

# Genera: From Cobordism to $K$ -theory

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## 1 Hirzebruch and genera

In the spring of 1953, Hirzebruch was at IAS in Princeton, thinking hard about something called the Riemann-Roch problem for analytic varieties. It said that a certain invariant having to do with complex analysis could be computed using topology. In May, Thom announced his now-famous results on cobordism, and Hirzebruch realized he had a way in. He made a first thrust towards a solution in July, and then finished it off in December. The key was to solve a related problem having to do with an invariant called the *signature* of a manifold. Let me tell that story first, and we'll get back to this 'Riemann-Roch' business later.

(1.1) Most of the invariants of manifolds we've met (excluding cobordism class) are invariants of the homotopy type of  $M$ - like its cohomology together with the ring structure and cohomology operations. Compact, oriented  $4n$ -manifolds have a bit more structure on their cohomology than just any old space. An orientation provides an identification  $H^{4n}(M, \mathbb{R}) \cong \mathbb{R}$  and Poincaré duality tells us that the cup product pairing

$$H^{2n}(M, \mathbb{R}) \otimes H^{2n}(M, \mathbb{R}) \longrightarrow H^{4n}(M, \mathbb{R}) \cong \mathbb{R}$$

is a non-degenerate, symmetric bilinear form. Whenever you have a symmetric bilinear form on a real vector space  $V$  it is possible to choose a basis  $e_1, \dots, e_d$  such that  $\langle e_i, e_j \rangle = 0$  whenever  $i \neq j$  and  $\langle e_i, e_i \rangle = \pm 1$ . Let  $n_+$  denote the number of times you get  $+1$  and  $n_-$  the number of times you get  $-1$ . Then the *signature* of the bilinear form is defined to be  $n_+ - n_-$  (it turns out to be independent of the choice of basis of this form.)

**Definition 1.1.** The *signature* of a compact, oriented  $4k$ -manifold  $M$  is defined to be the signature of the Poincaré duality pairing on  $H^{2k}(M, \mathbb{R})$ . If  $M$  has dimension not divisible by 4, we define its signature to be zero. The signature is denoted  $\sigma(M)$ .

**Example 1.2.** The symmetric bilinear form  $B$  on  $\mathbb{R}^2$  defined by  $B(x, y) = x_1y_2 + x_2y_1$  has signature 0.

**Example 1.3.** Recall that  $H^*(\mathbb{C}P^{2n}) = \mathbb{Z}[x]/x^{2n+1}$  where  $|x| = 2$ . Then  $H^{2n} = \mathbb{Z}\{x^n\}$  and the cup product pairing is  $x^n \cdot x^n = x^{2n}$ , which is the orientation class in  $H^{4n}$ . Thus  $\sigma(\mathbb{C}P^{2n}) = 1$ .

In one of Thom's earliest papers, he showed that the signature was actually a cobordism invariant.

**Proposition 1.4** (Thom). *The signature descends to a ring homomorphism:*

$$\sigma : MSO_* \longrightarrow \mathbb{Z}.$$

At this point let me make a simple observation. The right hand side of the homomorphism  $\sigma$  is torsion-free, so  $\sigma$  factors through the quotient of  $MSO_*$  by its torsion subgroup. So it's no loss of information to just consider the rational invariant:

$$MSO_* \otimes \mathbb{Q} \longrightarrow \mathbb{Q}.$$

The upshot here is that Thom knew a lot about  $MSO_* \otimes \mathbb{Q}$ . For example, he showed that the image of a compact, oriented manifold in  $MSO_* \otimes \mathbb{Q}$  is determined by its Pontryagin numbers. This leads to a natural question:

**Question 1.5.** What is the relationship between the signature and the Pontryagin numbers of a compact, oriented  $4n$ -manifold?

The case  $n = 1$  was dealt with by Thom himself, but the general question was settled by Hirzebruch.

(1.2) Let's deal with Thom's result first, because it contains the seed of the general strategy Hirzebruch used later. Here's the theorem.

**Theorem 2.6** (Thom). *Let  $M$  be a compact, oriented 4-manifold. Then*

$$\sigma(M) = \int_M \frac{p_1}{3},$$

where  $p_1$  is the first Pontryagin class of the tangent bundle  $TM$ .<sup>1</sup>

**Remark 2.7.** Note that this result has as a consequence the non-obvious fact that the first Pontryagin number of a compact, oriented 4-manifold is divisible by 3.

First of all: why did Thom expect a result like this to be true? Well, he had already computed that

$$MSO_4 \otimes \mathbb{Q} \cong \mathbb{Q}$$

Both the signature and the first Pontryagin number give linear maps  $MSO_4 \otimes \mathbb{Q} \cong \mathbb{Q} \rightarrow \mathbb{Q}$  so we know they must differ by a constant. To determine that constant, you need only evaluate  $\sigma(-)$  and  $\int_{(-)} p_1$  on a single, nontrivial cobordism class in dimension 4. The easiest one lying around is  $\mathbb{C}P^2$ . We've already computed that  $\sigma(\mathbb{C}P^2) = 1$ , so we just need to show that

$$\int_{\mathbb{C}P^2} p_1 = 3.$$

Since we'll need it later, we may as well compute all the Chern and Pontryagin classes of all the complex projective spaces.

(1.3) We'll need to know something about the tangent bundle of  $\mathbb{C}P^n$ . A tangent vector is supposed to tell me how to jiggle a point in  $\mathbb{C}P^n$ . But a point in  $\mathbb{C}P^n$  is a line  $\ell \subset \mathbb{C}^{n+1}$ . How do you jiggle a line? Well you pick a point on the line and tell me a direction to move it- the rest of the line follows along for the ride. A nice way to say this without picking a point is to say you've got a linear map

$$\ell \rightarrow \mathbb{C}^{n+1}$$

or, equivalently, you've got an  $(n + 1)$ -tuple of linear functionals on  $\ell$ ,  $(\varepsilon_0, \dots, \varepsilon_n)$ . Putting this together we get a surjective map:

$$(\ell^*) \oplus \dots \oplus (\ell^*) \rightarrow T_{[\ell]} \mathbb{C}P^n.$$

if we let  $\ell$  vary, then we get a surjective map of vector bundles:

$$\mathcal{O}(1)^{\oplus(n+1)} \rightarrow T\mathbb{C}P^n,$$

where  $\mathcal{O}(1)$  denotes the *dual* of the tautological bundle,  $\mathcal{O}(-1)$ .

Now, if I happened to choose my jiggle parallel to the line  $\ell$  itself, then that doesn't move the point in  $\mathbb{C}P^n$ . So the kernel is a copy of the trivial bundle  $\underline{1}$  which maps into  $\mathcal{O}(1)^{\oplus(n+1)}$  as the tuples which just scale  $\ell$ . So we've shown:

$$T\mathbb{C}P^n \cong \frac{\mathcal{O}(1)^{\oplus(n+1)}}{\underline{1}}.$$

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<sup>1</sup>I'm using  $\int_M$  to denote evaluation of a cohomology class on the fundamental class  $[M]$ .

It follows that there's an isomorphism of vector bundles:

$$TCP^n \oplus \underline{1} \cong \mathcal{O}(1)^{\oplus(n+1)}.$$

Now the only thing we should know is that

$$c(\mathcal{O}(1)) = 1 + x$$

where  $x \in H^2(\mathbb{C}P^2)$  is the preferred generator (i.e. restricts to the preferred generator of  $S^2 \cong \mathbb{C}P^1$  under the restriction map). This can be taken as one of the axioms of Chern classes. It follows from the Cartan formula that:

$$\begin{aligned} c(TCP^n) &= c(TCP^n \oplus 1) = c(\mathcal{O}(1))^{n+1} \\ &= (1+x)^{n+1} = 1 + (n+1)x + \binom{n+1}{2}x^2 + \cdots + \binom{n+1}{n}x^n \end{aligned}$$

So we know the Chern classes. The Pontryagin classes of a complex vector bundle are related to its Chern classes by the formula

$$(1 - p_1 + p_2 - \cdots \pm p_n) = (1 - c_1 + c_2 - \cdots \pm c_n)(1 + c_1 + \cdots + c_n).$$

It follows that the total Pontryagin class  $p(TCP^n)$  is just  $(1 - x^2)^{n+1}$ , so that the Pontryagin numbers are:

$$p_k = \binom{n+1}{k}, \quad 1 \leq k \leq n/2.$$

**Example 3.8.** The Pontryagin classes for the first few projective spaces are:

$$p(TCP^1) = 1, \quad p(TCP^2) = 1 + 3x^2, \quad p(TCP^3) = 1 + 4x^2, \quad p(TCP^4) = 1 + 5x^2 + 10x^4.$$

In particular, the first Pontryagin number of  $\mathbb{C}P^2$  is 3, which proves Thom's theorem.

(1.4) Hirzebruch's strategy for the general case rests on another description of the oriented cobordism ring, also due to Thom:

**Theorem 4.9** (Thom). *The natural map*

$$\mathbb{Q}[[\mathbb{C}P^2], [\mathbb{C}P^4], [\mathbb{C}P^6], \dots] \longrightarrow MSO_* \otimes \mathbb{Q}$$

*is an isomorphism of rings.*

So all Hirzebruch needed to do was define a ring homomorphism on  $MSO_* \otimes \mathbb{Q}$  in terms of Pontryagin numbers and check that it agrees with the signature on the even complex projective spaces.

Let's flesh that out a bit. Hirzebruch wants to define a homomorphism that looks like:

$$M \mapsto \int_M L_n(p_1, \dots, p_n)$$

for some polynomial in the Pontryagin classes  $L_n(p_1, \dots, p_n) \in H^{4n}(M)$ . If this is gonna be a *ring* homomorphism, then we'll need

$$\int_{M^{4k} \times N^{4r}} L_{k+r}(p_1, \dots, p_{k+r}) = \left( \int_M L_k(p'_1, \dots, p'_k) \right) \cdot \left( \int_M L_r(p''_1, \dots, p''_r) \right),$$

where  $p_i, p'_i$ , and  $p''_i$  are the Pontryagin classes of  $M \times N$ ,  $M$ , and  $N$ , respectively. Now, if we had some polynomials  $\{L_n\}$  in the Pontryagin classes where the coefficients didn't depend on  $M$ , then we could evaluate

them on any vector bundle. So this gives us a characteristic class. And how do we compute characteristic classes? We use the splitting principle!

So suppose we have an  $SO(2)$ -bundle  $V$ . Then  $p_i(V) = 0$  for  $i > 1$ , and our ‘total characteristic class’ looks like:

$$L(V) = 1 + L_1(p_1) + L_2(p_1, 0) + L_3(p_1, 0, 0) + \dots$$

In fact, let’s define once and for all the following formal power series:

$$L(u) = 1 + L_1(x^2) + L_2(x^2, 0) + \dots = 1 + \sum_{n \geq 1} L_n(x^2, 0, 0, \dots, 0).$$

where  $|x| = 2$  (which explains the squaring, because  $p_1$  has dimension 4). Then the multiplicative property above amounts to the following: if  $V = V_1 \oplus \dots \oplus V_n$  splits as a sum of line bundles, then

$$L_k(p_1(V), \dots, p_n(V)) = [L(u_1) \cdots L(u_n)]_{4k}.$$

I have to tell you how to interpret this formula. We think of the  $x_i$  as dummy variables of dimension 2, and identify the  $p_i$  with the elementary symmetric polynomials in the  $x_i^2$ . The  $[-]_k$  symbol denotes the operation ‘take the dimension  $k$  part’.

Now remember that  $T\mathbb{C}P^{2n} \oplus \underline{1} = \mathcal{O}(1)^{\oplus(2n+1)}$ . It follows that

$$L_n(T\mathbb{C}P^{2n} \oplus \underline{1}) = \text{coefficient of } x^{2n} \text{ in } L(u)^{2n+1}.$$

On the other hand, if we want the formula

$$\sigma(\mathbb{C}P^{2n}) = \int_{\mathbb{C}P^{2n}} L_n(p_1, \dots, p_n)$$

to hold, we are *forced* to take this coefficient to be 1.

Putting all of this together we see that all Hirzebruch needs to do is find a formal power series  $L(u)$  with the following properties:

- (i) It is even- so  $L(u)$  is a polynomial in  $x^2$ ,
- (ii) The coefficient of  $u^n$  in  $L(u)^{2n+1}$  is 1.

As it so happens: this *uniquely characterizes*  $L(u)$ . He used something called the Lagrange inversion formula to figure out that this formal power series is given by:

$$L(u) = \frac{u}{\tanh(u)} = 1 + \frac{1}{3}u^2 - \frac{1}{45}u^4 + \frac{2}{3^2 \cdot 5 \cdot 7}u^6 - \frac{1}{3^3 \cdot 5^2 \cdot 7}u^8 + \dots$$

Then he defines

$$L_n(p_1, \dots, p_n) := [L(u_1) \cdots L(u_n)]_{4n}$$

where the  $p_i$  are the elementary symmetric functions in the variables  $u_1^2, \dots, u_n^2$ . By construction, we get the following theorem:

**Theorem 4.10** (Hirzebruch). *The signature of a compact, oriented  $4n$ -manifold can be computed as*

$$\sigma(M) = \int_M L_n(p_1, \dots, p_n)$$

where  $L_n$  is the polynomial defined above.

**Example 4.11.**  $L_0 = 1$  by definition, and  $L_1(p_1)$  is  $\frac{1}{3}p_1$ , recovering Thom’s theorem.

**Example 4.12.** For  $L_2(p_1, p_2)$  I'll need to compute a bit of  $L(u_1)L(u_2)$ . It looks like:

$$\begin{aligned} L(u_1)L(u_2) &= 1 + \frac{1}{3}(u_1^2 + u_2^2) + \frac{1}{9}u_1^2u_2^2 + \frac{1}{45}(u_1^4 + u_2^4) + \dots \\ &= 1 + \frac{1}{3}p_1 + \frac{1}{9}p_2 - \frac{1}{45}(p_1^2 - 2p_2) + \dots \\ &= 1 + \frac{1}{3}p_1 + \frac{1}{45}(7p_2 - p_1^2) + \dots \end{aligned}$$

So  $L_2 = \frac{1}{45}(7p_2 - p_1^2)$ .

**Exercise 4.13.** Show that  $L_3 = \frac{1}{3^5 \cdot 5 \cdot 7} (62p_3 - 13p_1p_2 + 2p_1^3)$ .

This is supposed to convince you that you could compute these by hand for a ways and make a computer compute as many as you'd like.

Notice that the expression  $\int_M L_n(p_1, \dots, p_n)$  is some linear combination of the Pontryagin numbers with *rational* coefficients. On the other hand, the signature is clearly an integer. This has many ramifications. Here is a cute application:

**Example 4.14.** There is no oriented 12-manifold  $M$  with

$$H^k(M; \mathbb{Q}) = \begin{cases} \mathbb{Q} & k = 0, 6, 12 \\ 0 & \text{else} \end{cases}.$$

Indeed, by Poincaré duality, the pairing on the middle dimension, 6, has to be non-degenerate, so we necessarily have  $\sigma(M) = 1$ . On the other hand,  $p_1 = p_2 = 0$  because  $H^4 = H^8 = 0$ , so Hirzebruch's theorem implies  $1 = \frac{62}{3^5 \cdot 5 \cdot 7} p_3$ . But  $p_3$  is an integer and 62 does not divide  $3^5 \cdot 5 \cdot 7$ , so it's a no-go.

**Exercise 4.15.** Is there a compact, oriented manifold of dimension 8 with

$$H^k(M; \mathbb{Q}) = \begin{cases} \mathbb{Q} & k = 0, 4, 8 \\ 0 & \text{else} \end{cases} \quad ?$$

**Remark 4.16.** The Hirzebruch signature formula puts constraints on the possible homotopy types of manifolds. In fact, the Hirzebruch signature formula together with Poincaré duality essentially characterizes the possible homotopy types of  $4k$ -dimensional manifolds.

More explicitly: If  $X$  is a simply-connected, finite-dimensional  $CW$ -complex of dimension  $4n$  with  $n > 1$  equipped with an oriented vector bundle  $E$  and a fundamental class implementing Poincaré duality, then there exists a homotopy equivalence  $f : M \rightarrow X$  where  $M$  is a compact, oriented manifold with  $TM$  stably equivalent to  $f^*E$  if and only if  $X$  satisfies the conclusion of the Hirzebruch signature formula. This is a theorem of Browder.

**Remark 4.17.** Another application of integrality is due to Milnor, who used the Hirzebruch signature formula to define and study a diffeomorphism invariant of 7-manifolds. He used this to prove that there exist exotic 7-spheres. That was a surprise to pretty much everyone.

(1.5) The method of Hirzebruch applies in much greater generality. He came up with a recipe to find formulas for any sort of cobordism invariant. First, here's a fancy name for 'cobordism invariant':

**Definition 5.18.** A *genus* is any ring homomorphism

$$\Omega_* \otimes \mathbb{Q} \rightarrow R$$

where  $R$  is a  $\mathbb{Q}$ -algebra and  $\Omega_*$  are cobordism groups of manifolds with some extra structure on their stable normal bundle. Usually we take  $\Omega_* = MSO_*$  or  $\Omega_* = MU_*$ .

We have already seen that  $MSO_* \otimes \mathbb{Q} \cong \mathbb{Q}[[\mathbb{C}P^{2n}] | n \geq 1]$ . Recall that  $MU_*$  is the cobordism theory of manifolds with a complex structure on their stable normal bundle. These are called *stably almost complex* manifolds. Any complex manifold is an example of such a thing. The other calculation we'll need is the following:

**Theorem 5.19** (Milnor). *The natural map*

$$\mathbb{Q}[[\mathbb{C}P^n] | n \geq 1] \longrightarrow MU_* \otimes \mathbb{Q}$$

*is a ring isomorphism.*

**Corollary 5.20.** *The data of a genus  $\phi$  for oriented manifolds is equivalent to the sequence of numbers  $\{\phi(\mathbb{C}P^{2n})\}$ . The data of a genus  $\phi$  for stably almost complex manifolds is equivalent to the sequence of numbers  $\{\phi(\mathbb{C}P^n)\}$ .*

That's all well and good, but how do you actually *compute* the value of the genus on your favorite manifold? We want a formula in terms of Chern or Pontryagin numbers.

Hirzebruch proved, using the same methods described above (namely the Lagrange inversion formula) that one gets such a formula by doing the following.

First, record the sequence of numbers<sup>2</sup>  $\{\phi(\mathbb{C}P^n)\}$  in a power series, called the *logarithm of the genus*:

$$\log_\phi(x) = \sum_{n \geq 0} \phi(\mathbb{C}P^n) \frac{x^{n+1}}{n+1} = x + \phi(\mathbb{C}P^1) \frac{x^2}{2} + \dots$$

Then Hirzebruch defines the associated *characteristic series* by:

$$K_\phi(u) := \frac{u}{\log_\phi^{-1}(u)}.$$

From here he gets the associated *multiplicative sequence* by the formula:

$$K_{\phi,n}(c_1, c_2, \dots, c_n) = [K_\phi(u_1) \cdots K_\phi(u_n)]_{2n}.$$

If  $K_\phi(u)$  is *even*, i.e. is a power series in  $u^2$ , then he instead defines:

$$K'_{\phi,n}(p_1, \dots, p_n) = [K_\phi(u_1) \cdots K_\phi(u_n)]_{4n}.$$

Here the  $c_i$  are the elementary symmetric polynomials in the  $u_k$ , and the  $p_i$  are the elementary symmetric polynomials in the  $u_k^2$ . The theorem is then:

**Theorem 5.21** (Hirzebruch).

$$\begin{aligned} \phi(M) &= \int_M K_{\phi,n}(c_1, \dots, c_n). \\ \phi'(N) &= \in \text{Td}_n K'_{\phi,n}(p_1, \dots, p_n). \end{aligned}$$

where  $\phi$  (resp.  $\phi'$ ) is a genus for stably almost complex (resp. oriented) manifolds.

Now, if you start with some random sequence of numbers  $\phi(\mathbb{C}P^n)$ , this probably won't be an interesting theorem. But if  $\phi$  was a priori defined using some non-topological data, this gives you a topological way of computing this invariant.

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<sup>2</sup>We can treat the complex and oriented case simultaneously, since  $\phi(\mathbb{C}P^{2n+1}) = 0$  when  $\phi$  only depends on the oriented cobordism class.

**Remark 5.22.** In practice, you get theorems that look like:

$$\text{global analytic invariant} = \int_M \text{local topological invariant}.$$

A theorem like this is great for analysts and topologists. The left-hand side is usually hard to compute, but satisfies some obvious rigidity (like being an integer). The right-hand side is usually easy to compute for your favorite manifold, but not obviously rigid. A theorem like this is called an *index theorem*, and the über version was proved by Atiyah-Singer. Their original proof mimicked Hirzebruch's proof of the Riemann-Roch theorem, which we discuss below.

## 2 Grothendieck, Atiyah-Hirzebruch and $K$ -theory

(2.6) So that's the story of Hirzebruch and genera, but I was supposed to tell you about what all this has to do with  $K$ -theory. To understand that, I have to tell you why Hirzebruch was so interested in the signature formula in the first place.

You see, Hirzebruch was actually interested in *analytic varieties*. An analytic variety is like a smooth manifold except that you can make sense of *complex analytic functions* and not just smooth functions. (An analytic function is a function that locally looks like a convergent power series in many variables.)

Analytic varieties were and still are incredibly important mathematical objects with lots of beautiful structure. They were one of the motivations for the birth of topology. All the early pioneers of topology and algebraic topology- Riemann, Poincaré, and Lefschetz- were really interested in proving things about varieties.

In fact, our story starts with Riemann. He was really interested in *analytic* differential forms on complex surfaces (nowadays called Riemann surfaces). One of the upshots of working with analytic differential forms is that the vector space  $H^0(\Omega^1)$  of analytic differential forms is finite dimensional. Riemann asked: how do you compute its dimension? He gave half an answer, and his student Roch finished it off.

**Theorem 6.1** (Riemann-Roch). <sup>3</sup> *Let  $X$  be a compact, complex manifold of real dimension 2. Then  $X$  is, in particular, an orientable compact surface of some genus,  $g$ , and*

$$\dim H^0(\Omega^1) = g.$$

This is a pretty great theorem. The left-hand side makes heavy use of the complex structure- and the right hand side only depends on the *homotopy type* of  $X$ . Algebraic geometers got a lot of mileage out of this theorem, and, naturally, started to wonder what the story was for higher dimensional varieties.

Experience revealed that the correct generalization of the left hand side is:

$$\chi_{\text{an}}(X) = \sum (-1)^k \dim_{\mathbb{C}} H^0(\Omega^k).$$

Here  $H^0(\Omega^k)$  denotes the vector space of analytic  $k$ -forms, that is, analytic sections of the complex vector bundle  $\Lambda^k(T^*X)$  over  $X$ . This is sometimes called the *arithmetic genus* of  $X$ - it's a kind of analytic Euler characteristic. So the question was: what should the right-hand side say?

Todd, in the 30s, used reasoning very similar to Hirzebruch's to guess an answer in certain cases. Hirzebruch sharpened this reasoning as follows. He knew that

$$\chi_{\text{an}}(\mathbb{C}P^n) = 1.$$

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<sup>3</sup>This is actually just a special case of the Riemann-Roch theorem, but to state the more general version would take us too far afield.

Now we close our eyes and pretend that  $\chi_{\text{an}}$  is a cobordism invariant for stably almost complex manifolds (even though that makes no sense because there's no such thing as 'analytic differential forms' in that generality). Following Hirzebruch's recipe, we are led to form the logarithm:

$$\log_{\text{Td}}(x) = \sum_{n \geq 0} \frac{x^{n+1}}{n+1}.$$

This looks a lot like  $\log(1+x) = \sum (-1)^n \frac{x^{n+1}}{n+1}$ . In fact:

$$\log_{\text{Td}}(x) = -\log(1-x).$$

So:

$$\log_{\text{Td}}^{-1}(u) = 1 - e^{-u}, \quad K_{\text{Td}}(u) = \frac{u}{1 - e^{-u}}.$$

This gives a multiplicative sequence, and polynomials  $\{\text{Td}_n\}$  called the *Todd polynomials* and leads to a conjecture that Hirzebruch proves:

**Theorem 6.2** (Hirzebruch). *If  $X$  is a compact analytic variety, then*

$$\chi_{\text{an}}(X) = \int_X \text{Td}_n(c_1, \dots, c_n).$$

This is often called the *Hirzebruch-Riemann-Roch* theorem. Note that it does *not* follow from the previous theorem of Hirzebruch because the left-hand side is *not* a genus! On the other hand, it is a consequence of the theorem that  $\chi_{\text{hol}}$  extends to a genus on stably almost complex manifolds. This is called the *Todd genus*.

This raises a natural question:

**Question 6.3.** Is there a nice description of the left-hand side in the Hirzebruch-Riemann-Roch formula that makes sense for all stably almost complex manifolds?

If there were such a description, we would expect it to be integer-valued. So a preliminary question becomes:

**Question 6.4.** Is the Todd genus integer valued on stably almost complex manifolds?

And finally:

**Question 6.5.** Is there an analogous theorem for just plain-old smooth manifolds?

The answer to all of these questions turns out to be yes, but they weren't answered until after  $K$ -theory was invented. So, at long last, it's time to talk about  $K$ -theory.

**(2.7)** There are many things one can say about Grothendieck, but here is one: he loved to replace theorems about *objects* with theorems about *morphisms*. The original theorem about  $X$  would be the special case of the new theorem for the morphism  $X \rightarrow *$ .<sup>4</sup>

This didn't just lead to more powerful theorems, it often led to simpler proofs. If all you care about is islands, you'll be stuck on the one you're on; but if you start thinking about *bridges*, you can really go places.

His first great success with this technique was his generalization of the Riemann-Roch theorem. Recall that the shape of this theorem is:

$$\sum (-1)^k \dim H^0(\Omega_X^k) = \int_X \text{Td}_n(c_1, \dots, c_n).$$

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<sup>4</sup>It's said that when he was at his desk, he would repeatedly draw a vertical arrow  $X \rightarrow S$  while thinking.



Grothendieck noticed that the right hand side has a generalization for a map  $X \rightarrow S$ . For example, if  $X \rightarrow S$  is a fiber bundle, you can integrate a cohomology class along the fibers and get a class in  $H^*(S)$ . A quick way to formalize this that works in general is to use Poincaré duality.

Given a map  $X \rightarrow S$  of compact, oriented manifolds and a class  $x \in H^*(X)$ , define

$$f_{\#}(x) := Df_*(Dx) \in H^{*-d}(S),$$

where  $D$  denotes Poincaré duality and  $f_*$  is the usual pushforward in homology. The number  $d$  is the relative dimension  $\dim(X) - \dim(S)$  and comes from the dimension shifts in Poincaré duality.

So then Grothendieck needed to know: what's the generalization of the left hand side,

$$\chi_{\text{hol}}(X) = \sum (-1)^k \dim H^0(\Omega_X^k) ?$$

Well, the natural replacement of a vector space was a vector bundle. So he wanted something like an alternating sum of *vector bundles*, and then he needed a generalization of dimension that landed in cohomology instead of the integers. He solved the first problem using a standard mathematical trick: if you can't do something, declare that you can.

**Definition 7.6.** Given a commutative monoid,  $(M, \star)$ , define its *group-completion* of *Grothendieck group*,  $\text{gp}(M)$ , to be the free abelian group on the symbols  $[m]$  where  $m \in M$  modulo the relation  $[m \star n] = [m] + [n]$ .

Given a compact, analytic variety  $X$ , define<sup>5</sup>

$K_{\text{an}}(X)$  to be  $\text{gp}(\text{Vect}^{\text{an}}(X))$ , the Grothendieck group of (isomorphism classes of) complex analytic vector bundles on  $X$  under direct sum.

**Example 7.7.** The group completion of  $\mathbb{N}$  is  $\mathbb{Z}$ .

Grothendieck then used some serious algebraic geometry to define a pushforward for  $K$ -theory. That is, for any map  $f : X \rightarrow S$  between compact analytic varieties, he produced a homomorphism  $f_! : K_{\text{an}}(X) \rightarrow K_{\text{an}}(S)$ .

The correct generalization of dimension of a vector space is called the *Chern character*, which I don't want to go into. It's a ring map  $\text{ch} : K_{\text{an}}(X) \rightarrow H^*(X, \mathbb{Q})$ . In a letter to Serre in 1956, Grothendieck announced his now famous result.

**Theorem 7.8** (Grothendieck). *For any analytic vector bundle  $E$  on a compact<sup>6</sup> analytic variety  $X$ , and any map  $f : X \rightarrow S$ , we have:*

$$\text{ch}(f_!(E)) = f_{\#}(\text{ch}(E) \cdot \text{Td}_n(E)).$$

Hirzebruch's theorem is the special case when  $E$  is the trivial bundle of rank 1 and  $S$  is a point.

The nice thing about Grothendieck's version of the theorem is that, even if you were only interested in Hirzebruch's result, you get a simpler proof. Let  $X$  be a compact, analytic variety. Then you can embed  $X$  into some projective space  $\mathbb{C}P^n$  by a closed immersion<sup>7</sup>, so you get a diagram:

$$\begin{array}{ccc} X & \xrightarrow{i} & \mathbb{C}P^n \\ & \searrow \varepsilon & \downarrow p \\ & & * \end{array}$$

We were interested in  $\varepsilon_!(\mathbb{1}) = \chi_{\text{hol}}(X)$ . But, by functoriality this is the same as  $p_!(i_!(\mathbb{1}))$ . It turns out the theorem for projective space is really easy, so we are reduced to studying  $i_!$ , and that turns out to not be so bad.

<sup>5</sup>This isn't quite right. I need to ask that, for short exact sequences  $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ , we add the relation  $[E'] + [E''] = [E]$ . This relation is redundant in the topological case because every such exact sequence splits, topologically.

<sup>6</sup>I should probably say projective.

<sup>7</sup>Again, that's not really true. You need  $X$  to be *projective*, but this is an acceptable lie for now.

I realize this all seems pretty complicated and I've glossed over a bunch of things, but I needed to tell you what motivated the next chapter of the story.

**(2.8)** You can safely forget about analytic varieties and Chern characters and just remember that Grothendieck showed that it was a useful idea to take the group completion of the monoid of vector bundles and study pushforward maps. After Grothendieck publicly announced his result in 1957, Atiyah and Hirzebruch, motivated by the questions I listed previously, started to wonder if there was an analog of Grothendieck's results for smooth manifolds. In particular, they wanted to build an analog of  $K$ -theory and pushforwards in topology.

Before I tell you how that goes, let me give you another motivation for these weird pushforward maps. Having nice pushforward maps for the map to a *point* is essentially the same as building a genus with an integrality theorem that comes for free.

Indeed, suppose you have some homotopy invariant functor  $h : \mathbf{Spaces}^{op} \rightarrow \mathbf{Rings}$  equipped with pushforward maps  $\epsilon_1^X : h(X) \rightarrow h(*)$  whenever  $X$  has, say, an almost complex structure on its stable normal bundle. Then you could define a genus by:

$$X \mapsto \epsilon_1^X(1) \in h(*) \otimes \mathbb{Q}$$

and, by construction, it automatically lands in the image of  $h(*) \rightarrow h(*) \otimes \mathbb{Q}$ . This is also evidence of the usefulness of replacing the ring,  $R$ , that appears on the right hand side of a genus, by something like a *cohomology theory*. That is the modern point of view and people have had a lot of success with it.

Anyway- back to Atiyah and Hirzebruch. Their first task was straightforward: define  $K$ -theory in topology.

**Definition 8.9.** If  $X$  is a compact space, let  $K(X)$  (resp.  $KO(X)$ ) denote the group completion of the monoid  $\text{Vect}_{\mathbb{C}}(X)$  (resp.  $\text{Vect}_{\mathbb{R}}(X)$ ) of complex (resp. real) vector bundles under direct sum.

**Example 8.10.**  $K(*) = \mathbb{Z}$ .

**Example 8.11.**  $K(S^1) = \mathbb{Z}$ . Indeed, any complex vector bundle on  $S^1$  is trivial on the upper and lower half of the circle, and so determined by the transition function which is a map  $\{\pm 1\} \rightarrow GL_n(\mathbb{C})$ . Since  $GL_n(\mathbb{C})$  is connected, we can always deform this transition function to the trivial one. So  $\text{Vect}_{\mathbb{C}}(S^1) = 0$  and hence so is the group completion.

**Exercise 8.12.** Show that  $KO(S^1) = \mathbb{Z} \oplus \mathbb{Z}/2$  generated by the Möbius bundle and the trivial bundle.

This defines a contravariant functor on (compact) spaces which turns out to be homotopy invariant. When  $X$  is pointed we define  $\tilde{K}(X) := \ker(K(X) \rightarrow K(x_0))$  and similarly for  $\tilde{KO}(X)$ .

**Example 8.13.** The above examples show that  $\tilde{K}(S^1) = 0$  and  $\tilde{KO}(S^1) = \mathbb{Z}/2$ .

That was easy. Now we need to define the pushforward. Grothendieck's method didn't work in this case (it used facts about analytic varieties that have no obvious analogs for arbitrary smooth manifolds.) So Atiyah and Hirzebruch had to do something different. The pushforward in cohomology we defined above uses Poincaré duality, which also doesn't seem to have an analog for this  $K$ -theory (for example: what would replace homology in that formulation?) So, to motivate what Atiyah and Hirzebruch did, let me tell you how to define the pushforward in ordinary cohomology without using Poincaré duality.

**(2.9)** We're searching for a wrong-way map and we've already seen one in nature: the Pontryagin-Thom collapse map. Given an embedding of compact manifolds  $X \hookrightarrow Y$  we always get a collapse map going the other way:

$$Y \rightarrow X^\nu$$

where  $\nu$  is the normal bundle and  $X^\nu$  denotes the Thom space. For example, we can take  $X$  to be arbitrary and  $Y$  to be a high-dimensional sphere. Then the normal bundle represents the *stable normal bundle*, and we get a map:

$$S^N \rightarrow X^\nu.$$

This gives an element in  $\pi_N(X^\nu)$ . The Hurewicz map produces an element in  $H_N(X^\nu)$  and a choice of orientation of  $\nu$  gives a Thom isomorphism and hence an element in  $H_n(X)$ , where  $n = \dim(X)$ . What element could this be? It's the fundamental class! The same one that induces Poincaré duality.

So the Thom isomorphism knows about Poincaré duality. Maybe we can use it to build the pushforward. Start with a map

$$f : X \longrightarrow S$$

between compact, oriented manifolds. Choose an embedding  $X \hookrightarrow S^N$  so we get a factorization:

$$\begin{array}{ccc} X & \xrightarrow{i} & S^N \times S \\ & \searrow f & \downarrow p \\ & & S \end{array}$$

Now let  $n$  denote the dimension of  $X$ ,  $m$  denote the dimension of  $S$ ,  $\nu$  denote the normal bundle to  $i$  with a chosen orientation, and  $U_\nu \in \tilde{H}^{N+m-n}(X^\nu)$  the resulting Thom class.

**Definition 9.14.** In the above situation, the pushforward in cohomology is defined as the composite:

$$H^*(X) \xrightarrow{\cdot U_\nu} \tilde{H}^{*+N+m-n}(X^\nu) \xrightarrow{\text{collapse}} H^{*+N+m-n}(S^N \times S) \xleftarrow{\cong} H^*(S^N) \otimes H^{*+m-n}(S) \longrightarrow H^{*+m-n}(S)$$

where the final map caps with the fundamental class of  $S^N$ .

**Proposition 9.15.** *This doesn't depend on the choice of embedding and agrees with the other definition of the pushforward using Poincaré duality.*

This isn't so hard to prove. You can consider the case of an embedding and the projection  $S^N \times S \longrightarrow S$  separately, and it's just a matter of unwinding the definitions and using this fact about the relationship between the Thom isomorphism and the fundamental class.

**(2.10)** What's so great about this reinterpretation of the pushforward is that it makes sense for *any* functor  $h : \mathbf{Spaces} \longrightarrow \mathbf{Ab}$  as soon as you have a notion of Thom class and a Künneth isomorphism. In fact, you don't need the general Künneth isomorphism, you just need to know that the natural map

$$h(S^m) \otimes h(X) \longrightarrow h(S^m \times X)$$

is an isomorphism. Atiyah and Hirzebruch figured out how to do the Thom isomorphism, but they still needed a Künneth formula. Luckily, the missing ingredient had been supplied the same year that Grothendieck announced his Riemann-Roch theorem.

**(2.11)** As the apocryphal story goes, Bott and Milnor were hanging out by a chalkboard computing the first couple homotopy groups of  $U(n)$  for  $n \gg 0$ . Here's what they got:

$$\pi_0 U(n) = 0, \pi_1 U(n) = \mathbb{Z}, \pi_2 U(n) = 0, \pi_3 U(n) = \mathbb{Z}, \pi_4 U(n) = 0, \dots$$

and Bott was like "Hey, maybe it keeps going like that," and Milnor was like "Yeah, I'm not into idle speculation."<sup>8</sup>

So then Bott proved one of the most surprising results in mathematics, that has no right being true:

<sup>8</sup>This is probably not the real story. Bott himself, in his paper, says the result was inspired by the calculations of Toda... who actually made a mistake that contradicted the periodicity theorem. Bott noticed that Toda's calculation conflicted with recent work of Borel and Hirzebruch, guessed the mistake, and proved his theorem.

**Theorem 11.16** (Bott). *Let  $U = \bigcup U(n)$ . Then*

$$\pi_k U = \begin{cases} 0 & k \text{ even} \\ \mathbb{Z} & k \text{ odd.} \end{cases}$$

*Even better, there is a canonical homotopy equivalence  $\Omega U \cong BU \times \mathbb{Z}$  where  $BU = \bigcup BU(n)$  is the union of the infinite complex Grassmanians.*

What does this have to do with Atiyah and Hirzebruch's problem? Well, it turns out that the equivalence

$$[X, BU(n)] \cong \text{Vect}_{\mathbb{C},n}(X)$$

can be jazzed up to an equivalence:

$$[X, BU \times \mathbb{Z}]_* \cong \tilde{K}(X).$$

Now, a little basepoint and cofiber trickery together with the calculation  $K(S^2) = \mathbb{Z} \oplus \mathbb{Z}$  tells you that  $K(X \times S^2) \cong K(S^2) \otimes \tilde{K}(X_+ \wedge S^2)$ . So our Künneth formula follows from the calculation:

$$\begin{aligned} \tilde{K}(X_+ \wedge S^2) &\cong [X_+ \wedge S^2, BU \times \mathbb{Z}]_* \\ &\cong [X_+, \Omega^2(BU \times \mathbb{Z})]_* \\ &\cong [X_+, BU \times \mathbb{Z}]_* \\ &\cong K(X). \end{aligned}$$

(If you missed that, don't worry- it will be covered in the next lecture.)

This generalizes to even dimensional spheres, and that was enough for Atiyah and Hirzebruch to prove their differentiable Riemann-Roch formula. They got integrality theorems for the Todd genus, and even extended their arguments to the case of manifolds that had some extra structure on their stable normal bundle that was much weaker than a complex structure. This turned out to be a foreshadowing of Atiyah-Bott-Shapiro's work on Clifford algebras and spin structures and the  $\hat{A}$ -genus, which motivated the Atiyah-Singer index theorem and eventually led to the creation of such modern gadgets as 'topological modular forms' and the 'Witten genus.'

Everyone lived happily ever after.

## References

- [1] Atiyah, M., Bott, R., and Shapiro, A. *Clifford modules*. 1963.
- [2] Atiyah, M. and Hirzebruch, F. *Riemann-Roch theorems for differentiable manifolds*. 1959.
- [3] Atiyah, M. and Singer, M. *The index of elliptic operators on compact manifolds*. 1963.
- [4] Bott, R. *The stable homotopy of the classical groups*. 1957.
- [5] Borel, A. and Hirzebruch, F. *Characteristic classes and homogeneous spaces, I, II, III*. 1958.
- [6] Borel, A. and Serre, J.P. *Le théorème de Riemann-Roch*. 1958.
- [7] Hirzebruch, F. *On Steenrod's reduced powers, the index of inertia and the Todd genus*. 1953.
- [8] Hirzebruch, F. *On arithmetic genera of algebraic varieties*. 1954.
- [9] Hirzebruch, F. *Topological methods in algebraic geometry*. 1978.
- [10] Thom, R. *Espaces fibrés en sphères et carrés de Steenrod*. 1952.
- [11] Thom, R. *Variétés différentiables cobordantes*. 1953.
- [12] Thom, R. *Quelques propriétés globales des variétés différentiables*. 1954.