a generating family B for the topos (meaning that every object is a colimit of objects in the generating family) consisting of qc objects which is stable under fibre products, then:

[12] This notation is not ideal: the way we have defined things, faithful diamonds are not necessarily diamonds. Instead let's call faithful diamonds *fake diamonds*.

1. all objects of B are qcqs,

2. Z is qc if and only if it has a finite cover by objects of B, and

3. Z is qs if and only if for all $X, Y \in B$, $X, Y \to Z$, $X \times_Z Y$ is qc.

In our situation, can take B to be the class of $X^\diamondsuit$, where X is affinoid perfectoid. This is closed under fibre products because maps between analytic adic spaces are adic.

Warning: B is not stable under direct products. For example if $X = \operatorname{Spd} \mathbf{F}_p(\!(t^{1/p^\infty})\!)$, then $X \times X = \left(\mathbf{D}^*_{\mathbf{F}_p(\!(t^{1/p^\infty})\!)}\right)^\diamondsuit$ is not qc.

16.2 Spacial fake diamonds

Definition 16.2.1. A fake diamond $\mathcal{D}$ is *spatial*[13] if

1. $\mathcal{D}$ is quasiseparated,

2. $|\mathcal{D}|$ admits a neighborhood basis consisting of $|\mathcal{E}|$, where $\mathcal{E} \subset \mathcal{D}$ is qc open.

Remark 16.2.2.

1. For algebraic spaces, (1) implies (2); however (1) does not imply (2) in the context of fake diamonds. See Ex. 16.2.3 below.

2. If $\mathcal{D}$ is qc, then so is $|\mathcal{D}|$. Indeed, any open cover of $|\mathcal{D}|$ pulls back to a cover of $\mathcal{D}$. In particular a spatial diamond has lots of qc open subsets.

3. If $\mathcal{D}$ is qs, then so is any open subdiamond of $\mathcal{D}$. Thus if $\mathcal{D}$ is spatial, then so is any open subdiamond.

Example 16.2.3. Let K be a perfectoid field in characteristic p, and let $\mathcal{D} = \operatorname{Spd} K/\operatorname{Frob}^{\mathbf{Z}}$, so that $|\mathcal{D}|$ is one point. Then $\mathcal{D}$ isn't qs. Indeed if $X = Y = \operatorname{Spd} K$, then $X \times_{\mathcal{D}} Y$ is a disjoint union of $\mathbf{Z}$ copies of $\operatorname{Spd} K$, and so is not qc. In particular $\mathcal{D}$ isn't spatial. However, $\mathcal{D} \times \operatorname{Spd} \mathbf{F}_p(\!(t^{1/p^\infty})\!) = (\mathbf{D}^\times_K/\operatorname{Frob}^{\mathbf{Z}})^\diamondsuit$ *is* spatial.

Proposition 16.2.4. *Let $\mathcal{D}$ be a spatial qc fake diamond. Let $X^\diamondsuit \to \mathcal{D}$ be a faithful cover, with X qcqs. Let $R^\diamondsuit = X^\diamondsuit \times_{\mathcal{D}} X^\diamondsuit$, with R qcqs. Then $|\mathcal{D}| = \operatorname{Coeq}(|R| \rightrightarrows |X|)$ is a spectral space, and $|X| \to |\mathcal{D}|$ is a spectral map.*

[13] In the lectures, the term used was *nice*. Brian Conrad suggested the term *spatial* because of the analogy with algebraic spaces.

Proof. We need to construct many qc open subsets $U \subset |X|$ which are stable under the equivalence relation R. By (2) in Defn.16.2.1, we can just take the preimages of $|\mathcal{E}|$ for $\mathcal{E} \subset \mathcal{D}$ qc open. Since $\mathcal{E}$ is qc and $\mathcal{D}$ is qs, $\mathcal{E} \times_{\mathcal{D}} X \subset X$ is still qc, and so $\left|\mathcal{E} \times_{\mathcal{D}} X^{\diamondsuit}\right| \subset |X|$ is qc. □

Corollary 16.2.5. *If $\mathcal{D}$ is spatial, then $|\mathcal{D}|$ is locally spectral, and $\mathcal{D}$ is qc if and only if $|\mathcal{D}|$ is qc.*

To check whether a diamond is spatial, we can use the following proposition.

Proposition 16.2.6. *Let X be a spectral space, and $R \subset X \times X$ a spectral equivalence relation such that each $R \to X$ is open (and spectral). Then X/R is a spectral space, and $X \to X/R$ is spectral.*

Proof. We need to produce many $U \subset X$ which are qc open and R-stable. Let $s, t \colon R \to X$ be the maps to X. Let $V \subset X$ be any qc open. Then $s^{-1}(V) \subset R$ is qc open (since $R \to X$ is spectral), so $t(s^{-1}(V)) \subset X$ is qc open (since $R \to X$ is open) and R-stable. □

Remark 16.2.7. There are counterexamples to Prop. 16.2.6 if $R \to X$ is generalizing but not open.

Corollary 16.2.8. *Let $\mathcal{D}$ be a fake diamond. Assume there exists a faithful presentation $R^{\diamondsuit} \rightrightarrows X^{\diamondsuit} \to \mathcal{D}$, where R and X are qcqs, and that each $R \to X$ is open. Then $\mathcal{D}$ is spatial and qc.*

Proof. Since X is qc, $\mathcal{D}$ is qc. Since R is qc, $\mathcal{D}$ is qs. Then Prop. 16.2.6 shows that $|\mathcal{D}| = |X| \,/\, |R|$ is spectral, and $|X| \to |\mathcal{D}|$ is spectral. Any qc open $U \subset |\mathcal{D}|$ defines an open subdiamond $\mathcal{E} \subset \mathcal{D}$ covered by $\mathcal{E} \times_{\mathcal{D}} X^{\diamondsuit} \subset X^{\diamondsuit}$ (which is qc). Thus $\mathcal{E}$ itself is qc. □

Corollary 16.2.9. *If X is a qs analytic adic space over* $\operatorname{Spa} \mathbf{Z}_p$, *then $X^{\diamondsuit}$ is spatial.*

Proof. By Prop. 16.1.2, $\left|X^{\diamondsuit}\right| \cong |X|$; this implies that $\left|X^{\diamondsuit}\right|$ has a basis of opens $|U|$, where $U \subset X$ is qc open. By Prop. 16.1.4, these correspond to open subdiamonds $U^{\diamondsuit} \subset X^{\diamondsuit}$. □

We can now state the main theorem of today's lecture, which tells when a fake diamond is a "true" diamond.

Theorem 16.2.10. *Let $\mathcal{D}$ be a spatial qc fake diamond: this means there exists a faithful presentation $R^{\diamondsuit} \rightrightarrows X^{\diamondsuit} \to \mathcal{D}$ with R, X qcqs such that $|X| \to |D|$ is spectral. Assume moreover that*

1. *$X^\diamondsuit \to \mathcal{D}$ is a surjection on the qpf site, i.e., for all $Y^\diamondsuit \to \mathcal{D}$ there exists a qpf cover $Y' \to Y$ and a morphism $Y' \to X$ such that the diagram*

$$\begin{array}{ccc} (Y')^\diamondsuit & \longrightarrow & X^\diamondsuit \\ \downarrow & & \downarrow \\ Y^\diamondsuit & \longrightarrow & \mathcal{D} \end{array}$$

   commutes.

2. *For all $x \in |\mathcal{D}|$, there is a qpf morphism $S_x^\diamondsuit \to \mathcal{D}$ such that x lies in the image of $|S_x| \to |\mathcal{D}|$.*

Then $\mathcal{D}$ is a diamond.

Proof. It is enough to find a surjection $Y^\diamondsuit \to \mathcal{D}$ with profinite geometric fibres. Indeed, after passing to a qpf cover (using (1)), $Y^\diamondsuit \to \mathcal{D}$ factors through $X^\diamondsuit$. Since $X^\diamondsuit \to \mathcal{D}$ is relatively representable, so is $Y^\diamondsuit \to \mathcal{D}$, and this has profinite geometric fibres, and so is a relatively representable qpf surjection from a representable sheaf. Therefore $\mathcal{D}$ is a diamond.

Without loss of generality X is affinoid. Since we can replace X with any faithful cover, we may also assume that X is w-local. In that case, any qc open subset $U \subset X$ is affinoid (slightly nontrivial exercise). Let $\mathcal{D}^T = \mathcal{D} \times_{|\mathcal{D}|} |\mathcal{D}|^Z$. Explicitly, suppose $\{U_{ij}\}_{j \in J_i}$ is a basis of qc open covers of $|D|$; then $\mathcal{D}^T = \varprojlim \coprod_i \left(\coprod_{j \in J_i} U_{ij} \right)$. Then $\mathcal{D}^T$ has a cover by $X \times_{|\mathcal{D}|} |\mathcal{D}|^Z$, which is affinoid, as each $X \times_{|\mathcal{D}|} |U_{ij}|$ is.

Then $\mathcal{D}^T \to \mathcal{D}$ has profinite geometric fibres, so without loss of generality $|\mathcal{D}|$ is w-local.

Moreover we may assume that $\mathcal{D}$ has no nonsplit finite étale covers. The goal is now to show that $\mathcal{D}$ is representable.

First we check that all connected components are representable. Let $K \subset \mathcal{D}$ be a connected component, with $x \in K$ the unique closed point. Let $S_x^\diamondsuit = \mathrm{Spd}(C, C^+) \to \mathcal{D}$ be a qpf morphism such that x lies in the image of $|S_x|$. Here C is an algebraically closed nonarchimdean field and $C^+ \subset \mathcal{O}_C$ is an open valuation subring. Then the image of $|S_x|$ is exactly K.

We have $K = S_x^\diamondsuit / \underline{G}$ for some profinite group G. The product $S_x^\diamondsuit \times_K S_x^\diamondsuit$ is a profinite set of geometric points with a continuous group structure, which is to say it is a profinite group $G = \varprojlim G_i$, with G_i finite. Then $S_x^\diamondsuit \to K$ is a quotient by $\underline{G}$. For any open subgroup $H \subset G$, $S_x^\diamondsuit / \underline{H} \to K$ is finite étale.

Proposition 16.2.11. *Let $\mathcal{D}$ be a spatial qc fake diamond, and let $K = \bigcap_i U_i \subset |\mathcal{D}|$ be a pro-(qc open). Then*

$$K_{\text{fét}} = 2\text{-}\varinjlim (U_i)_{\text{fét}}.$$

Proof. Let $R^\diamondsuit \rightrightarrows X^\diamondsuit \to \mathcal{D}$ be as usual with R and X qcqs. Similar results hold for R and X by [Elk73] and [GR03], and then we can descend to $\mathcal{D}$. □

Thus $S_x^\diamondsuit/\underline{H} \to K$ extends to a finite étale cover $\widetilde{U}_i \to U_i$, with $U_i \subset |\mathcal{D}|$ open and closed. Then $\widetilde{U}_i \sqcup (\mathcal{D}\backslash U_i) \to \mathcal{D}$ is a finite étale cover, which is therefore split. Therefore $S_x^\diamondsuit/\underline{H} \to K$ is split, with only finitely many splittings (because K is connected). Now we can apply a "Mittag-Leffler" style argument: take the inverse limit over all H to get that $S_x^\diamondsuit \to K$ admits a section. Since $S_x = \operatorname{Spa}(C, C^+)$ is connected, we have that $S_x \stackrel{\sim}{\to} K$. Thus every connected component K really is a geometric "point" $\operatorname{Spd}(C(x), C(x)^+)$.

Finally, we show that $\mathcal{D}$ is representable. Let ϖ be a psuedo-uniformizer on X, and let "$\mathcal{O}_D^+/\varpi$" be the sheaf of rings on $|\mathcal{D}|$ equal to the equalizer of $\mathcal{O}_X^+/\varpi \rightrightarrows \mathcal{O}_R^+/\varpi$. The stalk of "$\mathcal{O}_D^+/\varpi$" at x is $\mathcal{O}_{C(x)}^+/\varpi$, where $(C(x), C(x)^+)$ is as above.

Let $A^+/\varpi = H^0(|\mathcal{D}|, \text{"}\mathcal{O}_D^+/\varpi\text{"})$. Then $\Phi\colon A^+/\varpi^{1/p} \stackrel{\sim}{\to} A^+/\varpi$ is an isomorphism. Let $A^+ = \varprojlim_\Phi A^+/\varpi$, $A = A^+[1/\varpi]$. Then $\operatorname{Spa}(A, A^+)$ represents $\mathcal{D}$. □

17 Drinfeld's lemma for diamonds, 4 November

17.1 The failure of $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$

This lecture is entirely about fundamental groups. For ease of notation we will omit mention of base points.

It is a basic fact that for connected topological spaces X and Y, the natural map $\pi_1(X \times Y) \to \pi_1(X) \times \pi_1(Y)$ is an isomorphism; let us call this the *Künneth theorem* for X and Y. Is the same result true if instead X and Y are varieties over a field k, and π_1 is interpreted as the étale fundamental group? Certainly the answer is no in general. For instance, suppose k is a prime field (*i.e.* $\mathbf{F}_p$ or $\mathbf{Q}$), and $X = Y = \operatorname{Spec} k$. Then $X \times Y = \operatorname{Spec} k$ once again, and the diagonal map $\operatorname{Gal}(\overline{k}/k) \to \operatorname{Gal}(\overline{k}/k) \times \operatorname{Gal}(\overline{k}/k)$ is not an isomorphism.

If k is an algebraically closed field of characteristic 0, then the answer becomes yes, by appeal to the Lefshetz principle: X can be descended to a finitely generated field, which can be embedded into $\mathbf{C}$, and then the étale topology of X can be related to the (usual) topology of the resulting complex-analytic space. The same principle shows that if k'/k is an extension of algebraically closed fields of characteristic 0, and X is a variety over k, then the natural map $\pi_1(X_{k'}) \to \pi_1(X)$ is an isomorphism. That is, such varieties satisfy *permanence of* π_1 *under (algebraically closed) base extension.*

So let us assume that k is an algebraically closed field of characteristic p. Both the Künneth theorem and the permanence of π_1 under base extension hold for *proper* varieties over k. But both properties can fail for non-proper varieties.

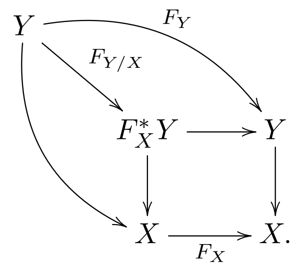
Example 17.1.1. Keep the assumption that k is an algebraically closed field of characteristic p. Let $X = \operatorname{Spec} R$ be an affine k-scheme. We have $\operatorname{Hom}(\pi_1(X), \mathbf{F}_p) = H^1_{\text{ét}}(X, \mathbf{Z}/p\mathbf{Z})$. By Artin-Schreier, this group is identified with the cokernel of the endomorphism $f \mapsto f^p - f$ of R. Generally (and particularly if $R = k[T]$) this group is not invariant under base extension, and therefore the same can be said about $\pi_1(X)$. Similarly, the Künneth theorem fails for $R = k[T]$.

The following lemma states that under mild hypotheses, the Künneth theorem holds when permanence of π_1 under base extension is satisfied for one of the factors.

Lemma 17.1.2 (EGA IV$_2$, 4.4.4)**.** *Let X and Y be schemes over an algebraically closed field k, with Y qcqs and X connected. Assume that for all algebraically closed extensions k'/k, $Y_{k'}$ is connected, and the natural map $\pi_1(Y_{k'}) \to \pi_1(Y)$ is an isomorphism. Then $\pi_1(X \times Y) \to \pi_1(X) \times \pi_1(Y)$ is an isomorphism.*

17.2 Drinfeld's lemma

Let us recall the notions of absolute and relative Frobenii. For a scheme $X/\mathbf{F}_p$, let $F_X \colon X \to X$ be the *absolute Frobenius map*: this is the identity on $|X|$ and the pth power map on the structure sheaf. For $f \colon Y \to X$ a morphism of schemes, we have the pullback $F_X^* Y = Y \times_{X, F_X} X$ (this is often denoted $Y^{(p)}$). The *relative Frobenius* $F_{Y/X} \colon Y \to F_X^* Y$ is the unique morphism making the following diagram commute:



A crucial fact is that $F_{Y/X}$ is an isomorphism when $Y \to X$ is étale.

Definition 17.2.1. Let $X_1, \dots, X_n$ be schemes of finite type over $\mathbf{F}_p$. Consider the *ith partial Frobenius*

$$F_i = 1 \times \cdots \times F_{X_i} \times \cdots \times 1 \colon X_1 \times \cdots \times X_n \to X_1 \times \cdots \times X_n.$$

Let

$$(X_1 \times \cdots \times X_n/\text{p.Fr.})_{\text{fét}}$$

(p.Fr.=partial Frobenius) be the category of finite étale maps $Y \to X_1 \times \cdots \times X_n$ equipped with commuting isomorphisms $\beta_i \colon Y \xrightarrow{\sim} F_i^* Y$ such that $\beta_n \circ \cdots \circ \beta_1 = F_{Y/X} \colon Y \xrightarrow{\sim} F_X^* Y$.

Remark 17.2.2. Strictly speaking, the notation $\beta_n \circ \cdots \circ \beta_1$ is an abuse: the morphism β_2 should be the pullback of β_2 through F_1, and so forth.

Remark 17.2.3. To give an object of this category it suffices to produce all but one of the β_i, by the product relation. Thus if $n = 2$, the category $(X_1 \times X_2/\text{p.Fr.})_{\text{fét}}$ is the category of finite étale morphisms $Y \to X_1 \times X_2$ equipped with an isomorphism $\beta \colon Y \xrightarrow{\sim} F_1^* Y$.

This forms a Galois category in the sense of SGA1, so that (after choosing a geometric point s of $X_1 \times \cdots \times X_n$) one can define the fundamental group $\pi_1(X_1 \times \cdots \times X_n/\text{p.Fr.})$; this is the automorphism of the fibre functor on $(X_1 \times \cdots X_n/\text{p.Fr.})_{\text{fét}}$ determined by s.

Theorem 17.2.4 (Drinfeld's lemma for schemes)**.** *Assume the X_i are connected. The natural map $\pi_1(X_1 \times \cdots \times X_n/p.Fr.) \to \pi_1(X_1) \times \cdots \times \pi_1(X_n)$ is an isomorphism.*

Example 17.2.5. If $X_1 = X_2 = \operatorname{Spec} \mathbf{F}_p$, then $(X_1 \times X_2/\text{p.Fr.})_{\text{fét}}$ is the category of finite étale covers of $\operatorname{Spec} \mathbf{F}_p$ equipped with one partial Frobenius; these are parametrized by $\widehat{\mathbf{Z}} \times \widehat{\mathbf{Z}}$.

The crucial step in the proof of Thm. 17.2.4 is to establish permanence of π_1 under extension of the base, once a relative Frobenius is added to the picture. Let $k/\mathbf{F}_p$ be algebraically closed. For a scheme $X/\mathbf{F}_p$, let $\overline{X} := X \otimes_{\mathbf{F}_p} k$; this has a relative Frobenius $F_{\overline{X}/k}\colon \overline{X} \to F_k^* \overline{X}$. One can then define a category $(\overline{X}/F_{\overline{X}/k})_{\text{fét}}$ and (after choosing a geometric point) a group $\pi_1(\overline{X}/F_{\overline{X}/k})$.

Lemma 17.2.6 ([Lau, Lemma 8.12])**.** *Let $X/\mathbf{F}_p$ be a scheme of finite type. Then $\pi_1(\overline{X}/F_{\overline{X}/k}) \to \pi_1(X)$ is an isomorphism; that is, there is an equivalence of categories between finite étale covers $Y_0 \to X$ and finite étale covers $Y \to \overline{X}$ equipped with an isomorphism $F^*_{\overline{X}/k} Y \xrightarrow{\sim} Y$.*

Proof. (Of Lemma 17.2.6, sketch.)

1. The category of finite-dimensional ϕ-modules (V, ϕ_V) over k is equivalent to the category of finite-dimensional $\mathbf{F}_p$-vector spaces, via $(V, \phi_V) \mapsto V^{\phi=1}$ and its inverse $V_0 \mapsto (V_0 \otimes k, 1 \otimes \phi_k)$. (Exercise)

2. Let X be projective over $\mathbf{F}_p$. Then there is an equivalence between pairs $(\mathcal{E}, \phi_{\mathcal{E}})$, where $\mathcal{E}$ is a coherent sheaf on $\overline{X}$, and $\phi_{\mathcal{E}}\colon F_k^* \mathcal{E} \xrightarrow{\sim} \mathcal{E}$, and coherent sheaves $\mathcal{E}_0/X$. (Describe everything in terms of graded modules, finite-dimensional over k (resp., $\mathbf{F}_p$) in each degree, then use (1). See [Lau, Lemma 8.1.1]. (This step fails if X is not projective.)

3. Without loss of generality in the lemma, X is affine. By cohomological descent, we can assume X is normal and connected. Choose an embedding $X \hookrightarrow X'$ into a normal projective $\mathbf{F}_p$-scheme. The following categories are equivalent:

   (a) $Y/\overline{X}$ finite étale with $F^*_{\overline{X}/Y} Y \xrightarrow{\sim} Y$, and

   (b) $Y/\overline{X}$ finite étale and $F_k^* Y \xrightarrow{\sim} Y$

   (c) $Y'/\overline{X}'$ finite normal, and $F_k^* Y' \xrightarrow{\sim} Y'$ such that Y' is étale over the open subset $\overline{X}$,

   (d) (using (2)) Y_0'/X' finite normal such that Y_0' is étale over X,

   (e) Y_0/X finite étale.

   (The proof of the equivalence of (b) and (c) uses the normalization of $\overline{X}'$ in Y.)

□

17.3 Drinfeld's lemma for diamonds

Let $\mathcal{D}$ be a diamond.

Definition 17.3.1. $\mathcal{D}$ is *connected* if $|\mathcal{D}|$ is.

For a connected diamond $\mathcal{D}$, finite étale covers of $\mathcal{D}$ form a Galois category, so for a geometric point $x \in \mathcal{D}(C, \mathcal{O}_C)$ we can define a profinite group $\pi_1(\mathcal{D}, x)$, such that finite $\pi_1(\mathcal{D}, x)$-sets are equivalent to finite étale covers $\mathcal{E} \to \mathcal{D}$.

We would like to replace all the connected schemes X_i appearing in Drinfeld's lemma with $\operatorname{Spd} \mathbf{Q}_p$. Even though $\mathbf{Q}_p$ has characteristic 0, its diamond $\operatorname{Spd} \mathbf{Q}_p$ admits an absolute Frobenius $F\colon \operatorname{Spd} \mathbf{Q}_p \to \operatorname{Spd} \mathbf{Q}_p$, because after all it is a sheaf on the category (Perf) of perfectoid affinoids in characteristic p, and there is an absolute Frobenius defined on these.

Let

$$(\operatorname{Spd} \mathbf{Q}_p \times \cdots \times \operatorname{Spd} \mathbf{Q}_p/\text{p.Fr.})_{\text{fét}}$$

be the category of finite étale covers $E \to (\operatorname{Spd} \mathbf{Q}_p)^n$ equipped with commuting isomorphisms $\beta_i\colon E \xrightarrow{\sim} F_i^* E$ (where F_i is the ith partial Frobenius), $i = 1, \ldots, n$ such that $\prod_i \beta_i = F_E\colon E \xrightarrow{\sim} E$. As above, this is the same as the category of finite étale covers $E \to (\operatorname{Spd} \mathbf{Q}_p)^n$ equipped with commuting isomorphisms $\beta_1, \ldots, \beta_{n-1}$. A new feature of this story is that the action of $F_1^Z \times \cdots F_{n-1}^{\mathbf{Z}}$ on $|(\operatorname{Spd} \mathbf{Q}_p)^n|$ is free and totally discontinuous. Thus the quotient $(\operatorname{Spd} \mathbf{Q}_p)^n/(F_1^{\mathbf{Z}} \times \cdots \times F_{n-1}^{\mathbf{Z}})$ really is a diamond; $((\operatorname{Spd} \mathbf{Q}_p)^n/\text{p.Fr.})_{\text{fét}}$ is simply the category of finite étale covers of it.

Example 17.3.2. Recall from Exmp. 10.1.6 that we have an isomorphism $\operatorname{Spd} \mathbf{Q}_p \times \operatorname{Spd} \mathbf{Q}_p \cong (\widetilde{\mathbf{D}}^*_{\mathbf{Q}_p})^\diamondsuit/\mathbf{Z}_p^\times$, where $\widetilde{\mathbf{D}}^*_{\mathbf{Q}_p}$ is the punctured perfectoid open unit disc over $\operatorname{Spa} \mathbf{Q}_p$, and the action of $a \in \mathbf{Z}_p^\times$ is through $t \mapsto (1+t)^a - 1$. Under this isomorphism, the partial Frobenius corresponds to $t \mapsto t^p$.

The version of Drinfeld's lemma we need is:

Theorem 17.3.3. $\pi_1((\operatorname{Spd} \mathbf{Q}_p)^n/\text{p.Fr.}) \cong G^n_{\mathbf{Q}_p}$.

In fact we prove something slightly more general. Let k be an algebraically closed field of characteristic p, with its discrete topology. Let $L = W(k)[1/p]$. We may define a relative Frobenius $F_{L/k}\colon \operatorname{Spd} L \to F_k^* \operatorname{Spd} L$. Formally, the definition of $F_{L/k}$ is the same as the definition of a relative Frobenius for schemes. Explicitly, if (R, R^+) is a perfectoid k-algebra and $S = \operatorname{Spa}(R, R^+)$, then $(\operatorname{Spd} L)(S)$ is the set of isomorphism classes of pairs

$(R^\sharp, \iota)$, where $R^\sharp$ is a perfectoid L-algebra and $\iota\colon R^{\sharp\flat} \xrightarrow{\sim} R$ is an isomorphism. Then $F_{L/k}(R^\sharp, \iota) = (R^\sharp, F \circ \iota)$, where $F\colon R \to R$ is the pth power Frobenius.

Similarly one can define the ith partial Frobenius $F_{L/k,i}$ on $(\operatorname{Spd} L)^n$ for $i = 1, \dots, n$. Consider the category

$$((\operatorname{Spd} L)^n/(F_{L/k,1}^{\mathbf{Z}} \times \cdots \times F_{L/k,n}^{\mathbf{Z}}))_{\text{fét}}$$

consisting of finite étale covers $E \to (\operatorname{Spd} L)^n$ together with commuting isomorphisms $\beta_i\colon E \xrightarrow{\sim} F_{L/k,i}^* E$.

Theorem 17.3.4. $\pi_1((\operatorname{Spd} L)^n/(F_{L/k,1}^{\mathbf{Z}} \times \cdots \times F_{L/k,n}^{\mathbf{Z}}))_{\text{fét}} \cong G_{\mathbf{Q}_p}^n$.

Thm. 17.3.4 implies Thm. 17.3.3 by taking $k = \overline{\mathbf{F}}_p$. But we really do need Thm. 17.3.4 in this generality to make the proof of Theorem 17.3.3 work.

Define $X = \operatorname{Spd} \mathbf{Q}_p/F^{\mathbf{Z}}$. Let $X_k = X \otimes_{\mathbf{F}_p} k = (\operatorname{Spd} L)/F_{L/k}^{\mathbf{Z}}$; in the spirit of Weil schemes in Deligne's Weil II, this is the "Weil version" of $\operatorname{Spd} \mathbf{Q}_p$.

Lemma 17.3.5. $\pi_1(X_k) \cong G_{\mathbf{Q}_p}$.

Proof. We have a canonical isomorphism $(\operatorname{Spd} L/F_{L/k}^{\mathbf{Z}})_{\text{fét}} \cong (\operatorname{Spd} L/F_k^{\mathbf{Z}})_{\text{fét}}$, arising from the fact that if M/L is étale then $F_{M/L}$ is an isomorphism. Thus it suffices to show that the category of pairs (M, ϕ_M), where M is a finite étale L-algebra and $\phi_M\colon \phi^* M \xrightarrow{\sim} M$ is an isomorphism, is equivalent to the category of finite étale M_0-algebras. For this one has to show there are enough ϕ-invariants; one can use the ϕ-stable lattice $\mathcal{O}_M$. ☐

The key lemma is:

Lemma 17.3.6. *For any algebraically closed nonarchimedean field $C/\mathbf{F}_p$, one can form the base change X_C, and then $\pi_1(X_C) \cong G_{\mathbf{Q}_p}$.*

This will be equivalent to the fact that the Fargues-Fontaine curve is simply connected! Note that

$$\begin{aligned}(X_C)_{\text{fét}} &= (\operatorname{Spd} \mathbf{Q}_p/F^{\mathbf{Z}}) \times_{\mathbf{F}_p} C)_{\text{fét}} \\ &= (\operatorname{Spd} \mathbf{Q}_p \times_{\mathbf{F}_p} (\operatorname{Spd} C/F_C^{\mathbf{Z}}))_{\text{fét}}.\end{aligned}$$

Lemma 17.3.7. *Let*

$$\mathcal{X}_{\text{FF}} = \mathcal{Y}_{(0,\infty)}/\phi^{\mathbf{Z}} = (\operatorname{Spa} W(\mathcal{O}_C) \backslash \{p[\varpi] = 0\})\,/\phi^{\mathbf{Z}}$$

be the adic Fargues-Fontaine curve. Then $\mathcal{X}_{\text{FF}}^\diamondsuit = \operatorname{Spd} \mathbf{Q}_p \times \operatorname{Spd} C/F_C^{\mathbf{Z}}$.

Proof. From the discussion of $\operatorname{Spd}\mathbf{Z}_p$, we know that $(\operatorname{Spa} W(\mathcal{O}_C)\setminus\{[\varpi]=0\})^\diamondsuit \cong \operatorname{Spd}\mathbf{Z}_p \times \operatorname{Spd} C$. Pass to $\{p \neq 0\}$ and quotient by $1 \times F_C$ to obtain the result. □

Lemma 17.3.8. *For any analytic affinoid adic space* $Y = \operatorname{Spa}(R, R^+)$ *over* $\operatorname{Spa}\mathbf{Z}_p$, $R_{\text{fét}} \cong Y^\diamondsuit_{\text{fét}}$. *Thus* $Y_{\text{fét}} \cong Y^\diamondsuit_{\text{fét}}$.

Proof. Take $(R, R^+) \to (\widetilde{R}, \widetilde{R}^+)$ a $\underline{G}$-cover such that $\widetilde{R}$ is perfectoid. Then we had defined $Y^\diamondsuit = \operatorname{Spd}(\widetilde{R}^\flat, \widetilde{R}^{\flat+})/\underline{G}$, so

$$Y^\diamondsuit_{\text{fét}} \cong \left\{\text{fin. ét. } \widetilde{R}^\flat\text{-algebras w. cts. } G\text{-descent datum}\right\}$$

which is in turn equivalent to finite étale $\widetilde{R}$-algebras together with a G-descent datum, which (using [GR03]) is equivalent to finite étale R-algebras. □

The equivalence in Lemma 17.3.8 globalizes to general analytic adic spaces over $\operatorname{Spa}\mathbf{Z}_p$, and so we obtain an isomorphism $(X_C)_{\text{fét}} \cong (\mathcal{X}_{\text{FF}})_{\text{fét}}$.

Lemma 17.3.9. $(\mathcal{X}_{\text{FF}})_{\text{fét}} \cong (X_{\text{FF}})_{\text{fét}}$, *where* X_{FF} *is the (schematic) Fargues-Fontaine curve.*

Proof. Use GAGA for the curve, Thm. 13.5.6. □

Finally, the proof of Lemma 17.3.6 the theorem of Fargues-Fontaine: $(X_{\text{FF}})_{\text{fét}} \xrightarrow{\sim} (\mathbf{Q}_p)_{\text{fét}}$.

The proof of Thm. 17.3.4 combines two facts. The first is that a sort of Stein factorization exists for certain morphisms of diamonds, Prop. 17.3.10. The second is the analogue of Lemma 17.1.2 for diamonds, Prop. **??**.

Proposition 17.3.10. *Let* $F, X \to \operatorname{Spd} k$ *be diamonds, with* $F \to \operatorname{Spd} k$ *qcqs. Assume that for all algebraically closed nonarchimedean fields* C/k, F_C *is connected, and* $\pi_1(F_C) \to \pi_1(F)$ *is an isomorphism. Let* $Y = F \times_k X$, *and let* $\widetilde{Y} \to Y$ *be finite étale. There exists a finite étale morphism* $\widetilde{X} \to X$ *fitting into the diagram*

$$\begin{array}{ccc} \widetilde{Y} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array} \tag{17.3.1}$$

such that $\widetilde{Y} \to \widetilde{X}$ *has geometrically connected fibres. Furthermore,* $\widetilde{X} \to X$ *is unique up to unique isomorphism.*

Remark 17.3.11. Prop. 17.3.10 can be interpreted as a Stein factorization of the morphism $\widetilde{Y} \to X$.

Proof. First we establish the claim of uniqueness. This will follow from the following universal property of the diagram in Eq. (17.3.1): if such a diagram exists, it is the initial object in the category of diagrams

$$\begin{array}{ccc} \widetilde{Y} & \longrightarrow & Z \\ \big\downarrow & & \big\downarrow{\scriptstyle \text{finite étale}} \\ Y & \longrightarrow & X. \end{array}$$

Suppose $\widetilde{X}$ and Z fit into diagrams as above, where $\widetilde{Y} \to \widetilde{X}$ has geometrically connected fibres; we will produce a unique morphism $\widetilde{X} \to Z$.

We claim that the image of $\left|\widetilde{Y}\right| \to \left|\widetilde{X} \times_X Z\right|$ is open and closed. Let $x \hookrightarrow X$ be a geometric point, with $x = \mathrm{Spa}(C, \mathcal{O}_C)$. Then $Y_x = F \times_k x = F_C$, and $\widetilde{Y}_x \cong \coprod_{i \in I} (F_C)_i$ for a finite set I, with each $(F_C)_i \to F_C$ finite étale and connected. Since $\pi_1(F_C) = \pi_1(F)$, there exists $F_i \to F$ finite étale and connected such that $(F_C)_i = F_i \times_k C$.

We need the following analogue of Lemma 7.4.6 for diamonds:

Lemma 17.3.12. *The functor*

$$2\text{-}\varinjlim_{U \ni x} (F \times_k U)_{\text{fét}} \to (F_C)_{\text{fét}},$$

where U runs over étale neighborhoods of x in X, is an equivalence.

Proof. We may assume that X is affinoid perfectoid. Then Lemma 7.4.6 implies the result when F is affinoid perfectoid.

We will show that any finite étale cover of F_C spreads out to $F \times_k U$ for some U. Let $R \rightrightarrows F' \to F$ be a presentation of F, with F' perfectoid, $F' \to F$ a qpf surjection and $R = F' \times_F F'$. Since $F \to \mathrm{Spd}\, k$ is qcqs, F' and R are qc over $\mathrm{Spd}\, k$, and so we assume they are affinoid. Let $Y \to F_C$ be a finite étale cover; then $Y \times_{F_C} F'_C \to F'_C$ is a finite étale cover, so by Lemma 7.4.6 it spreads out to a finite étale cover $Y'_U \to F' \times_k U$. Both pullbacks of Y'_U to $R \times_k U$ have the same fibre over x, so after shrinking U they must be isomorphic. Thus Y'_U descends to a finite étale cover $Y_U \to F \times_k U$. $\square$

Lemma 17.3.12 shows that there exists an étale neighborhood U of x such that

$$\widetilde{Y}_U \cong \coprod_{i \in I} F_i \times_k U_i,$$

where $U_i \cong U$. (Indeed, both sides are objects in the limit appearing in Lemma 17.3.12; since they have the same fibre at x, they become isomorphic after passing to a smaller U.)

Since $\widetilde{Y} \to \widetilde{X}$ has geometrically connected fibres, and $\widetilde{Y}_x = \coprod_{i \in I} (F_i)_C$, we must have $\widetilde{X}_x = \coprod_{i \in I} x$. Thus over U, the morphism $\widetilde{Y} \to \widetilde{X}$ looks like this:

$$\begin{array}{ccc} \widetilde{Y}_U & \xrightarrow{\sim} & \coprod_{i \in I} F_i \times_k U_i \\ \downarrow & & \downarrow \\ \widetilde{X}_U & \xrightarrow[\sim]{} & \coprod_{i \in I} U_i \end{array}$$

Meanwhile, possibly after shrinking U we have $Z_U = \coprod_{j \in J} U_j$, with J finite and $U_j \cong U$. Over U, the morphism $\widetilde{Y} \to Z$ looks like

$$\begin{array}{ccc} \widetilde{Y}_U & \xrightarrow{\sim} & \coprod_{i \in I} F_i \times_k U_i \\ \Bigg\downarrow & & \downarrow \\ & & \coprod_{i \in I} U_i \\ & & \Big\downarrow {\scriptstyle U_i \to U_{r(i)}} \\ Z_U & \xrightarrow[\sim]{} & \coprod_{j \in J} U_j \end{array}$$

for some function $r\colon I \to J$. Now consider the product $\widetilde{X} \times_X Z$. Over U, the morphism $\widetilde{Y} \to \widetilde{X} \times_X Z$ looks like

$$\begin{array}{ccc} \widetilde{Y}_U & \xrightarrow{\sim} & \coprod_{i \in I} F_i \times_k U_i \\ \Bigg\downarrow & & \downarrow \\ & & \coprod U_i \\ & & \Big\downarrow {\scriptstyle U_i \to U_{ir(i)}} \\ (\widetilde{X}_U \times_X Z)_U & \xrightarrow[\sim]{} & \coprod_{(i,j) \in I \times J} U_{ij}, \end{array}$$

where $U_{ij} = U_i \times_U U_j \cong U$. Suppose x' is a geometric point of $\widetilde{X} \times_X Z$ lying over x. We have $x' \in U_{ij}$ for some (i, j). If $j = r(i)$ there exists an open neighborhood of x', namely U_{ij}, which is contained in the image of $\widetilde{Y}$. On the other hand if $j \neq r(i)$, then U_{ij} is disjoint from the image of $\widetilde{Y}$. We conclude that the image of $\left|\widetilde{Y}\right| \to \left|\widetilde{X} \times_X Z\right|$ is open and closed.

Thus $\widetilde{Y} \to \widetilde{X} \times_X Z$ factors through an open subdiamond $W \subset \widetilde{X} \times_X Z$ for which $W_U \cong \coprod_{i \in I} U_{ir(i)}$ for sufficiently small étale $U \to X$. This shows that the projection $W \to \widetilde{X}$ is an isomorphism, and so we get a morphism $\widetilde{X} \to W \to Z$ as required.

Now we turn to existence. The idea is to build $\widetilde{X}$ locally over each geometric point and then glue using the uniqueness result.

We have already seen that there exists an cover of X by étale opens $U \in \mathcal{U}$ such that for each U, $\widetilde{Y}_U \cong \coprod_{i \in I} F_i \times_k U$, with $F_i \to F$ finite étale and connected. Let $\widetilde{X}_U = \coprod_{i \in I} U$. Then $\widetilde{X}_U$ fits into a diagram

$$\begin{CD} \widetilde{Y}_U @>>> \widetilde{X}_U \\ @VVV @VVV \\ Y_U @>>> U, \end{CD}$$

where $\widetilde{X}_U \to Y_U$ is finite étale and $\widetilde{Y}_U \to \widetilde{X}_U$ has geometrically connected fibres. If $V \in \mathcal{U}$, then both $\widetilde{X}_U \times_X V$ and $\widetilde{X}_V \times_X U$ fit into a diagram as above over $U \times_X V$, and so by our uniqueness result there exists a unique isomorphism $f_{UV} \colon \widetilde{X}_U \times_X V \xrightarrow{\sim} \widetilde{X}_V \times_X U$ making the appropriate diagram commue. The uniqueness implies that f_{UV} satisfies the cocycle condition, and so the $\widetilde{X}_U$ glue together to form the diamond $\widetilde{X}$ as required by the proposition. □

Proposition 17.3.13. *Let k be a discrete algebraically closed field of characteristic p. Let F and X be diamonds over k. Assume that:*

1. *X is connected.*
2. *$F \to \operatorname{Spd} k$ is qcqs.*
3. *For all algebraically closed nonarchimedean fields C/k, F_C is connected, and the map $\pi_1(F_C) \to \pi_1(F)$ is an isomorphism.*

Then $F \times_k X$ is also connected, and the map $\pi_1(F \times_k X) \to \pi_1(F) \times \pi_1(X)$ is an isomorphism.

Proof. Let $Y = F \times_k X$.

1. We show that Y is connected. Clearly $Y \neq \emptyset$. Assume $Y = Y_1 \sqcup Y_2$, with Y_i open and closed. Suppose $\overline{x} = \operatorname{Spa}(C, \mathcal{O}_C) \hookrightarrow X$ is a (rank 1) geometric point. By hypothesis (3), the fiber $Y_{\overline{x}}$ is connected, and thus $(Y_i)_{\overline{x}}$ is empty for one of the i, say $i = 1$. We claim there exists a

pro-étale neighborhood U of $\overline{x}$ in X such that $Y_1 \times_X U$ is also empty. Since étale maps are open, and since the images of geometric points are dense in X, the claim implies that $\{x \in X | (Y_1)_x = \emptyset\}$ is open, and similarly for Y_2. By hypothesis (1), X is connected, and therefore one of these sets is empty, and thus one of the Y_i is empty, showing that Y is connected.

The claim is local for the pro-étale topology on X, so we may assume that X is affinoid perfectoid. Since the morphism $F \to \operatorname{Spd} k$ is qc, so is the base change $Y = F \times_k X \to X$, and therefore Y is also qc. Choose a qpf cover $Z^\diamondsuit \to Y_1$ by a perfectoid space Z, which (because Y_1 is qc) may be assumed to be affinoid. Since

$$\overline{x} = \varprojlim_U U,$$

where U runs over étale neighborhoods of $\overline{x}$ in X, we have

$$Z_{\overline{x}} = \varprojlim_U Z \times_X U.$$

Since $Y_{\overline{x}} = \emptyset$, we have $Z_{\overline{x}} = \emptyset$ as well. On the level of topological spaces, we have here an empty inverse limit of spectral spaces along spectral maps, which implies that $Z \times_X U = \emptyset$ for some U (cf. the proof of Lemma 8.2.3), and therefore $Y_1 \times_X U$ is empty.

2. We show that $\pi_1(F \times_k X) \to \pi_1(X)$ is surjective. First we claim that for all finite étale $\widetilde{X} \to X$ with $\widetilde{X}$ connected, we have that $F \times \widetilde{X}$ is connected. This follows from (1) applied to $\widetilde{X}$. Now, a connected étale cover of X corresponds to a continuous transitive action of $\pi_1(X)$ on a finite set. So the claim is equivalent to saying that every such action restricts to a transitive action of $\pi_1(F \times_k X)$. It is a simple exercise to see that this is equivalent to the surjectivity of $\pi_1(F \times_k X) \to \pi_1(X)$.

3. Let $\overline{x} = \operatorname{Spa}(C, \mathcal{O}_C) \hookrightarrow X$ be a geometric point. Let $F_C = F \otimes_k \overline{x}$. We claim that the sequence

$$\pi_1(F_C) \to \pi_1(F \times_k X) \to \pi_1(X)$$

is exact in the middle.

Prop. 17.3.10 translates into the following fact about these groups: Given a finite quotient G of $\pi_1(F \times_k X)$, corresponding to a Galois cover $\widetilde{Y}$ of $Y = F \times_k X$, there exists a quotient $G \to H$, corresponding to a Galois cover $\widetilde{X} \to X$ as in Prop. 17.3.10. Consider the homomorphism $\pi_1(F_C) \to \ker(G \to H)$: the cosets of its image correspond to connected components in the fibre of $\widetilde{Y} \to \widetilde{X}$ over a geometric point. But the fibres of $\widetilde{Y} \to \widetilde{X}$ are geometrically connected, so $\pi_1(F_C) \to \ker(G \to H)$ is surjective. This suffices to prove the claim.

Putting together (1), (2) and (3), we have the following diagram of groups, where the top row is an exact sequence:

$$\begin{array}{ccccccc} \pi_1(F_C) & \longrightarrow & \pi_1(F \times_k X) & \longrightarrow & \pi_1(X) & \longrightarrow & 1 \\ & \searrow^{\sim} & \downarrow & & & & \\ & & \pi_1(F) & & & & \end{array}$$

This shows that $\pi_1(F \times_k X) \to \pi_1(F) \times \pi_1(X)$ is an isomorphism. $\square$

18 Examples of diamonds, 6 November

Today we will construct some interesting examples of diamonds, with a view towards defining the mixed-characteristic affine Grassmannian.

18.1 The self-product $\operatorname{Spd} \mathbf{Q}_p \times \operatorname{Spd} \mathbf{Q}_p$

We already encountered $\operatorname{Spd} \mathbf{Q}_p \times \operatorname{Spd} \mathbf{Q}_p$ in previous lectures. It is useful to keep in mind that a diamond can $\mathcal{D}$ can have multiple "incarnations", by which we mean that there are multiple presentations of $\mathcal{D}$ as $X^\diamondsuit/\underline{G}$, where X is an analytic adic space over $\operatorname{Spa} \mathbf{Z}_p$, and G is a profinite group. In the case of $\operatorname{Spd} \mathbf{Q}_p \times \operatorname{Spd} \mathbf{Q}_p$, there are (at least) the following two incarnations:

1. $X = \widetilde{\mathbf{D}}^*_{\mathbf{Q}_p}$, $G = \mathbf{Z}_p^\times$.
2. $X = \operatorname{Spa} W(\mathcal{O}_{C_p}) \backslash \left\{ p[p^\flat] = 0 \right\}$, $G = G_{\mathbf{Q}_p}$.

The first incarnation was discussed in Example 10.1.6. Recall that $\widetilde{\mathbf{D}}^*_{\mathbf{Q}_p} = \varprojlim \mathbf{D}^*_{\mathbf{Q}_p}$ (with transition map $T \mapsto (1+T)^p - 1$) is the punctured perfectoid open unit disc. This has an action of $\mathbf{Z}_p^\times$ via $t \mapsto (1+t)^a - 1$. Then we have an isomorphism of diamonds $\widetilde{\mathbf{D}}^{*,\diamondsuit}_{\mathbf{Q}_p}/\mathbf{Z}_p^\times \cong \operatorname{Spd} \mathbf{Q}_p \times \operatorname{Spd} \mathbf{Q}_p$. One of the partial Frobenii on $\operatorname{Spd} \mathbf{Q}_p \times \operatorname{Spd} \mathbf{Q}_p$ corresponds to the automorphism $t \mapsto (1+t)^p - 1$; let us define an action of $\mathbf{Q}_p^\times$ on $\widetilde{\mathbf{D}}^*_{\mathbf{Q}_p}$ by declaring that p acts as $t \mapsto (1+t)^p - 1$. The case $n = 2$ of Drinfeld's Lemma for diamonds

(Thm. 17.3.3) shows that $\pi_1(\widetilde{\mathbf{D}}^*_{\mathbf{Q}_p}/\mathbf{Q}_p^\times) \cong G_{\mathbf{Q}_p} \times G_{\mathbf{Q}_p}$. In particular if $C/\mathbf{Q}_p$ is complete an algebraically closed we have

$$\pi_1((\widetilde{\mathbf{D}}^*_C)^\diamondsuit/\mathbf{Q}_p^\times) \cong G_{\mathbf{Q}_p}.$$

Thus (rather surprisingly) $G_{\mathbf{Q}_p}$ can be realized as a *geometric fundamental group*, cf. [Wei13].

Consequently, to each finite extension $F/\mathbf{Q}_p$ of degree n there must correspond a connected $\mathbf{Q}_p^\times$-equivariant finite étale n-fold cover of $\widetilde{\mathbf{D}}_C^\times$; it is natural to ask what this cover is. Let $\varpi \in \mathcal{O}_F$ be a uniformizer, and let $\mathrm{LT}/\mathcal{O}_F$ be a *Lubin-Tate formal $\mathcal{O}_F$-module law*: this is a formal scheme isomorphic to $\operatorname{Spf} \mathcal{O}_F[\![T]\!]$ equipped with an $\mathcal{O}_F$-module structure, with the property that multiplication by ϖ sends T to a power series congruent to T^q modulo ϖ (here $q = \#\mathcal{O}_F/\varpi$). Then we can form the geometric generic fibre LT_C: this is an $\mathcal{O}_F$-module object in the category of adic spaces, whose underlying adic space is once again the open unit disc $\mathbf{D}_C$.

Now let $\widetilde{\mathrm{LT}}_C = \varprojlim_\varpi \mathrm{LT}_C$. Then $\widetilde{\mathrm{LT}}_C$ is an F-vector space object in the category of adic spaces, whose underlying adic space is the perfectoid open unit disc $\widetilde{\mathbf{D}}_C$.

One can define a *norm map* $N_{F/\mathbf{Q}_p}\colon \mathrm{LT}_C \to \mathbf{D}_C$, which is a (non-linear) morphism of pointed adic spaces. Its construction goes as follows. Let $\breve{F}$ be the completion of the maximal unramified extension of F; in fact the norm map is defined over $\mathcal{O}_{\breve{F}}$. We have that $\mathrm{LT}[p^\infty]_{\mathcal{O}_{\breve{F}}}$ is a p-divisible group of height n and dimension 1. By [Hed], the nth exterior power of $\mathrm{LT}[p^\infty]_{\mathcal{O}_{\breve{F}}}$ exists as a p-divisible group of dimension 1 and height 1, and so is isomorphic to $\mu_{p^\infty,\mathcal{O}_{\breve{F}}}$. Thus we have an alternating map $\lambda\colon \mathrm{LT}[p^\infty]^n_{\mathcal{O}_{\breve{F}}} \to \mu_{p^\infty,\mathcal{O}_{\breve{F}}}$. Let $\alpha_1, \dots, \alpha_n$ be a basis for $\mathcal{O}_F/\mathbf{Z}_p$, and let $N(x) = \lambda(\alpha_1 x, \dots, \alpha_n x)$, so that N is a morphism $\mathrm{LT}[p^\infty]_{\mathcal{O}_{\breve{F}}} \to \mu_{p^\infty,\mathcal{O}_{\breve{F}}}$. This induces a map of formal schemes $\lambda\colon LT^n_{\mathcal{O}_{\breve{F}}} \to \operatorname{Spf} \mathcal{O}_{\breve{F}}[\![t]\!]$, which becomes the desired norm map $N_{F/\mathbf{Q}_p}\colon \mathrm{LT}_C \to \mathbf{D}_C$ after passing to geometric generic fibres.

By construction we have $N_{F/\mathbf{Q}_p}(\alpha x) = N_{F/\mathbf{Q}_p}(\alpha) N_{F/\mathbf{Q}_p}(x)$ for all $\alpha \in \mathcal{O}_F^\times$. In particular $x \mapsto N_{F/\mathbf{Q}_p}(\varpi x)$ and $x \mapsto p N_{F/\mathbf{Q}_p}(x)$ agree up to an automorphism of $\mathbf{D}_C$. Therefore $N_{F/\mathbf{Q}_p}$ can be used to define a norm map $\widetilde{\mathrm{LT}} \to \mathbf{D}_C$, which is the desired finite étale cover.

The other incarnation of $\operatorname{Spd} \mathbf{Q}_p \times \operatorname{Spd} \mathbf{Q}_p$ as the quotient of an adic space by a profinite group will lead us into a discuss of the Fargues-Fontaine curve and Fontaine's period rings. Recall from Prop. 11.2.2 that "$SpaC_p^\flat \times \operatorname{Spa} \mathbf{Z}_p''$ is the analytic adic space $\mathcal{Y}_{[0,\infty)} = SpaW(\mathcal{O}_{\mathbf{C}_p^\flat}) \backslash \{[p^\flat]\} = 0$, and that its associated diamond is $\operatorname{Spd} C_p^\flat \times \operatorname{Spd} \mathbf{Z}_p$. After inverting p, we find that the diamond associated to $\mathcal{Y}_{(0,\infty)} = \operatorname{Spa} W(\mathcal{O}_{\mathbf{C}_p}^\flat) \setminus \{p[p^\flat] = 0\}$ is $\operatorname{Spd} C_p^\flat \times \operatorname{Spd} \mathbf{Q}_p$. It follows from this that $\mathcal{Y}_{(0,\infty)}^\diamondsuit / G_{\mathbf{Q}_p} = \operatorname{Spd} \mathbf{Q}_p \times \operatorname{Spd} \mathbf{Q}_p$.

Recall that the Fargues-Fontaine curve was defined as $\mathcal{X}_{\mathrm{FF}} = \mathcal{Y}_{(0,\infty)}/\phi^{\mathbf{Z}}$. It is a result of [FF11] that $\mathcal{X}_{\mathrm{FF}}$ is geometrically simply connected, which is to say that $\pi_1(\mathcal{X}_{\mathrm{FF}}) = G_{\mathbf{Q}_p}$. On the other hand, $\pi_1(\mathcal{X}_{\mathrm{FF}}) = \pi_1(X^\diamondsuit) = \pi_1(\operatorname{Spd} \mathbf{C}_p \times \operatorname{Spd} \mathbf{Q}_p)/(\text{p.Fr.})$, and so the simply-connectedness of $\mathcal{X}_{\mathrm{FF},C}$ was a special case of Drinfeld's Lemma in disguise.

18.2 Finite-Dimensional Banach Spaces, after Colmez

We recall the construction of the de Rham period ring. Let (R, R^+) be a perfectoid Huber pair. We have the surjective homomorphism $\theta \colon W(R^{\flat+}) \to R^+$, whose kernel is generated by a non-zero-divisor ξ. Let $\varpi^\flat \in R^\flat$ be a pseudouniformizer, so that $\varpi = (\varpi^\flat)^\sharp$ satisfies $\varpi^p | p$.

We get a surjection $\theta \colon W(R^{\flat+})[[\varpi^\flat]^{-1}] \to R = R^+[\varpi^{-1}]$. Let $B_{\mathrm{dR}}^+(R)$ be the ξ-adic completion of $W(R^{\flat+})[[\varpi^\flat]^{-1}]$. This comes with a canonical filtration $\operatorname{Fil}^i B_{\mathrm{dR}}^+ = \xi^i B_{\mathrm{dR}}^+$, whose associated gradeds are $\operatorname{gr}^i B_{\mathrm{dR}}^+ \cong \xi^i R$.

Philosophically, we think of $\operatorname{Spf} B_{\mathrm{dR}}^+(R)$ as the completion of "$\operatorname{Spec} \mathbf{Z} \times \operatorname{Spec} R$" along the graph of $\operatorname{Spec} R \hookrightarrow \operatorname{Spec} \mathbf{Z}$. The construction of $B_{\mathrm{dR}}^+(R)$ encompasses the following two important cases:

Example 18.2.1.

1. Let $R = C/\mathbf{Q}_p$ be algebraically closed and complete. Then $B_{\mathrm{dR}}^+(C)$ is the usual de Rham period ring of Fontaine. It is a complete DVR with residue field C. Therefore it is isomorphic to $C[\![\xi]\!]$, but there is no canonical isomorphism. (For instance, such an isomorphism would not respect the action of $G_{\mathbf{Q}_p}$ in the case $C = \mathbf{C}_p$.)
2. If R has characteristic p, then we can take $\xi = p$, and $B_{\mathrm{dR}}^+(R) = W(R)$.

Recall that $\operatorname{Spd} \mathbf{Q}_p$ is the functor on (Perf) which assigns to a perfectoid Huber pair (R, R^+) the set of un-tilts $(R^\sharp, R^{\sharp,+})$, where $R^\sharp$ has characteristic 0 (*i.e.* it is a $\mathbf{Q}_p$-algebra). We are about to define a few functors on (Perf) which are fibred over $\operatorname{Spd} \mathbf{Q}_p$. One may think of such a functor as being defined on the category of perfectoid Huber pairs (R, R^+) in characteristic 0.

Definition 18.2.2. Let $B_{\mathrm{dR}}^+/\operatorname{Fil}^i \to \operatorname{Spd} \mathbf{Q}_p$ be the functor on (Perf) which assigns to a characteristic 0 perfectoid Huber pair (R, R^+) the $\mathbf{Q}_p$-vector space $B_{\mathrm{dR}}^+(R)/\operatorname{Fil}^i B_{\mathrm{dR}}^+(R)$.

Proposition 18.2.3.

1. *$B_{\mathrm{dR}}^+/\operatorname{Fil}^i$ is a diamond.*

2. *$B^+_{\mathrm{dR}}/\operatorname{Fil}^i$ is a successive extension of forms of $\mathbf{A}^1$. More precisely, there are exact sequences of presheaves*

$$0 \to \operatorname{gr}^i B^+_{\mathrm{dR}} \to B^+_{\mathrm{dR}}/\operatorname{Fil}^{i+1} \to B^+_{\mathrm{dR}}/\operatorname{Fil}^i \to 0,$$

where $\operatorname{gr}^i B^+_{\mathrm{dR}} \cong (\mathbf{A}^1_{\mathbf{Q}_p})^\diamondsuit \otimes_{\underline{\mathbf{Q}}_p} \underline{\mathbf{Q}}_p(i)$, where $\underline{\mathbf{Q}}_p(i)$ is the ith Tate twist.

For instance, after base changing to $\mathbf{Q}_p^{\mathrm{cycl}}$ we have an exact sequence

$$0 \to (\mathbf{A}^1_{\mathbf{Q}_p^{\mathrm{cycl}}})^\diamondsuit \to B^+_{\mathrm{dR}}/\operatorname{Fil}^2 \times_{\operatorname{Spd}\mathbf{Q}_p} \operatorname{Spd}\mathbf{Q}_p^{\mathrm{cycl}} \to (\mathbf{A}^1_{\mathbf{Q}_p^{\mathrm{cycl}}})^\diamondsuit \to 0.$$

Thus $B^+_{\mathrm{dR}}/\operatorname{Fil}^2$ resembles a unipotent group. However we caution that $B^+_{\mathrm{dR}}/\operatorname{Fil}^2$ is not a rigid space–it really only exists as a diamond.

Proof. First we prove the claimed description of $\operatorname{gr}^i B^+_{\mathrm{dR}}$. Let $t = \log([\varepsilon]) \in B^+_{\mathrm{dR}}(\mathbf{Q}_p^{\mathrm{cycl}})$, where $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathbf{Q}_p^{\mathrm{cycl},\flat}$. One checks that the logarithm series converges, and that t generates the kernel of $\theta\colon B^+_{\mathrm{dR}}(\mathbf{Q}_p^{\mathrm{cycl}}) \to \mathbf{Q}_p^{\mathrm{cycl}}$. Then $\operatorname{gr}^i B^+_{\mathrm{dR}} \times_{\operatorname{Spd}\mathbf{Q}_p} \operatorname{Spd}\mathbf{Q}_p^{\mathrm{cycl}} \cong t^i(\mathbf{A}^1_{\mathbf{Q}_p^{\mathrm{cycl}}})^\diamondsuit$. Since the action of $\operatorname{Gal}(\mathbf{Q}_p^{\mathrm{cycl}}/\mathbf{Q}_p)$ on t is the cyclotomic character, we can descend the picture to $\operatorname{Spd}\mathbf{Q}_p$. The claim (2) is now clear from the description of $B^+_{\mathrm{dR}}(R^\sharp)$.

For (1), it is enough to prove that $B^+_{\mathrm{dR}}/\operatorname{Fil}^i \times_{\operatorname{Spd}\mathbf{Q}_p} \operatorname{Spd}\mathbf{Q}_p^{\mathrm{cycl}}$ is a diamond. We will now assume that the base is always $\operatorname{Spd}\mathbf{Q}_p^{\mathrm{cycl}}$, so that we can ignore Tate twists. We argue by induction on i, the case $i = 1$ being trivial. Consider the exact sequence of presheaves:

$$0 \longrightarrow B^+_{\mathrm{dR}}/\operatorname{Fil}^{i-1} \xrightarrow{\times t} B^+_{\mathrm{dR}}/\operatorname{Fil}^i \xrightarrow{\theta} (\mathbf{A}^1)^\diamondsuit \longrightarrow 0$$

We claim that θ is qpf-locally split. Proof: Let $X = \varprojlim_{T \mapsto T^p} \mathbf{A}^1_{\mathbf{Q}_p^{\mathrm{cycl}}}$; then the projection $X \to \mathbf{A}^1_{\mathbf{Q}_p^{\mathrm{cycl}}}$ is a perfectoid qpf cover. Let $T^\flat = (T, T^{1/p}, \dots) \in H^0(X, \mathcal{O}_X)$. Then $[T^\flat] \in (B^+_{\mathrm{dR}}/\operatorname{Fil}^i)(X)$ maps to T under θ; this means exactly that we have a morphism $X^\diamondsuit \to B^+_{\mathrm{dR}}/\operatorname{Fil}^i$ which makes the diagram commute:

$$\begin{array}{ccccccccc} & & & & & & X^\diamondsuit & & \\ & & & & & \swarrow & \downarrow & & \\ 0 & \longrightarrow & B^+_{\mathrm{dR}}/\operatorname{Fil}^{i-1} & \xrightarrow{\times t} & B^+_{\mathrm{dR}}/\operatorname{Fil}^i & \xrightarrow{\theta} & (\mathbf{A}^1)^\diamondsuit & \longrightarrow & 0 \end{array}$$

By the inductive hypothesis, $B^+_{\mathrm{dR}}/\operatorname{Fil}^{i-1}$ is a diamond. We have a qpf surjection $B^+_{\mathrm{dR}}/\operatorname{Fil}^i \times_{(\mathbf{A}^1)^\diamondsuit} X \cong B^+_{\mathrm{dR}}/\operatorname{Fil}^{i-1} \times X \to B^+_{\mathrm{dR}}/\operatorname{Fil}^i$, which shows that $B^+_{\mathrm{dR}}/\operatorname{Fil}^i$ is a diamond as well. $\square$

The sections of $B^+_{\mathrm{dR}}/\operatorname{Fil}^2$ over $\mathbf{C}_p$ form an infinite-dimensional $\mathbf{Q}_p$-vector space which is not a $\mathbf{C}_p$-vector space. Rather $(B^+_{\mathrm{dR}}/\operatorname{Fil}^2)(\mathbf{C}_p)$ is an extension of $\mathbf{C}_p$ by $\mathbf{C}_p$. In [Col02], $B^+_{\mathrm{dR}}/\operatorname{Fil}^2$ is called a Banach Space of Dimension 2 (note the capital letters).

Another family of such Banach Spaces is defined as follows.

Definition 18.2.4. Let (R, R^+) be an object of (Perf), and let $\varpi \in R$ be a pseudo-uniformizer.

1. Let $B(R, R^+)$ be the ring of global sections of the structure sheaf of $\operatorname{Spa} W(R^+)\backslash \{p[\varpi] = 0\}$.
2. Let $B^{\phi=p^n}(R, R^+) = B(R, R^+)^{\phi=p^n}$.

Remark 18.2.5.

1. The same functor on (Perf) can be constructed using the crystalline period ring $B^+_{\mathrm{cris}}(R^+/\varpi)$. (To define $B^+_{\mathrm{cris}}(R^+/\varpi)$, start with the universal PD thickening of the semiperfect ring R^+/ϖ, complete at p, and then invert p). Then $B^+_{\mathrm{cris}}(R^+/\varpi)^{\phi=p^n} = B(R, R^+)^{\phi=p^n}$.
2. If $n = 0$, then $B^{\phi=1} = \underline{\mathbf{Q}}_p$. (Thus $\underline{\mathbf{Q}}_p$ is the universal cover of the constant p-divisible group $\mathbf{Q}_p/\mathbf{Z}_p$.)
3. If $n = 1$, then $B(R, R^+)^{\phi=p} = B^+_{\mathrm{cris}}(R^+/\varpi)^{\phi=p}$. Since R^+/ϖ is semiperfect (meaning that Φ is surjective), a result in [SW13, §4.2] shows that $B^+_{\mathrm{cris}}(R^+/\varpi)^{\phi=p}$ is isomoprhic to the $\mathbf{Q}_p$-vector space of isogenies $\mathbf{Q}_p/\mathbf{Z}_p \to \mu_{p^\infty}$ over R^+/ϖ, which is in turn the same as $\widetilde{\mu}_{p^\infty}(R^+/\varpi)$. Therefore $B^{\phi=p} = \operatorname{Spd} \mathbf{F}_p[\![t^{1/p^\infty}]\!]$.
4. In both of the above cases $B^{\phi=p^n}$ is (once one picks a base) representable by an adic space, namely the universal cover of a p-divisible group. For $n > 1$, we should think of $B^{\phi=p^n}$ as the universal cover of a "formal group of slope n", cf. Scholze's talk in February at MSRI.

Proposition 18.2.6.

1. *$B^{\phi=p^n} \times \operatorname{Spd} \mathbf{Q}_p$ is a diamond.*
2. *There is a short exact sequence of qpf sheaves of $\mathbf{Q}_p$-vector spaces*

$$0 \longrightarrow \underline{\mathbf{Q}}_p(n) \longrightarrow B^{\phi=p^n} \times \operatorname{Spd} \mathbf{Q}_p \longrightarrow B^+_{\mathrm{dR}}/\operatorname{Fil}^n \longrightarrow 0$$

(the "fundamental exact sequence of p-adic Hodge theory").

Proof. Again we extend scalars to $\operatorname{Spd}\mathbf{Q}_p^{\mathrm{cycl}}$ (and descend later). Once again let $t = \log([\varepsilon]) \in B^{\phi=p}(\mathbf{Q}_p^{\mathrm{cycl}})$. The case $n = 0$ of (2) is trivial. For $n = 1$ we use the isomorphism $B^{\phi=p} \times \operatorname{Spd}\mathbf{Q}_p^{\mathrm{cycl}} = \widetilde{\mathbf{D}}^{\diamondsuit}$. This sequence gets identified with

$$0 \to \underline{\mathbf{Q}}_p t \to \widetilde{\mathbf{D}}^{\diamondsuit} \to \mathbf{G}_{\mathrm{a}}^{\diamondsuit} \to 0,$$

which is exact; in fact it is the universal cover of

$$0 \to \mu_{p^\infty} \to \mathbf{D}^{\diamondsuit} \to \mathbf{G}_{\mathrm{a}}^{\diamondsuit} \to 0$$

where "universal cover" just means take inverse limit under multiplication by p. It is enough to show that this latter sequence is exact, because multiplication by p is finite étale on $\mathbf{D}^{\diamondsuit}$. But in a small neighborhood of 0 in $\mathbf{D}^{\diamondsuit}$, log is invertible. Thus log is étale locally surjective – it is an *étale covering map* in the sense of [dJ95]. This can be used to prove exactness.

For $n > 1$ we apply induction. Consider the diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \underline{\mathbf{Q}}_p(n-1) & \xrightarrow{t^{n-1}} & B^{\phi=p^n} & \longrightarrow & B_{\mathrm{dR}}^{+}/\operatorname{Fil}^{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\cong} & & \downarrow & & \downarrow{\scriptstyle t} & & \\
0 & \longrightarrow & \underline{\mathbf{Q}}_p(n) & \xrightarrow[t^n]{} & B^{\phi=p^n} & \longrightarrow & B_{\mathrm{dR}}^{+}/\operatorname{Fil}^{n} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbf{G}_{\mathrm{a}} & \xrightarrow{=} & \mathbf{G}_{\mathrm{a}} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

We are given that the top row is exact. The middle row is left exact because t^n is not a zero divisor. For exactness of the column on the left: use an element $t' \in B^{\phi=p}$ for which $\theta(t') = 1$ (this exists pro-étale locally). Then right exactness of the middle row is a diagram chase. All that remains is exactness in the middle row. This will be equivalent to the exactness in the middle of

$$0 \to B^{\phi=p^{n-1}} \xrightarrow{t} B^{\phi=p^n} \to \mathbf{G}_a \to 0.$$

If $f \in B$ satsifies $\phi(f) = p^n f$, and $\theta(f) = 0$, then $(\theta \circ \phi^i)(f) = 0$ for all i. But t vanishes precisely at all roots of $\theta \circ \phi^i$ with multiplicity 1, which shows that f is divisible by t.

Now we turn to (1): we must show that $B^{\phi=p^n} \times \operatorname{Spd} \mathbf{Q}_p^{\mathrm{cycl}}$ is a diamond. The proof of (2) shows that there are qpf locally sections of the projection map $B^{\phi=p^n} \times \operatorname{Spa} \mathbf{Q}_p \to B_{\mathrm{dR}}^{+} / \operatorname{Fil}^n$. Then the same argument works as for $B_{\mathrm{dR}}^{+} / \operatorname{Fil}^n$ to show that $B^{\phi=p^n} \times \operatorname{Spd} \mathbf{Q}_p$ is a diamond. □

19 Moduli spaces of shtukas, 18 November

Today we discuss moduli spaces of local mixed-characteristic shtukas, and relate them to moduli of p-divisible groups in the simplest cases.

19.1 Local shtuka data

We first specify the data required to define a moduli space of local shtukas. These will resemble the data required to define moduli spaces of global (equal-characteristic) shtukas as introduced by Varshavsky, [Var04].

Definition 19.1.1. A *local shtuka datum* is a triple $(G, b, \{\overline{\mu}_i\})$, consisting of:

- A reductive group G defined over $\mathbf{Q}_p$ (and often assumed to be GL_r),
- A *σ-conjugacy class* $b \in B(G)$ in the sense of Kottwitz, [Kot85], see note below,
- A collection $\{\overline{\mu}_i\}$ of conjugacy classes of cocharacters $\mu_i \colon \mathbf{G}_{\mathrm{m}} \to G_{\overline{\mathbf{Q}}_p}$ for $i = 1, 2, \ldots, n$.

We recall here the definition of $B(G)$ for a reductive group G over a local field. (Kottwitz defines $B(G)$ in the global setting as well.) Let k be an algebraically closed field of characteristic p, let $L = W(k)[1/p]$, and as usual let $\sigma \in \operatorname{Aut} L$ be induced from the pth power Frobenius on k. One has an action of $G(L)$ on itself by σ-conjugation, defined for $h \in G(L)$ by $g \mapsto h^{-1} g \sigma(h)$. Let $B(G)$ be the set of σ-conjugacy classes in $G(L)$.

Theorem 19.1.2 (Kottwitz)**.** *The set $B(G)$ is independent of k.*

Example 19.1.3. In the case $G = \mathrm{GL}_r$, one has a bijection:

$$\begin{array}{rcl} B(G) & \stackrel{\sim}{\to} & \{\text{isocrystals } (N, \phi) \text{ over } L\} / \cong \\ g & \mapsto & (L^r, g\sigma). \end{array}$$

By the Dieudonné-Manin classification, isomorphism classes of isocrystals are in bijection with n-tuples $\lambda_1 \geq \cdots \geq \lambda_r$ of rational numbers satisfying the condition

$$\lambda \# \{i | \lambda_i = \lambda\} \in \mathbf{Z}.$$

These in turn are in bijection with the set of convex polygons in the plane whose breaks occur at integer lattice points, *i.e.*, Newton polygons. We say that $(\lambda_1, \ldots, \lambda_r)$ is the *Newton point* of b.

For general G, Kottwitz constructs a *Newton map*

$$\nu \colon B(G) \to (X_*(G) \otimes \mathbf{Q})_{\mathrm{dom}},$$

where the right-hand side is the set of dominant rational cocharacters; in the case of $G = \mathrm{GL}_r$ we may identify $X_*(\mathrm{GL}_r)$ with $\mathbf{Z}^r$, and then $\nu(b)$ is the Newton point of b.

Kottwitz also constructs a map

$$\varkappa \colon B(G) \to \pi_1(G_{\overline{\mathbf{Q}}_p})_\Gamma,$$

where $\Gamma = G_{\mathbf{Q}_p}$.

Remark 19.1.4. In the equal characteristic case, where $\mathbf{Q}_p$ is replaced with $k(\!(t)\!)$, we can think of $\varkappa$ as a map

$$G(k(\!(t)\!)) \to \pi_1(G)_\Gamma$$

which maps an algebraic loop to a topological loop.

Then

$$(\lambda, \kappa) \colon B(G) \hookrightarrow (X_*(G) \otimes \mathbf{Q})_{\mathrm{dom}} \bigoplus \pi_1(G_{\overline{\mathbf{Q}}_p})_\Gamma$$

is an injection, and one can describe the image. For $G = \mathrm{GL}_r$, $\kappa(g) = v_p(\det g) \in \mathbf{Z} = \pi_1(\mathrm{GL}_r)$ is already determined by the Newton point of g: $\kappa(g) = \sum_i \lambda_i$.

19.2 Vector bundles on the Fargues-Fontaine curve

Let S be a perfectoid space over $\mathrm{Spa}\, k$, where k is our algebraically closed field of characteristic p. Recall from Defn. 11.4.1 that a shtuka over S is a vector bundle $\mathcal{E}$ over "$S \times \mathrm{Spa}\, \mathbf{Z}_p$" equipped with an isomorphism $\phi_{\mathcal{E}} \colon \phi^*\mathcal{E} \to \mathcal{E}$ away from finitely many paws. Also recall from Prop. 11.3.2 that when $S = \mathrm{Spa}(R, R^+)$ is affinoid with pseudo-uniformizer ϖ, we have

$$\text{“}S \times \mathrm{Spa}\, \mathbf{Z}_p\text{”} = \mathrm{Spa}\, W(R^+) \backslash \{[\varpi] = 0\},$$

an adic space over L. Let us call this space $\mathcal{Y}_{[0,\infty)}(R, R^+)$. In the case that $(R, R^+) = (C, \mathcal{O}_C)$ is a geometric point, this agrees with the space $\mathcal{Y}_{[0,\infty)}$ from §12. We have a continuous map

$$\kappa\colon \mathcal{Y}_{[0,\infty)} \to [0, \infty),$$

the relative version of the map κ from §12.2.

Our intention is to extend the definition of a shtukas to general reductive groups G, by replacing the vector bundle $\mathcal{E}$ with a G-bundle. The first order of business is to classify G-shtukas with no paws in the case that $S = \mathrm{Spa}(C, \mathcal{O}_C)$ is a geometric point. If $\mathcal{E}$ is a G-shtuka over $\mathcal{Y}_{[0,\infty)}$ with no paws, then $\mathcal{E}|_{\mathcal{Y}_{(0,\infty)}}$ descends to a G-bundle on the quotient $\mathcal{X}_{\mathrm{FF}} = \mathcal{Y}_{(0,\infty)}/\phi^{\mathbf{Z}}$; these are classified by the following theorem.

Theorem 19.2.1 (Fargues-Fontaine for $G = \mathrm{GL}_r$, Fargues for all G)**.** *There is a bijection*

$$B(G) \xrightarrow{\sim} \{G\text{-}bundles\ over\ \mathcal{X}_{\mathrm{FF}}\} \,/ \cong .$$

Remark 19.2.2. In the case $G = \mathrm{GL}_r$, this is the classification of vector bundles on $\mathcal{X}_{\mathrm{FF}}$ from [FF11]. We recall its construction. An element $b \in B(\mathrm{GL}_r)$ corresponds to an isocrystal (N, ϕ) over L. We get a pair $(\mathcal{E}_b, \phi_{\mathcal{E}_b})$ over $\mathcal{Y}_{(0,\infty)}$, where

$$\mathcal{E}_b = \mathcal{O}_{\mathcal{Y}_{(0,\infty)}} \otimes_L N,\ \phi_{\mathcal{E}} = \phi \otimes \phi.$$

Now suppose that $(\mathcal{E}, \phi_{\mathcal{E}})$ is any shtuka over $Y_{[0,\infty)}$. For ρ large enough to avoid the paws, $(\mathcal{E}, \phi_{\mathcal{E}})|_{[\rho,\infty)}$ is a ϕ-module over $\mathcal{Y}_{[\rho,\infty)}$. By the "Frobenius pullback" trick discussed in §13.4, there exists a unique extension of $(\mathcal{E}, \phi_{\mathcal{E}})|_{[\rho,\infty)}$ to a ϕ-module defined on all of $\mathcal{Y}_{(0,\infty)}$. Thus by Thm. 19.2.1 there exists $b \in B(G)$ and an isomorphism

$$\iota\colon (\mathcal{E}, \phi_{\mathcal{E}})|_{[\rho,\infty)} \cong (\mathcal{E}_b, \phi_{\mathcal{E}_b})|_{[\rho,\infty)}. \tag{19.2.1}$$

Returning to the relative case, this discussion shows that if $(\mathcal{E}, \phi_{\mathcal{E}})$ a shtuka over S, then each geometric point of S determines an element $b \in B(G)$. Since $B(G)$ is discrete, this suggests that in order to define a nice moduli space of shtukas, we ought to fix a σ-conjugacy class $b \in B(G)$ in advance, and include ι as part of the moduli problem.

Remark 19.2.3. We note however that Fargues has a new formulation of geometric Langlands in mixed characteristic which requires working with a very stacky space Bun_G which classifies all G-bundles on the Fargues-Fontaine curve. The geometric points of Bun_G are then $B(G)$, but with each $b \in B(G)$ being an orbifold point whose automorphism group is the centralizer J_b of b in $G(L)$. Without specifying b in advance and rigidifying the moduli problem using ι, our moduli space of shtukas would be fibred over this Bun_G.

19.3 Definition of the moduli spaces of shtukas

A local shtuka datum includes a collection $\overline{\mu} = \{\mu_i\}$, where $\mu_i \colon \mathbf{G}_\mathrm{m} \to G_{\overline{\mathbf{Q}}_p}$ is a cocharacter up to conjugacy. The set of conjugacy classes of cocharacters is in bijection with $X_*(T)/W$, where $T \subset G_{\overline{\mathbf{Q}}_p}$ is a maximal torus and W is its Weyl group. Recall that if $G = \mathrm{GL}_r$, a cocharacter can be specified by a nonincreasing sequence of integers $k_1 \geq \cdots \geq k_r$.

Suppose that $C/\mathbf{Q}_p$ is complete and algebraically closed. Suppose V is an r-dimensional vector space over $B_{\mathrm{dR}} = B_{\mathrm{dR}}(C)$, and $\Lambda, \Lambda' \subset V$ are two B_{dR}^+-lattices. Recall that B_{dR} is a DVR with uniformizer ξ. By the theory of elementary divisors, there exists a basis $e_1, \dots, e_r$ for Λ and well-defined integers $k_1 \geq \cdots \geq k_r$ such that $\xi^{k_1} e_1, \dots, \xi^{k_r} e_r$ is a basis for Λ'. We say that Λ and Λ' are in *relative position* μ, where $\mu \colon \mathbf{G}_\mathrm{m} \to \mathrm{GL}_r$ is the cocharacter with $\mu(t) = \mathrm{diag}(t^{k_1}, \dots, t^{k_r})$. In this situation we write

$$\mathrm{inv}(\Lambda, \Lambda') = (k_1 \geq \cdots \geq k_r).$$

The notion can be defined for a general group G: if two G-torsors over B_{dR}^+ have the same generic fiber, their relative position is measured by a cocharacter of G which is well-defined up to conjugacy.

There is a dominance order on the set of cocharacters. For $G = \mathrm{GL}_r$, it corresponds to the *majorization order* on tuples $(k_1 \geq \cdots \geq k_r)$. This is defined by

$$(k_1 \geq \cdots \geq k_r) \geq_{\mathrm{maj}} (k_1' \geq \cdots \geq k_r')$$

if and only if $k_1 \geq k_1'$, $k_1 + k_2 \geq k_1' + k_2'$, etc., with $\sum_i k_i = \sum_i k_i'$.

Let $(\mathcal{E}, \phi_E)$ be a G-shtuka over a geometric point $S = \mathrm{Spa}(C, \mathcal{O}_C)$ with paws at $x_1, \dots, x_n \in \mathcal{Y}_{(0,\infty)}$, which correspond to un-tilts $C_1^\sharp, \dots, C_n^\sharp$. For $i = 1, \dots, n$, we have the $B_{\mathrm{dR}}(C_i^\sharp)$-lattices $\phi_{\mathcal{E}}(\phi^*\mathcal{E})^\wedge_{x_i}$ and $\mathcal{E}_{x_i}$, whose generic fibres are identified via $\phi_{\mathcal{E}}$. Let $(k_1(x_i) \geq \cdots \geq k_r(x_i)) = \mathrm{inv}((\phi^*\mathcal{E})^\wedge_{x_i}, \mathcal{E}_{x_i})$ be the r-tuple of integers measuring their relative position.

Definition 19.3.1. Let $\overline{\mu} = \{\overline{\mu}_i\}_{1 \leq i \leq n}$ be a collection of cocharacters, corresponding to the n-tuples $k_{i,1} \geq \cdots \geq k_{i,r}$. We say that $(\mathcal{E}, \phi_E)$ is *bounded by* $\overline{\mu}$ if

$$(k_1(x_i) \geq \cdots \geq k_r(x_i)) \leq_{\mathrm{maj}} \sum_{j, x_j = x_i} (k_{j,1} \geq \cdots \geq k_{j,r}).$$

If $(\mathcal{E}, \phi_{\mathcal{E}})$ is a shtuka over a perfectoid space S, we say that $(\mathcal{E}, \phi_{\mathcal{E}})$ is bounded by μ if it is so at all geometric points of S.

Remark 19.3.2. One must really take the sum on the right hand side to allow for the possibility that some of the paws may collide. This will ensure that the moduli space of shtukas we define will be partially proper. We also remark that this definition extends to general G.

We are now ready to define the moduli space of shtukas. Let $k = \overline{\mathbf{F}}_p$ and let $\breve{\mathbf{Q}}_p = W(k)[1/p]$. Let $(G, b, \{\overline{\mu_i}\})$ be a local Shtuka datum. For the moment assume that the μ_i are defined over $\mathbf{Q}_p$ (this can always be done if G is split).

Definition 19.3.3. Let $(\operatorname{Spd} \breve{\mathbf{Q}}_p)^n$ be the self-product of n copies of $\operatorname{Spd} \mathbf{Q}_p$ over $\operatorname{Spd} k$. We define a morphism $\operatorname{Sht}_{(G,b,\overline{\mu})} \to (\operatorname{Spd} \breve{\mathbf{Q}}_p)^n$ of functors on the category of perfectoid k-algebras as follows. If (R, R^+) is a perfectoid k-algebra together with n maps $x_i \colon \operatorname{Spd}(R, R^+) \to \operatorname{Spd} \breve{\mathbf{Q}}_p$, then the fiber of $\operatorname{Sht}_{(G,b,\overline{\mu})}(R, R^+)$ over $(x_1, \ldots, x_n)$ is the set of isomorphism classes of pairs $((\mathcal{E}, \phi_{\mathcal{E}}), \iota)$, where

- $(\mathcal{E}, \phi_{\mathcal{E}})$ is a G-shtuka over $\mathcal{Y}_{[0,\infty)}(R, R^+)$ with paws only at the x_i, such that $(\mathcal{E}, \phi_{\mathcal{E}})$ is bounded by $\overline{\mu}$, and
- $\iota \colon (\mathcal{E}, \phi_{\mathcal{E}})|_{[\rho,\infty)} \xrightarrow{\sim} (\mathcal{E}_b, \phi_{\mathcal{E}_b})|_{[\rho,\infty)}$ is an isomorphism for some sufficiently large ρ.

19.4 Relation to Rapoport-Zink spaces

Rapoport-Zink spaces are moduli of deformations of a fixed p-divisible group. After reviewing these, we will show that (over the generic fibre) the associated diamond of a Rapoport-Zink space is isomorphic to a moduli space of shtukas of the form $\operatorname{Sht}_{G,b,\{\overline{\mu}\}}$ with μ minuscule. In the case $G = \mathrm{GL}_r$, the minuscularity condition means that

$$\mu(t) = \operatorname{diag}(\underbrace{t, \ldots, t}_{d}, \underbrace{1, \ldots, 1}_{r-d})$$

This corresponds to the r-tuple $(1 \geq 1 \geq \cdots \geq 1 \geq 0 \geq \cdots \geq 0)$.

Fact: If $\operatorname{Sht}_{(\mathrm{GL}_r,b,\overline{\mu})} \neq \emptyset$ then b belongs to the set $B(\mathrm{GL}_r, \mu)$ of σ-conjugacy classes which are μ-*admissible*, meaning that the corresponding isocrystal is the Dieudonné module of a p-divisible group $\mathbf{X}$ over k of dimension d and height r. Put another way, b lies in $B(\mathrm{GL}_r, \mu)$ if and only if the

Newton polygon of b lies above the Hodge polygon associated to μ, which is the polygon linking $(0,0)$, $(r-d,0)$, and (r,d).

We recall from [RZ96] the definition and main properties of Rapoport-Zink spaces.

Definition 19.4.1. Assume that $b \in B(\mathrm{GL}_r, \mu)$. Let $\mathrm{Def}_{\mathbf{X}}$ be the functor which assigns to a formal scheme $S/\operatorname{Spf}\breve{\mathbf{Z}}_p$ the set of isomorphism classes of pairs (X, ρ), where X/S is a p-divisible group, and $\rho\colon X \times_S \overline{S} \xrightarrow{\sim} \mathbf{X} \times_{\overline{\mathbf{F}}_p} \overline{S}$ is a quasi-isogeny, where $\overline{S} = S \times_{\operatorname{Spf}\breve{\mathbf{Z}}_p} \operatorname{Spec}\overline{\mathbf{F}}_p$. (A *quasi-isogeny* is an isomorphism in the isogeny category; formally it is an isogeny divided by a power of p.)

Theorem 19.4.2 ([RZ96])**.** $\mathrm{Def}_{\mathcal{X}}$ *is representable by a formal scheme* $\mathcal{M}_{\mathbf{X}}$ *over* $\operatorname{Spf}\breve{\mathbf{Z}}_p$*, which is formally smooth and locally formally of finite type. Furthermore, all irreducible components of the special fibre of* $\mathcal{M}_{\mathbf{X}}$ *are proper over* $\operatorname{Spec} k$*.*

Remark 19.4.3. "Locally formally of finite type" means that locally $\mathcal{M}_{\mathbb{X}}$ is isomorphic to a formal scheme of the form $\operatorname{Spf}\breve{\mathbf{Z}}_p[\![T_1,\dots,T_m]\!]\langle U_1,\dots,U_\ell\rangle/I$.

Let $\mathcal{M}_{\mathbf{X},\breve{\mathbf{Q}}_p}$ be the generic fibre over $\operatorname{Spa}\breve{\mathbf{Q}}_p$. It will be useful to have a moduli interpretation of $\mathcal{M}_{\mathbf{X},\breve{\mathbf{Q}}_p}$

Proposition 19.4.4 ([SW13, Prop. 2.2.2])**.** *Let* $\mathrm{CAff}^{\mathrm{op}}_{\breve{\mathbf{Q}}_p}$ *be the category opposite to the category of complete Huber pairs over* $(\breve{\mathbf{Q}}_p, \breve{\mathbf{Z}}_p)$*. Then* $\mathcal{M}_{\mathbf{X}}$ *is the sheafification of the presheaf on* $\mathrm{CAff}^{op}_{\breve{\mathbf{Q}}_p}$ *defined by*

$$(R,R^+) \mapsto \varinjlim_{R_0 \subset R^+} \mathrm{Def}_{\mathcal{X}}(R_0),$$

where the limit runs over open and bounded $\breve{\mathbf{Z}}_p$*-subalgebras* $R_0 \subset R^+$*.*

Thus, to give a section of $\mathcal{M}_{\mathbf{X}}$ over (R,R^+) is to give a covering of $\operatorname{Spa}(R,R^+)$ by rational subsets $\operatorname{Spa}(R_i,R_i^+)$, and for each i a deformation $(X_i,\rho_i) \in \mathrm{Def}_{\mathbf{X}}(R_{i0})$ over an open and bounded $\breve{\mathbf{Z}}_p$-subalgebra $R_{i0} \subset R_i^+$, such that the (X_i,ρ_i) are compatible on overlaps.

Theorem 19.4.5. *There is an isomorphism* $\mathcal{M}^{\diamondsuit}_{\mathbf{X},\breve{\mathbf{Q}}_p} \cong \mathrm{Sht}_{(\mathrm{GL}_r, b, \overline{\mu})}$ *as diamonds over* $\operatorname{Spd}\breve{\mathbf{Q}}_p$*.*

Proof. (Sketch.) The crucial observation is that both spaces admit a period morphism to a Grassmannian $\mathrm{Grass}(d,r)$, and that there is a morphism $\mathcal{M}^{\diamondsuit}_{\mathbf{X},\breve{\mathbf{Q}}_p} \to \mathrm{Sht}_{(\mathrm{GL}_r,b,\overline{\mu})}$ lying over $\mathrm{Grass}(d,r)$.

On the Rapoport-Zink side we have a morphism of adic spaces

$$\pi_{\mathrm{GM}}\colon \mathcal{M}_{\mathbf{X},\breve{\mathbf{Q}}_p} \to \mathrm{Grass}(d,r)_{\breve{\mathbf{Q}}_p}$$

which arises from *Grothendieck-Messing periods*. In the case $d=1$ the existence of π_{GM} is due to Gross-Hopkins, [HG94], and for general d it is in [RZ96]. Here, $\mathrm{Grass}(d,r)$ is the variety of d-dimensional quotients of $N(\mathbf{X})[1/p]$, the (r-dimensional) rational Dieudonné module of $\mathbf{X}$.

We outline the construction of π_{GM}. Suppose (R,R^+) is a complete Huber pair over $(\breve{\mathbf{Q}}_p,\breve{\mathbf{Z}}_p)$. Let $R_0\subset R^+$ be an open and bounded subring, and let $(X,\rho)\in \mathrm{Def}_{\mathbf{X}}(R_0)$. In light of Prop. 19.4.4 it suffices to define π_{GM} on the section of $\mathcal{M}_{\mathbf{X}}(R,R^+)$ defined by (X,ρ).

Let EX be the universal vector extension of X. By Grothendieck-Messing theory, the quasi-isogeny ρ induces an isomorphism of locally free R-modules $(\mathrm{Lie}\, EX)[1/p] \to N(\mathbf{X})\otimes_{\breve{\mathbf{Z}}_p} R$. On the other hand we have the rank d quotient $\mathrm{Lie}\, EX \to \mathrm{Lie}\, X$. Combining these elements gives a rank d quotient of $N(\mathbf{X})\otimes_{\breve{\mathbf{Z}}_p} R$, which defines a section of $\mathrm{Grass}(d,r)$ over (R,R^+).

Grothendieck-Messing theory also shows that π_{GM} is étale. Over its image, it parametrizes $\mathbf{Z}_p$-lattices in a $\mathbf{Q}_p$-local system of rank r. Thus its geometric fibres are essentially $\mathrm{GL}_r(\mathbf{Q}_p)/\mathrm{GL}_r(\mathbf{Z}_p)$.

On the side of shtukas, we also have a morphism

$$\pi'_{GM}\colon \mathrm{Sht}_{(\mathrm{GL}_r,b,\overline{\mu})} \to \mathrm{Grass}(d,r)^{\diamondsuit}_{\breve{\mathbf{Q}}_p}.$$

If $(\mathcal{E},\phi_{\mathcal{E}})$ is a shtuka over (R,R^+) with a paw at x, together with an isomorphism between $\mathcal{E}$ and $\mathcal{E}_b$ over $\mathcal{Y}_{[r,\infty)}$ for some r, then

$$\phi_{\mathcal{E}}^{-1}\colon \mathcal{E} \to \phi^*\mathcal{E}$$

is a well-defined map, such that $\mathrm{coker}(\mathcal{E}^{\wedge}_{\phi(x)} \to (\phi^*\mathcal{E})^{\wedge}_x)$ is killed by the kernel of $W(R^+)\to R^{\sharp+}$ and is locally free of rank d over $R^\sharp$.

Choose r large enough so that $\phi(x)\in\mathcal{Y}_{[r,\infty)}$. Then $(\phi^*\mathcal{E})_x = \mathcal{E}_{\phi(x)} \cong N\otimes_{\breve{\mathbf{Q}}_p} R^\sharp$, and so we get a quotient $\mathrm{coker}(\mathcal{E}_x\to(\phi^*\mathcal{E})_x)$ of rank d of $N\otimes_{\breve{\mathbf{Q}}_p} R^\sharp$; that is, a point of $\mathrm{Grass}(d,r)^{\diamondsuit}_{\breve{\mathbf{Q}}_p}$.

What is the image of these π_{GM}? We only have to worry about geometric points, and rank 1 geometric points at that (by partial properness). Given a point $(C,\mathcal{O}_C)$ of $\mathrm{Grass}(d,r)^{\diamondsuit}_{\breve{\mathbf{Q}}_p}$, one gets a pair $(\mathcal{E}',\phi_{\mathcal{E}'})$ over $\mathcal{Y}_{(0,\infty)}$ by modifying $(\mathcal{E}_b, \phi_{\mathcal{E}_b})$ at $x, \phi^{-1}(x), \dots$ using this quotient of $N \otimes_{\breve{\mathbf{Q}}_p} C^\sharp$. Then $\phi_{\mathcal{E}}$ will be an isomorphism away from x. Now: when does a ϕ-module over $\mathcal{Y}_{(0,s]}$ extend to a ϕ-module over $\mathcal{Y}_{[0,s]}$, for s small?

Recall that for any $s > 0$, the category of ϕ-modules over $\mathcal{Y}_{[0,s]}$ is equivalent to the category of finite free $\mathbf{Z}_p$-modules, so that (after tensoring both categories with $\mathbf{Q}_p$), ϕ-modules over $\mathcal{Y}_{[0,s]}$ are equivalent to finite dimensional $\mathbf{Q}_p$-vector spaces, in turn equivalent to trivial ϕ-modules over $\mathcal{Y}_{[0,s]}$. Thus, any extension of a ϕ-module over $\mathcal{Y}_{[0,s]}$ is trivial. In that case, extensions are in bijection with $\mathbf{Z}_p$-lattices in $\mathbf{Q}_p$-vector spaces.

Definition 19.4.6. Let $\mathrm{Grass}(d,r)^{\mathrm{adm}}_{\breve{\mathbf{Q}}_p} \subset \mathrm{Grass}(d,r)_{\breve{\mathbf{Q}}_p}$ be the locus where the resulting ϕ-module over $\mathcal{Y}_{[0,s]}$ is trivial.

This is the image of π_{GM} on the space of shtukas.

Theorem 19.4.7 (Faltings)**.** *The image of* $\pi_{GM} \colon \mathcal{M}_{\mathbf{X},\breve{\mathbf{Q}}_p} \to \mathrm{Grass}(d,r)_{\breve{\mathbf{Q}}_p}$ *is the admissible locus.*

See also [SW13], where this is reproved using the classification of p-divisible groups over $\mathcal{O}_{C^\sharp}$ in terms of pairs (T, W). Also one sees there a map $\mathcal{M}^\diamondsuit_{\mathbf{X},\breve{\mathbf{Q}}_p} \to \mathrm{Sht}_{(\mathrm{GL}_r,b,\overline{\mu})}$ which compares with π_{GM}. They have the same image and the same fibres, and thus this is an isomorphism. $\square$

20 The mixed-characteristic Beilinson-Drinfeld Grassmannian, 20 November

In today's lecture we construct moduli spaces of shtukas, and announce the main theorem of the course, which is that these spaces are diamonds. In the case of shtukas with one paw, one gets the "local Shimura varieties" hypothesized in [RV]. But of course our spaces of shtukas may have multiple paws, so that one ought to see the sort of structure in their cohomology that appears in [Laf] in the global equal-characteristic setting.

20.1 Review of shtukas with one paw

Fix $C/\mathbf{Q}_p$ an algebraically closed nonarchimedean field. The following objects are associated with C:

1. The topological ring $A_{\mathrm{inf}} = W(\mathcal{O}_{C^\flat})$, which comes equipped with a surjective homomorphism $\theta \colon W(\mathcal{O}_{C^\flat}) \to C$, whose kernel is generated by an element ξ.

2. The DVR B_{dR}^+ and its fraction field B_{dR}, where B_{dR}^+ is the ξ-adic completion of $W(\mathcal{O}_{C^\flat})[1/p]$.

3. The Fargues-Fontaine curve X_{FF} defined in Defn. 13.5.3, with a distinguished point ∞ corresponding to the un-tilt C of $C^\flat$. We have. There is also the adic version $\mathcal{X}_{\mathrm{FF}}$; we have used x_C for its distinguished point. In either case, the completed local ring at the distinguished point is B_{dR}.

Proposition 20.1.1. *The following categories are equivalent.*

1. *Shtukas over* $\mathrm{Spa}\, C^\flat$ *with one (meromorphic) paw at* C.

2. *Pairs* (T, Ξ), *where* T *is a finite free* $\mathbf{Z}_p$*-module, and* $\Xi \subset T \otimes_{\mathbf{Z}_p} B_{\mathrm{dR}}$ *is a* B_{dR}^+*-lattice.*

3. *Breuil-Kisin modules over* A_{inf}.

4. *Quadruples* $(\mathcal{F}, \mathcal{F}', \beta, T)$, *where* $\mathcal{F}$ *and* $\mathcal{F}'$ *are vector bundles on the Fargues-Fontaine curve* X_{FF}, *and* $\beta \colon \mathcal{F}|_{X_{\mathrm{FF}} \backslash \{\infty\}} \xrightarrow{\sim} \mathcal{F}'|_{X_{\mathrm{FF}} \backslash \{\infty\}}$ *is an isomorphism, where* $\infty \in X_{\mathrm{FF}}(C)$ *corresponds to the un-tilt* C *of* $C^\flat$, $\mathcal{F}$ *is trivial, and* $T \subset H^0(X_{\mathrm{FF}}, \mathcal{F})$ *is a* $\mathbf{Z}_p$*-lattice.*

If the paw is minuscule, *which is to say that*

$$\xi(T \otimes_{\mathbf{Z}_p} B_{\mathrm{dR}}^+) \subset \Xi \subset T \otimes_{\mathbf{Z}_p} B_{\mathrm{dR}}^+,$$

then these categories are equivalent to the category of p*-divisible groups over* $\mathcal{O}_C$.

Proof. The equivalence between (1) and (2) is Thm. 12.4.4, and the equivalence between (2) and (3) is Thm. 12.5.1. Let us explain the equivalence between (1) and (4). Suppose $(\mathcal{E}, \phi_{\mathcal{E}})$ is a shtuka over $\mathrm{Spa}\, C^\flat$ with one paw at C. This means that $\mathcal{E}$ is a vector bundle on $\mathcal{Y}_{[0,\infty)}$ and $\phi_{\mathcal{E}} \colon \phi^*\mathcal{E} \to \mathcal{E}$ is an isomorphism away from x_C. The vector bundles $\mathcal{F}$ and $\mathcal{F}'$ on $\mathcal{X}_{\mathrm{FF}} = \mathcal{Y}_{[0,\infty)}/\phi^{\mathbf{Z}}$ come from descending $(\mathcal{E}, \phi_{\mathcal{E}})$ "on either side" of x_C, respectively, as we now explain.

First we treat the side of x_C close to the point $x_{C^\flat}$, where $p = 0$. The completed stalk of $\mathcal{E}$ over $x_{C^\flat}$ is a ϕ-module over the integral Robba ring $\widetilde{\mathcal{R}}^{\mathrm{int}} = \widehat{\mathcal{O}}_{\mathcal{Y}_{[0,\infty)}, x_{C^\flat}}$. By Thm. 12.3.4, the category of ϕ-modules over $\widetilde{\mathcal{R}}^{\mathrm{int}}$ is equivalent to the category of finite free $\mathbf{Z}_p$-modules. Thus there exists a finite free $\mathbf{Z}_p$-module T and an isomorphism of ϕ-modules $\widehat{\mathcal{E}}_{x_{C^\flat}} \xrightarrow{\sim} T \otimes_{\mathbf{Z}_p} \mathcal{O}_{\mathcal{Y}_{[0,\infty)}, x_{C^\flat}}$. This isomorphism spreads out to an isomorphism of ϕ-modules

$\widehat{\mathcal{E}}|_{[0,s]} \tilde{\to} T \otimes_{\mathbf{Z}_p} \mathcal{O}_{\mathcal{Y}_{[0,s]}}$ for some s small enough. Let $\mathcal{F} = T \otimes_{\mathbf{Z}_p} \mathcal{O}_{\mathcal{X}_{\mathrm{FF}}}$; then of course $\mathcal{F}$ is trivial and $T \subset H^0(\mathcal{X}_{\mathrm{FF}}, \mathcal{F})$ is a $\mathbf{Z}_p$-lattice.

Now we treat the other side of x_C. The map $\phi_{\mathcal{E}}$ restricts to an isomorphism $\phi^*\mathcal{E}|_{\mathcal{Y}_{[s,\infty)}} \tilde{\to} \mathcal{E}|_{\mathcal{Y}_{[s,\infty)}}$ for s large enough. Let $\mathcal{F}'$ be the descent of $(\mathcal{E}, \phi_{\mathcal{E}})|_{\mathcal{Y}_{[s,\infty)}}$.

The two descent procedures used to construct $\mathcal{F}$ and $\mathcal{F}'$ only involved modifications at x_C, so one has an isomorphism $\beta\colon \mathcal{F} \to \mathcal{F}'$ away from $\{x_C\}$. □

20.2 Moduli spaces of shtukas

As usual, we let $k = \overline{\mathbf{F}}_p$ and $\breve{\mathbf{Q}}_p = W(k)[1/p]$.

Let $G = \mathrm{GL}_r$, let $b \in B(G)$, and let $\overline{\mu}_1, \dots, \overline{\mu}_r$ be conjugacy classes of cocharacters of G. The class b corresponds to a ϕ-module $(\mathcal{E}_b, \phi_{\mathcal{E}_b})$ over $\mathcal{Y}_{(0,\infty)}$, where $\mathcal{E}_b = \mathcal{O}^r_{\mathcal{Y}_{(0,\infty)}}$ and $\phi_{\mathcal{E}_b} = g\phi$, where $g \in G(L)$ represents b.

Recall the definition of the morphism of functors on (Perf_k):

$$\mathrm{Sht}_{(\mathrm{GL}_r, b, \{\overline{\mu}_i\})} \to \mathrm{Spd}\, \breve{\mathbf{Q}}_p \times_k \cdots \times_k \mathrm{Spd}\, \breve{\mathbf{Q}}_p.$$

Given a perfectoid Huber pair (R, R^+) over k with morphisms $x_i\colon \mathrm{Spd}\, \breve{\mathbf{Q}}_p \to \mathrm{Spd}(R, R^+)$, it assigns the set of isomorphism classes of triples $(\mathcal{E}, \phi_{\mathcal{E}}, \iota)$ over $\mathcal{Y}_{[0,\infty)}$ of rank r with paws at the graphs of the x_i which are bounded by $\overline{\mu}_i$, together with an isomorphism $\iota\colon (\mathcal{E}, \phi_{\mathcal{E}}) \to (\mathcal{E}_b, \phi_{\mathcal{E}_b})$ near ∞.

So far $\mathrm{Sht}_{(\mathrm{GL}_r, b, \{\overline{\mu}_i\})}$ is just a functor, but in fact:

Proposition 20.2.1. $\mathrm{Sht}_{(\mathrm{GL}_r, b, \{\overline{\mu}_i\})}$ *is a sheaf on the faithful site.*

Proof. This is a matter of proving that triples $(\mathcal{E}, \phi_{\mathcal{E}}, \iota)$ glue on faithful covers. Once the vector bundle $\mathcal{E}$ glues, it will be easy to glue $\phi_{\mathcal{E}}$ and ι.

Recall that $\mathcal{Y}_{[0,\infty)}(R, R^+)$ is preperfectoid. Thus if K is a perfectoid field, say $K = \mathbf{Q}_p(p^{1/p^\infty})^\wedge$, then $\mathcal{Y}_{[0,\infty)} \times_{\mathrm{Spa}\, \mathbf{Z}_p} \mathrm{Spa}\, \mathcal{O}_K$ is perfectoid. If $\mathcal{E}$ is a vector bundle on $\mathcal{Y}_{[0,\infty)}$, then we can form $\mathcal{E} \widehat{\otimes}_{\mathbf{Z}_p} \mathcal{O}_K$; this comes equipped with an idempotent $e_{\mathcal{E}} = 1 \otimes e_{\mathcal{O}_K}$, where $e_{\mathcal{O}_K}\colon \mathcal{O}_K \to \mathbf{Z}_p$ is a $\mathbf{Z}_p$-linear splitting.

If $\mathcal{E} \widehat{\otimes}_{\mathbf{Z}_p} \mathcal{O}_K$ glues on faithful covers, then since $e_{\mathcal{E}}$ glues, $\mathcal{E}$ glues as well. Thus is suffices to show that vector bundles glue on faithful covers of perfectoid spaces. We now appeal to Lemma 20.2.2 below. □

Lemma 20.2.2. *The fibred category*

$$(R, R^+) \mapsto \{\textit{Finite projective } R\textit{-modules}\}$$

is a stack on the faithful site of (Perf).

Proof. Suppose $\widetilde{X} \to X$ is a surjective morphism of perfectoid affinoids, with $X = \operatorname{Spa}(R, R^+)$ and $\widetilde{X} = \operatorname{Spa}(\widetilde{R}, \widetilde{R}^+)$. We will show that the base change functor from finite projective R-modules to finite projective $\widetilde{R}$-modules equipped with a descent datum is an equivalence of categories.

Full faithfulness follows from the sheaf property of the structure presheaf on the faithful site, Thm. 15.3.5.

Essential surjectivity can be checked locally, as vector bundles glue over open covers by Thm. 5.5.8. Consider first the case where $R = K$ is a nonarchimedean local field. Then $\widetilde{R} = \widetilde{K}$ is a nonzero Banach $\widetilde{K}$-algebra, and we need to prove that projective $\widetilde{K}$-algebras with descent data descend to K.

Recall the proof of faithfully flat descent for modules (for instance, [Sta14, Descent, Prop. 3.9]). The same proof carries over to $\widetilde{K}/K$, except that we must check that the following lemma holds (in the context of $V = \widetilde{K}$:

Lemma 20.2.3. *Let C be a complex of K-Banach spaces, and let $V \neq 0$ be a K-Banach space. Then C is acyclic if and only if $C \widehat{\otimes}_K V$ is acyclic.*

Proof. Use the fact that $W \widehat{\otimes}_K V = \varinjlim_{V' \subset V} W \widehat{\otimes}_K V'$, where $V' \subset V$ runs over topologically countably generated subspaces. This reduces to proving the lemma for such V'. But if V' is topologically countably generated, then it is topologically free, [BGR84, §2.7, Thm. 4]. In that case, the lemma is easy. □

Thus we have established Lemma 20.2.2 over a point. Returning to the general case, suppose $(R, R^+) \to (\widetilde{R}, \widetilde{R}^+)$ is a faithful cover, and $\widetilde{M}/\widetilde{R}$ is a finite projective module equipped with a descent datum

$$\widetilde{M} \widehat{\otimes}_{\widetilde{R}, i_1} (\widetilde{R} \widehat{\otimes}_R \widetilde{R}) \cong \widetilde{M} \widehat{\otimes}_{\widetilde{R}, i_2} (\widetilde{R} \widehat{\otimes}_R \widetilde{R}),$$

where $i_1, i_2 \colon \widetilde{R} \rightrightarrows \widetilde{R} \widehat{\otimes}_R \widetilde{R}$ are the two obvious homomorphisms. We wish to descend $\widetilde{M}$ to M/R.

After replacing $\widetilde{X}$ with an open cover, we may assume that $\widetilde{M} = \widetilde{R}^r$ is free. The descent datum is given by a matrix $B \in \mathrm{GL}_r(\widetilde{R} \widehat{\otimes}_R \widetilde{R})$, which satisfies a cocycle condition. Pick any $x \in X$ with completed residue field $K(x)$. We can descend the fibre of $\widetilde{M}$ over x, so there exists $A_x \in GL_r(\widetilde{R} \widehat{\otimes}_R K(x))$ such that $B = \mathrm{pr}_1^*(A_x)\, \mathrm{pr}_2^*(A_x)^{-1} \in \mathrm{GL}_r(\widetilde{R} \widehat{\otimes}_R \widetilde{R} \widehat{\otimes}_R K(x))$. Approximate A_x by some $A_U \in \mathrm{GL}_r(\widetilde{R} \widehat{\otimes}_R \mathcal{O}_X(U))$. After conjugating by A_U, we may assume $B \in \mathrm{GL}_r(\widetilde{R}^+\widehat{\otimes}_{R^+}\mathcal{O}_X^+)$, and even that $B \equiv 1 \pmod{\varpi}$ for a pseudo-uniformizer $\varpi \in R$. Replacing X by U, we may assume that $B \equiv 1 \pmod{\varpi}$ to begin with.

Now $(B-1)/\varpi$ modulo ϖ satisfies the *additive* cocycle condition, so it lives in $\check{H}^1(\widetilde{X}/X, M_r(\mathcal{O}_X^+/\varpi))$, but this group is almost zero, see Thm. 15.3.5 and its proof. Thus we can conjugate B so as to assume that $B \equiv 1 \pmod{\varpi^{2-\varepsilon}}$ for some $\varepsilon > 0$. Continuing, we find that $B = 1$. □

20.3 Announcement of the main theorem, and the strategy of proof

Theorem 20.3.1. $\mathrm{Sht}_{(\mathrm{GL}_r,b,\{\overline{\mu}_i\})}$ *is a diamond.*

This is the main theorem of the course! It may not look interesting, but it implies that these spaces have cohomology. One can add level structures to the picture to obtain a tower of diamonds whose cohomology is a representation of $\mathrm{GL}_r(\mathbf{Q}_p) \times J_b$, where J_b is the centralizer of b in $G(L)$. This cohomology ought to realize instances of the local Langlands correspondence, along the lines of the Kottwitz conjecture for Rapoport-Zink spaces.

The strategy for proving Thm. 20.3.1 goes as follows.

1. We will define a morphism

$$\pi_{\mathrm{GM}}\colon \mathrm{Sht}_{(\mathrm{GL}_r,b,\{\overline{\mu}_i\})} \to \mathrm{Gr}^{\mathrm{BD}}_{(\mathrm{Spd}\,\breve{\mathbf{Q}}_p)^n,\{\overline{\mu}_i\}}$$

onto a *Beilinson-Drinfeld Grassmannian*. This period map records the modifications of the vector bundle $\mathcal{F}$ at the x_i. In the case of one minuscule paw, π_{GM} coincides with the Grothendieck-Messing period map of §19.4.

2. We will show that $\mathrm{Gr}^{\mathrm{BD}} = \mathrm{Gr}^{\mathrm{BD}}_{(\mathrm{Spd}\,\breve{\mathbf{Q}}_p)^n,\{\overline{\mu}_i\}}$ is a diamond, Thm. 21.3.7. We will do this over the next two lectures, by showing that $\mathrm{Gr}^{\mathrm{BD}}$ is a fake diamond satisfying the hypotheses of Thm. 16.2.10.

3. Finally we will show that π_{GM} is an étale cover of an open subdiamond $\mathrm{Gr}^{\mathrm{BD,adm}} \subset \mathrm{Gr}^{\mathrm{BD}}$. In particular $\mathrm{Sht}_{(\mathrm{GL}_r,b,\{\overline{\mu}_i\})}$ is a diamond.

20.4 Definition of the period morphism

Here we define the Beilinson-Drinfeld Grassmannian and the period morphism π_{GM}. For simplicity, we only define π_{GM} over the open subset $U \subset$

$(\mathrm{Spd}\,\breve{\mathbf{Q}}_p)^n$ where $\phi^m(x_i) \neq x_j$ for $m \neq 0$. (We permit $x_i = x_j$, though.) If we didn't work over U, we would have to change the target of π_{GM} slightly.

The Beilinson-Drinfeld Grassmannian makes sense as a functor lying over $(\mathrm{Spd}\,\mathbf{Z}_p)^n$, and so we define it in this generality.

Definition 20.4.1. $\mathrm{Gr}^{\mathrm{BD}}_{(\mathrm{Spd}\,\mathbf{Z}_p)^n,\{\overline{\mu}_i\}} \to (\mathrm{Spd}\,\mathbf{Z}_p)^n$ is the functor sending a perfectoid Huber pair (R, R^+) together with n morphisms $x_i\colon \mathrm{Spd}(R, R^+) \to \mathrm{Spd}\,\mathbf{Z}_p$ to the set of isomorphism classes of pairs $(\mathcal{G}, \iota)$, where $\mathcal{G}$ is a vector bundle over $\mathcal{Y}_{[0,\infty)} = \mathcal{Y}_{[0,\infty)}(R, R^+)$ and

$$\iota\colon \mathcal{O}^r_{\mathcal{Y}_{[0,\infty)}}|_{\mathcal{Y}_{[0,\infty)}\setminus \bigcup_i \Gamma_{x_i}} \cong \mathcal{G}|_{\mathcal{Y}_{[0,\infty)}\setminus \bigcup_i \Gamma_{x_i}}$$

is a trivialization of $\mathcal{G}$ away from the graphs of the x_i.

Remark 20.4.2. The same convention about collisions of paws from last time is in effect.

Now we define the morphism $\pi_{\mathrm{GM}}\colon \mathrm{Sht}_{(\mathrm{GL}_r,b,\{\overline{\mu}_i\})} \to \mathrm{Gr}^{\mathrm{BD}}_{(\mathrm{Spd}\,\mathbf{Q}_p)^n,\{\overline{\mu}_i\}}$. Suppose we are given a section of $\mathrm{Sht}_{(\mathrm{GL}_r,b,\{\overline{\mu}_i\})}$ over (R, R^+). This corresponds to a triple $(\mathcal{E}, \phi_{\mathcal{E}}, \iota)$ over (R, R^+), where $(\mathcal{E}, \phi_{\mathcal{E}})$ is a shtuka on $\mathcal{Y}(R, R^+)_{[0,\infty)}$ and $\iota\colon (\mathcal{E}, \phi_{\mathcal{E}}) \cong (\mathcal{E}_b, \phi_{\mathcal{E}_b})$ is an isomorphism over $\mathcal{Y}(R, R^+)_{[s,\infty)}$ for large enough s. By pulling this isomorphism back through ϕ, we can extend ι to an isomorphism over $\mathcal{Y}(R, R^+)_{[0,\infty)}\setminus \bigcup_{i,m\geq 0} \Gamma_{\phi^{-m}(x_i)}$.

By our assumption, $\phi^{-m}(x_i) \neq x_j$ for any $m > 0$. Let $\mathcal{F}$ be the vector bundle over $\mathcal{Y}_{[0,\infty)}$ obtained by modifiying $\mathcal{E}_b = \mathcal{O}^r_{\mathcal{Y}_{[0,\infty)}}$ at the x_i using $\mathcal{E}^{\wedge}_{x_i}$. Then $\mathcal{F}|_{\mathcal{Y}_{(0,\infty)}\setminus \bigcup \Gamma_{x_i}} \cong \mathcal{O}^r_{\mathcal{Y}_{(0,\infty)}\setminus \bigcup \Gamma_{x_i}}$ is a modification bounded by $\overline{\mu}_i$.

For s small, $\mathcal{F}|_{\mathcal{Y}_{(0,s]}} \cong \mathcal{O}^r|_{\mathcal{Y}_{(0,s]}}$, so we can extend $\mathcal{F}$ uniquely to $\mathcal{G}/\mathcal{Y}_{[0,\infty)}$ in such a way that $\mathcal{G} \cong \mathcal{O}^r$ over $\mathcal{Y}_{[0,\infty)}\setminus \bigcup_i \Gamma_{x_i}$. This $\mathcal{G}$ defines a section of $\mathrm{Gr}^{\mathrm{BD}}_{(\mathrm{Spd}\,\mathbf{Q}_p)^n,\{\overline{\mu}_i\}}$ over (R, R^+).

20.5 The admissible locus of $\mathrm{Gr}^{\mathrm{BD}}$

We explain how Thm. 20.3.1 follows from the fact that $\mathrm{Gr}^{\mathrm{BD}}$ is a diamond (Thm. 21.3.7). The first step is to identify the image of π_{GM}. Suppose we are given a section of $\mathrm{Gr}^{\mathrm{BD}} = \mathrm{Gr}^{\mathrm{BD}}_{(\mathrm{Spd}\,\mathbf{Q}_p)^n,\{\overline{\mu}_i\}}$ over (R, R^+), corresponding to a vector bundle $\mathcal{G}$ on $\mathcal{Y}_{[0,\infty)} = \mathcal{Y}(R, R^+)_{[0,\infty)}$ equipped with a trivialization $\iota\colon \mathcal{O}^r_{\mathcal{Y}_{[0,\infty)}} \to \mathcal{G}$ away from n paws $x_1, \dots, x_n$. For simplicity we assume that this section lies over the subset $U \subset (\mathrm{Spd}\,\mathbf{Q}_p)^n$ where $\phi^m(x_i) \neq x_j$ for all i, j and $m \geq 1$.

Now suppose $b \in B(\mathrm{GL}_r)$ is a σ-conjugacy class. We want to give a condition for when our given section of $\mathrm{Gr}^{\mathrm{BD}}_{(\mathrm{Spd}\,\mathbf{Q}_p)^n,\{\overline{\mu}_i\}}$ lies in the image of

$\pi_{\mathrm{GM}}\colon \mathrm{Sht}_{(\mathrm{GL}_r,b,\{\overline{\mu}_i\})} \to \mathrm{Gr}^{\mathrm{BD}}$. For this we attempt to reverse the construction in the definition of π_{GM}. Start with the ϕ-module $(\mathcal{E}_b, \phi_{\mathcal{E}_b})$ over $\mathcal{Y}$. Let $\mathcal{E}'$ be the vector bundle obtained by modifying $\mathcal{E}_b \cong \mathcal{O}^r_{\mathcal{Y}_{[0,\infty)}}$ at all translates $\phi^{-m}(x_i)$, $m \geq 0$, using $\phi^{-m}(\mathcal{G}^{\wedge}_{x_i})$. This process produces a pair $(\mathcal{E}', \phi_{\mathcal{E}'})$ over $\mathcal{Y}_{(0,\infty)}$, where $\phi_{\mathcal{E}'}\colon \phi^*\mathcal{E}' \to \mathcal{E}'$ is an isomorphism away from the x_i. Also note that $(\mathcal{E}', \phi_{\mathcal{E}'})$ is isomorphic to $(\mathcal{E}_b, \phi_{\mathcal{E}_b})$ near ∞.

Any extension of $(\mathcal{E}', \phi_{\mathcal{E}'})$ to all of $\mathcal{Y}_{[0,\infty)}$ would constitute a shtuka over (R, R^+) which lifts our given section of $\mathrm{Gr}^{\mathrm{BD}}$, but it is not at all guaranteed that such an extension exists. Let

$$\begin{aligned} \widetilde{\mathcal{R}}^{\mathrm{int}}_R &= \varinjlim H^0(\mathcal{Y}_{[0,r]}, \mathcal{O}_{\mathcal{Y}}) \\ \widetilde{\mathcal{R}}_R &= \varinjlim H^0(\mathcal{Y}_{(0,r]}, \mathcal{O}_{\mathcal{Y}}) \end{aligned}$$

(limits as $r \to 0$); these are the relative versions of the integral and extended Robba rings, cf. Defns. 12.3.1 and 13.4.3. We have here a ϕ-module over $\widetilde{\mathcal{R}}_R$ and are interested in the question of extension to $\widetilde{\mathcal{R}}^{\mathrm{int}}_R$.

First we recall the situation over a geometric point. There is a Dieudonné-Manin classification for ϕ-modules over the (absolute) extended Robba ring $\widetilde{\mathcal{R}}$. Whereas ϕ-modules over the (absolute) integral Robba ring $\widetilde{\mathcal{R}}^{\mathrm{int}}$ are trivial (Thm. 12.3.4). Thus a ϕ-module over $\widetilde{\mathcal{R}}$ extends to $\widetilde{\mathcal{R}}^{\mathrm{int}}$ exactly when its Newton polygon is trivial.

The behavior of Newton polygons attached to ϕ-modules over the relative extended Robba ring $\widetilde{\mathcal{R}}_R$ is controlled by the following (highly nontrivial) theorem.

Theorem 20.5.1 ([KL, Thm. 7.4.5])**.** *Let (M, ϕ_M) be a ϕ-module over $\widetilde{\mathcal{R}}_R$. The function which assigns to a geometric point $x \hookrightarrow \mathrm{Spa}(R, R^+)$ the Newton polygon of (M, ϕ_M) at x is lower semicontinuous.*

That is, the locus of $\mathrm{Spa}(R, R^+)$ where the Newton polygon lies above a given one is open. On the other hand, the locus where the endpoint of the Newton polygon is $(r, 0)$ is also open ([KL, Lemma 7.2.2]). We conclude from this that the locus where the Newton polygon is trivial is open.

Definition 20.5.2. The *admissible locus* $\mathrm{Gr}^{\mathrm{BD,adm}} \subset \mathrm{Gr}^{\mathrm{BD}}$ is the subfunctor defined by the condition that $(\mathcal{E}', \phi_{\mathcal{E}'})$ has trivial Newton polygon at all geometric points.

The preceding argument shows that $\mathrm{Gr}^{\mathrm{BD,adm}} \subset \mathrm{Gr}^{\mathrm{BD}}$ is an *open* subfunctor. Granting Thm. 21.3.7, we find that $\mathrm{Gr}^{\mathrm{BD,adm}}$ is a diamond.

We now apply another theorem from [KL], which is a relative version of Thm. 12.3.4:

Theorem 20.5.3 ([KL, Thms. 8.5.3 and 8.5.12]). *The following categories are equivalent:*

1. *ϕ-modules over $\widetilde{\mathcal{R}}_R$, and*
2. *Étale $\mathbf{Q}_p$-local systems on* $\operatorname{Spa}(R, R^+)$.

The equivalence sends a ϕ-module M to $\mathbf{V} =$ "$M^{\phi=1}$". Furthermore, extensions of M to $\widetilde{\mathcal{R}}_R^{\text{int}}$ are in equivalence with $\underline{\mathbf{Z}}_p$-lattices $\mathbf{L} \subset \mathbf{V}$.

Therefore there exists a $\mathbf{Q}_p$-local system $\mathbf{V}$ of rank r over $\mathrm{Gr}^{\mathrm{BD,adm}}$ corresponding to $(\mathcal{E}', \phi_{\mathcal{E}'})$. Given a section of $\mathrm{Gr}^{\mathrm{BD,adm}}$ over (R, R^+), the set of sections of $\mathrm{Sht}_{(\mathrm{GL}_r, b, \{\overline{\mu}_i\})}$ lying over it is the set of $\underline{\mathbf{Z}}_p$-lattices in $\mathbf{V}_{\operatorname{Spa}(R,R^+)}$.

It is now straightforward to check that $\pi_{\mathrm{GM}} \colon \mathrm{Sht}_{(\mathrm{GL}_r, b, \{\overline{\mu}_i\})} \to \mathrm{Gr}^{\mathrm{BD,adm}}$ is étale. This can be checked pro-étale locally on the target, so we pass to a covering $\{U_i\}$ of $\mathrm{Gr}^{\mathrm{BD,adm}}$ for which $\mathbf{V}|_{U_i} \cong \underline{\mathbf{Q}}_p^r$ is trivial. The set of $\underline{\mathbf{Z}}_p$-lattices in $\underline{\mathbf{Q}}_p^r$ is just the discrete set $\mathrm{GL}_n(\mathbf{Q}_p)/\mathrm{GL}_n(\mathbf{Z}_p)$. Therefore over U_i, π_{GM} is isomorphic to the projection $U_i \times \mathrm{GL}_n(\mathbf{Q}_p)/\mathrm{GL}_n(\mathbf{Z}_p) \to U_i$. It follows from this that $\mathrm{Sht}_{(\mathrm{GL}_r, b, \{\overline{\mu}_i\})}$ is a diamond; this completes the proof of Thm. 20.3.1.

Remark 20.5.4. From here it is easy to construct moduli spaces of shtukas with level structure: The space $\mathrm{Sht}_{(\mathrm{GL}_r, b, \{\overline{\mu}_i\})}$ comes equipped with a $\mathbf{Z}_p$-local system $\mathbf{L}$, and one obtains finite étale covers of it by trivializing $\mathbf{L}$ modulo p^m. (Or one could trivialize all of $\mathbf{L}$ to obtain a profinite étale cover.)

21 Conclusion of the proof, 25 November

Last time we reduced the proof of Thm. 20.3.1 (moduli of shtukas are diamonds) to proving Thm. 21.3.7 ($\mathrm{Gr}^{\mathrm{BD}}_{(\operatorname{Spd}\mathbf{Q}_p)^n, \{\overline{\mu_i}\}}$ is a diamond). Recall that $\mathrm{Gr}^{\mathrm{BD}}_{(\operatorname{Spd}\mathbf{Q}_p)^n, \{\overline{\mu_i}\}}$ parametrizes modifications of a trivial vector bundle at n points which are bounded by $\{\overline{\mu}_i\}$. Today we will show that a similar space is a diamond, namely the B_{dR}^+-Grassmannian, which parametrizes modifications of a trivial vector bundle at one point.

21.1 The B_{dR}^+-Grassmannian

We fix an integer $r \geq 1$.

Definition 21.1.1 (The B_{dR}^+-Grassmannian)**.** Let $\mathrm{Gr}^{B_{\mathrm{dR}}^+} \to \mathrm{Spd}\,\mathbf{Q}_p$ be the functor taking (R, R^+) with untilt $R^\sharp$ to the set of B_{dR}^+-lattices $M \subset B_{\mathrm{dR}}(R^\sharp)^r$.

For a conjugacy class $\overline{\mu}$ of cocharacters $\mu\colon \mathbf{G}_{\mathrm{m}} \to \mathrm{GL}_r$ (which corresponds to an r-tuple of integers $k_1 \geq \cdots \geq k_r$), we let $\mathrm{Gr}_{\overline{\mu}}^{B_{\mathrm{dR}}^+}$ be the set of M satisfying the condition

$$\mathrm{inv}\left(M_x, B_{\mathrm{dR}}^+(K(x)^\sharp)^r\right) \leq_{\mathrm{maj}} \overline{\mu}$$

for all $x \in \mathrm{Spa}(R, R^+)$.

The first order of business is to prove a sort of quasi-separatedness property for $\mathrm{Gr}^{B_{\mathrm{dR}}^+}$.

Lemma 21.1.2. *The diagonal* $\Delta\colon \mathrm{Gr}^{B_{\mathrm{dR}}^+} \to \mathrm{Gr}^{B_{\mathrm{dR}}^+} \times_{\mathrm{Spd}\,\mathbf{Q}_p} \mathrm{Gr}^{B_{\mathrm{dR}}^+}$ *is relatively representable and closed.*

Proof. Given $X = \mathrm{Spa}(R, R^+)$ with untilt $R^\sharp$, and two $B_{\mathrm{dR}}(R^\sharp)^+$-lattices $M_1, M_2 \subset B_{\mathrm{dR}}(R^\sharp)^r$, we want to show that $\{M_1 = M_2\}$ is closed and representable by an affinoid perfectoid space. It suffices to show this for $\{M_1 \subset M_2\}$.

Let $\xi \in B_{\mathrm{dR}}^+(R^\sharp)$ generate Fil^1. We have the loci $\left\{M_1 \subset \xi^{-i} M_2\right\}$ for $i \in \mathbf{Z}$. For $i \gg 0$, this is all of X. By induction we may assume $M_1 \subset \xi^{-1} M_2$. Then

$$\{M_1 \subset M_2\} = \bigcap_{m \in M_1} \left\{m \mapsto 0 \in \xi^{-1} M_2 / M_2\right\}$$

So it suffices to show that $\left\{m \mapsto 0 \in \xi^{-1} M_2/M_2\right\}$ is closed and representable by an affinoid perfectoid space.

The quotient $\xi^{-1} M_2 / M_2$ is a finite projective $R^\sharp$-module. After passing to an open cover of $\mathrm{Spa}(R, R^+)$ we may assume that $\xi^{-1} M_2/M_2 \cong (R^\sharp)^r$. Thus $\left\{m \mapsto 0 \in \xi^{-1} M_2/M_2\right\}$ is the vanishing locus of an r-tuple of elements of $R^\sharp$. Finally we are reduced to showing that the vanishing locus $\{f = 0\}$ of a single $f \in R^\sharp$ is closed and representable. But $\{f = 0\}$ is the intersection of the $\{|f| \leq |\varpi|^n\}$ for $n \geq 1$ (with $\varpi \in R$ a uniformizer, and each of these is rational (simple exercise), hence affinoid perfectoid. Thus the limit $\{f = 0\}$ is also affinoid perfectoid. The complement $\{f \neq 0\}$ is the union of loci $\{|\varpi|^n \leq |f| \neq 0\}$, which are open. □

21.2 The Demazure resolution

In the study of the usual Grassmannian variety G/B attached to a reductive group G, one defines a Schubert cell to be the closure of a B-orbit in

G/B. Generally, Schubert cells are singular varieties. Desingularizations of Schubert cells are constructed by Demazure, [Dem74]. We will make use of an analogue of this construction in the context of the B_{dR}^+-Grassmannian.

Definition 21.2.1. Suppose $\overline{\mu}$ corresponds to $(k_1 \geq \cdots \geq k_r)$, with $k_r \geq 0$. The *Demazure resolution*

$$\begin{array}{ccc} \mathrm{Dem}_{\overline{\mu}} & \longrightarrow & \mathrm{Gr}^{B_{\mathrm{dR}}^+} \\ & \searrow \quad \swarrow & \\ & \mathrm{Spd}\,\mathbf{Q}_p & \end{array}$$

sends a characteristic p perfectoid Huber pair (R, R^+) with untilt $R^\sharp$ to the functor

$$\left\{ M_{k_1} \subset M_{k_1 - 1} \subset \cdots \subset M_0 = B_{\mathrm{dR}}^+(R^\sharp)^r \right\} \mapsto M_{k_1} \in \mathrm{Gr}^{B_{\mathrm{dR}}^+}(R, R^+)$$

where for all $i = 1, \ldots, k_1 - 1$, $\xi M_i \subset M_{i+1} \subset M_i$, and M_i / M_{i+1} is a finite projective $R^\sharp$-module of rank j_i, where $k_{j_i} > i \geq k_{j_i + 1}$ (with convention $k_{r+1} = 0$).

The idea behind this definition is to write M_{k_1} as a series of successive minuscule modifications. Analyzing $\mathrm{Dem}_{\overline{\mu}}$ is easier than analyzing $\mathrm{Gr}^{B_{\mathrm{dR}}^+}$ directly. It is a succession of Grassmannian bundles.

Lemma 21.2.2. $\mathrm{Dem}_{\overline{\mu}}$ *is a qc diamond. (It is also spatial.)*

Proof. By induction it is enough to prove that if $X/\operatorname{Spd}\mathbf{Q}_p$ is a qc diamond (so that we get a sheaf $\mathcal{O}_X^\sharp$ on perfectoid spaces over X), and $\mathcal{E}/\mathcal{O}_X^\sharp$ is a locally free of finite rank, then $\mathrm{Grass}(d, \mathcal{E}) \to X$ is a qc diamond. Here $\mathrm{Grass}(d, \mathcal{E}) \to X$ associates to a morphism $\mathrm{Spa}(R, R^+) \to X$ the set of projective rank d quotients of $\mathcal{E}|_{\mathrm{Spa}(R,R^+)}$.

Since X is a qc diamond, we can choose a relatively representable $\widetilde{X}^\diamondsuit \to X$, where $\widetilde{X}$ is affinoid perfectoid. After replacing $\widetilde{X}$ with an open cover we may assume that $\mathcal{E}|_{\widetilde{X}} \cong (\mathcal{O}_{\widetilde{X}}^\sharp)^r$ is trivial. Then $\mathrm{Grass}(d, \mathcal{E}) \times_X \widetilde{X} \cong \mathrm{Grass}(d, r)^\diamondsuit \times_{\operatorname{Spd}\mathbf{Q}_p} \widetilde{X}$. Since $\mathrm{Grass}(d, r)$ is a classical qc rigid space, $\mathrm{Grass}(d, r)^\diamondsuit$ is a qc diamond, and thus so is $\mathrm{Grass}(d, \mathcal{E}) \times_X \widetilde{X}$. Thus $\mathrm{Grass}(d, \mathcal{E}) \times_X \widetilde{X}$ admits a relatively representable pro-étale cover by a qc perfectoid space; implying that $\mathrm{Grass}(d, \mathcal{E})$ does as well.

Now also $\mathrm{Dem}_{\overline{\mu}} \times_{\mathrm{Gr}^{B_{\mathrm{dR}}^+}} \mathrm{Spa}(R, R^+) \subset \mathrm{Dem}_{\overline{\mu}} \times_{\operatorname{Spd}\mathbf{Q}_p} \mathrm{Spa}(R, R^+)$ is a qc diamond. As the diagonal Δ of $\mathrm{Gr}^{B_{\mathrm{dR}}^+}$ is closed and relatively representable, this inclusion is closed and relatively representable, so that $\mathrm{Dem}_{\overline{\mu}} \times_{\mathrm{Spd}\,\mathbf{Q}_p} \mathrm{Spa}(R, R^+)$ is a qc diamond. □

Lemma 21.2.3. *Let $\mathcal{D} \to \mathrm{Spd}\,\mathbf{Q}_p$ be a qc diamond admitting a morphism to $\mathrm{Gr}^{B^+_{\mathrm{dR}}}$. Then $\mathrm{Dem}_{\overline{\mu}} \times_{\mathrm{Gr}^{B^+_{\mathrm{dR}}}} \mathcal{D}$ is also a qc diamond.*

Proof. This follows formally from the fact that $\mathrm{Dem}_{\mu} \times_{\mathrm{Spd}\,\mathbf{Q}_p} \mathcal{D}$ is a qc diamond and Lemma 21.1.2. □

Lemma 21.2.4. *Let (R, R^+) be a perfectoid Huber pair in characteristic p with un-tilt $R^\sharp$, and let $M \subset B_{\mathrm{dR}}(R^\sharp)^r$ be a $B^+_{\mathrm{dR}}(R^\sharp)$-lattice. The image of*

$$\mathrm{Dem}_{\overline{\mu}} \times_{\mathrm{Gr}^{B^+_{\mathrm{dR}}}} \mathrm{Spd}(R, R^+) \to \mathrm{Spd}(R, R^+)$$

is the locus $\left\{x \middle| \mathrm{inv}(M_x, (B^+_{\mathrm{dR}})^r) \leq \overline{\mu}\right\}$. (Over the locus where one has an equality, this is an isomorphism.)

Lemma 21.2.5. *Let $M \subset B_{\mathrm{dR}}(R^\sharp)^r$ be a $B^+_{\mathrm{dR}}(R^\sharp)$-lattice. The function*

$$\begin{array}{ccc} \mathrm{Spa}(R, R^+) & \text{to} & \mathbf{Z}^r_{dom} = \{k_1 \geq \cdots \geq k_r\} \\ x & \mapsto & \mathrm{inv}(M_x, (B^+_{\mathrm{dR}}(K(x)^\sharp)^+)) \end{array}$$

is lower semicontinuous.

Proof. Fix $\overline{\mu}$, which corresponds to $k_1 \geq \cdots \geq k_r$. Let Z be the locus of $x \in \mathrm{Spa}(R, R^+)$ where $\mathrm{inv}(M_x, (B^+_{\mathrm{dR}}(K(x)^\sharp)^r) \leq \overline{\mu}$. We want Z to be closed. Rescaling by a power of ξ, we may assume that $k_r \geq 0$.

By Lemma 21.2.4, Z is the image of $\mathrm{Dem}_{\overline{\mu}} \times_{\mathrm{Gr}^{B^+_{\mathrm{dR}}}} \mathrm{Spd}(R, R^+) \to \mathrm{Spd}(R, R^+)$. By Lemma 21.2.2, this fibre product is a qc diamond, and so Z is the image of a morphism $\mathrm{Spa}(S, S^+) \to \mathrm{Spa}(R, R^+)$. This is a spectral map of spectral spaces, which implies that Z is pro-constructible. Also, since the functor $\mathrm{Dem}_{\overline{\mu}} \times_{\mathrm{Gr}^{B^+_{\mathrm{dR}}}} \mathrm{Spd}(R, R^+) \to \mathrm{Spd}(R, R^+)$ does not depend on the ring of integral elements, the image Z is closed under specializations. Now we apply the fact that any specializing pro-constructible subset of a spectral space is closed, [Sta14, Topology, Lemma 22.5(2)]. □

21.3 $\mathrm{Gr}^{B^+_{\mathrm{dR}}}_{\overline{\mu}}$ is a diamond

As before, let $\overline{\mu}$ correspond to $(k_1 \geq \cdots \geq k_r)$. To prove that $\mathrm{Gr}^{B^+_{\mathrm{dR}}}_{\overline{\mu}}$ is a qc spatial diamond, we may assume that $k_r \geq 0$. Let $N = k_1 + \cdots + k_r$, and let $\mathrm{Gr}_N^{B_{\mathrm{dR}}^+} = \mathrm{Gr}_{(N,0,\ldots,0)}^{B_{\mathrm{dR}}^+}$. Then $\mathrm{Gr}_{\overline{\mu}}^{B_{\mathrm{dR}}^+} \subset \mathrm{Gr}_N^{B_{\mathrm{dR}}^+}$ is closed, so we are reduced to proving the claim for $\mathrm{Gr}_N^{B_{\mathrm{dR}}^+}$.

Note that $\mathrm{Gr}_N^{B_{\mathrm{dR}}^+} = \left\{ M \subset (B_{\mathrm{dR}}^+)^r \,|\, \det_{B_{\mathrm{dR}}^+} M = \xi^n B_{\mathrm{dR}} \right\}$. It is enough to show that $\mathrm{Gr}_{N,\mathbf{C}_p} := \mathrm{Gr}_N^{B_{\mathrm{dR}}^+} \times_{\mathrm{Spd}\,\mathbf{Q}_p} \mathrm{Spd}\,\mathbf{C}_p$ is a qc spacial diamond. We may now fix $\xi = p - [p^\flat]$. We need the following restatement of Thm. 16.2.10.

Theorem 21.3.1. *Let $\mathcal{D}$ be a qc spatial fake diamond. Assume that*

1. *$\Delta\colon \mathcal{D} \to \mathcal{D} \times \mathcal{D}$ is relatively representable.*
2. *For all $x \in |\mathcal{D}|$, there exists a perfectoid space S_x and a qpf map $S_x^\diamondsuit \to \mathcal{D}$ such that the image of $\left|S_x^\diamondsuit\right| \to |\mathcal{D}|$ contains x.*

Then $\mathcal{D}$ is a qc spatial diamond.

We intend to apply Thm. 21.3.1 to $\mathcal{D} = \mathrm{Gr}_N^{B_{\mathrm{dR}}^+}$. Hypothesis (1) is Lemma 21.1.2. Let $X/\,\mathrm{Spd}\,\mathbf{C}_p$ be the functor

$$X(R, R^+) = \left\{ B \in M_r(W(R^+)) \,|\, \det B \in \xi^N W(R^+)^\times \right\}.$$

Lemma 21.3.2. *X is an affinoid perfectoid space.*

Proof. First we observe that the functor $(R, R^+) \mapsto M_r(W(R^+))$ is representable by an infinite-dimensional closed unit ball $B^\infty_{\mathbf{C}_p}$ and is thus perfectoid. For an element $f \in W(R^+)$, the condition that $f \equiv 0 \pmod{\xi}$ is closed and relatively representable, as it is equivalent to the condition that $f^\sharp = 0 \in R^{\sharp +} = W(R^+)/\xi$. The condition that f is invertible is a rational subset, as it is equivalent to $\left\{ \left|\overline{f}\right| = 1 \right\}$, where $\overline{f} = f \pmod{p} \in R^+$. This suffices. □

Let $\lambda\colon X \to \mathrm{Gr}_{N,\mathbf{C}_p}^{B_{\mathrm{dR}}^+}$ be the map which sends B to $B((B_{\mathrm{dR}}^+)^r)$.

Lemma 21.3.3. *λ is a surjection of sheaves on the faithful site.*

Proof. Assume first that $(R, R^+) = (C, C^+)$ is a geometric point (possibly of higher rank). Given $M \subset (B_{\mathrm{dR}}^+)^r$ which is finite projective with $\det M = \xi^N B_{\mathrm{dR}}^+$, let $L = M \cap W(C^+)^r \subset (B_{\mathrm{dR}}^+)^r$. Then L is a $W(C^+)$-module such that $L[\xi^{-1}] = W(C^+)[\xi^{-1}]^r$, and $L[1/p]^\wedge_\xi \cong M$. So L is finite and projective away from the closed point of $\mathrm{Spa}\,W(C^+)$.

By our results on extending vector bundles, we get that L is actually a finite projective $W(C^+)$-module (as it is the space of global sections of restriction to the punctured spectrum). Fixing an isomorphism $L \cong W(C^+)^r$, get a matrix $B \in M_r(W(C^+))$ as desired.

For general (R, R^+), we use the faithful cover $\mathrm{Spa}(S, S^+) \to \mathrm{Spa}(R, R^+)$, where

$$S^+ = \prod_{x \in \mathrm{Spa}(R,R^+)} \widehat{K(x)^+}$$

and $S = S^+[1/p]$. Then one has a section of λ over this cover. $\square$

Corollary 21.3.4. $\mathrm{Gr}_N^{B_{\mathrm{dR}}^+}$ *is a qc fake diamond.*

In order to apply Thm. 21.3.1, we need to show that Gr_N is spatial. Let $Y = X \times_{\mathrm{Gr}_{N,\mathbf{C}_p}^{B_{\mathrm{dR}}^+}} X$. This is closed and relatively representable in $X \times_{\mathbf{C}_p} X$, so that Y is affinoid perfectoid. Let $L^+ \mathrm{GL}_r \,/\, \mathrm{Spd}\, \mathbf{C}_p$ be the functor sending (R, R^+) to $\mathrm{GL}_r(W(R^+))$; this is also affinoid perfectoid.

Lemma 21.3.5. $X \times_{\mathbf{C}_p} L^+ \mathrm{GL}_r \xrightarrow{\sim} Y$.

Proof. $W(R^+) \subset B_{\mathrm{dR}}^+(R^\sharp)$ is injective (check on geometric points). We need to show that if $B_1, B_2 \in X(R, R^+)$ give the same point of $\mathrm{Gr}^{B_{\mathrm{dR}}^+}$, then their ratio $B_1 B_2^{-1}$, a priori in $\mathrm{GL}_r(B_{\mathrm{dR}}(R^\sharp))$, actually lives in $\mathrm{GL}_r(W(R^+))$. This can be checked on geometric points (of whatever rank), and there it follows from the uniqueness of the lattice L constructed above. $\square$

Finally we can verify that $\mathrm{Gr}_{N,\mathbf{C}_p}^{B_{\mathrm{dR}}^+}$ is spatial. It is enough to show that $Y \to X$ is open. But $Y = X \times_{\mathbf{C}_p} L^+ \mathrm{GL}_r \subset X \times_{\mathbf{C}_p} B_{\mathbf{C}_p}^{\infty} \to X$ is the composition of open maps, hence it is open. For openness of the second map, one can show that if $f' \colon Y' \to X'$ is a flat map of rigid spaces, then f' is open.

To finish the proof that $\mathrm{Gr}_{N,\mathbf{C}_p}^{B_{\mathrm{dR}}^+}$ is a qc spatial diamond, we need to check the punctual criterion (2). Let $x \in \left|\mathrm{Gr}_{N,\mathbf{C}_p}^{b_{\mathrm{dR}}^+}\right|$. We want to produce $S_x \to \mathrm{Gr}_{N,\mathbf{C}_p}^{B_{\mathrm{dR}}^+}$, where S_x is representable and x lies in the image. This can be deduced from the Demazure resolution coming from the μ which describes x. We have:

Theorem 21.3.6. $\mathrm{Gr}_{\overline{\mu}}^{B_{\mathrm{dR}}^+}$ *is a qc spatial diamond.*

With the same proof, one shows that:

Theorem 21.3.7. $\mathrm{Gr}^{\mathrm{BD}}_{(\mathrm{Spd}\,\mathbf{Z}_p)^n \setminus \{s\}, \{\overline{\mu}_i\}}$ *is a qc spatial diamond.*

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