ACTA ARITHMETICA
XV (1969)

Sums involving Farey fractions

by

J. Lehner and M. Newman (College Park, Md.)

1. Let F_n be the Farey series of order $n, n \geqslant 1$. Then F_n consists of all fractions h/k arranged in ascending order, where $0 \leqslant h \leqslant k \leqslant n$, $(h, k) = 1$. For example, the Farey series of order 5 is

$$\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}.$$

Let h/k be a term of F_n, $h/k < 1$. Then the term that follows h/k in F_n will be denoted by h'/k'. The set of pairs (k, k'), where h/k runs over all terms of F_n less than 1, will be denoted by Q_n. Thus Q_5 consists of the pairs

$$(1,5),\ (5,4),\ (4,3),\ (3,5),\ (5,2),\ (2,5),\ (5,3),\ (3,4),\ (4,5),\ (5,1).$$

H. Gupta has proved in his paper [1] that

$$(1) \qquad \sum_{r=1}^{\infty} \sum_{\substack{1 \leqslant k \leqslant r \\ (k,r)=1}} \frac{1}{r^2(r+k)} = \frac{3}{4}.$$

This sum is connected with the Farey fractions. In fact we show in what follows that

$$(2) \qquad \frac{1}{2} \sum_{(k,k') \in Q_n} \frac{1}{kk'(k+k')} = \frac{3}{4} - \sum_{r=1}^{n} \sum_{\substack{1 \leqslant k \leqslant r \\ (k,r)=1}} \frac{1}{r^2(r+k)}.$$

This identity reduces a sum involving Farey fractions to one which does not. More generally we show in Section 2 that for an arbitrary function f,

$$(3) \qquad \sum_{(k,k') \in Q_n} f(k, k') = f(1,1) + \sum_{r=2}^{n} \sum_{\substack{1 \leqslant k \leqslant r \\ (k,r)=1}} \{f(k, r) + f(r, k) - f(k, r-k)\}.$$

In Section 3 we discuss the asymptotic behaviour of the left member of (3) as $n \to \infty$, for certain functions f. We show for example that the sum in (2) is asymptotic to c/n, where $c = 6\log 2/\pi^2$; and that it is asymptotic to $c_{\alpha,\beta}n^{\alpha+\beta+2}$ for $f = k^{\alpha}k'^{\beta}$, where α, β are non-negative and $c_{\alpha,\beta}$ is a certain constant which is determined precisely. From the latter result we can determine bounds for the sum when f is any polynomial in k, k'; and even its exact asymptotic behaviour, in many cases. Finally, some additional results of a similar nature are quoted without proof.

We remark parenthetically that the sum in (2) admits an interesting interpretation. Let $\|x\| = \min(\{x\}, 1-\{x\})$ represent the distance from x to its nearest integer, and consider the n functions

$$y_r = \|rx\|, \quad 1 \leqslant r \leqslant n.$$

Then these functions vanish in $[0, 1]$ if and only if x belongs to F_n; and if $h/k, h'/k'$ are consecutive terms of F_n, the triangle whose sides are the lines $y = 0, y = kx-h$, and $y = -k'x+h'$ has area $1/2kk'(k+k')$. Thus the sum in (2) represents the total area of all such triangles. The configuration is of interest in questions of diophantine approximation.

2. Assume from now on that $n > 1$. It is known that $h/k, h'/k'$ are consecutive terms of F_n if and only if

$$0 \leqslant h \leqslant k \leqslant n, \quad 0 \leqslant h' \leqslant k' \leqslant n, \quad h/k < h'/k', \quad h'k - hk' = 1.$$

A further result is that $k+k' > n$, so that

$$n+1 \leqslant k+k' \leqslant 2n-1.$$

(For these and other results on Farey fractions, see [2].)

Thus the conditions

$$0 < a, b \leqslant n, \quad (a, b) = 1, \quad n+1 \leqslant a+b \leqslant 2n-1, \tag{4}$$

are certainly necessary for (a, b) to belong to Q_n. But they are also sufficient. Indeed, we can find integers c, d such that

$$0 \leqslant c \leqslant a-1, \quad 1 \leqslant d \leqslant b, \quad ad-bc = 1.$$

This easily implies that $c/a, d/b$ are consecutive terms of F_n, so that $(a, b) \in Q_n$.

Thus we have proved

LEMMA 1. *Q_n consists of all pairs (a, b) which satisfy* (4).

This lemma leads directly to the proof of the transformation equation (3). Define

$$S_n = \sum_{(k,k')\in Q_n} f(k, k'), \tag{5}$$

where f is an arbitrary function. Then we have

THEOREM 1. *Let $n > 1$. Then*

$$S_n = f(1, 1) + \sum_{r=2}^{n} \sum_{\substack{1\leqslant k\leqslant r\\(k,r)=1}} \{f(k, r)+f(r, k)-f(k, r-k)\}.$$

Proof. Observe that Q_r consists of the coprime pairs (x, y) lying in or on the triangle bounded by the lines $x+y = r+1$, $x = r$, $y = r$. Hence the points of Q_r not in Q_{r-1} are the coprime pairs (x, y) on the line segments $x = r$, $0 \leqslant y \leqslant r$; $y = r$, $0 \leqslant x \leqslant r$; and the points of Q_{r-1} not in Q_r are the coprime pairs (x, y) on the line segment $x+y = r$, $0 \leqslant x$, $y \leqslant r$. This implies that

$$S_r - S_{r-1} = \sum_{\substack{1\leqslant k\leqslant r\\(k,r)=1}} \{f(k, r)+f(r, k)-f(k, r-k)\}. \tag{6}$$

The result now follows by summation on r.

Another result of this kind (but not as useful as the previous one) is

$$S_n = \sum_{r=1}^{n} \sum_{\substack{n-r+1\leqslant k\leqslant n\\(k,r)=1}} f(r, k). \tag{7}$$

3. For our first application of Theorem 1 we turn to the problem treated by Gupta, and choose $f(k, k') = 1/kk'(k+k')$. Since

$$\sum_{(k,k')\in Q_n} \frac{1}{kk'} = \sum_{(k,k')\in Q_n} \left(\frac{h'}{k'} - \frac{h}{k}\right) = 1,$$

and $n+1 \leqslant k+k' \leqslant 2n-1$, it follows that

$$\frac{1}{2n-1} \leqslant S_n \leqslant \frac{1}{n+1},$$

where

$$S_n = \sum_{(k,k')\in Q_n} \frac{1}{kk'(k+k')}. \tag{8}$$

Thus $S_n = O(1/n)$. We shall prove

THEOREM 2. *Suppose that S_n is defined by* (8). *Then*

$$\frac{n}{2} S_n = \frac{6\log 2}{\pi^2} + O\left(\frac{\log n}{n}\right). \tag{9}$$

Consequently,

$$\lim_{n\to\infty} \frac{n}{2} S_n = \frac{6\log 2}{\pi^2}. \tag{10}$$

Proof. From (6) we obtain

$$S_r - S_{r-1} = \sum_{\substack{1\leqslant k\leqslant r\\(k,r)=1}} \left\{\frac{1}{kr(k+r)} + \frac{1}{rk(r+k)} - \frac{1}{k(r-k)r}\right\}$$

$$= \sum_{\substack{1\leqslant k\leqslant r\\(k,r)=1}} \frac{1}{r^2}\left\{\frac{1}{k} - \frac{2}{r+k} - \frac{1}{r-k}\right\} = -2\sum_{\substack{1\leqslant k\leqslant r\\(k,r)=1}} \frac{1}{r^2(r+k)}.$$

Since $f(1,1)=\frac{1}{2}$, Theorem 1 implies that

$$\frac{1}{2}S_n = \frac{3}{4} - \sum_{r=1}^{n}\sum_{\substack{1\leqslant k\leqslant r\\(k,r)=1}} \frac{1}{r^2(r+k)}.$$

Since $S_n = O(1/n)$, this yields Gupta's result (1) when $n\to\infty$. It follows that

$$\tfrac{1}{2}S_n = \sum_{r>n} A_r,$$

where

$$A_r = \sum_{\substack{1\leqslant k\leqslant r\\(k,r)=1}} \frac{1}{r^2(r+k)}.$$

We now transform A_r. Since

$$\sum_{d\mid(k,r)} \mu(d) = \begin{cases}1, & (k,r)=1,\\ 0, & \text{otherwise},\end{cases}$$

we have

$$A_r = \sum_{k=1}^{r}\sum_{d\mid(r,k)} \frac{\mu(d)}{r^2(r+k)}.$$

Replacing k by hd, we find that

$$A_r = \sum_{d\mid r}\sum_{1\leqslant h\leqslant r/d} \frac{\mu(d)}{r^2(r+hd)};$$

and replacing d by r/d, we find that

$$A_r = \sum_{d\mid r}\sum_{1\leqslant h\leqslant d} \frac{d\mu(r/d)}{r^3(h+d)}.$$

Now

$$\sum_{h=1}^{d}\frac{1}{h+d} = \log 2 + O\left(\frac{1}{d}\right).$$

Hence

$$A_r = \sum_{d\mid r}\frac{d\mu(r/d)}{r^3}\left\{\log 2 + O\left(\frac{1}{d}\right)\right\} = \log 2\cdot\frac{\varphi(r)}{r^3} + O\left(\frac{1}{r^3}\sum_{d\mid r}|\mu(d)|\right).$$

Now

$$\sum_{r=1}^{n}\sum_{d\mid r}|\mu(d)| = \sum_{r=1}^{n}\sum_{s=1}^{\infty}|\mu(r)|\left\{\left[\frac{r}{s}\right]-\left[\frac{r-1}{s}\right]\right\} \tag{11}$$

$$= \sum_{r=1}^{n}|\mu(r)|\left[\frac{n}{r}\right] = O(n\log n),$$

and

$$\sum_{r=1}^{n}\varphi(r) = \frac{3n^2}{\pi^2} + O(n\log n). \tag{12}$$

(See [2], for example, for formula (12).)

Using these formulas it follows by partial summation that

$$\sum_{r>n}\frac{1}{r^3}\sum_{d\mid r}|\mu(d)| = O\left(\frac{\log n}{n^2}\right),$$

and that

$$\sum_{r>n}\frac{\varphi(r)}{r^3} = \frac{6}{n\pi^2} + O\left(\frac{\log n}{n^2}\right).$$

These now yield (9), and (10) is an immediate consequence of (9). This completes the proof of the theorem.

Before going on to the second application of Theorem 1, we state two other lemmas.

LEMMA 2. *Suppose that $a\geqslant 0$. Then*

$$\sum_{r=1}^{n} r^a\varphi(r) = \frac{6}{\pi^2}\cdot\frac{n^{a+2}}{a+2} + O(n^{a+1}\log n). \tag{13}$$

Proof. Lemma 2 follows directly from (12) by partial summation.

LEMMA 3. *Suppose that $a, b\geqslant 0$. Then*

$$\sum_{r=1}^{n} r^a(n-r)^b = \frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(2+a+b)}\, n^{a+b+1} + O(n^{a+b}). \tag{14}$$

Proof. Lemma 3 follows by comparing the sum with

$$\int_0^n r^a(n-r)^b\,dr = \frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(2+a+b)}\, n^{a+b+1},$$

via the Euler-Maclaurin summation formula.

The restriction that $a, b \geqslant 0$ in these lemmas can of course be weakened.

We now go on to our second application of Theorem 1. We choose $f(k, k') = k^\alpha k'^\beta$, where $\alpha, \beta \geqslant 0$. Then (6) becomes

(15) $$S_r - S_{r-1} = \sum_{\substack{1\leqslant k\leqslant r\\ (k,r)=1}} \{k^\alpha r^\beta + r^\alpha k^\beta - k^\alpha (r-k)^\beta\},$$

where now

(16) $$S_n = \sum_{(k,k')\in S_n} k^\alpha k'^\beta .$$

We find as before that

$$\sum_{\substack{1\leqslant k\leqslant r\\ (k,r)=1}} k^\alpha = \frac{r^\alpha \varphi(r)}{\alpha+1} + r^\alpha O\Big(\sum_{d|r} |\mu(d)|\Big),$$

and

$$\sum_{\substack{1\leqslant k\leqslant r\\ (k,r)=1}} k^\beta = \frac{r^\beta \varphi(r)}{\beta+1} + r^\beta O\Big(\sum_{d|r} |\mu(d)|\Big).$$

Also, Lemma 3 implies that

$$\sum_{\substack{1\leqslant k< r\\ (k,r)=1}} k^\alpha (r-k)^\beta = \sum_{1\leqslant k\leqslant r}\ \sum_{d|(k,r)} k^\alpha (r-k)^\beta \mu(d) = \sum_{d|r}\ \sum_{1\leqslant k\leqslant r/d} d^{\alpha+\beta}\mu(d) k^\alpha \Big(\frac{r}{d}-k\Big)^\beta$$

$$= \sum_{d|r} d^{\alpha+\beta}\mu(d) \left\{\frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(2+\alpha+\beta)} \Big(\frac{r}{d}\Big)^{\alpha+\beta+1} + O\bigg(\Big(\frac{r}{d}\Big)^{\alpha+\beta}\bigg)\right\}$$

$$= \frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(2+\alpha+\beta)} r^{\alpha+\beta}\varphi(r) + r^{\alpha+\beta} O\Big(\sum_{d|r} |\mu(d)|\Big).$$

Putting these together we get that

(17) $$S_r - S_{r-1} = K_{\alpha,\beta} r^{\alpha+\beta}\varphi(r) + r^{\alpha+\beta} O\Big(\sum_{d|r} |\mu(d)|\Big),$$

where

$$K_{\alpha,\beta} = \frac{1}{1+\alpha} + \frac{1}{1+\beta} - \frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(2+\alpha+\beta)}.$$

Now if we sum (17) over r and use (11) and Lemma 2, we get

THEOREM 3. *Suppose that $\alpha, \beta \geqslant 0$, and that S_n is defined by* (16). *Then*

(18) $$S_n = c_{\alpha,\beta} n^{\alpha+\beta+2} + O(n^{\alpha+\beta+1}\log n),$$

where

$$c_{\alpha,\beta} = \frac{6}{\pi^2}\left\{\frac{1}{(1+\alpha)(1+\beta)} - \frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(3+\alpha+\beta)}\right\}.$$

As an example of the above, we have

$$\sum_{(k,k')\in Q_n} \sqrt{kk'} = \frac{32-3\pi}{12\pi^2} n^3 + O(n^2\log n).$$

Finally we mention some special results which are derivable in the same way:

(19) $$\sum_{(k,k')\in Q_n} \frac{k}{k'} = \frac{3}{2}\sum_{r=1}^{n}\varphi(r) - \frac{1}{2} = \frac{9n^2}{2\pi^2} + O(n\log n),$$

(20) $$\sum_{(k,k')\in Q_n} \frac{1}{k+k'} = \frac{6}{\pi^2}(2\log 2 - 1)n + O\big((\log n)^2\big),$$

(21) $$\sum_{(k,k')\in Q_n} \frac{kk'}{k+k'} = \frac{(11-12\log 2)}{3\pi^2} n^3 + O(n^2\log n).$$

References

[1] H. Gupta, *An identity*, Research Bulletin of the Panjab University, 15 (1964), pp. 347-349.

[2] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford 1938.

UNIVERSITY OF MARYLAND
NATIONAL BUREAU OF STANDARDS

Reçu par la Rédaction le 29. 3. 1968