Basics of Wavelets

References: I. Daubechies (Ten Lectures on Wavelets; Orthonormal Bases of Compactly Supported Wavelets)

Also: Y. Meyer, S. Mallat

Outline:

1. Need for time-frequency localization
2. Orthonormal wavelet bases: examples
3. Meyer wavelet
4. Orthonormal wavelets and multiresolution analysis

1. Introduction

Signal:

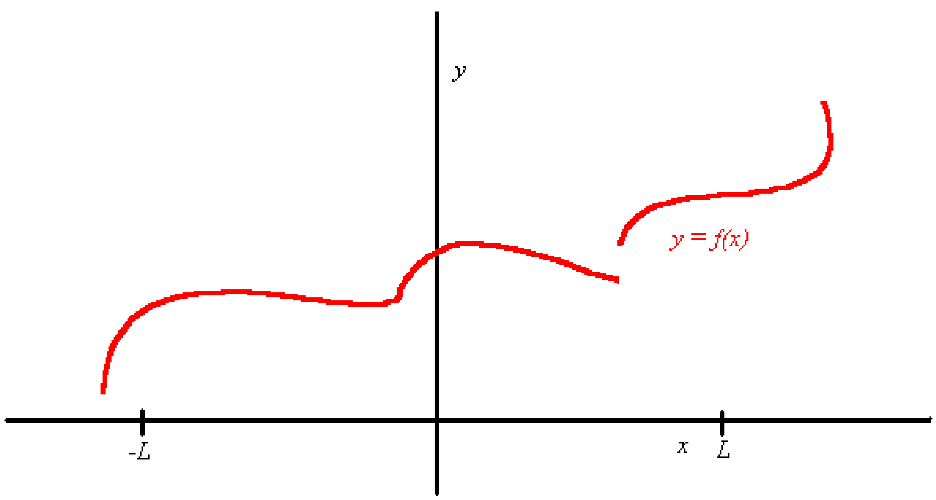


fig 1

Interested in "frequency content" of signal, locally in time. E.G., what is the frequency content in the interval [.5, .6]?

Standard techniques: write in Fourier series as sum of sines and cosines: given function defined on $[-L, L]$ as above:

$f(x) =$

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx(\pi/L) + b_n \sin nx(\pi/L)$$

$(a_{n,} b_n$ constants)

$$a_n = \frac{1}{L}\int_{-L}^{L} dx\, f(x) \cos nx(\pi/L)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} dx\, f(x) \sin\, nx\, (\pi/L)$$

(generally f is complex-valued and a_n, b_n are complex numbers).

FOURIER SERIES:

Consider function $f(x)$ defined on $[-L, L]$.

Let $L^2[-L, L] =$ square integrable functions

$$= \left\{ f : [-L, L] \to \mathbb{C} \middle| \int_{-L}^{L} dx\, |f^2(x)| < \infty \right\}$$

where $\mathbb{C} =$ complex numbers. Then L^2 forms a Hilbert space.

Basis for Hilbert space:

$$\left\{ \frac{1}{\sqrt{L}} \cos nx(\pi/L),\ \frac{1}{\sqrt{L}} \sin nx(\pi/L) \right\}_{N=1}^{\infty}$$

(together with the constant function $1/\sqrt{2L}$).

These vectors form an orthonormal basis for L^2 (constants $1/\sqrt{L}$ give length 1).

1. Recall the complex form of Fourier series:

Equivalent representation:

Can use Euler's formula $e^{ib} = \cos b + i \sin b$. Can show similarly that the family

$$\left\{ \frac{1}{\sqrt{2L}} e^{inx(\pi/L)} \right\}_{n=-\infty}^{n=\infty} = \left\{ \frac{1}{\sqrt{2L}} \cos nx(\pi/L) + \frac{i}{\sqrt{2L}} \sin nx(\pi/L) \right\}_{n=-\infty}^{\infty}$$

is orthonormal basis for L^2.

Function $f(x)$ can be written

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \phi_n(x)$$

where

$$c_n = \langle \phi_{n,} f \rangle = \int_{-L}^{L} dx\, \overline{\phi_n(x)} f(x),$$

where

$$\phi_n(x) = n^{th} \text{ basis element } = \frac{1}{\sqrt{2L}} e^{inx(\pi/L)}.$$

2. Recall derivation of the Fourier transform from Fourier series:

We start with function $f(x)$ on $(-L, L)$:

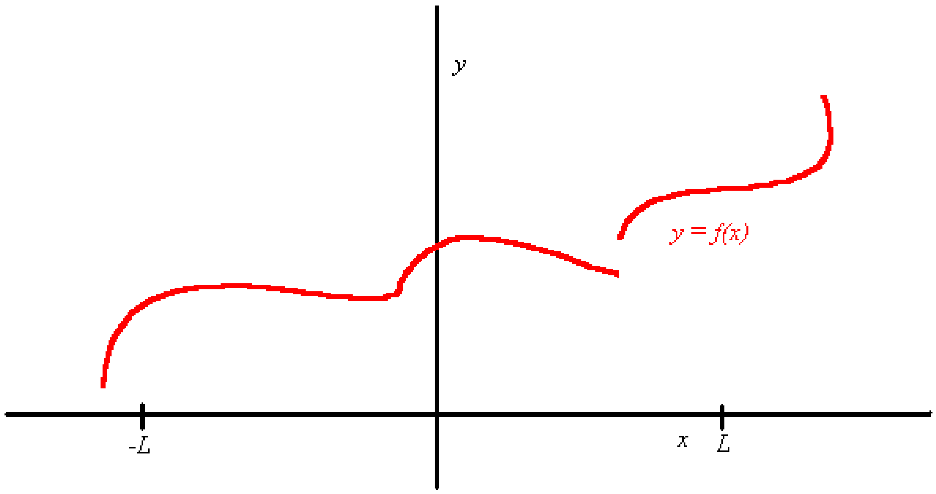


fig 2

$$f(x)=\sum_{n=-\infty}^{\infty} c_n\phi_n(x)=\sum_{n=-\infty}^{\infty} c_n e^{inx(\pi/L)}/\sqrt{2L}$$

Let $\xi_n = n\pi/L$; let $\Delta\xi = \pi/L$;

let $c(\xi_n) = c_n\sqrt{2\pi}/\left(\sqrt{2L}\,\Delta\xi\right)$.

Then:

$$\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx(\pi/L)}/\sqrt{2L} \\
&= \sum_{n=-\infty}^{\infty} (c_n/\sqrt{2L})\, e^{ix\xi_n} \\
&= \sum_{n=-\infty}^{\infty} c_n/\left(\sqrt{2L}\Delta\xi\right) e^{ix\xi_n}\Delta\xi\,. \\
&= \frac{1}{\sqrt{2\pi}} \quad \sum_{n=-\infty}^{\infty} c_n\sqrt{2\pi}/\left(\sqrt{2L}\Delta\xi\right) e^{ix\xi_n}\Delta\xi. \\
&= \frac{1}{\sqrt{2\pi}} \quad \sum_{n=-\infty}^{\infty} c(\xi_n) e^{ix\xi_n}\Delta\xi.
\end{aligned}$$

Note as $L\to\infty$, we have $\Delta\xi\to 0$, and

$$
\begin{aligned}
c(\xi_n) &= c_n\sqrt{2\pi}/\left(\sqrt{2L}\Delta\xi\right) \\
&= \int_{-L}^{L} dx\, f(x)\,\overline{\phi_n(x)}\cdot\sqrt{2\pi}/\left(\sqrt{2L}\,\Delta\xi\right) \\
&= \int_{-L}^{L} dx\, f(x)\ \frac{\sqrt{2\pi}}{2L\Delta\xi}\, e^{-inx(\pi/L)} \\
&= \int_{-L}^{L} dx\, f(x)\ \frac{\sqrt{2\pi}}{2L(\pi/L)}\, e^{-inx(\pi/L)} \\
&= \frac{1}{\sqrt{2\pi}}\int_{-L}^{L} dx\, f(x)\ \ e^{-ix\xi_n}\,.
\end{aligned}
$$

Now (informally) take the limit $L \to \infty$. The interval becomes

$$[-L, L] \to (-\infty, \infty).$$

We have

$$f(x) = \frac{1}{\sqrt{2\pi}} \quad \sum_{n=-\infty}^{\infty} c(\xi_n)\, e^{ix\xi_n}\Delta\xi$$

$$\overrightarrow{L \to \infty} \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c(\xi)e^{ix\xi}\ d\xi$$

Finally, from above

$$c(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-L}^{L} dx\, f(x)\ e^{ix\xi}$$

$$\overrightarrow{L \to \infty}\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, f(x)\ \ e^{ix\xi}.$$

Thus, the informal arguments give that in the limit, we can write

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c(\xi)\, e^{ix\xi}\, d\xi,$$

where $c(\xi)$ (called *Fourier transform* of f) is

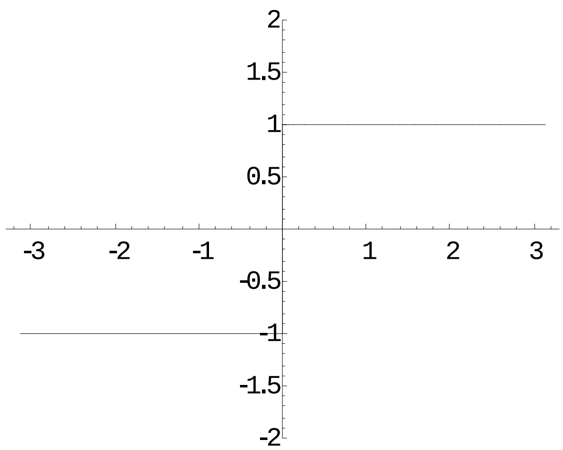
$$c(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x)\, e^{ix\xi}$$

(like Fourier series with sums replaced by integrals over the real line).

Note: can prove that writing $f(x)$ in the above integral form is valid for arbitrary $f \in L^2(-\infty, \infty)$.

3. FREQUENCY CONTENT AND GIBBS PHENOMENON

Consider Fourier series of a function $f(x)$ which is discontinuous at $x = 0$. E.g. if $f(x) = \frac{|x|}{x}$:



the first few partial sums of the Fourier series of f look like this:

5 terms:

$$\frac{4\sin x}{\pi} + \frac{4\sin 3x}{3\pi} + \frac{4\sin 5x}{5\pi}$$

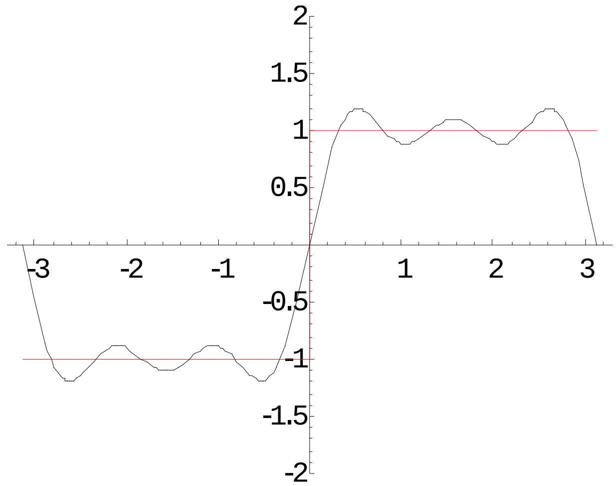
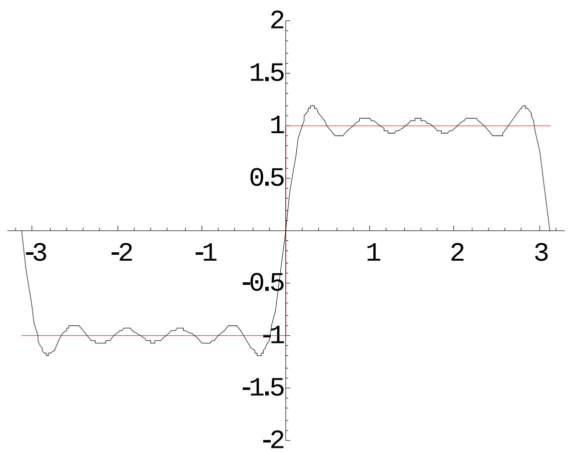


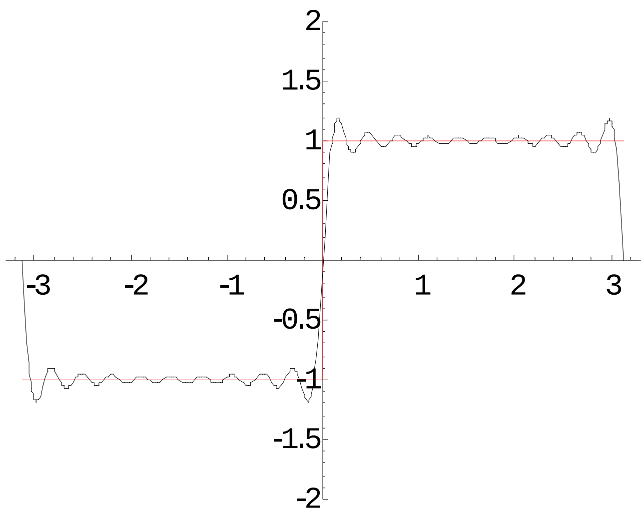
fig 3

10 terms of Fourier series:

$$\frac{4\sin x}{\pi} + \frac{4\sin 3x}{3\pi} + \frac{4\sin 5x}{5\pi} + \ldots + \frac{4\sin 10x}{10\pi}$$



20 terms:



40 terms:

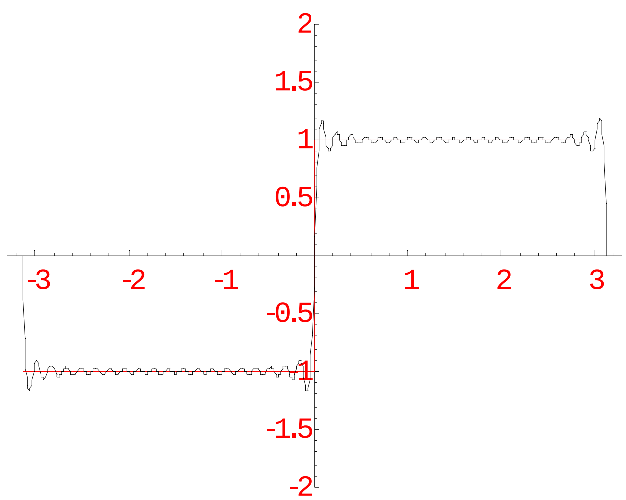


fig 3

Note that there are larger errors appearing near the “singularity"

Specifically: “overshoot” of about 9% of the jump near singularity no matter how many terms we take!

In general, singularities (discontinuities in $f(x)$ or its derivatives) cause high frequency components so that the Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{inx} / \sqrt{2\pi}$$

has large c_n for n large (bad for convergence).

But notice that the singularities are in only one point, but cause all the c_n to be too large.

Wavelets deal with the problem of localization of singularities, since they are localized.

Advantages of Fourier series:

- "Frequency content" displayed in sizes of the coefficients a_k and b_k.

- Easy to write derivatives of f in terms of series (and use to solve differential equations)

Fourier series are a natural for differentiation.

Equivalently, sines and cosines are "eigenvectors" of the derivative operator $\frac{d}{dx}$.

Disadvantages:

- Usual Fourier transform or series not well-adapted for time-frequency analysis (i.e., if high frequencies are there, then we have large a_k and b_k for $k = 100$. But what part of the function has the high frequencies? Where $x < 0$? Where $2 < x < 3$?

Possible solution:

Sliding Fourier transform -

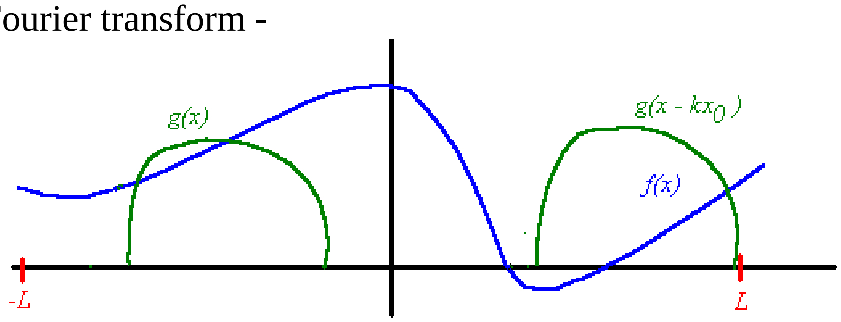


fig 4

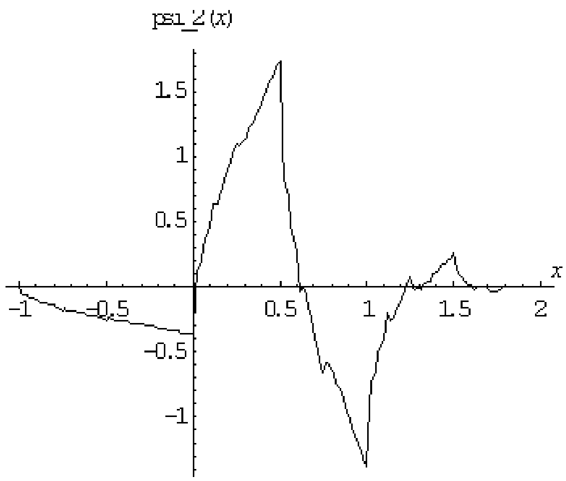
Thus first multiply $f(x)$ by a “window” $g(x - kx_0)$, and then take a look at the Fourier series or take Fourier transform: look at

$$\int_{-L}^{L} dx\, f(x)\, g_{jk}(x) = \int_{-L}^{L} dx\, f(x)\; g(x - kx_0)e^{ij\omega x} \equiv c_{jk}$$

Note however: the functions $g_{jk}(x) = g(x - kx_0)e^{ij\omega x}$ are not orthonormal like sines and cosines; do not form a nice basis as in Fourier series; need something better.

4. The wavelet transform

Try: Wavelet transform - first fix an appropriate function $h(x)$.



Then form all possible translations by integers, and all possible “stretchings" by powers of 2:

$$h_{jk}(x) = 2^{j/2} h(2^j x - k)$$

($2^{j/2}$ is just a normalization constant)

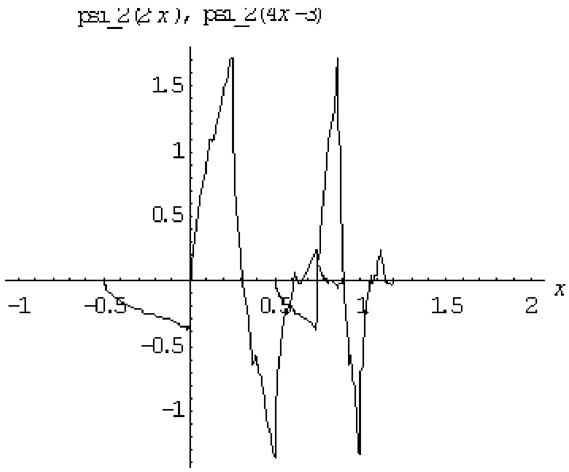


fig. 5: $h(2x)$ and $h(4x-3)$

Let

$$c_{jk} = \int dx\, f(x)\, h_{jk}(x).$$

If h chosen properly, then can get back f from the c_{jk}:

$$f(x) = \sum_{j,k} c_{jk}\, h_{jk}(x)$$

These new functions and coefficients are easier to manage. Sometimes much better.

Advantages over windowed Fourier transform:

- Coefficients c_{jk} are all real
- For high frequencies (j large), the functions $h_{jk}(t)$ have good localization (they get thinner as $j \to \infty$; see above diagram). Thus short lived (i.e. of small duration in x) high frequency components can be seen from wavelet analysis, but not from windowed Fourier transform.

Note h_{jk} has width of order 2^{-j}, and is centered about $k2^{-j}$ (see diagram earlier).

DISCRETE WAVELET EXPANSIONS:

Take a basic function $h(x)$ (the basic wavelet);

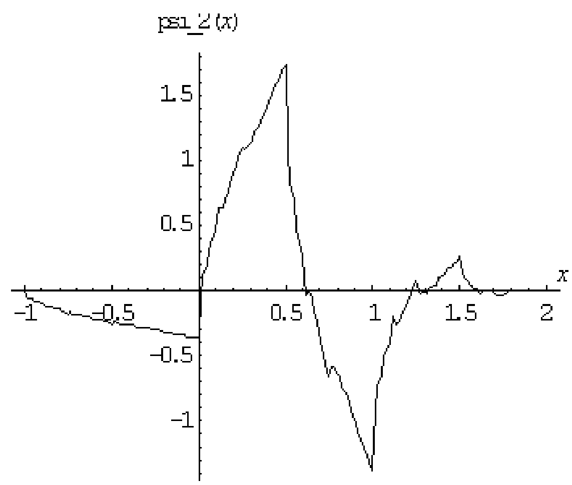


fig 6

let

$$h_{jk}(x) = 2^{j/2} h(2^j x - k).$$

Form discrete wavelet coefficients:

$$c_{jk} = \int dx\, f(x)\, h_{jk}(x) \equiv \langle f,\, h_{jk} \rangle.$$

Questions:

- Do the coefficients c_{jk} characterize f?
- Can we expand f in an expansion of the h_{jk}?
- What properties must h have for this to happen?
- How can we reconstruct f in a numerically stable way from knowing c_{jk}?

We will show: It is possible to find a function h such that the functions h_{jk} form such a perfect basis for the functions on $\mathbb{R}$.

That is, the functions h_{jk} are orthonormal:

$$\langle h_{jk}, h_{j'k'} \rangle \equiv \int h_{jk}(x) h_{j'k'}(x) dx = 0$$

unless $j = j'$ and $k = k'$.

And any function $f(x)$ can be represented by the functions h_{jk}:

$$f(x) = \sum_{j,k} c_{jk} h_{jk}(x).$$

So: just like Fourier series, but the h_{jk} have better properties (e.g., they are non-zero only on a small sub-interval, i.e., compactly supported)

5. A SIMPLE EXAMPLE: HAAR WAVELETS

Motivation: suppose we have a basic function

$$\phi(x) = \begin{cases} 1 \text{ if } 0 \le x \le 1 \\ 0 \text{ otherwise} \end{cases} = \text{basic "pixel".}$$

We wish to build all other functions out of this pixel and translates $\phi(x-k)$

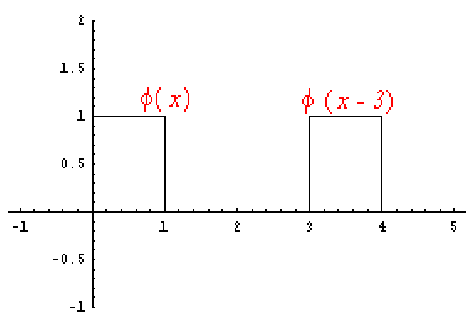


fig 7: ϕ and its translates

Linear combinations of the $\phi(x-k)$:

$$f(x) = 2\phi(x) + 3\phi(x-1) - 2\phi(x-2) + 4\phi(x-3)$$

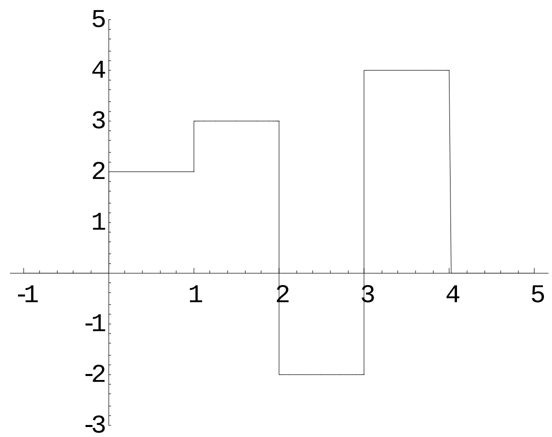


fig 8: a linear combination of $\phi(x-k)$

[Note that any function which is constant on the integers can be written in such a form:]

Given function $f(x)$, approximate $f(x)$ by a linear combination of $\phi(x-k)$:

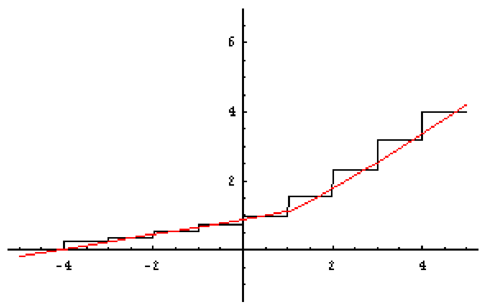


fig 9: approximation of $f(x)$ using the pixel $\phi(x)$ and its translates.

Define V_0 = all square integrable functions of the form

$$g(x) = \sum_k a_k \phi(x-k)$$

= all square integrable functions which are constant on integer intervals

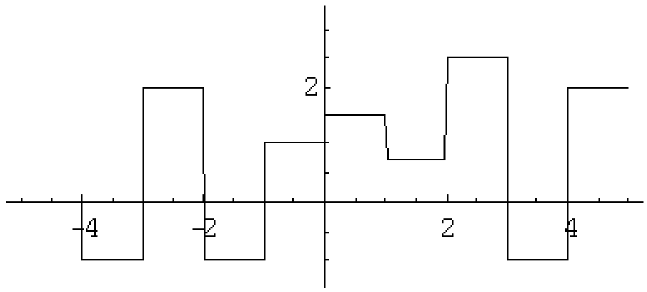


fig 10: a function in V_0

To get better approximations, shrink the pixel :

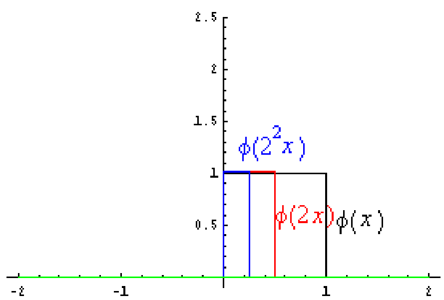


fig 11: $\phi(x)$, $\phi(2x)$, and $\phi(2^2x)$

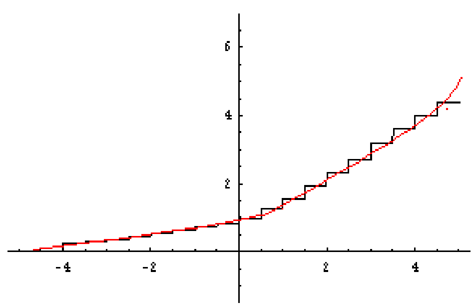


fig 12: approximation of $f(x)$ by translates of $\phi(2x)$.

Define

V_1 = all square integrable functions of the form

$$g(x) = \sum_k a_k \phi(2x - k)$$

$=$ all square integrable functions which are constant on all half-integers

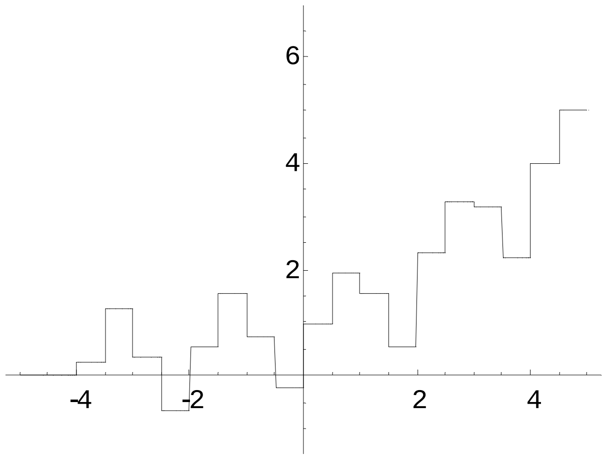


fig 13: Function in V_1

Define V_2 = sq. int. functions

$$g(x) = \sum_k a_k \phi(2^2 x - k)$$

= sq. int. fns which are constant on quarter integer intervals

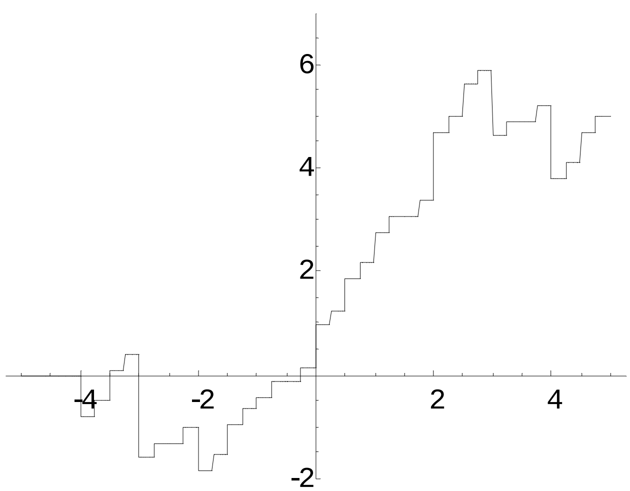


fig 14: function in V_2

Generally define V_j = all square integrable functions of the form

$$g(x) = \sum_k a_k \phi(2^j x - k)$$

= all square integrable functions which are constant on 2^{-j} length intervals

[note if j is negative the intervals are of length greater than 1].

2. Haar Wavelets, General Theory

1. The Haar wavelet

Now define the desired wavelet $\psi(x)$

$$\equiv \begin{cases} 1 \quad \text{if } \ 0 \le x \le 1/2 \quad -1 \ \text{ if } 1/2 \le x < 1 \\ \qquad\qquad\qquad\qquad\quad 0 \quad \text{otherwise} \end{cases}$$

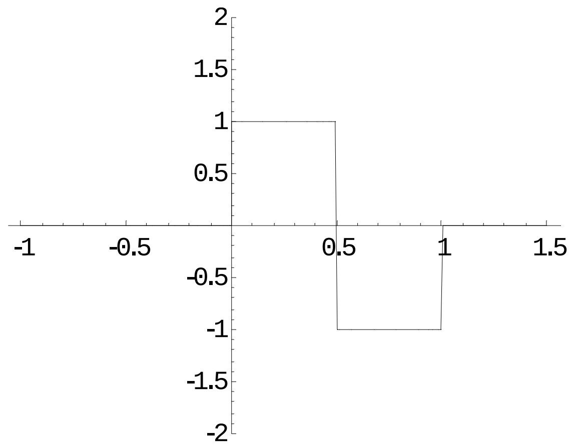


fig 15: $\psi(x)$

Now define family of Haar wavelets by translating:

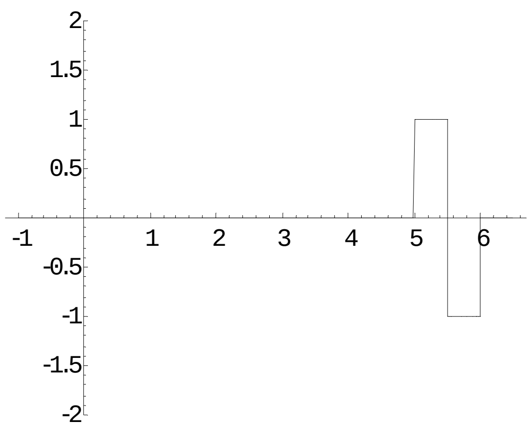


fig 16 : $\psi(x-5) = \psi_{0,5}$

and stretching:

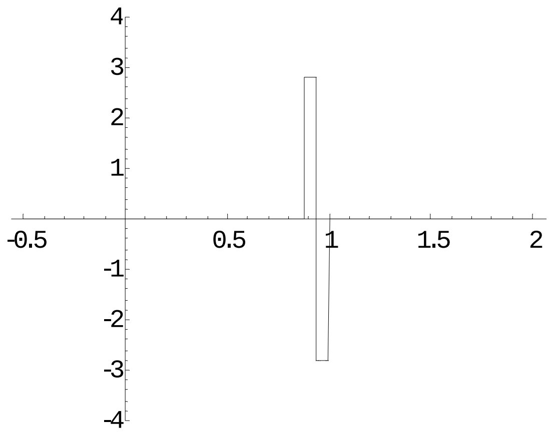


fig 17: $2^{3/2}\psi(2^3x-7)=\psi_{3,7}$

In general:

$$\psi_{jk} \equiv 2^{j/2}\,\psi(2^jx-k)$$

Show Haar wavelets are orthogonal, i.e.,

$$\left\langle \psi_{jk},\, \psi_{j'k'} \right\rangle \equiv \int_{-\infty}^{\infty} dx\, \psi_{jk}(x)\psi_{j'k'}(x) = 0$$

if $\quad j \neq j' \quad$ or $\quad k \neq k'$:

(i) if $\quad j = j', k \neq k'$:

$$\left\langle \psi_{jk}, \psi_{j'k'} \right\rangle = 0$$

because $\quad \psi_{jk} = 0 \quad$ wherever $\quad \psi_{jk'} \neq 0 \quad$ and vice-versa.

(ii) if $\quad j \neq j'$:

$$\left\langle \psi_{jk}, \psi_{j'k'} \right\rangle = 0$$

because:

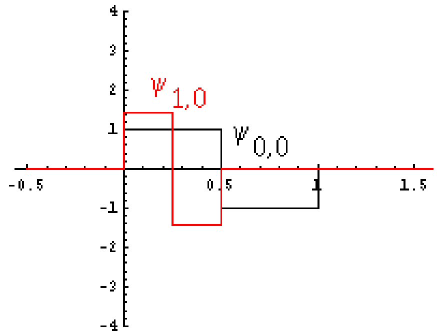


fig 18

$\Rightarrow$ integral $\langle \psi_{jk}, \psi_{j'k'} \rangle$ is 0.

2. Can any function be represented as a combination of Haar wavelets?

[A general approach:]

Recall:

$$V_j = \text{square int. functions of form } \sum_k a_k \phi(2^j x - k)$$

$=$ square int. functions constant on dyadic intervals of length 2^{-j}.

[note if j is negative the intervals are of length greater than 1:]

V_{-1} = functions constant on intervals of length 2

V_{-2} = functions constant on intervals of length 4

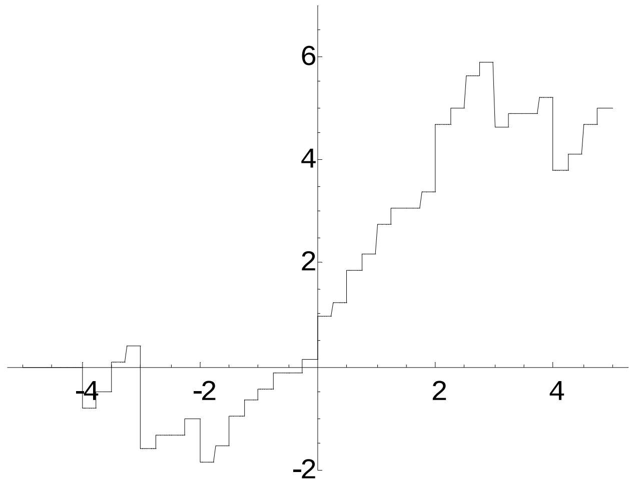


fig. 19: function in V_j $(j = 2)$

We have:

(a) $\ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset V_3 \ldots$

[i.e., piecewise constant on integers $\Rightarrow$ piecewise constant on half-integers, etc.]

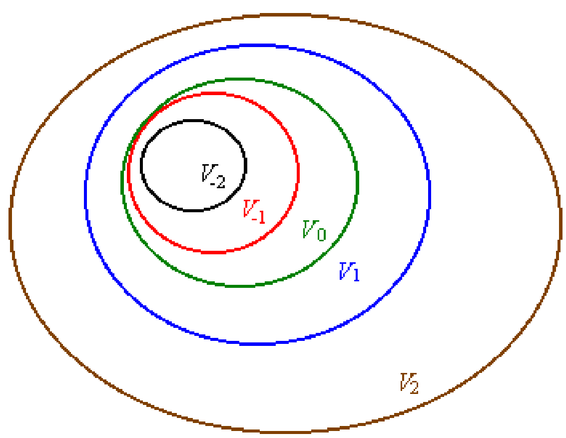


Fig. 20: Relationship of the nested spaces V_j

(b) $\bigcap\limits_n V_n = \{0\}$ (only 0 function in all spaces) [if a function is in all the spaces, then it must be constant on arbitrarily large intervals $\Rightarrow$ must be everywhere constant; also must be square integrable; so must be 0].

(c) $\bigcup_n V_n$ is dense in $L^2(\mathbb{R})$

[i.e. the collection of all functions of this form can approximate any function $f(x)$]

Proof: First consider a function of the form $f(x) = \chi_{[a,b]}(x)$. Assume that $a = k/2^n - a_1$, and $b = \ell/2^n + b_1$, where $a_1, b_1 < 1/2^n$.

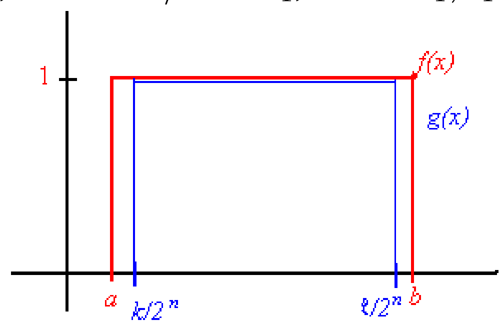


fig 21: Relationship of a, b with $k/2^n$ and $\ell/2^n$.

Let

$$g(x) = \chi_{[k/2^n, \ell/2^n]}(x) \in \bigcup_j V_j.$$

Then

$$\|f - g\| = \int dx\,(f-g)^2 = \text{area under } f - g \le 2/2^n$$

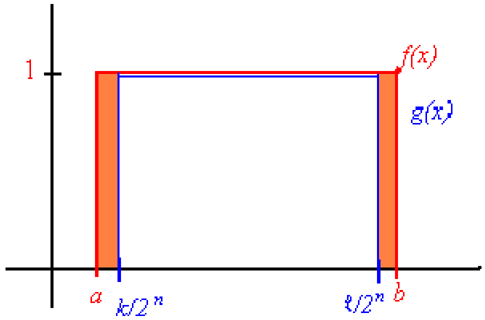


fig 22: area under $f - g$

Since n is arbitrary, $\| f - g \|$ can be made arbitrarily small. Thus arbitrary char. functions f can be well-approximated by functions $g(x) \in \bigcup_i V_i$.

Now if $f(x)$ is a step function:

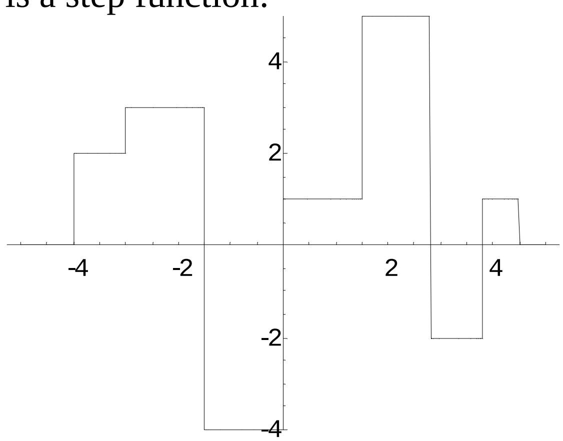


fig 23: step function

We can write

$$f(x) = \sum_i c_i \chi_{[a_i, b_i]}(x) = \text{ linear combination of char. functions.}$$

So by above argument, step functions f can be approximated arbitrarily well by $g \in \bigcup_j V_j$.

Now step functions are dense in $L^2(\mathbb{R})$ (see R&S, problem II.2), so that $\bigcup_j V_j$ must be dense in $L^2(\mathbb{R})$. $\square$

(d) $f(x) \in V_n \quad \Rightarrow \quad f(2x) \in V_{n+1}$

[because a function constant on intervals of length 2^{-n} when shrunk is constant on intervals of length 2^{-n-1}]

(e) $f(x) \in V_0 \quad \Rightarrow \quad f(x-k) \in V_0$

[i.e. translating a function by an integer does not change that it is constant on integer intervals]

(f) There is an orthogonal basis for the space V_0 in the family of functions

$$\phi_{0k} \equiv \phi(x-k)$$

where k varies over the integers. This function ϕ is (in this case) $\phi = \chi_{[0,1]}(x)$.

ϕ is called a *scaling function.*

Definition: A sequence of spaces $\{V_j\}$ together with a scaling function ϕ which generates V_0 so that (a) - (f) above are satisfied, is called a *multiresolution analysis.*

3. Some more Hilbert space theory

Recall: Two subspaces M_1 and M_2 of vector space V are *orthogonal* if every vector $w_1 \in M_1$ is perpendicular to every vector $w_2 \in M_2$.

Ex: Consider $V = L^2(-\pi, \pi)$. Then let

$$M_1 = \{f(x) : f(x) = \sum_{n=0}^{\infty} a_n \cos nx\}$$

be the set of Fourier series with cosine functions only. Let

$$M_2 = \{f(x) : f(x) = \sum_{n=1}^{\infty} b_n \sin nx\}$$

be the set of Fourier series with sin functions only.

Then if $f_1 = \sum_{n=1}^{\infty} a_n \cos\, nx \in M_1$ and if $f_2 = \sum_{k=1}^{\infty} b_n \sin kx \in M_2$, then using usual arguments:

$$\langle f_1, f_2 \rangle = \sum_{n=1}^{\infty} a_n b_k \langle \cos nx, \sin kx \rangle = 0$$

Thus M_1 is orthogonal to M_2.

Recall: **A** vector space V is a *direct sum* $M_1 \oplus M_2$ of subspaces M_1, M_2 if every vector $v \in V$ can be written uniquely as a sum of vectors $w_1 \in M_1$ and $w_2 \in M_2$.

V is an *orthogonal* direct sum $M_1 \oplus M_2$ if the above holds and in addition M_1 and M_2 are orthogonal to each other.

Ex: If $V = \mathbb{R}^3$, and

$$M_1 = x - y \,\text{plane} = \{(x, y, 0) : x, y \in \mathbb{R}\}$$

$$M_2 = z - \text{axis} = \{(0, 0, z) : z \in \mathbb{R}\},$$

then every vector $(x, y, z) \in V$ can be written uniquely as a sum of $(x, y, 0) \in M_1$ and $(0, 0, z) \in M_2$, so that V is an orthogonal direct sum $M_1 \oplus M_2$.

Ex: $V = L^2[-\pi, \pi]$. Then every function $f(x)$ can be written uniquely as

$$f(x) = \sum_{n=0}^{\infty} a_n \cos nx + \sum_{k=1}^{\infty} b_n \sin kx$$

[note first sum in M_1 and second in M_2]

Thus $L^2 = M_1 \oplus M_2$ is an orthogonal direct sum. Note: not hard to show that

$$M_1 = \text{ even functions in } L^2$$

$$M_2 = \text{odd functions in } L^2$$

[thus L^2 is an orthogonal direct sum of even functions and odd functions]

Theorem 1: *If V is a Hilbert space and if $M_1 \perp M_2$ and $V = M_1 + M_2$, i.e., $\forall\ v \in V\ \exists\ m_i \in M_i$ s.t. $v = m_1 + m_2$, then $V = M_1 \oplus M_2$ is an orthogonal direct sum of M_1 and M_2*

Pf: In exercises.

Note: no assumption of uniqueness of v_i necessary above.

Def: If $V = W_1 \oplus W_2$ is an orthogonal direct sum, we also write

$$W_1 = V \ominus W_2; \qquad W_2 = V \ominus W_1.$$

Recall: Given a subspace $M \subset V$,

$M^\perp$ = vectors which are perpendicular to everything in M

$$= \{v \in V : v \perp w \forall w \in W\}$$

Ex: If $V = \mathbb{R}^3$, and $W = x$-y plane, then $W^\perp = z$-axis

Ex: If $V = L^2$, then if $W =$ even functions, $W^\perp =$ odd functions.

Pf. exercise

Recall (R&S, Theorem II.3)**:** Given a complete inner product space V and a complete subspace M, then V is an orthogonal direct sum of M and $M^\perp$

4. Back to wavelets:

Recall:

- $V_j =$ functions constant on dyadic intervals$[k2^{-j}, (k+1)2^{-j}]$.
- $\ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \ldots$

Since $V_0 \subset V_1$, there is a subspace $W_0 = V_1 \ominus V_0$ such that $V_0 \oplus W_0 = V_1$.

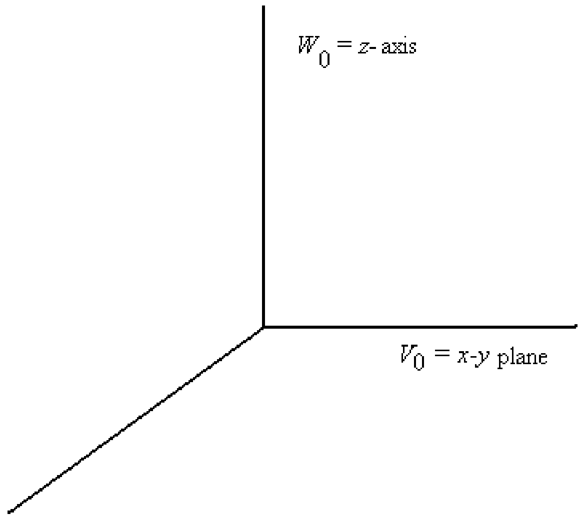


fig 24: Relationship of V_0 and W_0 : V_0 as the x-y plane and W_0 as the z axis ... $V_0 \oplus W_0 = V_1 = \mathbb{R}^3$.

Similarly define

$$W_1 = V_2 \ominus V_1.$$

Generally:

$$W_{j-1} = V_j \ominus V_{j-1}$$

Then relationships are:

$$\begin{array}{c} \ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \ldots \\ W_{-2} \quad W_{-1} \quad W_0 \quad W_1 \end{array}$$

Also note, say, for V_3:

$$V_3 = V_2 \oplus W_2$$

$$= V_1 \oplus W_1 \oplus W_2$$

$$= V_0 \oplus W_0 \oplus W_1 \oplus W_2$$

$$= V_{-1} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2$$

Thus if $v_3 \in V_3$, we have:

$$
\begin{aligned}
v_3 &= v_2 + w_2 \\
&= v_1 + w_1 + w_2 \\
&= v_0 + w_0 + w_1 + w_2 \\
&= v_{-1} + w_{-1} + w_0 + w_1 + w_2,
\end{aligned}
$$

with $v_i \in V_i$ and $w_i \in W_i$.

[successively decomposing the v into another v and a w].

In general :

$$v_3 = v_{-n} + \sum_{k=-n}^{2} w_k. \tag{1}$$

Now let $n \to \infty$. Since all vectors in above sum orthogonal, we have (see exercises):

$$\|v_3\|^2 = \|v_{-n}\|^2 + \sum_{k=-n}^{2} \|w_k\|^2.$$

Thus

$$\sum_{k=-n}^{2} \|w_k\|^2 \leq \|v_3\|^2$$

$\forall\, n$, so

$$\sum_{k=-\infty}^{2} \|w_k\|^2 < \infty.$$

Lemma: In a Hilbert space H, if w_k are orthogonal vectors and the sum $\sum_k \|w_k\|^2 < \infty$, then the sum $\sum_k w_k$ converges.

Pf: We can show that the sum $\sum_{k=1}^{N} w_k$ forms a Cauchy sequence by noting if $N > M$:

$$\|\sum_{k=1}^{N} w_k - \sum_{k=1}^{M} w_k\|^2 = \|\sum_{k=M+1}^{N} w_k\|^2 = \sum_{k=M+1}^{N} \|w_k\|^2 \underset{N,M\to\infty}{\longrightarrow} 0.$$

Thus we have a Cauchy sequence. The sequence must converge (H is complete), and so $\sum_{k=1}^{\infty} w_k$ exists. $\square$

From above:

$$v_{-n} = v_3 - \sum_{k=-n}^{2} w_k \, .$$

Letting $n \to \infty$, get $v_{-n} \underset{n\to\infty}{\longrightarrow} v_3 - \sum_{k=-\infty}^{2} w_k$. Thus vectors v_{-n} have limit as $n \to \infty$: $v_{-n} \to v_{-\infty}$.

But notice

$$v_{-n} \in V_{-n} \subset V_{-n+1} \subset V_{-n+2} \dots$$

$\Rightarrow$

$$v_{-n} \in V_{-n} \cap V_{-n+1} \cap V_{-n+2} \dots = \bigcap_{k=-n}^{\infty} V_k$$

Thus $v_{-\infty} \in$ intersection of all V_n's $=$

$$\Rightarrow v_{-\infty} = 0 \quad \text{(by condition (b) on spaces } V_j\text{).}$$

Thus taking the limit as $n \to \infty$ in (1) :

$$v_3 = v_{-n} + \sum_{k=-n}^{2} w_k \qquad (1)$$

get

$$v_3 = \sum_{k=-\infty}^{2} w_k \, .$$

So by definition of direct sum:

$$V_3 = \ldots W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 = \bigoplus_{k=-\infty}^{2} W_k$$

i.e., every vector in V_3 can be uniquely expressed as a sum of vectors in the W_j. Further this is an orthogonal direct sum since W_j's orthogonal.

Generally:

$$V_n = \ldots W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \ldots \oplus W_{n-1} = \bigoplus_{k=-\infty}^{n-1} W_k$$

Now note

$$L^2 = V_3 \oplus V_3^{\perp}$$

$$= V_4 \oplus V_4^{\perp}$$

$$= V_3 \oplus W_3 \oplus V_4^{\perp}.$$

Thus comparing above get

$$V_3^{\perp} = W_3 \oplus V_4^{\perp}.$$

Similarly,

$$V_4^{\perp} = W_4 \oplus V_5^{\perp}.$$

So

$$V_3^{\perp} = W_3 \oplus W_4 \oplus V_5^{\perp}.$$

Generally:

$$V_3^{\perp} = W_3 \oplus W_4 \oplus W_5 \oplus \ldots \oplus W_n \oplus V_{n+1}^{\perp}.$$

Letting $n \to \infty$ and using same arguments, we see that the $V_{n+1}^{\perp}$ components "go to 0" as $n \to \infty$, so that

$$V_3^{\perp} = W_3 \oplus W_4 \oplus W_5 \oplus \ldots$$

Thus:

$$L^2 = V_3 \oplus V_3^{\perp} = \ldots W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \ldots$$

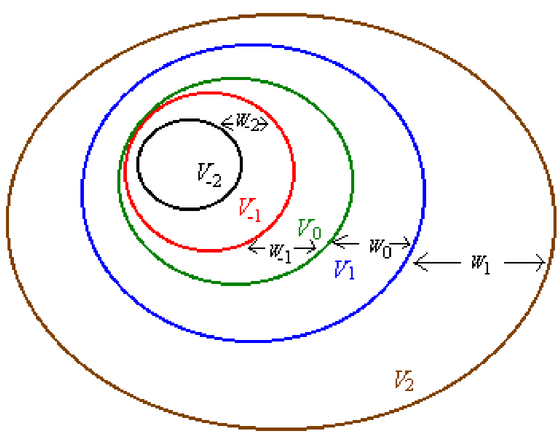
[Thus every function in L^2 can be uniquely written as a sum of functions in the W_j's].

Thus:

Theorem: Every vector $v \in L^2(-\infty, \infty)$ can be uniquely expressed as a sum

$$\sum_{j=-\infty}^{\infty} w_j \text{ where } w_j \in W_j .$$

Conclusion - relationship of V_j and W_j:



5. What are the W_j spaces?

Consider W_0.

Claim: $W_0 = A \equiv$ functions which are constant on half-integers and take equal and opposite values on half of each integer interval.

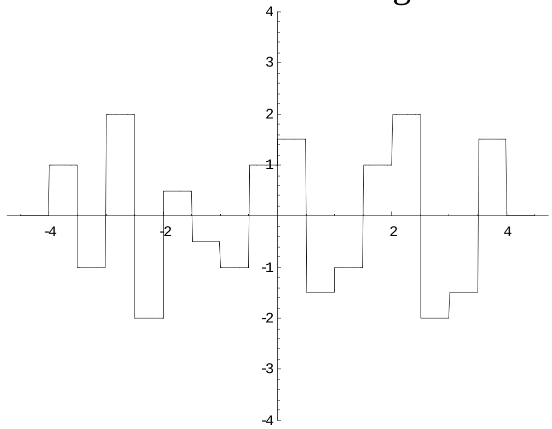


fig 24: Typical function in A

Proof: Will show that with above definition of A,

$$V_0 \oplus A = V_1,$$

and that V_0 and A are orthogonal. Then it will follow that

$$A = V_1 \ominus V_0 \equiv W_0,$$

First to show V_0 and A are orthogonal: let $f \in V_0$ and $g \in A$. Then f looks like:

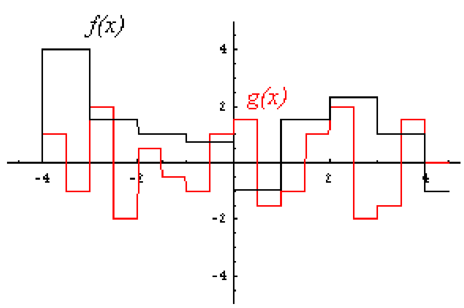


fig 24: $f(x) \in V_0$; $g(x) \in W_0$

Thus

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x)dx = \left(\int_{-2}^{-1} + \int_{-1}^{0} + \int_{0}^{1} + \int_{1}^{2} + \dots \right) f(x)g(x)dx$$

$$= 0$$

since $f(x)g(x)$ takes on equal and opposite values on each half of every integer interval above, and so integrates to 0 on each interval.

Thus f and g orthogonal, and so V_0 and A are orthogonal.

Next will show that if $f \in V_1$, then $f = f_0 + g_0$, where $f_0 \in V_0$ and $g_0 \in A$ (which is all that's left to show).

Let $f \in V_1$. Then f is constant on half integer intervals:

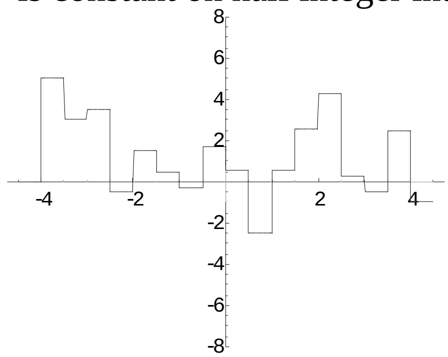


fig 25: $f(x) \in V_1$

Define f_0 to be the function which is constant on each integer interval, and whose value is the *average* of the two values of $f(x)$ on that interval:

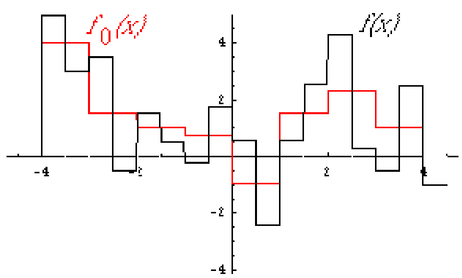


fig 26: $f_0(x)$ as related to $f(x)$.

Then clearly $f_0(x)$ is constant on integer intervals, and so is in V_0.

Now define $g_0(x) = f(x) - f_0(x)$:

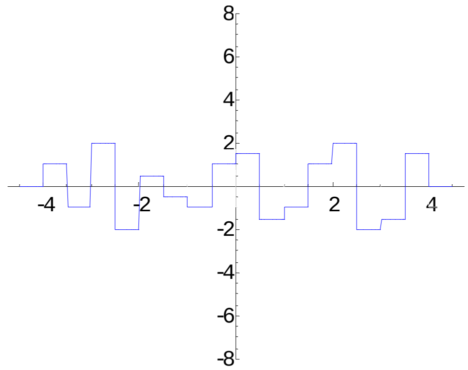


fig 27: $g_0(x) = f(x) - f_0(x)$

Then clearly g_0 takes on equal and opposite values on each half of every integer interval, and so is in A. Thus we have: for $f(x) \in V_1$,

$$f(x) = f_0(x) + g_0(x),$$

where $f_0 \in V_0$ and $g_0 \in A$. Thus $V_1 = V_0 \oplus A$ by Theorem 1 above.

Thus $A = V_1 \ominus V_0 = W_0$, so $A = W_0$.

Thus $W_0 =$ functions which take on equal and opposite values on each half of an integer interval, as desired. □

Similarly, can show:

W_j = functions which take on equal and opposite values on each half of the dyadic interval of length 2^{-j} and are square integrable:

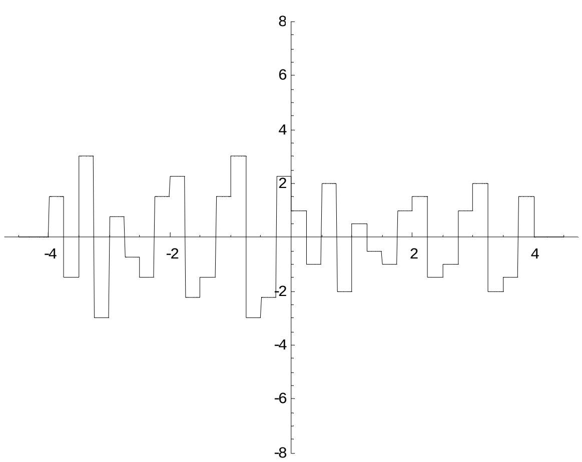


fig 28: typical function in W_j $(j = 1)$

6 What is a basis for the space W_j?

Consider

W_0 = functions which take equal and opposite values on each integer interval

What is a basis for this space? Let

$$\psi(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1/2 \\ -1 & \text{if } \ 1/2 < x \le 1 \end{cases}.$$

Claim a basis for W_0 is $\{\psi(x-k)\}_{k=-\infty}^{\infty}$.

Note linear combinations of $\psi(x-k)$ look like:

$$g(x) = 2\psi(x) + 3\psi(x-1) - 2\psi(x-2) + 2\psi(x-3).$$

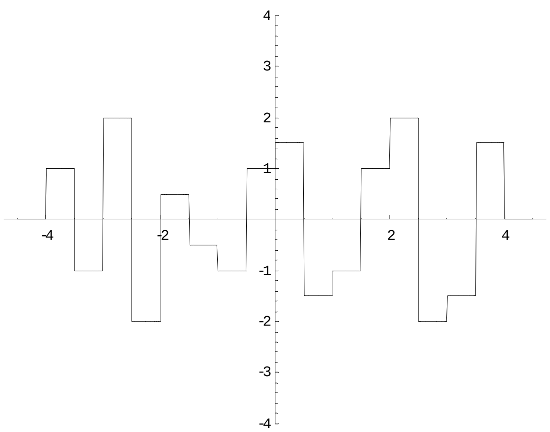


fig 29: graph of $g(x)$.

I.E., linear combinations of translates $\psi(x-k)$ = functions equal and opposite on each half of every integer interval.

Can easily conclude:

$$W_0 = \text{ functions in } L^2 \text{ equal and opposite on integer intervals}$$

$$= \text{ functions in } L^2 \text{ which are linear combinations of translates } \psi(x-k).$$

Also easily seen translates $\psi(x-k)$ are orthonormal.

Conclude: $\{\psi(x-k)\}$ form orthonormal basis for W_0.

Similarly can show $\{2^{1/2}\psi(2x-k)\}_k$ form orthonormal basis for W_1.

$\{2^{2/2}\psi(2^2x-k)\}_k$ form orthonormal basis for W_2.

Generally,

$$\{2^{j/2}\psi(2^jx-k)\}_{k=-\infty}^{\infty} \text{ form orthonormal basis for } W_j.$$

Define $\psi_{jk}(x) = 2^{j/2}\psi(2^jx-k)$.

Recall every function $f \in L^2$ can be written

$$f = \sum_j w_j$$

where $w_j \in W_j$. But each w_j can be written

$$w_j = \sum_k a_k \psi_{jk}(x)$$

[note j fixed above replace a_k by a_{jk} since need to keep track of j].

so:

$$f = \sum_j \sum_k a_{jk}\psi_{jk}(x).$$

Furthermore we have shown the ψ_{jk} orthonormal. Conclude they form orthonormal basis for L^2.

7. Example of a wavelet expansion:

Let $f(x) = \begin{cases} x^2 & \text{if } 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$. Find wavelet expansion.

8. Some more Fourier analysis:

Recall Fourier transform (use ω instead of ξ for Fourier variable):

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(\omega) e^{ix\omega}\, d\omega$$

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x)\; e^{-ix\omega}.$$

[earlier had $\widehat{f}(\omega) = c(\omega)$]

Write $\widehat{f}(\omega)$ = Fourier transform of $\widehat{f}(\omega) = \mathcal{F}(f)$.

9. Plancherel theorem:

Plancharel Theorem:
(i) The Fourier transform is a one to one correspondence from L^2 to itself.

That is, for every function $f(x) \in L^2$ there is a unique L^2 function which is its Fourier transform, and for every function $\widehat{g}(\omega) \in L^2$ there is a unique L^2 function which it is the Fourier transform of.

(ii) The Fourier transform preserves inner products, i.e., if $\widehat{f}$ is the FT of f and $\widehat{g}$ is the FT of g, then $\langle \widehat{f}(\omega), \widehat{g}(\omega)\rangle = \langle f(x), g(x)\rangle$.

(iii) Thus

$$\|f(x)\|^2 = \|\widehat{f}(\omega)\|^2 .$$

Now for a function $f \in L^2[-\pi, \pi]$, consider the Fourier series of f, given by

$$\sum_{k=-\infty}^{\infty} c_k\, e^{ikx}.$$

The above theorem has analog on $[-\pi,\pi]$. Theorem below follows immediately from fact that $\{e^{inx}/\sqrt{2\pi}\}_{n=-\infty}^{\infty}$ form orthonormal basis for $L^2[-\pi,\pi]$.

Plancharel Theorem for Fourier series:

(i) The correspondence between functions $f \in L^2[-\pi,\pi]$ and the coefficients $\{c_k\}$ of their Fourier series is a one to one correspondence, if we restrict $\sum_k c_k^2 < \infty$. That is, for every $f \in L^2[-\pi,\pi]$ there is a unique series of square summable Fourier coefficients $\{c_k\}$ of f such that $\sum_k |c_k|^2 < \infty$. Conversely for every square summable sequence $\{c_k\}$ there is a unique function $f \in L^2[-\pi,\pi]$ such that $\{c_k\}$ are the coefficients of the Fourier series of f.

(ii) Furthermore, $\sum_k \|c_k\|^2 = \frac{1}{2\pi}\|f(x)\|^2$

3. General Wavelet Constructions

1. Other constructions:

Suppose we use another "pixel" function $\phi(x)$:

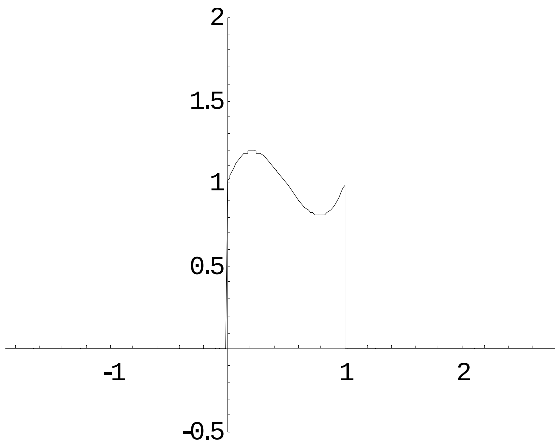


fig 30: another pixel function

Can we use this to build approximations to other functions? Consider linear combination:

$$2\phi(x) + 3\phi(x-1) - 2\phi(x-2) + \phi(x-3)$$

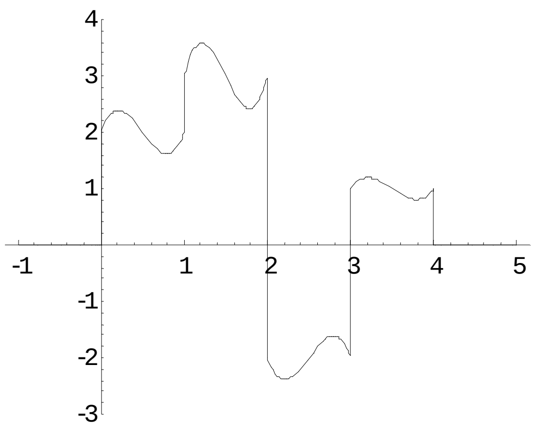


fig 31: graph of linear combination of translates of ϕ

Note we can try to approximate functions with other pixel functions.

Question: Can we repeat the above process with this pixel (scaling) function? What would be the corresponding wavelet?

Assumptions: $|\phi(x)|$ has finite integral and $\int \phi(x)dx \neq 0$.

More general construction:

As before define $V_0 =$ all L^2 linear combinations of ϕ and its translates:

$$= \{f(x) = \sum_k a_k\, \phi_{0k}(x) \Big| \, a_k \in \mathbb{R};\ f \in L^2\}. \tag{2}$$

with

$$\phi_{0k}(x) = \phi(x-k).$$

and

$$V_1 = \{f(x) = \sum_k a_k\, \phi_{1k}(x) |\, a_k \in \mathbb{R};\ f \in L^2\}. \tag{3}$$

$$\phi_{1k}(x) = 2^{1/2}\, \phi(2x-k)$$

etc.

We want the same theory as earlier.

[Note V_0 no longer piecewise constant functions]

Recall condition

(d) $f(x) \in V_n \Rightarrow f(2x) \in V_{n+1}$

This is automatically true by definition of V_n, since if $f(x) \in V_0$, then f has the form of an element of (2). Then $f(2x)$ has form of an element of (3), and $f(2x) \in V_1$.

Similarly can be shown that (d) holds for any pair of spaces V_n and V_{n+1} of above form.

2. Some basic properties of F.T.:

Assume that $\widehat{f} = \mathcal{F}(f)$. Then

(a) $\mathcal{F}(f(x-c))(\omega) = e^{-i\omega c}\,\widehat{f}(\omega)$
(b) $\mathcal{F}(f(cx)) = \frac{1}{c}\widehat{f}(\omega/c)$

Proofs: Exercises.

3. Orthogonality of the ϕ's:

Another property of V_j :

(f) The basis $\{\phi(x-k)\}$ for V_0 is orthogonal, i.e. $\langle \phi(x-k), \phi(x-\ell)\rangle = 0$ for $k \neq \ell$.

Not automatic. Let $\mathcal{F}(f) \equiv$ F.T. of $f \equiv \widehat{f}(\omega)$.

Require a condition on ϕ of the following sort: if $k \neq \ell$, then (note use ω as Fourier variable) :

$$0 = \langle \phi(x-k), \phi(x-\ell)\rangle = \langle \mathcal{F}(\phi(x-k)), \mathcal{F}(\phi(x-\ell))\rangle$$
$$= \langle e^{-i\omega k}\,\widehat{\phi}\,(\omega), e^{-i\omega\ell}\,\widehat{\phi}(\omega)\rangle$$
$$= \int_{-\infty}^{\infty} e^{i\omega(k-\ell)}|\widehat{\phi}(\omega)|^2\,d\omega$$

Thus conclude if $m \neq 0$,

$$0 = \int_{-\infty}^{\infty} e^{im\omega}|\widehat{\phi}(\omega)|^2\,d\omega$$

$$= \left(\ldots\int_{-4\pi}^{-2\pi} + \int_{-2\pi}^{0\pi} + \int_{0\pi}^{2\pi} + \right) e^{im\omega}|\widehat{\phi}(\omega)|^2\,d\omega$$

$$= \sum_{n=-\infty}^{\infty}\int_{n\cdot 2\pi}^{(n+1)\cdot 2\pi} e^{im\omega}|\widehat{\phi}(\omega)|^2\,d\omega$$

$$= \sum_{n=-\infty}^{\infty} \int_0^{2\pi} e^{im\omega} |\widehat{\phi}(\omega - 2n\pi)|^2 \, d\omega$$

$$= \int_0^{2\pi} e^{im\omega} \sum_{n=-\infty}^{\infty} |\widehat{\phi}(\omega - 2n\pi)|^2 \, d\omega$$

[since we can show that the integral of the absolute sum converges because $\sum_{n=-\infty}^{\infty} |\widehat{\phi}(\omega - 2n\pi)|^2 \, d\omega$ absolutely integrable; see exercises]

Conclude function $\sum_{n=-\infty}^{\infty} |\widehat{\phi}(\omega - 2n\pi)|^2$ on $[0, 2\pi]$ is in L^2 because it has square summable Fourier coefficients (in fact they are 0 if $m \neq 0$).

Further $\sum_{n=-\infty}^{\infty} |\widehat{\phi}(\omega - 2n\,\pi\,)|^2$ is 2π- periodic in ω, and has a Fourier series

$$\sum_{n=-\infty}^{\infty} |\widehat{\phi}(\omega - 2n\,\pi\,)|^2 = \sum_{m=-\infty}^{\infty} c_m \, e^{im\omega},$$

where

$$c_m = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\omega} \sum_{n=-\infty}^{\infty} |\widehat{\phi}(\omega - 2n\pi)|^2 \, d\omega = 0 \qquad \text{if } m \neq 0.$$

And

$$c_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-im\omega} |\widehat{\phi}(\omega)|^2 \, d\omega \Big|_{m=0} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\widehat{\phi}(\omega)|^2 \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\phi\,(x)|^2 \, dx = \frac{1}{2\pi}.$$

Thus

$$\sum_{n=-\infty}^{\infty} |\widehat{\phi}(\omega - 2n\pi)|^2 = \sum_{m=-\infty}^{\infty} c_m e^{imx} = \frac{1}{2\pi}.$$

This condition equivalent to orthonormality of $\{\phi(x-k)\}$.

$\mathbf{V_0 \subset V_1}$:

Recall the condition

(a) $V_0 \subset V_1$

What must be true of ϕ for this to hold in general? This says that every function in V_0 is in V_1. Thus since $\phi(x) \in V_0$, it follows $\phi(x) \in V_1$, i.e.

$$\phi(x) = \text{ linear combination of translates of } \sqrt{2}\,\phi(2x)$$

$$= \sum_k h_k\, \phi_{1k}(x) \tag{4}$$

$$\phi_{1k}(x) = 2^{1/2}\phi(2x-k)$$

[recall normalization constant $\sqrt{2}$ is so we have unit L^2 norm].

Ex: If $\phi(x) =$ Haar wavelet, then

$$\phi(x) = \phi(2x) + \phi(2x-1)$$

$$= \frac{1}{\sqrt{2}}\,\phi_{10}(x) + \frac{1}{\sqrt{2}}\,\phi_{11}(x)$$

$$= h_{10}\phi_{10}(x) + h_{11}\phi_{11}(x)$$

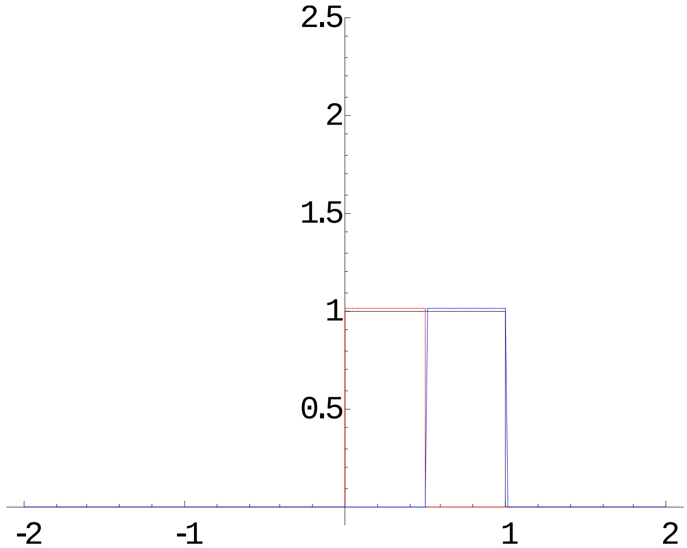


fig 32: $\phi(x) = \phi(2x) + \phi(2x-1)$

Thus in this case all h's are 0 except h_{10} and h_{11};

$$h_{10} = \frac{1}{\sqrt{2}}; \qquad h_{11} = \frac{1}{\sqrt{2}}.$$

Note in general that since this is an orthonormal expansion,

$$\sum_k h_k^2 = \|\phi(x)\|^2 < \infty.$$

4. What must be true of the scaling function for (1) above to hold?

Thus in general we have:

$$\phi(x) = \sum_{k=-\infty}^{\infty} h_k \phi_{1k}(x) = \lim_{N\to\infty} \sum_{k=-N}^{N} h_k \phi_{1k}(x) \tag{3}$$

in L^2 norm. Denote

$$\sum_{k=-N}^{N} h_k \phi_{1k}(x) \equiv F_N(x)$$

Specifically,

$$\|\phi(x) - \sum_{k=-N}^{N} h_k \phi_{1k}(x)\| \ \to \ 0.$$

[recall $\mathcal{F}$ is Fourier transform]

Corollary of Plancherel Theorem:

Corollary: *The Fourier transform is a bounded linear transformation. In particular, if the sequence of functions* $\{F_N(x)\}$ *converges in* L^2 *norm, then*

$$\mathcal{F}(\lim_{n\to\infty} F_N)(\omega) \ = \ \lim_{N\to\infty} \mathcal{F}(F_N)(\omega)$$

in L^2 *norm, i.e., Fourier transforms commute with limits.*

Thus since ∞ sums are limits and $\mathcal{F}$ is linear:

$$\mathcal{F}\left(\sum_{K=-\infty}^{\infty} h_k \phi_{1k}(x) \right) = \sum_{k=-\infty}^{\infty} h_k \ \mathcal{F}(\phi_{1k}(\omega))$$

[i.e., $\mathcal{F}$ commutes with ∞ sums]

Let $\mathcal{F}(\phi)(\omega) = \widehat{\phi}(\omega)$. Then generally:

$$\mathcal{F}(\phi_{jk})(\omega) = \mathcal{F}(2^{j/2}\phi(2^j x - k))(\omega)$$

$$= 2^{j/2}\,\mathcal{F}(\phi(2^j x - k))(\omega)$$

[recall dilation properties of Fourier transform earlier]

$$= \ 2^{j/2}\,\frac{1}{2^j}\,\mathcal{F}(\phi(x-k))(\omega/2^j)$$

[recall translation by k pulls out an $e^{-i\omega k}$]

$$= 2^{-j/2}\,e^{-i\omega k/2^j}\,\mathcal{F}(\phi(x))(\omega/2^j)$$

$$= 2^{-j/2}\,e^{-i\omega k/2^j}\,\widehat{\phi}(\omega/2^j)$$

Specifically for $j = 1$:

$$\mathcal{F}(\phi_{1k})(\omega) = \sqrt{2}\, e^{-i\omega k/2}\, \frac{1}{2}\, \widehat{\phi}(\omega/2)$$

Recall (3):

$$\phi(x) = \sum_{k=-\infty}^{\infty} h_k \phi_{1k}(x)$$

Fourier transforming both sides:

$$\begin{aligned} \widehat{\phi}(\omega) &= \mathcal{F}(\phi)(x) \\ &= \mathcal{F}\Big(\sum_{k=-\infty}^{\infty} h_k\, \phi_{1k}(x) \Big) \\ &= \sum_{k=-\infty}^{\infty} h_k\, \frac{1}{\sqrt{2}}\; e^{-ik(\omega/2)}\, \widehat{\phi}(\omega/2) \end{aligned} \tag{5}$$

Define

$$m(\omega/2) \;=\; \sum_{k=-\infty}^{\infty} h_k\, \frac{1}{\sqrt{2}}\, e^{-ik(\omega/2)} \tag{6}$$

note m is 2π- periodic $-$ Fourier series of $m(\omega/2)$ given above.

Note $m(\omega) \in L^2[0, 2\pi]$, since $\sum_k h_k^2 < \infty$.

Thus by (5):

$$\widehat{\phi}(\omega) = m(\omega/2)\, \widehat{\phi}(\omega/2).$$

with $m(\,\cdot\,)$ a 2π-periodic L^2 function.

[Note: This condition exactly summarizes our original demand that $V_0 \subset V_1$!]

Note if $V_0 \subset V_1$, then it follows (same arguments) that $V_1 \subset V_2$, and $V_j \subset V_{j+1}$ in general.

5. Some preliminaries:

Given a Hilbert space H and a closed subspace V, for $f \in H$ write

$$f = v + v^{\perp}$$

where $v \in V$ and $v^{\perp} \in V^{\perp}$.

Definition: The operator P defined by

$$Pf = P(v + v^{\perp}) = v$$

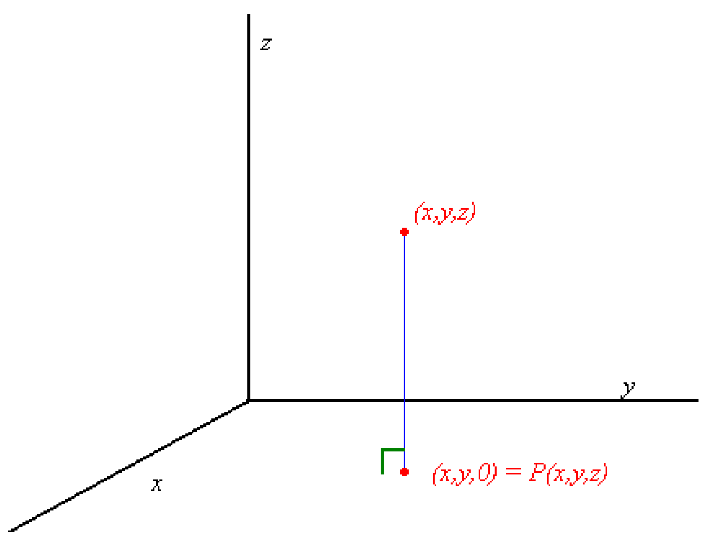
is the *orthogonal projection* onto V.

Note P is a bounded linear operator (see exercises).

Easy to check that $\|P\| = 1$ if $P \neq 0$ (see exercises).

Ex: $V = \mathbb{R}^3$. $P(x, y, z) = (x, y, 0) =$ is the orthogonal projection onto the $x{-}y$ plane.

$P(x, y, z) = (0, 0, z) =$ orthogonal projection onto z axis.



Ex: $V \subset L^2[-\pi, \pi]$ is the even functions. Then for $f \in L^2$

$$Pf(x) = f_{\text{even}}(x) = \frac{f(x) + f(-x)}{2}$$

(see exercises).

6. How to construct the wavelet?

Recall we have now given conditions on the scaling function:

Condition

(a) $\ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset V_3 \ldots$

is equivalent to:

(i) $$\widehat{\phi}(\omega) = m_0(\omega/2)\widehat{\phi}(\omega/2),$$

where m_0 is a function of period 2π.

Condition

(f) There is an orthogonal basis for the space V_0 in the family of functions

$$\phi_{0k} \equiv \phi(x-k)$$

is equivalent to:

(ii) $$\sum_k |\widehat{\phi}(\omega + 2\pi k)|^2 = \frac{1}{2\pi}$$

Condition

(b) $\bigcap_n V_n = \{0\}$

can also be shown to follow from (ii) as follows:

Proposition: If $\phi \in L^2(\mathbb{R})$ and satisfies (ii), then $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

Proof: Denote C_c to be compactly supported continuous functions. Let $f \in \bigcap_{j\in\mathbb{Z}} V_j$. Let $\epsilon > 0$ be arbitrarily small. By arguments as in problem II.2 in R&S, C_c is dense in $L^2(\mathbb{R})$, so that there exists an $\tilde{f} \in C_c$ with

$$\|f - \tilde{f}\| < \epsilon,$$

with $\|\cdot\|$ denoting L^2 norm. Let

$$P_j = \text{orthogonal projection onto } V_j.$$

Then since $f \in V_j$:

$$\|f - P_j\tilde{f}\| = \|P_j f - P_j\tilde{f}\| = \|P_j(f - \tilde{f})\| \le \|f - \tilde{f}\| \le \epsilon.$$

Thus by triangle inequality

$$\|f\| \le \|f - P_j\tilde{f}\| + \|P_j\tilde{f}\| \le \epsilon + \|P_j\tilde{f}\|. \qquad (7)$$

Since $P_j\tilde{f} \in V_j$, we have

$$P_j\tilde{f} = \sum_k c_{jk}\phi_{jk}(x).$$

where $c_{jk} = \langle \phi_{jk}, f\rangle$ (recall $\{\phi_{jk}(x)\}_{k=-\infty}^{\infty}$ is an orthonormal basis for V_j).

Thus if $\|f\|_\infty = \sup_x |f(x)|$,

$$\|P_j\tilde{f}\|^2 = \sum_k |c_{jk}|^2 = \sum_k |\langle \phi_{jk}, \tilde{f}\rangle|^2$$

$$= \sum_k \left| \int \overline{\phi_{jk}(x)}\, \tilde{f}(x)\, dx \right|^2$$

[assuming $\tilde{f}$ is supported in $[-R, R]$]

$$\le 2^j \|\tilde{f}\|_\infty^2 \sum_k \left(\int_{[-R,R]} 1 \cdot |\phi(2^j x - k)| dx \right)^2$$

[using Schwartz inequality $\langle a(x) b(x)\rangle \le \|a(x)\| \|b(x)\|$]

$$\le 2^j \|\tilde{f}\|_\infty^2 \sum_k \int_{[-R,R]} 1^2\, dx \int_{[-R,R]} |\phi(2^j x - k)|^2 dx$$

$$= 2^j \|\tilde{f}\|_\infty^2 2R \sum_k \int_{[-R,R]} |\phi(2^j x - k)|^2 dx$$

$$\overset{y=2^j x-k}{=} \|\tilde{f}\|_\infty^2 \, 2R \int_{S_{R,j}} |\phi(y)|^2 dy$$

[where $S_{R,j} = \cup_{k\in\mathbb{Z}}[k - 2^j R, \, k + 2^j R]$ (note we replaced $k \to -k$ in the union) assuming j large and negative, so $2^{-j}R < \frac{1}{2}$. Note that then the k sum becomes a sum over disjoint intervals after the change of variables above, and we therefore replace a sum over k by a union over these intervals, as above]

$$= \|f\|_\infty^2 \, 2R \int \chi_{S_{R,j}}(y) |\phi(y)|^2 dy \underset{j \to -\infty}{\longrightarrow} 0$$

by the dominated convergence theorem, since if $y \notin \mathbb{Z}$, $\chi_{S_{R,j}}(y) \underset{j \to \infty}{\longrightarrow} 0$. Thus by (7), we have for j large and negative and all $\epsilon > 0$:

$$\|f\| \leq \|f - P_j\tilde{f}\| + \|P_j\tilde{f}\| \leq \epsilon + \|P_j\tilde{f}\| \leq 2\epsilon.$$

Thus $\|f\| = 0$ and $f = 0$. ☐

Condition

(c) $\underset{n}{\cup} V_n$ is dense in $L^2(\mathbb{R})$

also follows from (ii):

Proposition: If $\phi \in L^2(\mathbb{R})$ and satisfies (ii), then $\overline{\underset{j\in\mathbb{Z}}{\cup} V_j} = L^2(\mathbb{R})$.

Proof: Similarly technical proof.

Condition

(d) $f(x) \in V_n \;\; \Rightarrow \;\; f(2x) \in V_{n+1}$

is automatic from the definition of the V_n.

Condition

(e) $f(x) \in V_0 \;\; \Rightarrow \;\; f(x-k) \; \in \; V_0$

is also automatic from definition.

Thus we conclude:

Theorem: Conditions (i) and (ii) above are necessary and sufficient for the spaces $\{V_j\}$ and scaling function ϕ to form a multiresolution analysis.

Thus if (i), (ii) are satisfied for ϕ and we define the spaces V_j as usual, the spaces will satisfy properties (a) - (f) of a multiresolution analysis.

Recall: orthonormality of translates $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ is equivalent to:

(ii) $$\sum_k |\widehat{\phi}(\omega+2\pi k)|^2 = \frac{1}{2\pi}$$

Rewrite (ii):

$$\sum_k |m_0(\omega/2+\pi k)|^2 \, |\widehat{\phi}(\omega/2+\pi k)|^2 = \tfrac{1}{2\pi}$$

$$\Rightarrow \tfrac{1}{2\pi} = \sum_k |m_0(\omega'+\pi k)|^2 \, |\widehat{\phi}(\omega'+\pi k)|^2$$

$[\omega' = \omega/2]$

$$= \sum_{k \text{ even}} |m_0(\omega'+\pi k)|^2 |\widehat{\phi}(\omega'+\pi k)|^2 + \sum_{k \text{ odd}} |m_0(\omega'+\pi k)|^2 |\widehat{\phi}(\omega'+\pi k)|^2$$

$$= \sum_k |m_0(\omega'+\pi\cdot 2k)|^2 \, |\widehat{\phi}(\omega'+\pi\cdot 2k)|^2 + \sum_k |m_0(\omega'+\pi(2k+1))|^2 \, |\widehat{\phi}(\omega'+\pi(2k+1)|^2$$

$$\overset{m_0 \text{ periodic}}{=} |m_0(\omega')|^2 \sum_k |\widehat{\phi}(\omega'+2\pi k)|^2 + |m_0(\omega'+\pi)|^2 \sum_k |\widehat{\phi}(\omega'+\pi+2\pi k)|^2$$

$$\overset{\text{by (ii)}}{=} |m_0(\omega')|^2 \cdot \tfrac{1}{2\pi} + |m_0(\omega'+\pi)|^2 \cdot \tfrac{1}{2\pi}.$$

This implies that

$$|m_0(\omega')|^2 + |m_0(\omega' + \pi)|^2 = 1. \tag{8}$$

What about wavelets? Recall we define $W_j = V_{j+1} \ominus V_j$. We now know that $\{\phi_{jk}(x)\}$ form basis for V_j. The wavelets ψ_{jk} will form basis for W_j.

4. More on General Constructions

1. What are ψ_{jk}?

[Recall norms and inner products of functions are preserved when we take Fourier transform. Let's take FT to see.]

Note if we find $W_0 = V_1 \ominus V_0$, then we will be done.

[Let's look at Fourier transforms of functions in these spaces:]

Note that if $f \in V_0$, then

$$f(x) = \sum_k a_k \phi(x-k) = \sum_k a_k \, \phi_{0k}(x) \qquad (9)$$

gives by F.T.:

$$\widehat{f}(\omega) = \sum_k a_k \, \mathcal{F}(\phi_{0k}(x)) = \sum_k a_k e^{-ik\omega} \widehat{\phi}(\omega) \equiv m_f(\omega) \widehat{\phi}(\omega) \qquad (10)$$

where

$$m_f(\omega) \equiv \sum_k a_k e^{-ik\omega}.$$

is a 2π periodic $L^2[0, 2\pi]$ function which depends on f. In fact reversing argument shows (9) and (10) are equivalent.

Similarly can show under Fourier transform that $g \in V_1$ equivalent to:

$$\widehat{g}(\omega) = m_g(\omega/2) \, \widehat{\phi}(\omega/2). \qquad (11)$$

with $m_g(\,\cdot\,)$ some other 2π periodic function on $L^2[0, 2\pi]$.

Notice functions m_f and m_g both have period 2π (look at their Fourier series). Also note above steps are reversible, so equation (10) implies (9) by reverse argument.

Thus:

$$f \in V_1 \;\Leftrightarrow\; \widehat{f} = m_f(\omega/2)\,\widehat{\phi}(\omega/2)$$

Recall: we want to characterize $f \in W_0$; such an f has the property that $f \in V_1$ and $f \perp V_0$.

Now note:

$$f \perp V_0 \;\Leftrightarrow\; f \perp \phi_{0k}\;\forall\, k \Leftrightarrow \widehat{f} \perp \widehat{\phi}_{0k},$$

$$\Leftrightarrow \int_{-\infty}^{\infty} \widehat{f}(\omega)\, e^{i\omega k}\, \overline{\widehat{\phi}(\omega)}\, d\omega = 0$$

$$\Leftrightarrow \quad 0 = \int_{-\infty}^{\infty} \widehat{f}(\omega)\, e^{i\omega k}\, \overline{\widehat{\phi}(\omega)}\, d\omega = \sum_m \int_{2\pi m}^{2\pi(m+1)} \widehat{f}(\omega)\, e^{i\omega k}\, \overline{\widehat{\phi}(\omega)}$$

$$= \sum_m \int_0^{2\pi} \widehat{f}(\omega + 2\pi m) e^{ik(\omega+2\pi m)} \overline{\widehat{\phi}(\omega + 2\pi m)}\, d\omega$$

$$= \int_0^{2\pi} e^{ik\omega} \sum_m \widehat{f}(\omega + 2\pi m)\, \overline{\widehat{\phi}(\omega + 2\pi m)}\, d\omega\,.$$

where above identities hold for all k.

Hence [viewing sum as some function of ω]

$$\sum_m \widehat{f}(\omega + 2\pi m)\, \overline{\widehat{\phi}(\omega + 2\pi m)} = 0.$$

Thus:

$$0 = \sum_m \widehat{f}(\omega + 2\pi m)\, \overline{\widehat{\phi}(\omega + 2\pi m)}$$

$$= \sum_m m_f((\omega + 2\pi m)/2) \widehat{\phi}((\omega + 2\pi m)/2) \overline{m_0((\omega + 2\pi m)/2) \widehat{\phi}((\omega + 2\pi m)/2)}$$

$$= \sum_m m_f(\omega/2 + \pi m) \widehat{\phi}(\omega/2 + m)\, \overline{m_0(\omega/2 + \pi m) \widehat{\phi}(\omega/2 + \pi m)}$$

$$= \sum_{m \text{ even}} + \sum_{m \text{ odd}} m_f(\omega/2 + \pi m)\, \widehat{\phi}(\omega/2 + \pi m)$$

$$\times \quad \overline{m_0(\omega/2 + \pi m)\widehat{\phi}(\omega/2 + \pi m)}$$

$$= \sum_m m_f(\omega/2 + 2\pi m)\widehat{\phi}(\omega/2 + 2\pi m)\overline{m_0(\omega/2 + 2\pi m)\widehat{\phi}(\omega/2 + 2\pi m)}$$

$$+ \sum_m m_f(\omega/2 + \pi + 2\pi m)\widehat{\phi}(\omega/2+\pi+2\pi m)$$

$$\times \quad \overline{m_0(\omega/2 + \pi + 2\pi m)\widehat{\phi}(\omega/2 + \pi + 2\pi m)}$$

$$= m_f(\omega/2)\overline{m_0(\omega/2)}\sum_m \widehat{\phi}(\omega/2 + 2\pi m)\, \overline{\widehat{\phi}(\omega/2 + 2\pi m)}$$

$$+ m_f(\omega/2 + \pi)\overline{m_0(\omega/2 + \pi)}\sum_m \widehat{\phi}(\omega/2 + \pi + 2\pi m)\overline{\widehat{\phi}(\omega/2 + \pi + 2\pi m)}$$

$$= m_f(\omega/2)\overline{m_0(\omega/2)}\sum_m |\widehat{\phi}(\omega/2 + 2\pi m)|^2$$

$$+ m_f(\omega/2 + \pi)\overline{m_0(\omega/2 + \pi)}\sum_m |\widehat{\phi}(\omega/2 + \pi + 2\pi m)|^2$$

$$= (m_f(\omega/2)\overline{m_0(\omega/2)} \cdot \tfrac{1}{2\pi} + m_f(\omega/2 + \pi)\overline{m_0(\omega/2 + \pi)}) \cdot \tfrac{1}{2\pi}$$

(3) $$\Rightarrow \; m_f(\omega')\overline{m_0(\omega')} + m_f(\omega' + \pi)\overline{m_0(\omega' + \pi)} = 0$$

Thus (note $m_0(\omega')$ and $m_0(\omega' + \pi)$ cannot vanish together); let $\omega' \to \omega$:

$$m_f(\omega) = -\frac{m_f(\omega + \pi)}{\overline{m_0(\omega)}}\, \overline{m_0(\omega + \pi)} \equiv \lambda(\omega)\, \overline{m_0(\omega + \pi)}\,, \qquad (12)$$

where

$$\lambda(\omega) \equiv -\frac{m_f(\omega + \pi)}{\overline{m_0(\omega)}}$$

and so $\lambda(\omega)$ is 2π periodic. Also,

$$\lambda(\omega) + \lambda(\omega+\pi) = -\frac{m_f(\omega+\pi)}{\overline{m_0(\omega)}} - \frac{m_f(\omega+2\pi)}{\overline{m_0(\omega+\pi)}} \qquad (13)$$

$$\overset{\text{combining fractions and using (3)}}{=} \quad 0.$$

Define $\nu(2\omega) = \lambda(\omega)\, e^{-i\omega}$.

Then

$$\nu(2\omega+2\pi) = \lambda(\omega+\pi)\, e^{-i(\omega+\pi)}$$

$$= -\lambda(\omega)e^{-i\omega}e^{-i\pi} = \lambda(\omega)e^{-i\omega} = \nu(2\omega)$$

so ν has period 2π.

Thus $\widehat{f}(\omega) = m_f(\omega/2)\widehat{\phi}(\omega/2) = \lambda(\omega/2)\overline{m_0(\omega/2+\pi)}\,\widehat{\phi}(\omega/2)$

$$= \nu(\omega)\, e^{i\omega/2}\, \overline{m_0(\omega/2+\pi)}\, \widehat{\phi}(\omega/2).$$

Thus we define the wavelet $\psi(x)$ by its Fourier transform:

$$\widehat{\psi}(\omega) = e^{i\omega/2}\, \overline{m_0(\omega/2+\pi)}\, \widehat{\phi}(\omega/2) \qquad (14)$$

Thus

$$\widehat{f}(\omega) = \nu(\omega)\widehat{\psi}(\omega).$$

Going back in Fourier transform, we would get (compare with how we got $\widehat{f}(\omega) = m_f(\omega)\widehat{\phi}(\omega)$)

$$f(x) = \sum_k a_k\, \psi(x-k). \qquad (15)$$

where a_k are coefficients of the Fourier series of $\nu(\omega)$, i.e.,

$$\nu(\omega) = \sum_k a_k e^{ik\omega}.$$

To justify process of Fourier transformation as above, need to also show that the coefficients a_k are square summable (i.e. $\sum_k |a_k|^2 < \infty$), since we do not know whether Fourier transform properties which we have used in getting (15) are valid otherwise.

Note since a_k are coefficients of Fourier series of ν, we just need to show ν is square integrable on $[0, 2\pi]$ (recall this is equivalent to the a_k being square summable). To show that ν is square integrable, note that with m_f as in (0):

$$\infty \overset{\text{use } m_f \in L^2[0,2\pi]}{>} \int_0^{2\pi} d\omega\, |m_f(\omega)|^2$$

$$\overset{\text{by (12)}}{=} \int_0^{2\pi} d\omega\, |\lambda(\omega)|^2\, |m_0(\omega+\pi)|^2$$

$$= \left(\int_0^{\pi} + \int_\pi^{2\pi}\right) d\omega\, |\lambda(\omega)|^2\, |m_0(\omega+\pi)|^2$$

[substitute $\omega\,' = \omega - \pi$ in second integral; then rename $\omega\,' = \omega$ again]

$$= \int_0^{\pi} d\omega\, |\lambda(\omega)|^2\, |m_0(\omega+\pi)|^2 + \int_0^{\pi} d\omega\, |\lambda(\omega+\pi)|^2\, |m_0(\omega+2\pi)|^2$$

[recall that by periodicity $|m_0(\omega+2\pi)|^2 = |m_0(\omega)|^2$ and use (13)]

$$= \int_0^{\pi} d\omega\, |\lambda(\omega)|^2\, \left(|m_0(\omega+\pi)|^2 + |m_0(\omega)|^2\right)$$

$$\overset{\text{use (8)}}{=} \int_0^{\pi} d\omega\, |\lambda(\omega)|^2$$

$$= \int_0^{\pi} d\omega\, |\nu(2\omega)|^2$$

$$\overset{\omega' = 2\omega}{=} \frac{1}{2}\int_0^{2\pi} d\omega\, |\nu(\omega)|^2$$

Thus we have that $\infty > \int_0^{2\pi} d\omega\, |\nu(\omega)|^2$, so that ν is square integrable, as desired.

This was only thing left to show $\psi(2x-k)$ span W_0. Wish to show also orthonormal. Use almost exactly the same argument as was used to show the same for $\phi(x-k)$:

$$\sum_k |\widehat{\psi}(\omega+2\pi k)|^2 \overset{\text{use (14)}}{=} \sum_k |m_0(\omega/2+\pi k+\pi)|^2 |\widehat{\phi}(\omega/2+\pi k)|^2$$

[now break up the sum into even and odd k again and use the same method as before]

$$= \left(\sum_{k \text{ even}} + \sum_{k \text{ odd}} \right) |m_0(\omega/2+\pi k+\pi)|^2 |\widehat{\phi}(\omega/2+\pi k)|^2$$

$$= \sum_k |m_0(\omega/2+\pi\cdot 2k+\pi)|^2 |\widehat{\phi}(\omega/2+\pi\cdot 2k)|^2$$

$$+ \sum_k |m_0(\omega/2+\pi\cdot(2k+1)+\pi)|^2 |\widehat{\phi}(\omega/2+\pi\cdot(2k+1))|^2 \qquad (16)$$

$$= |m_0(\omega/2+\pi)|^2 \sum_k |\widehat{\phi}(\omega/2+\pi\cdot 2k)|^2$$

$$+ |m_0(\omega/2)|^2 \sum_k |\widehat{\phi}(\omega/2+\pi\cdot(2k+1))|^2$$

$$\overset{\text{using (ii) above again}}{=} \left(|m_0(\omega/2+\pi)|^2 + |m_0(\omega/2)|^2\right)\cdot \frac{1}{2\pi}$$

$$= \frac{1}{2\pi}$$

By same arguments as used for $\phi(x-k)$, it follows by (16) $\psi(x-k)$ orthonormal.

This proves our choice of ψ gives a basis for W_0 as desired. Specifically,

$$\psi_{0k}(x) = \psi(x-k)$$

form an orthogonal basis for W_0 (in fact can show their length is 1 so they are orthonormal).

In same way as for ϕ, can show immediately that since functions in W_j are functions in W_0 stretched by factor 2^j, the functions

$$\psi_{jk}(x) = 2^{j/2}\,\psi(2^j x - k)$$

form a basis for W_j (j fixed, k varies).

Since L^2 $=$ direct sum of the W_j spaces, conclude functions $\{\psi_{jk}(x)\}_{j,k=-\infty}^{\infty}$ over all integers j and k form orthonormal basis for L^2.

Conclusion:
If we start with a pixel function $\phi(x)$, which satisfies

(i) $\widehat{\phi}(\omega) = m_0(\omega/2)\widehat{\phi}(\omega/2)$ (with m_0 some 2π-periodic function)

(ii) $\sum\limits_k |\phi(\omega + 2\pi k|^2 = \frac{1}{2\pi}$

then the set of spaces V_j form a multiresolution analysis, i.e., satisfy properties (a) - (f) from earlier.

Further, if define function $\psi(x)$ with Fourier transform:

$$\widehat{\psi}(\omega) = e^{i\omega/2}\,\overline{m_0(\omega/2+\pi)}\,\widehat{\phi}(\omega/2) \qquad (17)$$

(here m_0 is from (i) above), then

$$\psi_{jk}(x) = 2^{j/2}\psi\,(2^j x - k).$$

form orthonormal basis for L^2

[Next we'll construct some wavelets]

2. Additional remarks:

Note further that (17) has another interpretation without Fourier transform :

Recall the two scale equation:

$$\phi(x) = \sum_k h_k \phi_{1k}(x).$$

Also then we have (see eq. (5)) that if

$$m_0(\omega) = \sum_k \frac{h_k}{\sqrt{2}} e^{-ik\omega},$$

then:

$$\widehat{\phi}(\omega) = m_0(\omega/2)\widehat{\phi}(\omega/2).$$

Then we have from (17):

$$\begin{aligned}
\widehat{\psi}(\omega) &= e^{i\omega/2} \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{2}} \overline{h_k} e^{ik(\omega/2+\pi)} \widehat{\phi}(\omega/2) \\
&= e^{i\omega/2} \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{2}} \overline{h_k} e^{ik\pi} e^{ik\omega/2} \widehat{\phi}(\omega/2) \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{2}} \overline{h_k} (-1)^k e^{i(k+1)\omega/2} \widehat{\phi}(\omega/2)
\end{aligned}$$

Inverse Fourier transforming:

$$\begin{aligned}
\psi(x) &= \mathcal{F}^{-1}(\widehat{\psi}(\omega)) \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{2}} \overline{h}_k (-1)^k \mathcal{F}^{-1}(e^{i(k+1)\omega/2} \widehat{\phi}(\omega/2)) \\
&= \sum_{k=-\infty}^{\infty} \frac{\overline{h}_k}{\sqrt{2}} (-1)^k 2\phi(2x + (k+1)) \\
&= \sum_{k=-\infty}^{\infty} \frac{\overline{h}_{k-1}}{\sqrt{2}} (-1)^{k-1} \sqrt{2}\sqrt{2}\phi(2x + k)) \\
&= \sum_{k=-\infty}^{\infty} \overline{h}_{-k-1} (-1)^{-k-1} \phi_{1k}(x) \\
&= \sum_{k=-\infty}^{\infty} g_k \phi_{1k}(x)
\end{aligned}$$

where

$$g_k = \overline{h}_{-1-k}(-1)^{-k-1} = \overline{h}_{-1-k}(-1)^{k+1} \overset{\text{standard form}}{=} \overline{h}_{-1-k}(-1)^{k-1},$$

and (recall) h_k defined by

$$\phi(x) = \sum_k h_k \phi_{1k}(x).$$

3. Some comments on the scaling function:

Recall

$$\widehat{\phi}(\omega) = m_0(\omega/2)\,\widehat{\phi}(\omega/2)$$

from earlier. This stated that the Fourier transform of ϕ and its stretched version are related by some function $m_0(\omega/2)$, where m_0 is a periodic function of period 2π.

Lemma: The Fourier transform of an integrable function is continuous.

Proof: exercise

Assumption: $\phi(x)$ (the scaling function) is integrable (i.e., its absolute value has a finite integral).

Fact: Under our assumptions, it can be shown that $\int_{-\infty}^{\infty} dx\, \phi(x) = 1$
[proof is an exercise]

Consequence: A consequence of the above assumption is that the Fourier transform $\widehat{\phi}(\omega)$ satisfies:

$$\widehat{\phi}(0) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, \phi(x)\, e^{-i\cdot 0x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, \phi(x) = \frac{1}{\sqrt{2\pi}}.$$

Now recall we had

$$\widehat{\phi}(\omega) = m_0(\omega/2)\widehat{\phi}(\omega/2) \tag{18}$$

for some periodic function m_0. Replacing ω by $\omega/2$ above:

$$\widehat{\phi}(\omega/2) = m_0(\omega/4)\widehat{\phi}(\omega/4);$$

Plugging into (18):

$$\widehat{\phi}(\omega) = m_0(\omega/2)m_0(\omega/4)\widehat{\phi}(\omega/4). \tag{19}$$

Now taking (18) and replacing ω by $\omega/4$, and then plugging into (19):

$$\widehat{\phi}(\omega) = m_0(\omega/2)m_0(\omega/4)m_0(\omega/8)\widehat{\phi}(\omega/8).$$

Continuing this way n times, we get:

$$\widehat{\phi}(\omega) = m_0(\omega/2)m_0(\omega/4)m_0(\omega/8)\dots m_0(\omega/2^n)\widehat{\phi}(\omega/2^n).$$

or:

$$\widehat{\phi}(\omega) = \left(\prod_{j=1}^{n} m_0(\omega/2^j)\right)\widehat{\phi}(\omega/2^n)$$

$$\Rightarrow$$

$$\frac{\widehat{\phi}(\omega)}{\widehat{\phi}(\omega/2^n)} = \prod_{j=1}^{n} m_0(\omega/2^j). \tag{20}$$

Now let $n \to \infty$ on both sides of equation. Since $\widehat{\phi}$ is continuous (above assumption), we get

$$\widehat{\phi}(\omega/2^n) \underset{n\to\infty}{\longrightarrow} \widehat{\phi}(0) = \frac{1}{\sqrt{2\pi}}.$$

Since the left side of (20) converges as $n \to \infty$, the right side also converges. After letting $n \to \infty$ on both sides of (20):

$$\frac{\widehat{\phi}(\omega)}{\widehat{\phi}(0)} = \prod_{j=1}^{\infty} m_0(\omega/2^j),$$

$$\Rightarrow$$

$$\widehat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}} \prod_{j=1}^{\infty} m_0(\omega/2^j).$$

Conclusion: If we can find $m_0(\omega)$, we can find the scaling function ϕ.

4. Examples of wavelet constructions using this technique:

Haar wavelets: Recall that we chose the scaling function

$$\phi(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases},$$

and then we defined spaces V_j.

From ϕ we constructed the wavelet ψ whose translates and dilates form a basis for L^2.

Such constructions can be made automatic if we use above observations.

Note first in Haar case:

$$\widehat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-i\omega x}\, dx = -\frac{1}{\sqrt{2\pi}} \frac{e^{-i\omega x}}{i\omega}\Big|_0^1 = \frac{1}{\sqrt{2\pi}} \left[-\frac{e^{-i\omega}}{i\omega} + \frac{1}{i\omega} \right]$$

$$= -\frac{2}{\sqrt{2\pi}\,\omega} e^{-i\omega/2} \left(\frac{e^{-i\omega/2}}{2i} - \frac{e^{i\omega/2}}{2i} \right)$$

$$= \frac{2}{\sqrt{2\pi}\,\omega} e^{-i\omega/2} \sin \omega/2.$$

For Haar wavelets we can find $m_0(\omega)$ from:

$$\widehat{\phi}(\omega) = m_0(\omega/2)\widehat{\phi}(\omega/2),$$

so

$$m_0(\omega/2) = \frac{\widehat{\phi}(\omega)}{\widehat{\phi}(\omega/2)} = \frac{1}{2}\, e^{-i\omega/4}\, \frac{\sin \omega/2}{\sin \omega/4}$$

$$= \frac{1}{2}\, e^{-i\omega/4}\, \frac{\sin\, (2 \cdot \omega/4)}{\sin \omega/4}$$

$$= \frac{1}{2}\, e^{-i\omega/4}\, \frac{2 \sin \omega/4 \cos\, \omega/4}{\sin \omega/4}$$

$$= \quad \frac{1}{2}\, e^{-i\omega/4}\, 2 \cos\, \omega/4$$

$$= \; e^{-i\omega/4} \cos\, \omega/4.$$

Recall wavelet Fourier transform is:

(4) $\quad \widehat{\psi}(\omega) = e^{i\omega/2}\, \overline{m_0(\omega/2+\pi)}\, \widehat{\phi}(\omega/2)$

In this case

$$\widehat{\psi}(\omega) = e^{i\omega/2}\, e^{i(\omega/4+\pi/2)} \cos\, (\omega/4+\pi/2)\, \frac{4}{\sqrt{2\pi}\,\omega}\, e^{-i\omega/4} \sin \omega/4.$$

[using

$\cos\, (\omega/4 + \pi/2) = \cos \omega/4 \cos \pi/2 \; - \; \sin \omega/4 \sin \pi/2 \; = \; - \sin \omega/4]$

$$= -\, \frac{4i}{\sqrt{2\pi}\,\omega}\, e^{i\omega/2} \sin^2(\omega/4)$$

Can check (below) this indeed is Fourier transform of usual Haar wavelet ψ, except the complex conjugate (which means the original wavelet is reflected about 0, i.e., translated and negated, which still yields a basis for W_0).

To check this, recall Haar wavelet:

$$\psi(x) = \begin{cases} 1 & \text{if } 0 \le x < 1/2 \\ -1 & \text{if } 1/2 \le x < 1 \end{cases}$$

Thus:

$$\widehat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \psi(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\Big(\int_0^{1/2} + \int_{1/2}^{1}\Big)\, \psi(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^{1/2} e^{-i\omega x}\, dx - \frac{1}{\sqrt{2\pi}}\int_{1/2}^{1} e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\Big(- \frac{e^{-i\omega/2}}{i\omega} + \frac{1}{i\omega}\Big) - \frac{1}{\sqrt{2\pi}}\Big(- \frac{e^{-i\omega}}{i\omega} + \frac{e^{-i\omega/2}}{i\omega}\Big)$$

$$= -\frac{2e^{-i\omega/2}}{\sqrt{2\pi}\, i\omega} + \frac{e^{-i\omega} + 1}{\sqrt{2\pi}\, i\omega}$$

$$= \frac{2}{\sqrt{2\pi}\, i\omega}\left(- e^{-i\omega/2} + e^{-i\omega/2}\, \frac{(e^{-i\omega/2} + e^{i\omega/2})}{2}\right)$$

$$= \frac{2}{\sqrt{2\pi}\, i\omega}\left(- e^{-i\omega/2} + e^{-i\omega/2} \cos \omega/2\right)$$

$$= \frac{2}{\sqrt{2\pi}\, i\omega}\Big(- e^{-i\omega/2} + e^{-i\omega/2} \cos 2 \cdot \omega/4\Big)$$

[using $\cos 2x = 1 - 2 \sin^2 x$]

$$= \frac{2}{\sqrt{2\pi}\, i\omega}\Big(- e^{-i\omega/2} + e^{-i\omega/2}(1 - 2 \sin^2\omega/4)\Big)$$

$$= \frac{-\,4}{\sqrt{2\pi}\, i\omega}\Big(e^{-i\omega/2} \sin^2\omega/4\Big)$$

$$= \quad \frac{4i}{\sqrt{2\pi}\,\omega}\left(e^{-i\omega/2}\sin^2\omega/4\right)$$

5. Constructing Wavelets

1. Meyer wavelets: another example -

Scaling function:

$$\widehat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}} \begin{cases} 1 \quad \text{if} \quad |\omega| \leq 2\pi/3 \\ \cos\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\omega| - 1\right)\right] \quad \text{if} \quad 2\pi/3 \leq |\omega| \leq 4\pi/3 \\ 0 \;\; \text{otherwise} \end{cases} .$$

[error in Daubechies : $3/4\pi$ instead of $3/2\pi$ inside ν]

where ν is any infinitely differentiable non-negative function satisfying

$$\nu(x) = \begin{cases} 0 \quad \text{if } x \leq 0 \\ 1 \quad \text{if } x \geq 1 \\ \text{smooth transition in } \nu \text{ from } 0 \text{ to } 1 \text{ as } x \text{ goes from } 0 \text{ to } 1 \end{cases}$$

and

$$\nu(x) + \nu(1-x) = 1.$$

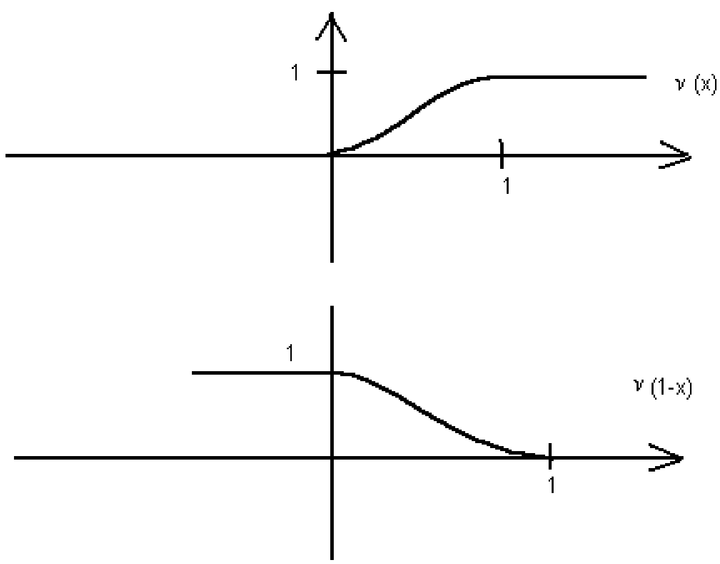


fig 33: $\nu(x)$ and $\nu(1-x)$

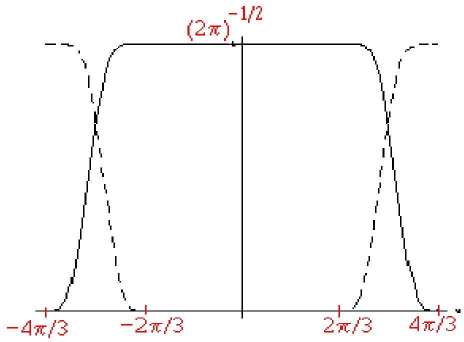


fig 34: Fourier transform $\widehat{\phi}(\omega)$ of the Meyer scaling function

Need to verify necessary properties for a scaling function:
(i)

$$\sum_k |\widehat{\phi}(\omega + 2\pi k)|^2 = \frac{1}{2\pi} \tag{21}$$

To see this, consider the two possible ranges of values of ω:

(a) $|\omega + 2\pi k_1| \leq 2\pi/3$ for some k_1. In that case (see diagram above):

$$\widehat{\phi}(\omega + 2\pi k_1) = \frac{1}{\sqrt{2\pi}}; \qquad \widehat{\phi}(\omega + 2\pi k) = 0 \text{ if } k \neq k_1$$

since if $|\omega + 2\pi k_1| \leq 2\pi/3$, then $|\omega + 2\pi k| \geq 4\pi/3$ for $k \neq k_1$. Thus (21) holds because there is only one non-zero term in that sum.

(b) $2\pi/3 \leq \omega + 2\pi k_1 \leq 4\pi/3$ for some k_1. In this case we also have

$$-4\pi/3 \leq \omega + 2\pi(k_1 - 1) \leq -2\pi/3.$$

Also, for all values $k \neq k_1$ or $k_1 - 1$, can calculate that

$$2\pi k \notin [-4\pi/3, 4\pi/3],$$

so

$$\widehat{\phi}(\omega + 2\pi k) = 0.$$

So sum has only two non-zero terms:

$$
\begin{aligned}
&2\pi\sum_k |\widehat{\phi}(\omega+2\pi k)|^2 = 2\pi\Big(|\widehat{\phi}(\omega+2\pi k_1)|^2 + |\widehat{\phi}(\omega+2\pi(k_1-1)|^2\Big).\\
&= \cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\omega+2\pi k_1|-1\right)\right] + \cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\omega+2\pi(k_1-1)|-1\right)\right]\\
&= \cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega+3k_1-1\right)\right] + \cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}(-(\omega+2\pi(k_1-1)))-1\right)\right]\\
&= \cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega+3k_1-1\right)\right] + \cos^2\left[\frac{\pi}{2}\nu\left(-\frac{3}{2\pi}\omega-3k_1+2\right)\right]\\
&= \cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega+3k_1-1\right)\right] + \cos^2\left[\frac{\pi}{2}\left(1-\nu\left(1-\left(-\frac{3}{2\pi}\omega-3k_1+2\right)\right)\right)\right]\\
&= \cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega+3k_1-1\right)\right] + \cos^2\left[\frac{\pi}{2}-\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega+3k_1-1\right)\right]\\
&= \cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega+3k_1-1\right)\right] + \sin^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega+3k_1-1\right)\right]\\
&= 1
\end{aligned}
$$

Note that above $|\omega+2\pi(k_1-1)| = -(\omega+2\pi(k_1-1))$, since quantity in parentheses always negative for our range of ω. In next to last equality have used $\cos\left(\frac{\pi}{2}-x\right) = \sin x$.

Note since cases (a), (b) cover all possibilities for ω (since they cover a range of size 2π for $\omega+2\pi k_1$), we are finished proving (21).

Also need to verify:

(ii)

$$\widehat{\phi}(\omega) = m_0(\omega/2)\widehat{\phi}(\omega/2)$$

for some 2π-periodic $m_0(\omega/2)$. Indeed, looking at pictures:

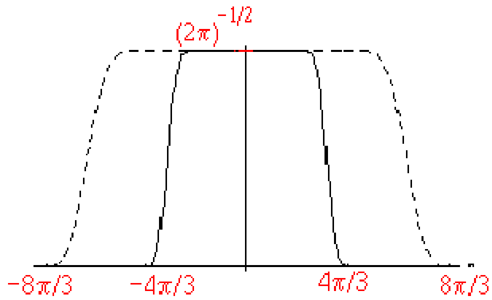


fig 35: $\widehat{\phi}(\omega)$ and $\widehat{\phi}(\omega/2)$ (----)

ratio of these two looks like:

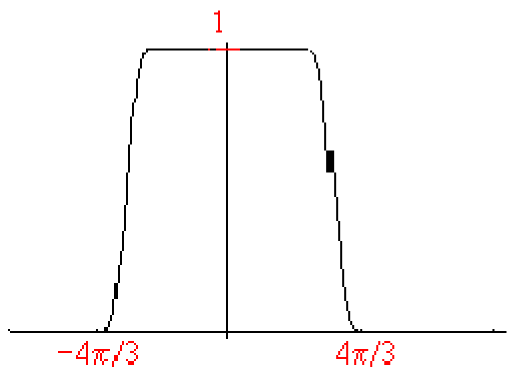


fig. 36: $\widehat{\phi(\omega)}/\widehat{\phi}(\omega/2) = \sqrt{2\pi}\,\widehat{\phi}(\omega)$ in the interval $[-2\pi, 2\pi]$.

Note since ratio $\widehat{\phi}(\omega)/\widehat{\phi}(\omega/2) = \sqrt{2\pi}\,\widehat{\phi}(\omega)$ in $[-2\pi, 2\pi]$, we can define

$$m_0(\omega/2) = \frac{\widehat{\phi}(\omega)}{\widehat{\phi}(\omega/2)} = \sqrt{2\pi}\,\widehat{\phi}(\omega) \qquad (22)$$

if $\omega \in [-2\pi, 2\pi]$.

Definition ambiguous when numerator and denominator are 0.

Definition also ambiguous for $\omega \notin [-2\pi, 2\pi]$ since numerator and denominator both 0. So define $m_0(\omega/2)$ by periodic extension of above for all real ω.

How to do that? Just add all possible translates of the bump $\widehat{\phi}(\omega)$ to make it 4π-periodic:

$$m_0(\omega/2) = \sqrt{2\pi} \sum_k \widehat{\phi}\,(\omega + 4\pi k).$$

Check:

$$m_0(\omega/2)\widehat{\phi}(\omega/2) = \sqrt{2\pi} \sum_k \widehat{\phi}(\omega + 4\pi k)\widehat{\phi}(\omega/2)$$

$$= \sqrt{2\pi}\,\widehat{\phi}(\omega)\widehat{\phi}(\omega/2)$$

$$= \widehat{\phi}(\omega)$$

where we have used the fact that $\widehat{\phi}(\omega + 4\pi k)$ has no overlap with $\widehat{\phi}(\omega/2)$ if $k \neq 0$.

[So we expect a full MRA.]

2. Construction of the Meyer wavelet

Standard construction:

$$\widehat{\psi}(\omega) = e^{i\omega/2}\,\overline{m_0(\omega/2 + \pi)}\,\widehat{\phi}(\omega/2)$$

$$= e^{i\omega/2} \sum_k \overline{\widehat{\phi}(\omega + 2\pi(2k+1))}\widehat{\phi}(\omega/2)$$

$$= e^{i\omega/2}\left[\widehat{\phi}(\omega + 2\pi) + \widehat{\phi}(\omega - 2\pi)\right]\widehat{\phi}(\omega/2)$$

[supports of 2d and 3d factors do not overlap for other values of k; note $\overline{\widehat{\phi}} = \widehat{\phi}$ since $\widehat{\phi}$ is real]

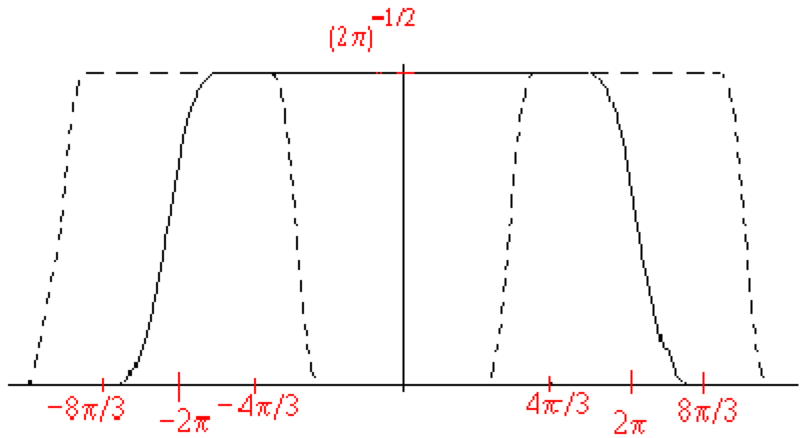


fig 37: $\widehat{\phi}(\omega+2\pi)+\widehat{\phi}(\omega-2\pi)$ and $\widehat{\phi}(\omega/2)$ (dashed)

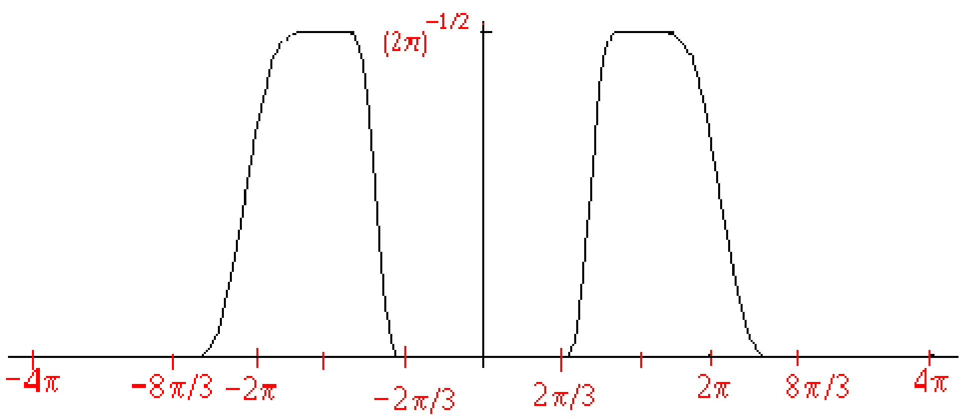


fig 38: $\left[\widehat{\phi}(\omega+2\pi)+\widehat{\phi}(\omega-2\pi)\right]\widehat{\phi}(\omega/2)$

Thus have 2 distinct regions:

(a) For $2\pi/3 \leq \omega \leq 4\pi/3$ we see in diagram that

$$
\begin{aligned}
e^{-i\omega/2}\widehat{\psi}(\omega) &= \sqrt{2\pi}\left[\widehat{\phi}(\omega+2\pi)+\widehat{\phi}(\omega-2\pi)\right]\widehat{\phi}(\omega/2) \\
&= \widehat{\phi}(\omega-2\pi) \\
&= \frac{1}{\sqrt{2\pi}}\cos\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\omega-2\pi|-1\right)\right] \\
&= \frac{1}{\sqrt{2\pi}}\cos\left[\frac{\pi}{2}\nu\left(-\frac{3}{2\pi}(\omega-2\pi)-1\right)\right] \\
&= \frac{1}{\sqrt{2\pi}}\cos\left[\frac{\pi}{2}\nu\left(-\frac{3}{2\pi}\omega+2\right)\right] \\
&= \frac{1}{\sqrt{2\pi}}\cos\left[\frac{\pi}{2}\left[1-\nu\left(1-\left(-\frac{3}{2\pi}\omega+2\right)\right)\right]\right] \\
&= \frac{1}{\sqrt{2\pi}}\cos\left[\frac{\pi}{2}\left[1-\nu\left(\frac{3}{2\pi}\omega-1\right)\right]\right] \\
&= \frac{1}{\sqrt{2\pi}}\sin\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega-1\right)\right]
\end{aligned}
$$

So by symmetry same is true in $-2\pi/3 \le \omega \le -4\pi/3$, so replace ω by $|\omega|$ above to get:

$$
e^{-i\omega/2}\widehat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}}\sin\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\omega|-1\right)\right] \quad \text{for} \quad 2\pi/3 \le |\omega| \le 4\pi/3
$$

(b) For $4\pi/3 \le \omega \le 8\pi/3$, we see from diagram (note $2\pi/3 \le \omega/2 \le 4\pi/3$):

$$
\begin{aligned}
e^{-i\omega/2}\widehat{\psi}(\omega) &= \sqrt{2\pi}\left[\widehat{\phi}(\omega+2\pi)+\widehat{\phi}(\omega-2\pi)\right]\widehat{\phi}(\omega/2) \\
&= \widehat{\phi}(\omega/2) \\
&= \frac{1}{\sqrt{2\pi}}\cos\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega/2-1\right)\right] \\
&= \frac{1}{\sqrt{2\pi}}\cos\left[\frac{\pi}{2}\nu\left(\frac{3}{4\pi}\omega-1\right)\right]
\end{aligned}
$$

Again by symmetry same is true in $-8\pi/3 \leq \omega \leq -4\pi/3$, so replace ω by $|\omega|$:

$$e^{-i\omega/2}\widehat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}}\cos\left[\frac{\pi}{2}\nu\left(\frac{3}{4\pi}|\omega| - 1\right)\right] \quad \text{for} \quad 4\pi/3 \leq |\omega| \leq 8\pi/3$$

Thus:

$$\widehat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}}\begin{cases} e^{i\omega/2}\sin\left[\frac{\pi}{2}\nu(\frac{3}{2\pi}|\omega| - 1\right], & \text{if } \; 2\pi/3 \leq |\omega| \leq 4\pi/3 \\ e^{i\omega/2}\cos\left[\frac{\pi}{2}\nu(\frac{3}{4\pi}|\omega| - 1)\right], & \text{if } \; 4\pi/3 \leq |\omega| \leq 8\pi/3 \\ 0 \quad \text{otherwise} \end{cases}$$

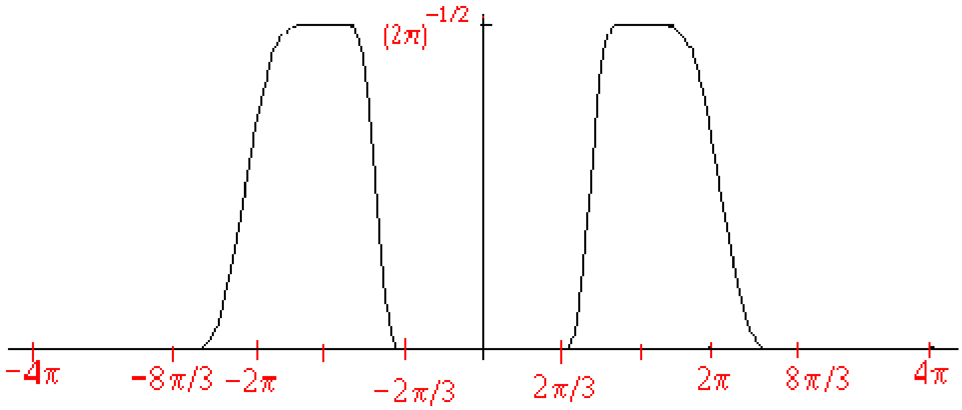


Fig. 39: The wavelet Fourier transform $|\widehat{\psi}(\omega)|$

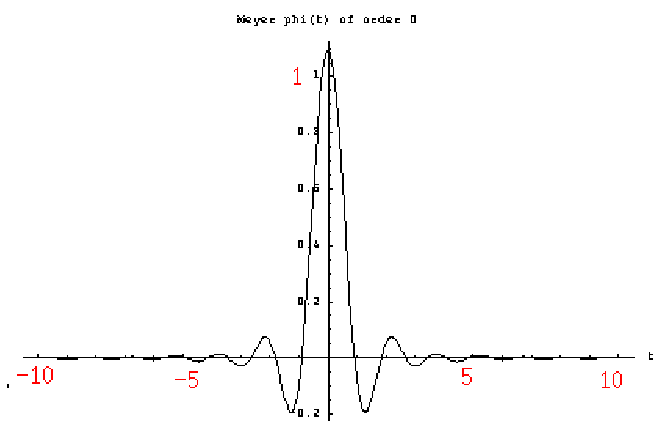


Fig. 40: The Meyer wavelet $\psi(x)$

3. Properties of the Meyer wavelet

Note: If ν is chosen as above and has all derivatives 0 at $\pi/2$, can check that $\widehat{\psi}(\omega)$ is:

• infinitely differentiable (since it is a composition of infinitely differentiable functions), and one can check that all derivatives are 0 from both sides at the break. For example, the derivatives coming in from the left at $\omega = \frac{2\pi}{3}$ are:

$$\frac{d^n}{d\omega^n}\widehat{\psi}(\omega^-)\Big|_{\omega=\frac{2\pi}{3}} = 0$$

and similarly

$$\frac{d^n}{d\omega^n}\widehat{\psi}(\omega^+)\Big|_{\omega=\frac{2\pi}{3}} = 0$$

(proof in exercises).

• supported (non-zero) on a finite interval

Lemma:
(a) If a function $\psi(x)$ has n derivatives which are integrable, then the Fourier transform satisfies

$$|\widehat{\psi}(\omega)| \leq K(1 + |\omega|)^{-n}. \tag{23}$$

Conversely, if (23) holds, then $\psi(x)$ has at least $n - 2$ derivatives.

(b) Equivalently, if $\widehat{\psi}(\omega)$ has n integrable derivatives, then

$$|\psi(x)| \leq K(1 + |x|)^{-n} \tag{24}$$

Conversely, if (24) holds, then $\widehat{\psi}(\omega)$ has at least $n - 2$ derivatives.

Proof: in exercises.

Thus: $\psi(x)$

• Decays at ∞ faster than any inverse power of x

- Is infinitely differentiable

Claim:

$$\psi_{jk}(x) = 2^{j/2}\,\psi(2^j x - k)$$

form an orthonormal basis for $L^2(\mathbb{R})$.

- Check (only to verify above results - we already know this to be true from our theory):

$$\int_{-\infty}^{\infty} |\psi(x)|^2\,dx = \int_{-\infty}^{\infty} |\widehat{\psi}(\omega)|^2\,d\omega = 1$$

Pf:

$$\int_{-\infty}^{\infty} |\widehat{\psi}(\omega)|^2\ d\omega = \frac{1}{2\pi}\left(\int_{\frac{2\pi}{3}\le|\omega|\le\frac{4\pi}{3}} d\omega\ \sin^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\omega|-1\right)\right]\right.$$
$$\left. + \int_{\frac{4\pi}{3}\le|\omega|\le\frac{8\pi}{3}} d\omega\ \cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{4\pi}|\omega|-1\right)\right]\right)$$

[getting rid of the $|\cdot|$ and doubling; changing vars. in second integral]

$$= \frac{1}{\pi}\left(\int_{\frac{2\pi}{3}\le\omega\le\frac{4\pi}{3}} d\omega\ \sin^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega-1\right)\right]\right.$$
$$\left. + 2\int_{\frac{2\pi}{3}\le\omega\le\frac{4\pi}{3}} d\omega\,\cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega-1\right)\right]\right)$$

$$= \frac{1}{\pi}\left(\int_{\frac{2\pi}{3}\le\omega\le\frac{4\pi}{3}} d\omega\ \left\{\sin^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega-1\right)\right] + 2\cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega-1\right)\right]\right\}\right)$$

$$= \frac{1}{\pi}\left(\int_{\frac{2\pi}{3}\le\omega\le\frac{4\pi}{3}} d\omega\left\{1 + \cos^2\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\omega-1\right)\right]\right\}\right)$$

[letting $s = \frac{3}{2\pi}\omega - 1 \;\Rightarrow\; \omega = 2\pi/3(s+1)$]

$$= \frac{2}{3}\left(\int_0^1 ds\Big(1+\cos^2\left[\frac{\pi}{2}\nu(s)\right]\Big)\right)$$

$$= \frac{2}{3}\left(\int_0^{1/2} ds\Big(1+\cos^2\left[\frac{\pi}{2}\nu(s)\right]\Big) + \int_{1/2}^{1} ds\Big(1+\cos^2\left[\frac{\pi}{2}\nu(s)\right]\Big)\right)$$

$$= \frac{2}{3}\left(\int_0^{1/2} ds\Big(1+\cos^2\left[\frac{\pi}{2}\nu(s)\right]\Big) + \int_0^{1/2} ds\Big(1+\cos^2\left[\frac{\pi}{2}\nu(s+1/2)\right]\Big)\right)$$

[using $\nu(s+1/2) = 1-\nu(1/2-s)$]

$$= \frac{2}{3}\left(\int_0^{1/2} ds\Big(1+\cos^2\left[\frac{\pi}{2}\nu(s)\right]\Big) + \int_0^{1/2} ds\Big(1+\cos^2\left[\frac{\pi}{2}(1-\nu(1/2-s))\right]\Big)\right)$$

$$= \frac{2}{3}\left(\int_0^{1/2} ds\Big(1+\cos^2\left[\frac{\pi}{2}\nu(s)\right]\Big) + \int_0^{1/2} ds\Big(1+\sin^2\left[\frac{\pi}{2}\nu(1/2-s)\right]\Big)\right)$$

$$\overset{s\to 1/2-s}{=} \frac{2}{3}\left(\int_0^{1/2} ds\Big(1+\cos^2\left[\frac{\pi}{2}\nu(s)\right]\Big) + \int_0^{1/2} ds\Big(1+\sin^2\left[\frac{\pi}{2}\nu(s)\right]\Big)\right)$$

$$= \frac{2}{3}\left(\int_0^{1/2} ds(2+1)\right) = 1$$

• To show in another way that they form an orthonormal basis, sufficient to show that for arbitrary $f \in L^2(\mathbb{R})$,

$$\sum_{j,k}^{\infty} |\langle \psi_{jk}, f\rangle|^2 = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

[this is a basic analytic theorem].

Now note:

$$\sum_{j,k}^{\infty} |\langle \psi_{jk}, f \rangle|^2 = \sum_{j,k}^{\infty} \left| \int dx\, \overline{\psi_{jk}(x)} f(x) dx \right|^2$$

$$= \sum_{j,k}^{\infty} \left| \int d\omega\, \widehat{f}(\omega) \overline{\widehat{\psi}_{jk}(\omega)} \right|^2 .$$

Note if

$$\psi_{jk}(x) = 2^{j/2}\, \psi(2^j x - k).$$

Then as usual:

$$\widehat{\psi}_{jk}(\omega) = 2^{-j/2}\, \widehat{\psi}(2^{-j}\omega)\, e^{-i2^{-j}k\omega}.$$

Plug this in above and can do calculation to show (we won't do the calculation):

$$\sum_{j,k}^{\infty} |\langle f, \psi_{jk} \rangle|^2 = \int_{-\infty}^{\infty} dx\, |f(x)|^2 ,$$

as desired.

CONCLUSION:

The wavelets

$$\psi_{jk}(x) \equiv 2^{j/2}\, \psi(2^j x - k)$$

form an orthonormal basis for the square integrable functions on the real line.

4. Daubechies wavelets:

Recall that one way we have defined wavelets is by starting with the scaling (pixel) function $\widehat{\phi}(x)$. Recall it satisfies:

$$\widehat{\phi}(\omega) = m_0(\omega/2)\widehat{\phi}\,(\omega/2)$$

for all ω, where $m_0(\omega)$ is some periodic function. If we use m_0 as the starting point, recall we can write

$$\widehat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}} \prod_{j=1}^{\infty} m_0(\omega/2^j). \tag{25}$$

Recall m_0 is periodic, and so has Fourier series:

$$m_0(\omega) = \sum_k a_k \, e^{-ik\omega} \, .$$

If m_0 satisfies $|m_0(\omega)|^2 + |m_0(\omega + \pi)|^2 = 1$, then it is a candidate for construction of wavelets and scaling functions.

For Haar wavelets, recall $m_0(\omega) = e^{i\omega/2} \cos \omega/2$, so we could plug into (25) to get $\widehat{\phi}$, and then use previous formulas to get wavelet $\psi(x)$.

If we *start* with a function $m_0(\omega)$, when does (25) lead to a genuine wavelet? Check conditions:

(1)

$$\widehat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}} \prod_{j=1}^{\infty} m_0(\omega/2^j)$$

$$= \frac{1}{\sqrt{2\pi}} m_0(\omega/2) \prod_{j=2}^{\infty} m_0(\omega/2^j)$$

$$= m_0(\omega/2) \frac{1}{\sqrt{2\pi}} \prod_{j=1}^{\infty} m_0(\omega/2^{j+1})$$

$$= m_0(\omega/2)\widehat{\phi}(\omega/2) \tag{26}$$

Recall this implies that $V_j \subset V_{j+1}$ where

$$V_j = \left\{ \sum_{k=-\infty}^{\infty} a_k \phi_{jk}(x) \middle| \sum_k |a_k|^2 < \infty \right\}$$

(usual definition) with $\phi_{jk}(x) = 2^{j/2}\phi(2^j x - k)$

(2) The second condition we need to check is that translates of ϕ orthonormal, i.e.,

$$\sum_k |\widehat{\phi}(\omega + 2\pi k)|^2 = \frac{1}{2\pi}.$$

If

$$m_0(\omega) = \text{finite Fourier series} = \sum_{k=-N}^{N} a_k e^{-i\omega k} = \text{trigonometric polynomial}$$

there is a simple condition which guarantees condition (2) holds.

Theorem (Cohen, 1990): If the trigonometric polynomial m_0 satisfies $m_0(0) = 1$ and

$$|m_0(\omega)|^2 + |m_0(\omega + \pi)|^2 = 1 \qquad (27)$$

(our standard condition on m_0), and also $m_0(\omega) \neq 0$ for $|\omega| \leq \pi/3$, then condition (2) above is satisfied by

$$\widehat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}} \prod_{j=1}^{\infty} m_0(\omega/2^j)$$

Proof: Daubechies, Chapter 6.

Since condition **(1)** is also automatically satisfied, this means ϕ is a scaling function which will lead to a full orthonormal basis using our algorithm for constructing wavelets.

Another choice of m_0 is:

$$m_0(\omega) = \frac{1}{8}[(1+\sqrt{3}) + (3+\sqrt{3})e^{-i\omega} + (3-\sqrt{3})e^{-2i\omega} + (1-\sqrt{3})\, e^{-3i\omega}]$$

(Fourier series with finite number of terms).

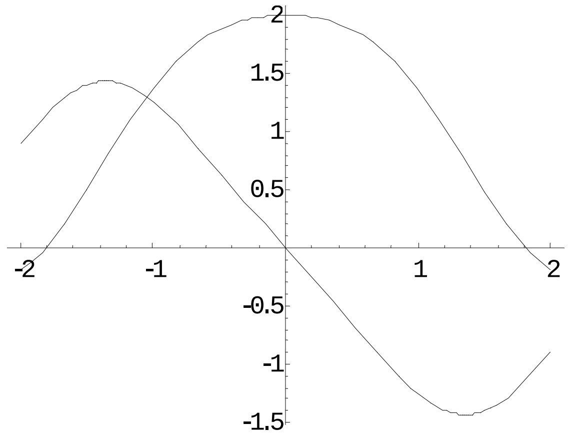


Fig 41: Real (symmetric) and imaginary (antisymmetric) parts of $m_0(\omega)$

To check Cohen's theorem satisfied:

(i) Equation (27) satisfied (see exercises).

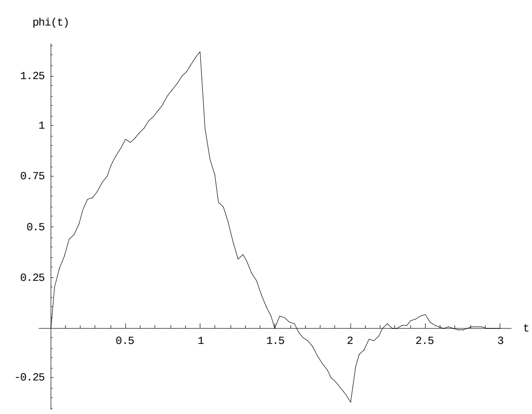
(ii) If $m_0(\omega) = \text{Re}\, m_0(\omega) + i\ \text{Im}\, m_0(\omega)$,

$$|m_0(\omega)|^2 = |\text{Re}\ m_0(\omega)|^2 + |\text{Im}\, m_0(\omega)|^2 \neq 0$$

for $|\omega| \leq \pi/3$, as can be seen from graph above.

So: conditions of Cohen's theorem are satisfied.

In this case if we define scaling function ϕ by computing infinite product (25) (perhaps numerically), and then use our standard procedure to construct wavelet $\psi(x)$, we get:



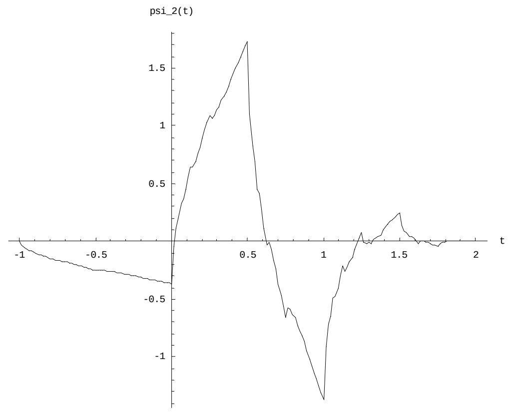


fig 42: pictures of ϕ and ψ

Note meaning of m_0: In terms of the original wavelet, this states

$$\phi(x) = \frac{1}{4}\left[(1+\sqrt{3})\phi(2x) + (3+\sqrt{3})\phi(2x-1) \right.$$
$$\left. + (3-\sqrt{3})\phi(2x-2) + (1-\sqrt{3})\phi(2x-3)\right]$$

(see (26) above). Note this equation gives the information we need on ϕ, since it determines $m_0(\omega)$.

6. Examples, applications

1. Other examples

Note again it is possible to get other wavelets this way: If we demand

$$\phi(x) = .226\,\phi(2x) + .854\,\phi(2x-1) + 1.24\,\phi(2x-2)$$

$$+\ .196\,\phi(2x-3)\ -1.434\,\phi(2x-4) - .046\,\phi(2x-5)$$

$$+\ .110\,\phi(2x-6) - .008\,\phi(2x-7) - .018\,\phi(2x-8)$$

$$+\ .004\,\phi(2x-9) \tag{28}$$

Then this results with an $m_0(\omega)$

$$m_0(\omega) = .113 + .427\,e^{i\omega} + .512\,e^{2i\omega} + .098e^{3\omega} + \ldots + .002e^{9i\omega}.$$

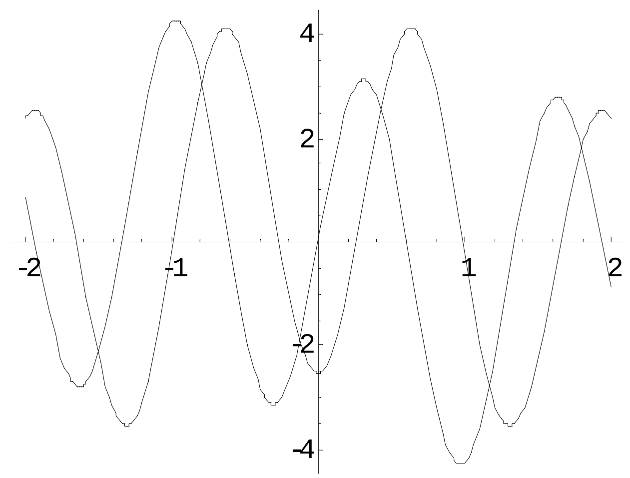


Fig 43: Real (symmetric) and imaginary parts of m_0; note condition (ii) of Cohen's theorem is satisfied.

Can check it satisfies condition (ii) of Cohen's theorem and resulting ϕ is obtained:

$$\widehat{\phi}(\omega) = \prod_{j=1}^{\infty} m_0(\omega/2^j).$$

It satisfies required properties (a) - (f) of a multiresolution analysis. Corresponding scaling function ${}_5\phi(x)$ and wavelet ${}_5\psi(x)$ are below

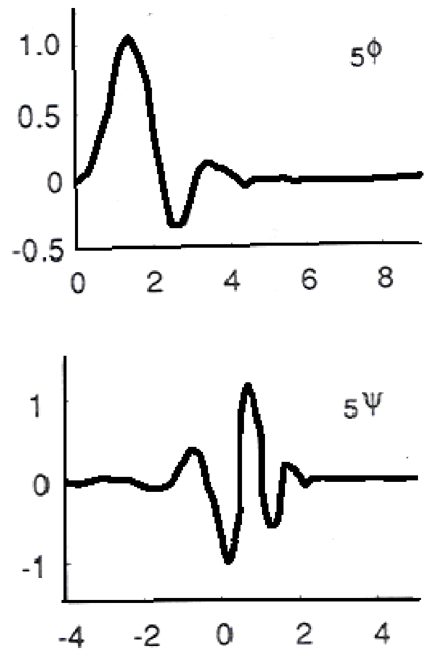


Fig 44: Scaling function and wavelet for the above ϕ choice

NOTE: Can show that if there is a finite number of terms on the right side of (28), then corresponding wavelet and scaling function are compactly supported.

2. Numerical uses of wavelets

Note that once we have an orthonormal wavelet basis $\{\psi_{jk}\}$, can write any function:

$$f(x) = \sum_{j,k} a_{jk}\, \psi_{jk}(x),$$

with $a_{jk} = (f, \psi_{jk})$. Numerically, can find $a_{jk} = \langle \psi_{jk}, f \rangle$ using numerical integration to evaluate inner product.

With Daubechies and other wavelets, there are no closed form for the wavelets, so above integrations must be performed on the computer.

But there are very efficient methods of doing this: in order to get *all* the wavelets ψ_{jk} into the computer, we just need to input one - all others are rescalings and translations of the original one.

There are efficient algorithms to get coefficients a_{jk}; more details in Daubechies' book.

3. SOME GENERAL PROPERTIES OF ORTHONORMAL WAVELET BASES:

Theorem: If the basic wavelet $\psi(x)$ has exponential decay, then ψ cannot be infinitely differentiable.

(in particular, if ψ has compact support, then ψ cannot be infinitely differentiable).

Proof: Daubechies, Chapter 5.

Compactly Supported Wavelets:

So far we are able to get wavelets

$$\psi_{jk}(x) = 2^{j/2}\,\psi(2^j x - k)$$

which form an orthonormal basis for L^2. Note Haar wavelets had compact support. When will wavelets be compactly supported in general?

Recall we assume that given basic scale space V_0, that we have scaling (pixel) function ϕ such that $\{\phi(x-k)\}_k$ form basis for V_0.

Recall

- $V_0 \subset V_1$,
- $\phi(x) \in V_0 \quad \Rightarrow \quad \phi(x) \in V_1$
- $\sqrt{2}\,\phi(2x) \in V_1$
- $\{\sqrt{2}\,\phi(2x-k)\}_{k=1}^{\infty}$ form a basis for V_1

Recall since $\phi(x) \in V_1$, we have for some choice of h_k:

$$\phi(x) = \sum_0^\infty h_k \sqrt{2}\ \phi(2x-k).$$

Constants h_k relate the space V_0 to V_1.

We will see that:

Theorem:

finitely many $h_k \neq 0 \quad \Leftrightarrow \quad \psi,\ \phi$ have compact support.

Proof:

$\Leftarrow:$ Assume ϕ has compact support. Then note since $\sqrt{2}\phi(2x-\ell)$ are orthonormal,

$$h_\ell = \int \sqrt{2}\phi(2x-\ell)\phi(x)dx$$

$= 0$ for all but a finite number of ℓ :

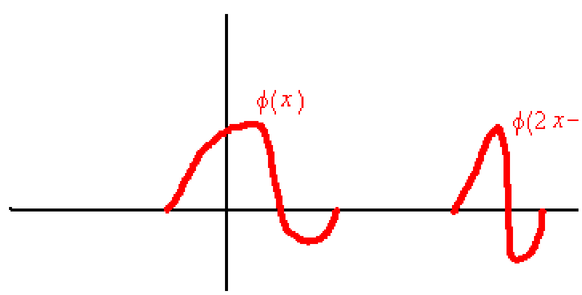


fig 45 : Note $h_l =$ integral of product $= 0$ for all but finite number of ℓ

To prove $\Rightarrow$: (rough sketch only)

Assume that h_k are 0 for all but a finite number of k. Then need to show $\phi(x)$ has compact support.

Strategy of proof: look at $\widehat{\phi}(\omega)$.

Recall we defined

$$m_0(\omega) = \sum_k \frac{h_k}{\sqrt{2}} e^{-ik\omega}$$

Recall:

$$\widehat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}} \prod_{j=1}^{\infty} m_0(2^{-j}\omega).$$

- From this show that $\widehat{\phi}(\omega)$ extends to an analytic function of ω in whole complex plane satisfying:

$$|\widehat{\phi}(\omega)| \leq C(1 + |\omega|)^M \, e^{N|\mathrm{Im}\,\omega|}$$

for constants M and N.

- This implies by Paley-Wiener type theorems that $\phi(x) = F^{-1}\,(\widehat{\phi})$ is compactly supported. □

4. GENERIC PRESCRIPTION FOR COMPACTLY SUPPORTED WAVELETS:

- Start with finite sequence of numbers h_k (define how V_0 will be related to V_1)

- Construct

$$m_0(\omega) = \sum_k \frac{h_k}{\sqrt{2}} \, e^{-ik\omega}$$

check that it satisfies Cohen's theorem conditions :

$$|m_0(\omega)| \neq 0 \text{ for } |\omega| \leq \pi/3.$$

and

$$|m_0(\omega)|^2 + |m_0(\omega + \pi)|^2 = 1.$$

- Construct

$$\frac{1}{\sqrt{2\pi}}\prod_{j=1}^{\infty} m_0(2^{-j}\omega) = \widehat{\phi}(\omega)$$

- Construct Fourier transform of wavelet by:

$$\widehat{\psi}(\omega) \equiv e^{i\omega/2}\,\overline{m_0(\omega/2+\pi)}\,\widehat{\phi}(\omega/2),$$

- Take inverse Fourier transform to get $\psi(x) =$ wavelet

5. SOME FURTHER PROPERTIES OF WAVELET EXPANSIONS

QUESTION: Do wavelet expansions actually converge to the function being expanded at individual points x?

Assume that scaling function ϕ is bounded by an integrable decreasing function. Then:

Theorem: If f is a square integrable function, then the wavelet expansion of f

$$f(x) = \sum_{j,k}^{\infty} a_{jk}\,\psi_{jk}(x)$$

converges to the function f almost everywhere (i.e., except on a set of measure 0).

QUESTION: How fast do wavelet expansions converge to the function f?

ANSWER: That depends on how "regular" the wavelet ψ is. More particularly it depends exactly on the Fourier transform of ψ:

Theorem: In d dimensions, the wavelet expansion

$$f(x) = \sum_{j,k} a_{jk}\,\psi_{jk}(x)$$

converges to a smooth f in such a way that the partial sum

$$\sum_{j \le N, k} a_{jk} \psi_{jk}(x)$$

differs from $f(x)$ at each x by at most $C \cdot 2^{-Ns}$, iff

$$\int |\widehat{\psi}(\omega)|^2 \, |\omega|^{-2s-d} \, d\omega < \infty.$$

6. CONTINUOUS WAVELET TRANSFORMS

Consider a function $\psi(x) \in L^2$ (i.e., ψ is square integrable), such that $\psi(x)$ decays fast enough at ∞ (faster than $1/x^2$), and such that

$$\int_{-\infty}^{\infty} \psi(x) \, dx = 0.$$

Then we can define an integral wavelet expansion (integrals instead of sums) using re-scalings of $\psi(x)$:

Define rescaled functions

$$\psi_{a,b}(x) \equiv |a|^{1/2} \, \psi(a(x-b)).$$

[note $a \to 1/a$ in definition of Daubechies]

Here $a, b \in \mathbb{R}$. Thus a measures how much ψ has been stretched (dilation parameter), and b measures how much ψ has been moved to the right (translation parameter).

New point: dilation parameter a and translation parameter b can take on any real value.

Now define wavelet expansions in this case (analogous to Fourier transform -- called wavelet transform): given $f \in L^2(\mathbb{R})$, we define the transform (assuming that ψ is real)

$$(Wf)(a,b) = \int dx\, f(x)\, \overline{|a|^{1/2}\psi(a(x-b))}$$

$$= \int dx\, f(x)\, \overline{\psi_{a,b}(x)}$$

$$= \langle \psi_{a,b}, f \rangle$$

How to recover f from $(Wf)(a,b)$?

Claim:

$$f(x) = C \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} da\, db\ \ (Wf)(a,b)\, \psi_{a,b}(x)$$

where

$$C^{-1} = -2\pi \int d\omega\, |\omega|^{-1} |\widehat{\psi}(\omega)|^2.$$

Pf. of claim (sketch; details in Daubechies, Ch. 2):

We will show that for any $g(x) \in L^2$,

$$\big\langle g(x), f(x) \big\rangle = \big\langle g(x), C \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} da\, db\, (Wf)(a,b)\, \psi_{a,b}(x) \big\rangle$$

To see this, note that

$$\big\langle g(x), f(x) \big\rangle = \int_{-\infty}^{\infty} \overline{g(x)}\, f(x)\, dx$$

$$= \int_{-\infty}^{\infty} d\omega\ \overline{\widehat{g}\,(\omega)}\, \widehat{f}(\omega)$$

[use “Plancherel Theorem” for wavelet transforms]

$$= C \int\!\!\int da\, db\ \overline{(Wg)(a,b)}(Wf)(a,b)$$

$$= C \int\int da\, db\, \langle g(x), \psi_{a,b}(x)\rangle (Wf)(a,b)(x)$$

$$= \left\langle g(x), C \int\int da\, db\, (Wf)(a,b)\psi_{a,b}(x) \right\rangle,$$

as desired, completing the proof.

Thus we know how to recover $f(x)$ from $Wf(a,b)$ (analogous to recovering $f(x)$ from $\widehat{f}(\omega)$ in Fourier transform).

QUESTION: What sorts of functions are $(Wf)(a,b)$? For some choices of ψ, these are spaces of analytic functions.

7. Convolutions:

Definition: The *convolution* of two functions $f(x)$ and $g(x)$ is defined to be

$$f(x)*g(x) \equiv \int_{-\infty}^{\infty} f(x-y)g(y)dy.$$

Theorem 2: The convolution is commutative: $f*g = g*f$

Proof: Exercise.

Theorem 3: The Fourier transform of a convolution is a product. Specifically,

$$\mathcal{F}(f(x)*g(x)) = \sqrt{2\pi}\,\widehat{f}(\omega)\widehat{g}(\omega)$$

Proof: Exercise.

Lemma 4: For any function f, $\mathcal{F}(f(-x)) = \overline{\widehat{f}(\omega)}$

Proof: Exercise.

8. APPLICATION OF INTEGRAL WAVELET TRANSFORM: IMAGE RECONSTRUCTION (S. Mallat)

Dyadic wavelet transform: a variation on continuous wavelet transform.

Now define new dilation only by powers of 2; arbitrary translations:

$$\psi_{j,b}(x) = 2^j \psi(2^j(x-b))$$

Define

$$\psi_j(x) = 2^j \psi(2^j x).$$

(Still allow $b \in \mathbb{R}$ to take all values, but restrict $a = 2^j$.)

Define this dyadic (partially discrete) wavelet transform by:

$$(Wf)(j,b) = \int f(x)\, \psi_{j,b}(x)\, dx$$

i.e., usual set of wavelet coefficients, except that b is continuous.

Note:

$$(Wf)(j,b) = \int f(x)\, \psi_{j,b}(x)\, dx$$

$$= \int dx\, f(x)\, 2^j \psi(2^j(x-b))$$

$$= \int dx\, f(x)\, \psi_j(x-b)$$

$$= (f*\psi_j)(b)$$

(a convolution) where as above

$$\psi_j(x) = 2^j \psi(2^j x) = \text{ shrinking of } \psi \text{ by a factor } 2^j.$$

New assumption: Fourier transform $\widehat{\psi}(\omega)$ satisfies

$$\sum_{j=-\infty}^{\infty} |\widehat{\psi}(2^j \omega)|^2 = \frac{1}{2\pi}.$$

Now: given $f(x)$, consider dyadic wavelet transform; $a = 2^j$ only:

Can show under our assumptions that can recover f in this case too: Recovery formula for f is:

$$f(x) = \sum_{j=-\infty}^{\infty} (Wf)(j,x) * \psi_j(-x)$$

(convolution in variable x). It is easy to check that this is correct: if $\mathcal{F}$ denotes Fourier transform:

$$\mathcal{F}\left(\sum_{j=-\infty}^{\infty} (Wf)(j,x) * \psi_j(-x) \right) = \mathcal{F}\left(\sum_{j=-\infty}^{\infty} f(x) * \psi_j(x) * \psi_j(-x) \right)$$

$$= \sum_{j=-\infty}^{\infty} \mathcal{F}(f(x) * \psi_j(x) * \psi_j(-x))$$

$$= 2\pi \sum_{j=-\infty}^{\infty} \widehat{f}(\omega)\, \widehat{\psi}_j(\omega)\, \overline{\widehat{\psi}_j(\omega)}$$

$$= 2\pi \sum_{j=-\infty}^{\infty} \widehat{f}(\omega)\, \widehat{\psi}(2^{-j}\omega)\, \overline{\widehat{\psi}(2^{-j}\omega)}\,.$$

$$= 2\pi \sum_{j=-\infty}^{\infty} \widehat{f}(\omega) |\widehat{\psi}(2^{-j}\omega)|^2$$

$$= \widehat{f}(\omega)\, 2\pi \sum_{j=-\infty}^{\infty} |\widehat{\psi}(2^{-j}\omega)|^2$$

$$= \widehat{f}(\omega).$$

QUESTION: Given $f(x)$, what sort of function is the wavelet transform $(Wf)(j,b)$, as a function of j and b?

Let $V =$ the collection of possible functions $(Wf)(j,b) =$ collection of possible wavelet transforms. When is an arbitrary function $g(j,b)$ a wavelet transform?

Can check that g must satisfy a so-called reproducing kernel equation: $g(j,b)$ is the wavelet transform of some function iff

$$g(j,b) = (Kg)(j,b) \equiv \sum_{\ell=-\infty}^{\infty} \psi_j(b) * \psi_\ell(-b) * g(\ell,b)$$

[this equation defines Kg; note convolution is in b.]

Back to recovering f from wavelet transform:

Thus we can recover f as a sum of f at different scales:

$$f = \sum_{j=-\infty}^{\infty} (Wf)(j,x) * \psi_j(-x).$$

Since ψ is a known function, we can recover f from the sequence of functions. Assume $a(x)$ is a cubic B-spline:

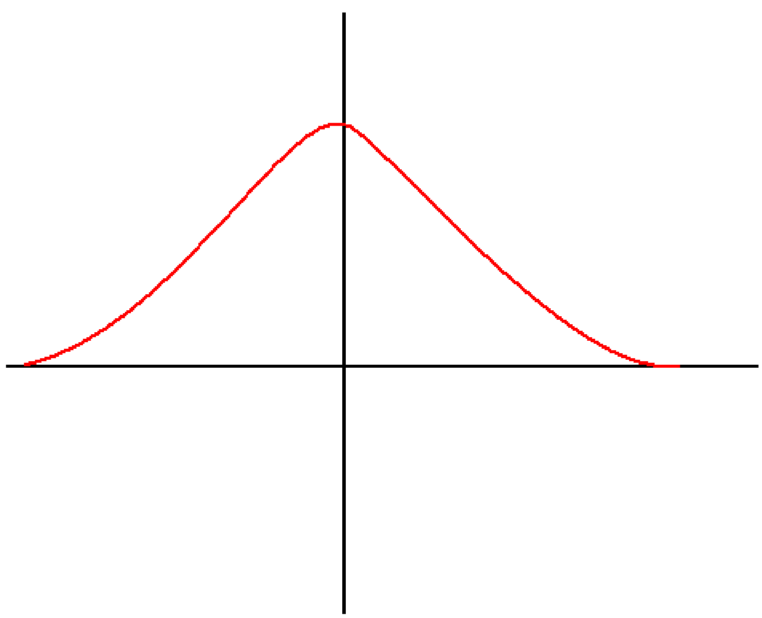


Fig. 46: A cubic B-spline $a(x)$ is a symmetric compactly supported piecewise cubic polynomial function whose transition points are twice continuously differentiable

Now let the wavelet be its first derivative: $\psi(x) = \frac{d}{dx}a(x)$

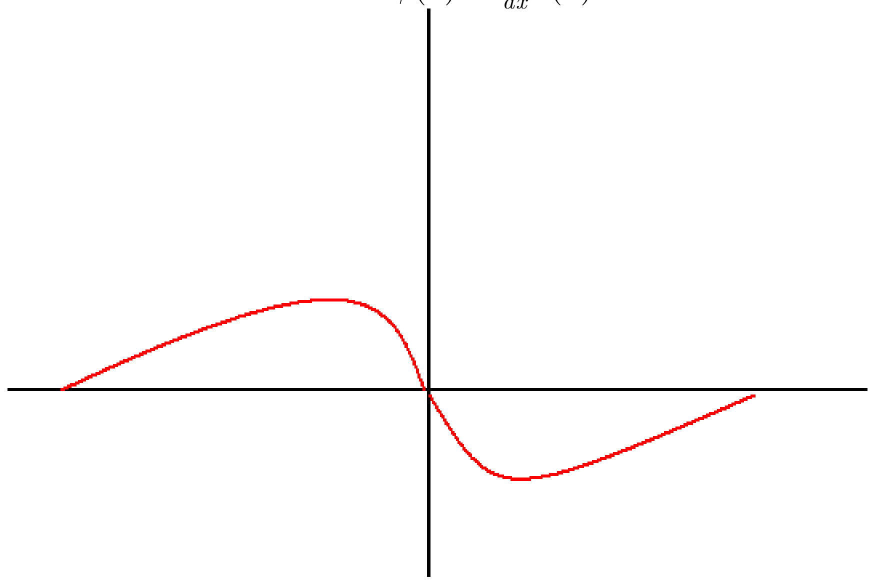
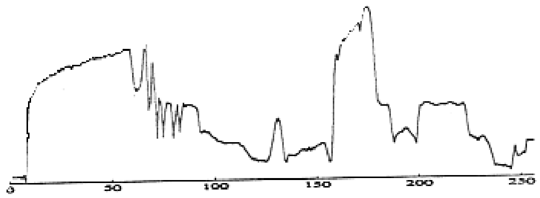


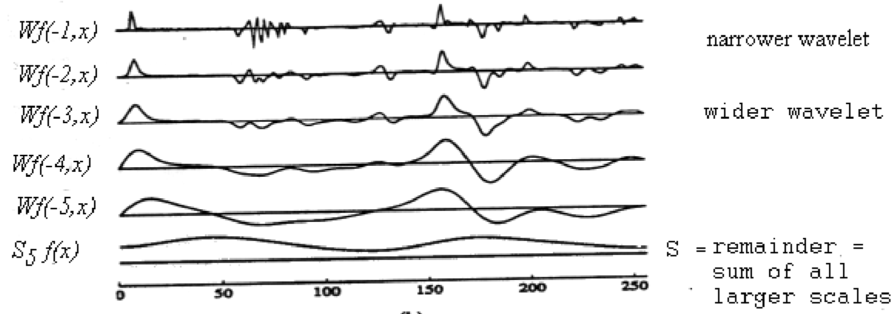
Fig 47: $\psi(x) = \frac{d}{dx}a(x)$ is the wavelet

Using the wavelet $\psi(x)$:

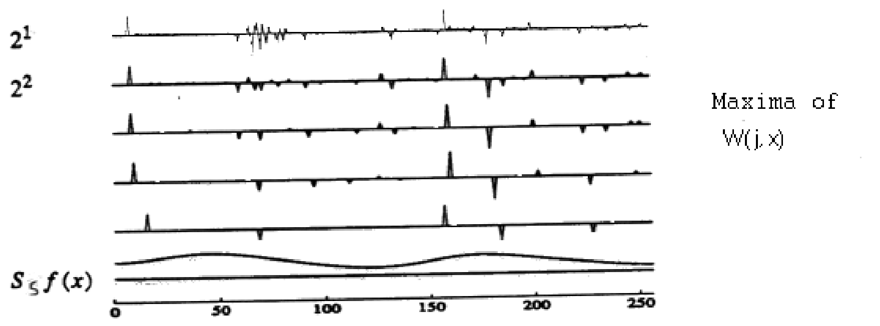
$$(Wf)(-2,x)$$
$$(Wf)(-1,x)$$
$$(Wf)(0,x)$$
$$(Wf)(1,x)$$
$$(Wf)(2,x)$$
$$(Wf)(3,x)$$

To see that these pieces of f represent f at different scales, look at example:





(b)



So: one can recover f from knowing the functions

$$(Wf)(j, x).$$

This is a lot of functions. What advantage of storing f in such a large number of functions? We can compress the data.

CONJECTURE: We can recover f not from knowing all of the functions $W(j, x)$, but just from knowing their maxima and minima.

Meyer has proved this conjecture false strictly speaking certain choices of ψ (including the above derivative $\psi(x)$ of the cubic spline). It has been proved true for another choice, the derivative of a Gaussian.

$$\psi(x) = \frac{d}{dx} e^{-x^2}$$

However, for either choice of ψ numerically it is possible to recover $f(x)$ from knowing only the maxima and minima of the functions $W(j, x)$.

Numerical method:

Assume that we are given only the maxima and minima points of the function $W(j,x)$ for each j. How to recover f?

Given f, first take its wavelet transform; get $W(j,x)$. Define

$\Gamma =$ set of all functions $g(j,x)$ which have the same set of maxima and minima (in x) as $W(j,x)$ for each j.

$V =$ set of all $g(j,x)$ which are wavelet transforms of some function of x.

Idea is: the true wavelet transform $Wf(j,x)$ of our given function $f(x)$ is in Γ (i.e. has the same maxima as itself) and is in V (i.e., in the collection of functions which are wavelet transforms).

Thus

$$Wf \in \Gamma \cap V.$$

intuitive picture:

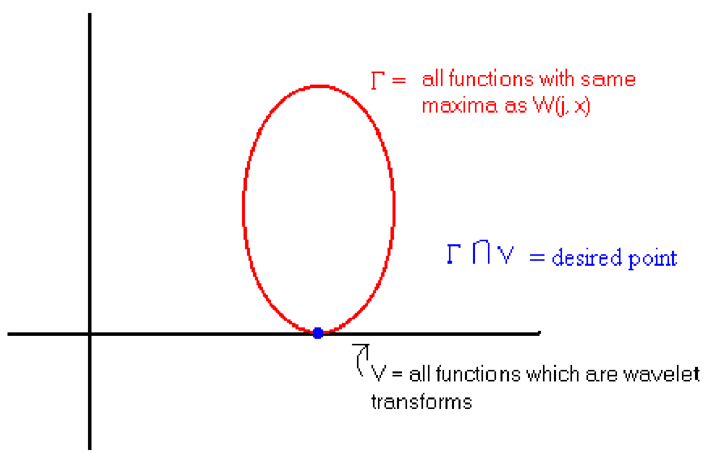


fig 48

Thus if we know just the maxima of $Wf(j,x)$, we can try to find $Wf(j,x)$

That is:

1. We know maxima of $Wf(j,x)$, so
2. know $\Gamma =$ all functions with same maxima as $Wf(j,x)$

3. Find $Wf(j,x)$ as "unique" point in Γ which is also a wavelet transform, i.e., unique point in $\Gamma \cap V$:

Algorithm:

1. Start with only the maxima information about $W(j,x)$. Call M the maxima information.

2. Make initial guess using function $g_1(j,x)$ which has the same maxima as $W(j,x)$.

3. Find closest function in $V =$ set of wavelet transforms to $g_1(j,x)$. Call this function $g_2(j,x)$.

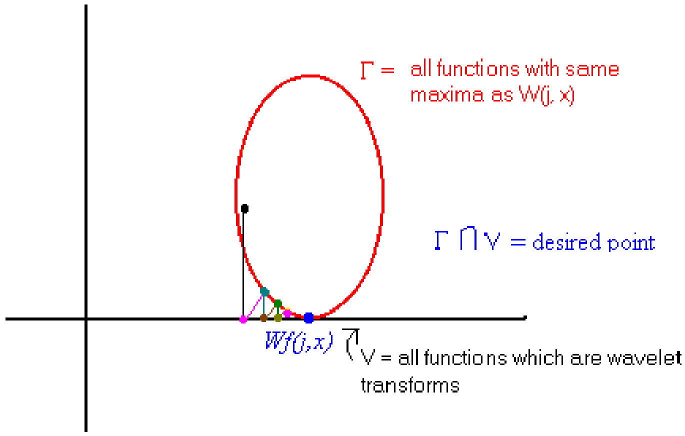
4. Find closest function in $\Gamma =$ functions with same maxima as M to $g_2(j,x)$. Call this function $g_3(j,x)$.

5. Find closest function in V to $g_3(j,x)$; call this $g_4(j,x)$.

6. Find closest function in Γ to g_4; call this g_5.

7. Continue this way: at each stage j find the closest function g_j to g_{j-1} in
the space V or Γ (alternatingly).

Eventually the $g_j(j,x) \underset{j \to \infty}{\longrightarrow} Wf(j,x)$ as desired.



CONCLUSION: We can recover the wavelet transform $Wf(j,x)$ of a function just by knowing its maxima in x.

THE POINT: Compression. We can store the maxima of Wf using a lot less memory.

APPLICATION: Compression of images:

Fig. 9: *The upper left is the original lady image. The upper right image is a reconstruction from the maxima representation shown in the second column of fig. 8. This reconstruction is performed with 8 iterations and the noise to signal ratio is* $6.6\ 10^{-2}$. *The lower left and lower right images have been reconstructed from the maxima representation shown respectively in the third and fourth column of fig. 8 (thresholding by the factors 4 and 8). The light textures have disappeared but the strong edges and textures remain unchanged.*

Fig. 49

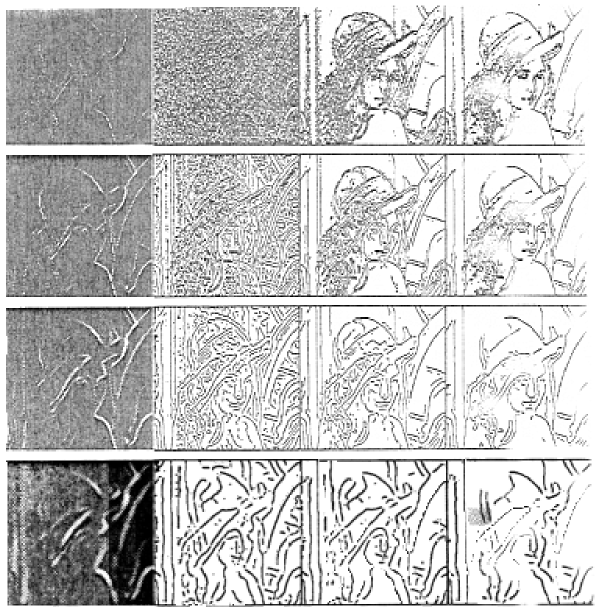


Fig. 8: *The first column gives the modulus images $M_{2^j}f(x,y)$ for $1 \le j \le 4$ of the lady image shown at the top left of fig. 9. The second column displays the position of the maxima of $M_{2^j}f(x,y)$. The third and fourth columns display the position of the local maxima whose amplitude are respectively larger than 4 and 8. The maxima that have been removed correspond essentially to the noise and the light texture irregularities.*

Fig. 50

9. Wavelets and Wavelet Transforms in Two Dimensions

Multiresolution analysis and wavelets can be generalized to higher dimensions. Usual choice for a two-dimensional scaling function or wavelet is a product of two one-dimensional functions. For example,

$$\phi_2(x, y) = \phi(x)\phi(y)$$

and scaling equation has form

$$\phi(x, y) = \sum_{k,l} h_{kl} \cdot 2\phi(2x - k, 2y - l).$$

Since $\phi(x)$ and $\phi(y)$ both satisfy the sclaing equation

$$\phi(x) = \sum_{k} h_k \cdot \sqrt{2}\phi(2x - k),$$

we have $h_{kl} = h_k h_l$. Thus two dimensional scaling equation is product of two one dimensional scaling equations.

We can proceed analogously to construct wavelets using products of one-dimensional functions. However, unlike one-dimensional case, we have three rather than one basic wavelet. They are:

$$\psi^{(I)}(x,y) = \phi(x)\psi(y)$$

$$\psi^{(II)}(x,y) = \psi(x)\phi(y)$$

$$\psi^{(III)}(x,y) = \psi(x)\psi(y).$$

The generalization of the one-dimensional wavelet equation leads to the following relations:

$$\psi^{(I)}(x,y) = \sum_{k,l} g_{kl}^{(I)} \cdot 2\phi(2x-k, 2y-l)$$

$$\psi^{(II)}(x,y) = \sum_{k,l} g_{kl}^{(II)} \cdot 2\phi(2x-k, 2y-l)$$

$$\psi^{(III)}(x,y) = \sum_{k,l} g_{kl}^{(III)} \cdot 2\phi(2x-k, 2y-l)$$

where $g_{kl}^{(I)} = h_k g_l, \quad g_{kl}^{(II)} = g_k h_l,$ and $g_{kl}^{(II)} = g_k g_l.$

We can generate two-dimensional scaling functions and wavelets using the functions ScalingFunction and Wavelet then taking the product. For example, here we plot the Haar wavelets in two dimensions. Various translated and dilated versions of the wavelets can be plotted similarly.

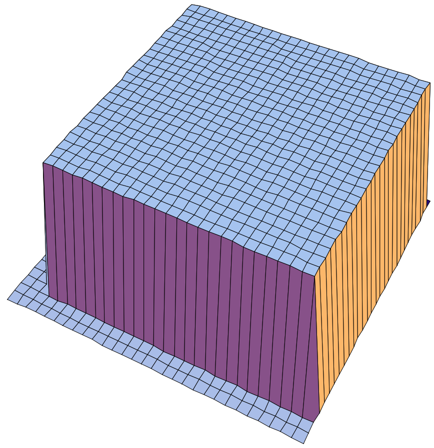


Fig. 5

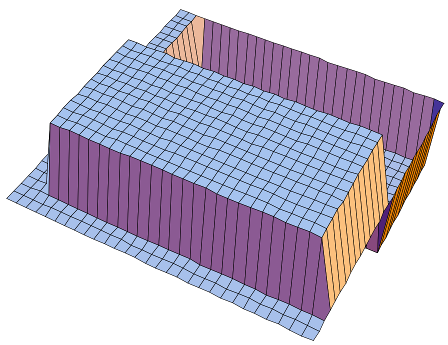
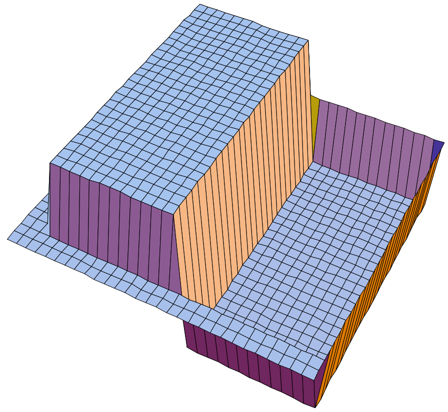


Fig. 52: Haar wavelet $\psi^{(I)}(x, y)$



F (II) (I)

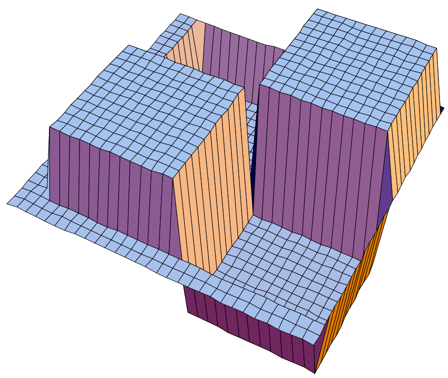


Fig. 54: Third wavelet $\psi^{(III)}(x,y)$

As example of another wavelet, here is so-called "least asymmetric wavelet" of order 8 in two dimensions :

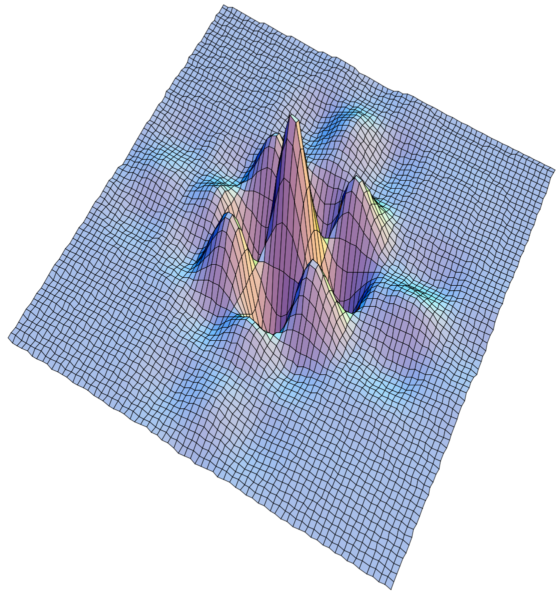


Fig. 55: Least asymmetric wavelet of order 8