получаем $K = O(\ln P)$ и

$$K^2 P^{\frac{2\varkappa+1}{4\varkappa+3}} \gg K^{\frac{4\varkappa+2}{4\varkappa+3}} K^2 g^{\frac{2\varkappa+1}{2\varkappa+2}K}, \qquad K^2 g^{\frac{2\varkappa+1}{2\varkappa+2}K} \ll K^{\frac{4\varkappa+4}{4\varkappa+3}} P^{\frac{2\varkappa+1}{4\varkappa+3}},$$

т.е.

$$N_\gamma(P) = \gamma P + O\left(P^{\frac{2\varkappa+1}{4\varkappa+3}} (\ln P)^{\frac{4\varkappa+4}{4\varkappa+3}}\right) = \gamma P + O\left(P^{\frac{1}{2}-\frac{1}{8\varkappa+6}} (\ln P)^i\right),$$

что и требуется доказать.

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Получено 31. 7. 1971 (182)

ACTA ARITHMETICA
XXI (1972)

On some problems of W. Sierpiński

by

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Dedicated to the memory of my teacher
Professor Wacław Sierpiński

A composite natural number n is said to be a *pseudoprime* if $n \mid 2^n - 2$.

The most important theorems on pseudoprimes which answer to questions raised by Sierpiński are:

1. Every arithmetical progression $ax+b$ $(x = 1, 2, \ldots)$, where $(a, b) = 1$ contains an infinite number of pseudoprimes (Rotkiewicz [4] and [5]).
2. Let a, b be fixed coprime positive integers. If $D > 0$ is given and $x > x_0(a, D)$, there exists at least one pseudoprime P satisfying: $P \equiv b \pmod{a}, x < P < x\exp\left\{\frac{\log x}{(\log\log x)^D}\right\}$ (Halberstam and Rotkiewicz [1]).
3. There exist infinitely many squarefree pseudoprimes divisible by an arbitrary given prime p (Rotkiewicz [3]).
4. There exist infinitely many arithmetic progressions formed of four pseudoprimes (Rotkiewicz [10]).
5. There exist infinitely many pseudoprimes which are at the same time triangular (Rotkiewicz [6] and [9]).
6. There exist infinitely many pseudoprimes which are at the same time pentagonal (Rotkiewicz [8] and [9]).

In 1965 (during a seminar which the author attended) W. Sierpiński raised the question whether there exist pseudoprimes which are at the same time tetrahedral. (A tetrahedral number is one of the form $\frac{n(n+1)(n+2)}{6}$). The answer to this question is in the affirmative.

Here we shall prove the following

Theorem 1. *If the numbers $8n+1$, $12n+1$ and $24n+1$ are primes and the numbers $12n+1$ and $24n+1$ are of the form x^2+27y^2, then the tetrahedral number T_{24n+1} is a pseudoprime number.*

Proof. We have

$$T_{24n+1} = \frac{(24n+1)(24n+2)(24n+3)}{6} = (24n+1)(12n+1)(8n+1).$$

Since $8n+1$ and $24n+1$ are primes $\equiv 1 \pmod 8$, we have

$$\left(\frac{2}{8n+1}\right) = \left(\frac{2}{24n+1}\right) = 1.$$

On the other hand, since the numbers $12n+1$ and $24n+1$ are primes of the form x^2+27y^2, 2 is a cubic residue of primes $12n+1$ and $24n+1$. Hence 2 is a residue of the 2nd, 3rd and the 6th degrees of the numbers $8n+1$, $12n+1$ and $24n+1$, respectively. Thus

$$8n+1 \mid 2^{4n}-1, \quad 12n+1 \mid 2^{4n}-1, \quad 24n+1 \mid 2^{4n}-1,$$

whence

$$(8n+1)(12n+1)(24n+1) \mid 2^{4n}-1 \mid 2^{(8n+1)(12n+1)(24n+1)-1}-1$$

and T_{24n+1} is a pseudoprime number.

For $1 \leqslant n \leqslant 2000$ there exist 30 values for which the numbers $8n+1$, $12n+1$ and $24n+1$ are simultaneously primes, but only 3 which satisfy the assumptions of Theorem 1. These numbers we get for $n = 1179$, 1274, 1895.

For $n = 1179$ we have

$$8n+1 = 9433, \quad 12n+1 = 14149 = 107^2+27\cdot 10^2,$$
$$24n+1 = 28297 = 163^2+27\cdot 8^2.$$

For $n = 1274$ we have

$$8n+1 = 10193, \quad 12n+1 = 15289 = 67^2+27\cdot 20^2,$$
$$24n+1 = 30577 = 97^2+27\cdot 28^2.$$

For $n = 1895$ we have

$$8n+1 = 15161, \quad 12n+1 = 22741 = 67^2+27\cdot 26^2,$$
$$24n+1 = 45481 = 173^2+27\cdot 24^2.$$

Thus the tetrahedral numbers:

$$T_{28297} = 3776730328549, \ T_{30577} = 4765143438329, \ T_{45481} = 15680770945781$$

are pseudoprimes.

Although I cannot deduce from the hypothesis H of A. Schinzel (Schinzel and Sierpiński [13]) concerning primes that there exist infinitely many tetrahedral pseudoprimes, I can prove the following theorem:

THEOREM 2. *From the hypothesis* H *of A. Schinzel concerning primes it follows that there exist infinitely many pseudoprimes of the form*

$$\frac{T_n}{4} = \frac{n(n+1)(n+2)}{24}.$$

Proof. From the hypothesis H concerning primes it follows that there exist infinitely many natural numbers n such that $12n+1$, $18n+1$ and $36n+1$ are at the same time primes. Let $12n+1$, $18n+1$ and $36n+1$ be prime numbers. Since $27 \mid 2^{18}-1$, we have

$$(1) \qquad 27(12n+1)(18n+1)(36n+1) \mid 2^{36n}-1.$$

Let $N = \dfrac{2^{36n}-1}{27}$. We shall prove that the number $\frac{1}{4}T_{72N+2}$ is a pseudoprime number. As is easy to see, we have:

$$18N+1 = \frac{2^{36n+1}+1}{3}, \quad 24N+1 = \frac{2^{3(12n+1)}+1}{9}, \quad 36N+1 = \frac{2^{2(18n+1)}-1}{3}$$

and from (1) it follows that

$$18N = \frac{2(2^{36n}-1)}{3} \equiv 0 \left(\operatorname{mod} 2\cdot 9\,(12n+1)(18n+1)(36n+1)\right),$$
$$24N = \frac{2^3(2^{36n}-1)}{9} \equiv 0 \left(\operatorname{mod} 8\cdot 3\,(12n+1)(18n+1)(36n+1)\right),$$
$$36N = \frac{4(2^{36n}-1)}{3} \equiv 0 \left(\operatorname{mod} 4\cdot 9\,(12n+1)(18n+1)(36n+1)\right),$$

whence

$$\tfrac{1}{4}T_{72N+2} = (18N+1)(24N+1)(36N+1) \equiv 1 \left(\operatorname{mod} 2\cdot 3\,(12n+1)(18n+1)(36n+1)\right),$$

and thus

$$\frac{1}{4}T_{72N+2} = \left(\frac{2^{36n+1}+1}{3}\right)\left(\frac{2^{3(12n+1)}+1}{9}\right)\left(\frac{2^{2(18n+1)}-1}{3}\right) \Bigm| 2^{\frac{1}{4}T_{72N+2}-1}-1,$$

and the number $\frac{1}{4}T_{72N+2}$ is a pseudoprime number.

EXAMPLE. For $n = 1$ the numbers $12n+1 = 13$, $18n+1 = 19$, $36n+1 = 37$ are prime numbers. Then

$$18N+1 = \frac{2^{37}+1}{3}, \quad 24N+1 = \frac{2^{39}+1}{9}, \quad 36N+1 = \frac{2^{38}-1}{3}$$

and the number

$$\frac{1}{4}T_{\frac{2^{39}-2}{3}} = \frac{(2^{37}+1)(2^{38}-1)(2^{39}+1)}{81}$$

is a pseudoprime number.

Now we shall consider pseudoprimes which are at the same time k-gonal numbers.

The *n-th k-gonal number* N_n^k is defined to be

$$N_n^k = \frac{n[(k-2)(n-1)+2]}{2}.$$

We shall prove the following

THEOREM 3. *From the hypothesis* H *it follows that for* $k = 3, 5, 6, 8, 10, 14, 18$ *there exist infinitely many k-gonal pseudoprimes which are products of two different primes.*

Proof. 1) Let $k = 3$. We have $N_n^3 = \dfrac{n(n+1)}{2}$, $N_{2n-1}^3 = (2n-1)n$. From the hypothesis H it follows that there exist infinitely many natural numbers x for which each of the numbers $4x+1$ and $8x+1$ is a prime. Then $\left(\dfrac{2}{8x+1}\right) = 1$, whence $8x+1 \mid 2^{4x}-1$ and $(4x+1)(8x+1) \mid 2^{4x}-1$ and the number $N_{8x+1}^3 = (8x+1)(4x+1)$ is a pseudoprime number.

2) Let $k = 5$. We have $N_n^5 = \dfrac{n[3(n-1)+2]}{2}$, whence

$$N_{2n-1}^5 = \frac{(2n-1)[3(2n-2)+2]}{2} = (2n-1)(3n-2).$$

From the hypothesis H it follows that there exist infinitely many natural numbers y for which each of the numbers $8(3y^2+y+9)+1$ and $12(3y^2+y+9)+1$ is a prime. Since 2 is a quadratic residue of the number $8(3y^2+y+9)+1$, we have $8(3y^2+y+9)+1 \mid 2^{4(3y^2+y+9)}-1$. Since $12(3y^2+y+9)+1 = (6y+1)^2+27\cdot 2^2$, 2 is a cubic residue of the prime $12(3y^2+y+9)+1$, we have $12(3y^2+y+9)+1 \mid 2^{4(3y^2+y+9)}-1$. Thus

$$N_{8(3y^2+y+9)+1}^5 = [8(3y^2+y+9)+1][12(3y^2+y+9)+1] \mid$$
$$\mid 2^{4(3y^2+y+9)}-1 \mid 2^{[8(3y^2+y+9)+1][12(3y^2+y+9)+1]-1}-1$$

and $N_{8(3y^2+y+9)+1}^5$ is a pseudoprime number.

3) Let $k = 6$. We have $N_n^6 = n(2n-1)$ and the proof of Theorem 3 in this case is the same as in the case 1).

4) Let $k = 8$. We have $N_n^8 = n(3n-2)$. From the hypothesis H it follows that there exist infinitely many natural numbers y for which each of the numbers $3y^2+2y+10$ and $3(3y^2+2y+10)-2 = (3y+1)^2+27\cdot 1^2$ is a prime. Since 2 is a cubic residue of the prime $(3y+1)^2+27\cdot 1^2$, we have $3(3y^2+2y+10)-2 \mid 2^{3y^2+2y+9}-1$. Since also $3y^2+2y+10 \mid 2^{3y^2+2y+9}-1$, we have $N_n^8 \mid 2^{n-1}-1 \mid 2^{N_n^8-1}-1$ for $n = 3y^2+2y+10$. Thus N_n^8 for $n = 3y^2+2y+10$ is a pseudoprime number.

5) Let $k = 10$. We have $N_n^{10} = n[4(n-1)+1] = n(4n-3)$. From the hypothesis H it follows that there exist infinitely many values of y for which each of the numbers $4y^2+2y+17$ and $4(4y^2+2y+17)-3 = (4y+1)^2+64\cdot 1^2$ is a prime. Since 2 is a residue of the 4th degree of the prime number $(4y+1)^2+64$, we have $(4y+1)^2+64 \mid 2^{4y^2+2y+16}-1$. Since also $4y^2+2y+17 \mid 2^{4y^2+2y+16}-1$, we have $N_n^{10} \mid 2^{n-1}-1 \mid 2^{N_n^{10}-1}-1$ for $n = 4y^2+2y+17$ and the number N_n^{10} is a pseudoprime for $n = 4y^2+2y+17$.

6) $k = 14$, $\dfrac{k-2}{2} = 6$, $N_n^{14} = n[6(n-1)+1] = n(6n-5)$. From the hypothesis H it follows that there exist infinitely many natural numbers y for which each of the numbers $24y^2+4y+73$ and $6(24y^2+4y+73)-5 = (12y+1)^2+27\cdot 16$ is a prime. Since 2 is a residue of 6th degree for the prime number $(12y+1)^2+27\cdot 4^2$, we have $6n-5 \mid 2^{n-1}-1$ for $n = 24y^2+4y+73$. Since also $n \mid 2^{n-1}-1$ for $n = 24y^2+4y+73$, we have $N_n^{14} \mid 2^{n-1}-1 \mid 2^{N_n^{14}-1}-1$ for $n = 24y^2+4y+73$. This proves Theorem 3 for $k = 14$.

7) $k = 18$, $\dfrac{k-2}{2} = 8$, $N_n^{18} = n[8(n-1)+1] = n(8n-7)$. From the hypothesis H it follows that there exist infinitely many natural numbers y for which each of the numbers $8y^2+2y+33$ and $8(8y^2+2y+33)-7 = (8y+1)^2+256\cdot 1^2$ is a prime. Since 2 is a residue of the 8th degree of the prime number $(8y+1)^2+256\cdot 1^2$ and $8n-7 \mid 2^{n-1}-1$ for $n = 8y^2+2y+33$. Since also $n \mid 2^{n-1}-1$ for $n = 8y^2+2y+33$, we have $n(8n-7) \mid 2^{n-1}-1 \mid 2^{n(8n-7)-1}-1$ for $n = 8y^2+2y+33$ and $n(8n-7)$ is a pseudoprime number.

This completes the proof of Theorem 3.

Let $P(x)$ denote the number of pseudoprimes $\leqslant x$. K. Szymiczek [16] has proved the following theorem:

If k is a natural number and x is sufficiently large, then

$$P(x) > \frac{1}{4}\{\log x + \log\log x + \ldots + \underbrace{\log\log\ldots\log x}_{n \text{ times iterated logarithm}}\}.$$

I have proved (Rotkiewicz [12]) the following much stronger theorem: $P(x) > \dfrac{5}{8}\log_2 x$ ($\log_2 x$ denotes logarithm at the base 2). Here we shall prove the following:

THEOREM 4. *Let $P_1(x)$ denote the number of pseudoprimes which are $\equiv 1 \pmod{n}$, $\leqslant x$, where n is a given natural number > 6.*

Then $P_1(x) \geqslant \dfrac{\log_2 x}{2n}$.

Proof. A factor m of 2^n-1 is said to be *primitive* if it does not divide any of the numbers 2^k-1, $k=1,2,\dots,n-1$.

By Theorem 1 of the paper [12] the number $2^{2n}-1$ for $n>6$ has at least one primitive composite factor of the form $nk+1$. As is easy to see, if $nk+1$ is a composite divisor of 2^n-1, then $nk+1\,|\,2^n-1\,|\,2^{nk}-1\,|\,2^{nk+1}-2$ and $nk+1$ is a pseudoprime number.

Let us calculate the number of pseudoprimes which are $\equiv 1 \pmod n$, $\leqslant x = 2^{\log_2 x}$. Let $n>6$. For every $k\geqslant 1$ the number $2^{2nk}-1$ has a composite primitive factor of the form $nk+1$ and to different values of k correspond different pseudoprimes of the form $nk+1$. By the above argument, there are at least $\dfrac{\log_2 x}{2n}$ pseudoprimes of the form $nk+1$ which are $\leqslant x = 2^{\log_2 x}$.

THEOREM 5. *Let a, b be fixed coprime positive integers. Let $P_a(x)$ denote the number of pseudoprimes which are $\equiv b \pmod a$, $\leqslant x$.*

Then $P_a(x) \gg \dfrac{\log x}{a^c \log\log x}$, *where c denotes an absolute constant.*

Proof. Let us calculate the number of pseudoprimes which are $\equiv b \pmod a$, $\leqslant x = 2^{\log_2 x}$. Let q, q_1 be any two distinct odd primes satisfying the conditions

$$q_1 \nmid a, \quad q \equiv 1 \bigl(\bmod\, aq_1\varphi(aq_1)\bigr)$$

and let m be any (odd) integer such that

$$m \equiv b \pmod a, \qquad m \equiv 1+q_1 \pmod{q_1^2}, \qquad m \equiv 1 \pmod{q^2}.$$

By Lemma 3 of my paper [11] for every prime $p \equiv m \pmod{aq^2q_1^2}$ there exists a pseudoprime number $<2^p$ and to different primes correspond different pseudoprimes $<2^p$. Thus the number of pseudoprimes $\leqslant x$, $\equiv b \pmod a$ is $\geqslant$ the number of primes $p\leqslant \log_2 x$ such that $p\equiv m \pmod{aq^2q_1^2}$, where q_1 is the least prime such that $q_1\nmid a$ and q is the least prime $\equiv 1 \bigl(\bmod\, aq_1\varphi(aq_1)\bigr)$.

The number of primes $\leqslant \log_2 x$, $p\equiv m \pmod{aq^2q_1^2}$ is $\sim \dfrac{\log_2 x}{\varphi(aq^2q_1^2)\log\log_2 x}$. Let q denote the least prime $\equiv 1 \bigl(\bmod\, aq_1\varphi(aq_1)\bigr)$. We have $aq^2q_1^2 < a^c$, where c denotes an absolute constant. The number of primes $p\leqslant \log_2 x$, $p\equiv m \pmod{aq^2q_1^2}$ is thus $\gg \dfrac{\log_2 x}{a^c\log\log_2 x} \gg \dfrac{\log x}{a^c \log\log x}$ and the number of pseudoprimes p, $p\leqslant x$, $\equiv b \pmod a$ is also $\gg \dfrac{\log x}{a^c\log\log x}$. This completes the proof of Theorem 5.

THEOREM 6. *Let $a\bar{x}^2+b\bar{x}\bar{y}+c\bar{y}^2$ be a primitive quadratic form (positive or indefinite) having a fundamental discriminant and belonging to the principal genus. For even b, let the quadratic form $a\bar{x}^2+b\bar{x}\bar{y}+c\bar{y}^2$ satisfy the following weaker assumptions:*

a) $a>0$, $(a,b,c)=1$,

b) $d=b^2-4ac$ *is not divisible by an odd square* >1,

c) $\left(\dfrac{a}{p_i}\right)=1$ *for* $p_i\nmid a$, $p_i|d$; $\left(\dfrac{c}{p_i}\right)=1$ *for* $p_i\nmid c$, $p_i|d$,

d) $a\equiv 1 \bmod 4$ *or* $c\equiv 1\bmod 4$ *or* $a+b+c\equiv 1 \bmod 4$ *and let* $\overline{P(x)}$ *denote the number of pseudoprimes of the form* $a\bar{x}^2+b\bar{x}\bar{y}+c\bar{y}^2$, $\leqslant x$. *Then*

$$\overline{P(x)} \gg \frac{\log x}{\log\log x}.$$

Proof. Let $d=b^2-4ac=\pm 2^\beta d_1$, $3^\gamma \| d_1$, let the numbers a, b, c satisfy the above conditions, let q be the least prime $\equiv 1 \bmod m\varphi(m)$, where $m=2^{3+\beta}3^{\gamma+2}$ and let p be a prime such that $p=a\bar{x}^2+b\bar{x}\bar{y}+c\bar{y}^2$, $p=2^{\alpha+1}3^3d_1q^2z+2^\alpha 3^2 d_1+1$ $(\alpha=2$ or $\alpha=3)$. Then by Theorem 15 of my paper [11] there exists a pseudoprime of the form $a\bar{x}^2+b\bar{x}\bar{y}+c\bar{y}^2$ less than 2^p. From the proof of Theorem 15 it follows also that to different primes p correspond different pseudoprimes of the form $a\bar{x}^2+b\bar{x}\bar{y}+c\bar{y}^2$, $\leqslant x$.

The number of primes which satisfy the above conditions and which are less than $x=2^{\log_2 x}$ is $\gg \dfrac{\log x}{\log\log x}$. Thus the number of pseudoprimes $\leqslant x$, of the form $a\bar{x}^2+b\bar{x}\bar{y}+c\bar{y}^2$ is also $\gg \dfrac{\log x}{\log\log x}$.

This completes the proof of Theorem 6.

Let p_n denote the nth pseudoprime. In 1965 (during a seminar) W. Sierpiński put forward the following problem "What can we tell about $\varliminf_{n=\infty}(p_{n+1}-p_n)$?"

Here we shall prove the following:

THEOREM 7. $\varliminf_{n=\infty} \dfrac{(p_{n+1}-p_n)}{p_n}=0$.

Proof. Let n denote an odd positive integer and p a prime of the form $\varphi\bigl(n(2^{2n}-1)\bigr)k+1$ greater than $2^{2n}-1$. By Lemma 2 of my paper [7] the number $N=\dfrac{2^{np}-1}{2^n-1}$ is a pseudoprime. Similarly we can prove that the number $N_1=\dfrac{2^{np}+1}{2^n+1}$ is also pseudoprime. We have

$$\frac{2^{np}-1}{2^n-1}-\frac{2^{np}+1}{2^n+1}=\frac{2^{np}+1}{2^n+1}\left(\frac{2^n+1}{2^{np}+1}\cdot\frac{2^{np}-1}{2^n-1}-1\right).$$

Since

$$\lim_{n=\infty}\left(\frac{2^n+1}{2^{np}+1}\cdot\frac{2^{np}-1}{2^n-1}-1\right)=0,$$

we have

$$\frac{2^{np}-1}{2^n-1}-\frac{2^{np}+1}{2^n+1}<\varepsilon\frac{2^{np}+1}{2^n+1}$$

for every $\varepsilon>0$ and sufficiently large n. From this Theorem 7 follows.

Recently Lieuwens [2] has noted that perfect numbers come into the picture of absolutely pseudoprime numbers.

We call a positive integer n an *absolutely pseudoprime number* if $a^m\equiv a \pmod m$ for every a. These numbers are also called *Carmichael numbers*. Lieuwens [2] has proved the following theorem:

If n is a perfect number and $n_1, n_2, \ldots, n_k$ are all divisors of n, then

$$m=\prod_{i=1}^{k}(n_i nh+1)$$

is an absolutely pseudoprime number if $p_i=n_i nh+1$ is a prime for $i=1,2,\ldots,k$.

Here we shall prove the following

THEOREM 8. *If $n_i|n$ and $n_i\neq n_j$ for $i\neq j$, nn_1+1, $nn_2x+1, \ldots$ $\ldots, nn_kx+1$ are primes and $n|n_1+n_2+\ldots+n_k$, then the number*

$$m=(nn_1x+1)(nn_2x+1)\ldots(nn_kx+1)$$

is an absolutely pseudoprime number.

Proof. Let $n|n_1+n_2+\ldots+n_k$. We have

$$(nn_1x+1)(nn_2x+1)\ldots(nn_kx+1)-1$$
$$\equiv n(n_1+\ldots+n_k)x \pmod{n^2x}\equiv 0 \pmod{n^2x}.$$

Since $nn_ix|n^2x$ for $i=1,2,\ldots,k$, we have $nn_ix|m-1$ for $i=1,2,\ldots,k$. Thus m is an absolutely pseudoprime number.

This theorem gives us the connection between absolutely pseudoprimes, perfect numbers, multiply perfect numbers and practical numbers.

Natural numbers n such that $\sigma(n)=mn$, where m is a natural number >1, are called P_m *numbers* or *multiply perfect numbers*. A natural number n is said to be a *practical number* if every natural number $\leqslant n$ is a sum of different divisors of the number n. For a necessary and sufficient condition for a natural number n to be a practical number see Sierpiński [14] and Stewart [15].

In a similar way to that followed in the proof of Theorem 8 we can prove the following

THEOREM 9. *If $n_i|n$, $8|n$, $n_i\neq n_j$ for $i\neq j$ and if the numbers nn_1x+1, $nn_2x+1, \ldots, nn_kx+1$ are primes, $\frac{n}{2}|n_1+n_2+\ldots+n_k$, then the number*

$$m=(nn_1x+1)(nn_2x+1)\ldots(nn_kx+1)$$

is a pseudoprime number.

EXAMPLE. Let $n=24$, $n_1=1$, $n_2=2$, $n_3=3$, $n_4=4$, $n_5=6$, $n_6=8$, $n_7=12$, $n_8=24$. Then

$$(24\cdot 1x+1)(24\cdot 2x+1)(24\cdot 3x+1)(24\cdot 4x+1)(24\cdot 6x+1)(24\cdot 8x+1)\times$$
$$\times(24\cdot 12x+1)(24\cdot 24x+1)$$

is a pseudoprime number if each of the numbers $24\cdot 1x+1$, $24\cdot 2x+1$, $24\cdot 3x+1$, $24\cdot 4x+1$, $24\cdot 6x+1$, $24\cdot 8x+1$, $24\cdot 12x+1$, $24\cdot 24x+1$ is a prime number.

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Received on 31. 7. 1971 (202)