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Graduate program in Logic, Algorithms and Computation

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Algebraic and combinatorial techniques in rigidity theory

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Contents

Chapter 1

Introduction

By $\mathbb{E}^d$ we denote the d-dimensional Euclidean space i.e. $\mathbb{R}^d$ equipped with the Euclidean metric. A realization (or an embedding) of a graph G in $\mathbb{E}^d$ is obtained by specifying the location of the vertices of the graph in $\mathbb{E}^d$. Two realizations of the $K_{3,3}$ graph in $\mathbb{E}^2$ are illustrated below:

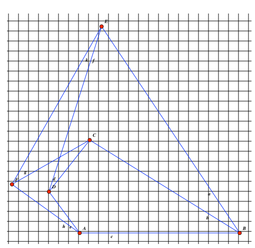

Figure 1.1: A realization of the $K_{3,3}$ where the vertices lie on a quadric, which is decomposed into two lines.

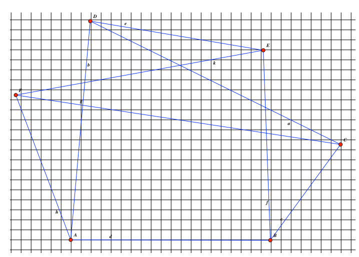

Figure 1.2: A realization of the $K_{3,3}$ where the vertices lie on a irreducible quadric.

Given a graph G its configuration space $\mathcal{C}^d(G)$ is defined to be the space of all realizations of G in $\mathbb{E}^d$. A realization of a graph G will be called flexible iff there exists a continuous deformation which preserves the edge lengths. Otherwise the realization will be called rigid. Both of the realizations of the $K_{3,3}$ illustrated above are flexible. Four snapshots of the length-preserving deformation of the realization in Figure 1.1, are illustrated in Table 1.1.

Notice however, that in both of the realizations illustrated in Figures 1.1 and 1.2, it appears that the positions of the vertices are rather carefully contrived and in some sense they do not represent the typical behavior of a realization of the $K_{3,3}$ in $\mathbb{E}^2$. It is thus natural to ask what will happen if we restrict our attention to realizations of the $K_{3,3}$ where the vertices are placed in "general position" (generic realizations). The somewhat surprising answer is that all generic realizations of the $K_{3,3}$ in $\mathbb{E}^2$ are rigid. In view of this, the characterization of the $K_{3,3}$ as being generically rigid in $\mathbb{E}^2$ seems fully justified.

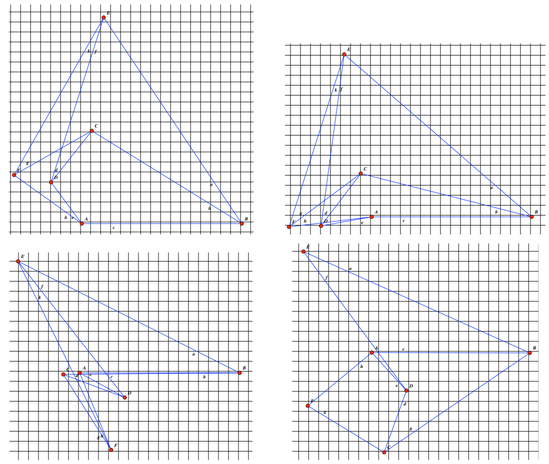

Table 1.1: Four snapshots (top left to bottom right) of the length preserving deformation of a realization of the $K_{3,3}$.

Going to the other extreme consider the $K_{2,2}$ graph. It is not difficult to see that this graph is be generically flexible in $\mathbb{E}^2$ i.e. all of its generic realizations are flexible in $\mathbb{E}^2$. An illustration of a flex of generic realization of the $K_{2,2}$ graph can be seen in Table 1.2.

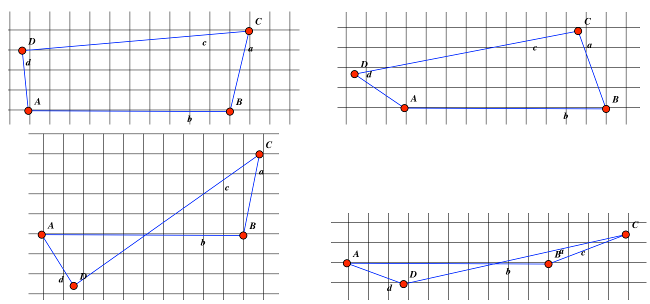

Table 1.2: Four snapshots (top left to bottom right) of the length preserving deformation of a realization of the $K_{2,2}$ graph.

Summarizing, we have seen so far that the $K_{3,3}$ is generically rigid in $\mathbb{E}^2$, whereas the $K_{2,2}$ is generically flexible in $\mathbb{E}^2$. A surprising fact, is that a similar situation holds for any arbitrary graph G i.e. every graph G is either generically rigid or generically flexible in $\mathbb{E}^d$. In order to prove this, one shows that the generic realizations of G form an open and dense subset of $\mathcal{C}^d(G)$ and moreover, that the generic realizations of a given graph G are either all rigid or all flexible in $\mathbb{E}^d$. A graph G is called generically rigid (flexible) in $\mathbb{E}^d$ iff one, and thus all, of its generic realizations is rigid (flexible) in $\mathbb{E}^d$. Intuitively, a graph which is generically rigid in $\mathbb{E}^d$ should be thought as a graph for which an arbitrary realization in $\mathbb{E}^d$ will be rigid, with high probability. For a detailed account of the basic notions of the theory of rigid graphs, the reader should consult Chapter 2.

Notice that the addition of extra edges to a generically rigid graph will not affect its generic behavior. Consequently, it seems reasonable to focus our study on the class of generically minimally rigid graphs i.e. graphs that become generically flexible once any edge is removed.

The preceding discussion suggests that if we ignore some singular realizations, minimal rigidity can be considered as a property of the underlying graph rather than a property of the geometry of a specific realization and thus there should be some purely combinatorial way for detecting it. Indeed, this is the case for $d = 1$, where a graph G is generically minimally rigid in $\mathbb{E}$ iff G is a tree. Moreover, this is also true for $d = 2$, where a graph $G = (V, E)$ is generically minimally rigid in $\mathbb{E}^2$ (or Laman) iff $|E| = 2|V| - 3$ and additionally, all of its induced subgraphs on $2 \leq k < |V|$ vertices have $\leq 2k - 3$ edges. On the other hand for $d \geq 3$ the plot thickens. A combinatorial characterization of generically minimally rigid graphs in $\mathbb{E}^3$ has proven to be elusive, and remains one of the most important open questions in the area of rigidity theory. An extensive study of generic rigidity in $\mathbb{E}^2$ and $\mathbb{E}^3$ can be found in Chapters 3 and 4, respectively.

Laman graphs exhibit a rich combinatorial structure and possess a number of equivalent characterizations. One of the most important ones, is that they coincide with the class of graphs that have a Henneberg 2-sequence i.e. graphs that can be constructed inductively starting for a triangle, followed by a succession of two allowable operations, the so-called Henneberg I and Henneberg II steps. We will make use of this fact in Chapter 6, where we will need to generate all Laman graphs with $n \leq 10$ vertices. It is also worth noting that for $d \geq 3$, there is no analogous procedure that enables us to construct all generically minimally rigid graphs in $\mathbb{E}^d$. The topic of the inductive constructions of generically minimally rigid graphs is treated in detail in Chapter 5.

Our contribution

Given a generically rigid graph in $\mathbb{E}^d$, it follows by its definition, that for generic edge lengths it can be embedded into Euclidean d-space in a finite number number of ways, modulo rigid motions (translations and rotations). The problem we are dealing with in Chapter 6, is to compute tight bounds on the number of non-congruent embeddings of Laman graphs and of the graphs that correspond to 1-skeleta of convex simplicial polyhedra in $\mathbb{E}^2$ and $\mathbb{E}^3$, respectively.

A crucial observation is to made here: the problem of computing the number of non-congruent realizations of a graph G can be formulated as an algebraic one. Specifically, given generic edge lengths l_{ij}, $ij \in E$, we can construct a polynomial system whose real solutions correspond to all possible non-congruent realizations of G. For a representative case when $d = 2$, see the system below:

$$\begin{cases} x_i = a_i,\ y_i = b_i, & i = 1,2, \quad a_i, b_i \in \mathbb{R}, \\ (x_i - x_j)^2 + (y_i - y_j)^2 = l_{ij}^2, & ij \in E - \{12\} \end{cases}$$

This is a square $2n \times 2n$ polynomial system in the unknowns $x_1, y_1, \ldots, x_n, y_n$, where (x_i, y_i) corresponds to the coordinates of the vertex i in the embedding. Notice that we have assumed without loss of generality that there exists an edge between vertices 1 and 2 and this has been fixed in order to discard translations and rotations as solutions of our system.

In order to bound the number of embeddings of rigid graphs, we have developed specialized software that constructs all Laman graphs and all 1-skeleta of simplicial polyhedra in $\mathbb{E}^3$ with $n \leq 10$. Our computational platform is SAGE [1]. We exploit the fact that these graphs admit inductive constructions, and construct all of them using the Henneberg operations. The latter were implemented, using SAGE's interpreter, in Python. After we construct all the graphs, we classify them up to isomorhism using SAGE's interface for N.I.C.E., an open-source isomorphism check engine. Then, for each graph we set up its corresponding polynomial system and for each system we bound the number of its (complex) solutions by computing its mixed volume (BKK bound).

Our main contribution is twofold: first, we derive an improved lower bound in $\mathbb{E}^2$ and the first non-trivial lower bound in $\mathbb{E}^3$:

$$32^{\lfloor (n-2)/4 \rfloor} \simeq 2.37^n,\ n \geq 10, \ \text{ and } \ 16^{\lfloor (n-3)/3 \rfloor} \simeq 2.52^n,\ n \geq 9,$$

by designing a $K_{3,3}$ caterpillar and a cyclohexane caterpillar, respectively. The way these bounds are derived is simple enough to allow for improvements.

Second, we give tight bounds for $n = 7, 8$ in $\mathbb{E}^2$ and $n = 6, 7$ in $\mathbb{E}^3$. These bounds are important for applications and may lead to tighter lower bounds. We also reduce the existing gap for $n = 9, 10$ in $\mathbb{E}^2$ and for $n = 8, 9, 10$ in $\mathbb{E}^3$ (see Tables 6.2 and 6.3).

Our results appeared in preliminary form in the 25th European Workshop on Computational Geometry (EuroCG'09) [27]. A state of the art account of our work can be found in [26], a paper that has been accepted to the 17th International Symposium on Graph Drawing, which has published conference proceedings, included in the Springer-Verlag series Lecture Notes in Computer Science.

[1] `http://www.sagemath.org/`

Chapter 2

Rigidity of frameworks

2.1 Rigidity of frameworks

In the classic literature of rigidity theory, a graph $G = (V, E)$ is usually referred to as an *abstract framework*. We will use both of these terms interchangeably. By $\mathbb{E}^d$ we denote the d-dimensional Euclidean space i.e. $\mathbb{R}^d$ equipped with the Euclidean metric and by $\mathrm{Eucl}(d)$ we denote the group of rigid motions (translations and rotations) of d-space. A matrix $A \in \mathbb{R}^{d\times d}$ is called orthogonal iff $AA^T = A^T A = I$. The set of all orthogonal $d \times d$ matrices, forms a group under multiplication, denoted by $O(d)$. The following Theorem is a standard result of Linear Algebra.

Theorem 2.1.1. *Any rigid motion $T \in Eucl(d)$ has the form $T(x) = Ax + b$, where $A \in O(d)$ and $b \in \mathbb{R}^d$.*

We continue with some basic definitions.

Definition 2.1.2. *A realization of an abstract framework in $\mathbb{E}^d$ consists of a graph $G = (V, E)$ together with a map $p : V \mapsto \mathbb{R}^d$, where $p(i) = p_i = (p_i^1, \ldots, p_i^d) \in \mathbb{R}^d$ should be interpreted as the point in d-space to which the i-th vertex of G is assigned to. A realization of G will be denoted by $G(p)$, where $p = (p_1, \ldots, p_{|V|})$. Sometimes, this will be also referred to as a framework or an embedding of G in $\mathbb{E}^d$.*

Definition 2.1.3. *By $C^d(G)$ we will denote the space of all realizations in $\mathbb{E}^d$ of the abstract framework $G = (V, E)$ i.e.*

$$C^d(G) = \left\{ p = (p_1, \ldots, p_{|V|}) \in \mathbb{E}^{d|V|} \mid G(p) \text{ is a realization of } G \text{ in } \mathbb{E}^d \right\}$$

The space $C^d(G)$ will also be referred to as the configuration space of G. As a simple example, let G be K_3 graph. Then $C^2(G)$ consists of all triples of points in the plane, with the property that the distances between them satisfy the triangle inequality.

Definition 2.1.4. *Every abstract framework $G = (V, E)$ determines a map, called the rigidity map, defined as*

$$f_G : C^d(G) \mapsto \mathbb{R}^{|E|} \text{ where } f_G(p) = (\ldots, ||p_i - p_j||^2, \ldots)$$

and the edges of G are ordered lexicographically.

Definition 2.1.5. *Two frameworks $G(p), G(q)$ are called equivalent iff*

$$||p_i - p_j|| = ||q_i - q_j||, \ \forall ij \in E$$

Notice that for a given framework $G(p)$, the set of realizations that are equivalent to it is just $f_G^{-1}\big(f_G(p)\big)$.

Definition 2.1.6. *Two frameworks $G(p), G(q)$ are called congruent iff*

$$||p_i - p_j|| = ||q_i - q_j||, \ \forall i, j \in V$$

Equivalently, two frameworks $G(p), G(q)$ are congruent iff the map

$$T : \mathbb{E}^d \mapsto \mathbb{E}^d \text{ with } T(p_i) = q_i, \ \forall i \in V$$

can be extended to a rigid motion of $\mathbb{E}^d$. For an example of equivalent and congruent frameworks see Figure 2.1.

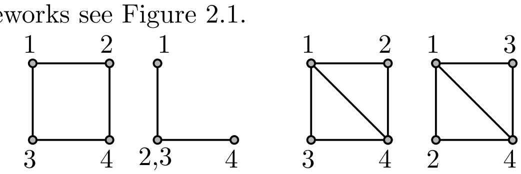

Figure 2.1: Two equivalent and two congruent frameworks.

Definition 2.1.7. *Let $p = (p_1, \ldots, p_n) \in \mathbb{E}^{d|V|}$. By $\mathcal{M}_p$ we will denote the set of points in $\mathbb{E}^{d|V|}$ that are congruent to p i.e.*

$$\mathcal{M}_p = \Big\{ q \in \mathbb{E}^{d|V|} \ \big| \ q = T(p), \ \textit{where } T \in Eucl(d) \Big\}$$

The set $\mathcal{M}_p$ can be shown to be a smooth manifold for "almost all" choices of p and moreover if points $p_1, \ldots, p_{|V|}$ are affinely independent then $\mathcal{M}_p$ is $d(d+1)/2$ dimensional since it arises from the $d(d-1)/2$ dimensional manifold of orthogonal transformations of $\mathbb{E}^d$ and the d dimensional manifold of translations of $\mathbb{E}^d$. It is clear that $\mathcal{M}_p \subseteq f_G^{-1}\big(f_G(p)\big)$ and it is exactly the nature of this inclusion near p that determines the rigidity or flexibility of $G(p)$.

Definition 2.1.8. *Let $G = (V, E)$ be an abstract framework. The framework $G(p)$ is rigid in $\mathbb{E}^d$ iff there exists a neighborhood U of p in $\mathbb{E}^{d|V|}$ such that*

$$\mathcal{M}_p \cap U = f_G^{-1}\big(f_G(p)\big) \cap U$$

Equivalently, the framework $G(p)$ is rigid iff $[p]$ is isolated in $f_G^{-1}\big(f_G(p)\big)/\mathcal{M}_p$.

Definition 2.1.9. *A flex of a framework $G(p)$ is a function*

$$\chi = (\chi_1, \ldots, \chi_{|V|}) : [0,1] \mapsto \mathbb{R}^{d|V|}$$

that satisfies the following three conditions:

- χ *is continuous*
- $\chi(0) = p$
- $\chi(t) \in f_G^{-1}\big(f_G(p)\big) - \mathcal{M}_p, \ \forall t \in (0,1]$

Example 2.1.10. *Let $G = (\{v_1, v_2, v_3, v_4\}, \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\})$ and $G(p)$ the framework seen in Figure 2.2, where $p(v_1) = (0,0), p(v_2) = (1,0), p(v_3) = (1,1), p(v_4) = (0,1)$. To prevent the square from moving in the plane by translations and rotations we fix two vertices at points $(0,0)$ and $(1,0)$. Now imagine that the two vertical rods start to move simultaneously, in a clockwise direction, with the same (constant) speed. This defines a flex of the square framework, which is depicted in Figure 2.2 and it is described by the family of solutions:*

$$x_3(t) = \left(1+t, \sqrt{1-t^2}\right), \ x_4(t) = \left(t, \sqrt{1-t^2}\right), \ \textit{for } t \in [0,1]$$

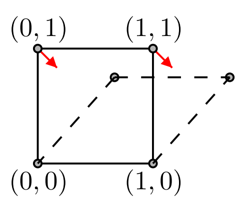

Figure 2.2: A flex of the square framework.

Definition 2.1.11. *A framework $G(p)$ will be called flexible iff it has a flex χ.*

Are the notions of non-rigidity and flexibility equivalent? One would expect the answer to be affirmative, as is it indeed the case by the following Theorem.

Theorem 2.1.12. [4, Proposition 1] *Let $G = (V, E)$ be an abstract framework. The following are equivalent:*

- $G(p)$ *is not rigid in* $\mathbb{E}^d$
- $G(p)$ *is flexible in* $\mathbb{E}^d$
- *there exists a flex χ with $\chi(0) = p$ such that $\chi(t) \notin \mathcal{M}_p$ for some $t \in (0,1]$*

2.2 Infinitesimal rigidity of frameworks

Let χ be a flex of a framework $G(p)$. It can be shown using "elementary" differential geometry [5] that the existence of a flex for $G(p)$ implies the existence of a smooth flex for $G(p)$. So from now on, we will assume our flexes to be differentiable functions.

By the definition of a flex we have that

$$||\chi_i(t) - \chi_j(t))|| = ||p_i - p_j||, \forall t \in [0, 1]$$

so

$$(\chi_i(t) - \chi_j(t)) \cdot (\chi_i(t) - \chi_j(t)) = ||p_i - p_j||^2, \forall t \in [0, 1]$$

Differentiating and evaluating at $t = 0$ we obtain that

$$(\chi_i(0) - \chi_j(0)) \cdot (\chi_i^{'}(0) - \chi_j^{'}(0)) = 0, \forall ij \in E$$

and since $\chi_i(0) = p_i$ it follows that

$$(p_i - p_j) \cdot (\chi_i^{'}(0) - \chi_j^{'}(0)) = 0, \forall ij \in E \tag{2.1}$$

The coefficient matrix of the system above, will play an important role and a special name is reserved for it.

Definition 2.2.1. *The coefficient matrix of the system of equations (2.1) is called the rigidity matrix of the framework $G(p)$ and it will be denoted by $\mathcal{R}(p)$.*

Equivalently, the rigidity matrix of a framework $G(p)$ can be also defined as the Jacobian matrix of the rigidity map f_G, multiplied by 1/2. It follows from its definition, that the dimension of the rigidity matrix is $|E| \times d|V|$. The archetypical form of the rigidity matrix can be seen in table 2.1.

$$\begin{bmatrix} . & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & . \\ . & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & . \\ 0 & \cdots & (p_i^1 - p_j^1) & \cdots & (p_i^d - p_j^d) & \cdots & (p_j^1 - p_i^1) & \cdots & (p_j^d - p_i^d) & 0 \\ . & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & . \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & . \end{bmatrix}$$

Table 2.1: The archetypical form of the rigidity matrix.

It is clear that the rigidity matrix will be large even for small examples so it is common to use a more "compact" notation, by assigning d-dimensional vector entries to a $|E| \times |V|$ matrix. An example can be seen in table 2.2.

In terms of the rigidity matrix, the system of equations (2.1) can be re-expressed as

$$R(p) \cdot (\chi_1^{'}(0), \ldots, \chi_{|V|}^{'}(0) = 0$$

i.e.

$$(\chi_1^{'}(0), \ldots, \chi_{|V|}^{'}(0)) \in \text{ Ker } \mathcal{R}(p)$$

$$\mathcal{R}(p)=\begin{bmatrix} . & \dots & \dots & \dots & \dots & \dots & . \\ . & \dots & \dots & \dots & \dots & \dots & . \\ 0 & \dots & (p_i-p_j) & \dots & (p_j-p_i) & \dots & 0 \\ . & \dots & \dots & \dots & \dots & \dots & . \\ . & \dots & \dots & \dots & \dots & \dots & . \end{bmatrix}$$

Table 2.2: A more usual "compact" notation for the rigidity matrix.

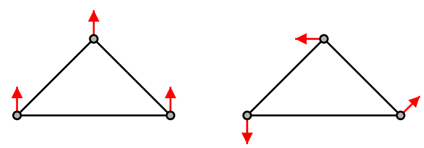

Figure 2.3: Two infinitesimal flexes of a triangular framework, that correspond to a translation and a rotation, respectively.

Definition 2.2.2. *Any element* $(p_1^{'},\dots,p_{|V|}^{'})\in Ker\ \mathcal{R}(p)$ *will be called an infinitesimal motion (or infinitesimal flex) of the framework* $G(p)$.

For examples of infinitesimal flexes, see Figures 2.3 and 2.4.

Notice that there will always exist some trivial infinitesimal motions, namely the initial velocity assignments that correspond to rigid motions of $\mathbb{E}^d$. We proceed by giving the precise definition of the infinitesimal flexes that correspond to rigid motions and by computing the dimension of the space spanned by them.

Definition 2.2.3. *A flex* $\chi=(\chi_1,\dots,\chi_{|V|})$ *of a framework* $G(p)$ *will be called trivial flex iff there exists a family* $\{T_t|\ t\in[0,1]\}$ *of rigid motions of* $\mathbb{E}^d$ *such that:*

- $T_0 = Id$
- $\chi_i(t)=T_t(p_i),\ \forall t\in(0,1]$

See Figure 2.5 for an example of a trivial flex.

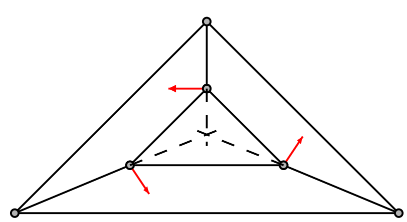

Figure 2.4: A infinitesimal flex of a realization of the triangular prism graph in $\mathbb{E}^2$, that does not correspond to a rigid motion of $\mathbb{E}^2$.

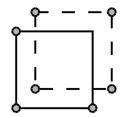

Figure 2.5: A trivial flex of a square framework.

Definition 2.2.4. *An infinitesimal flex* $(p_1^{'}, \dots, p_{|V|}^{'})$ *of a framework* $G(p)$ *will be called trivial iff it is the derivative at* $t = 0$ *of some trivial flex* χ *of* $G(p)$.

Now, let χ be a trivial flex of $G(p)$ and let $\{T_t| \ t \in [0, 1]\}$ be the family of rigid motions of $\mathbb{E}^d$ as above. By Theorem 2.1.1 it follows that for each $t \in [0, 1]$ there exists a matrix $A_t \in O(d)$ and a vector $b_t \in \mathbb{R}^d$ such that

$$T_t(x) = A_t x + b_t, \ \forall x \in \mathbb{R}^d$$

and since $T_0 = Id$, it follows that $A_0 = I$ and $b_0 = 0$.

By definition, $(\chi_1^{'}(0), \dots, \chi_{|V|}^{'}(0))$ is a trivial infinitesimal flex of framework $G(p)$ that satisfies

$$\chi_i^{'}(0) = \frac{d}{dt} T_t(p_i)\Big|_{t=0} = \frac{d}{dt} A_t\Big|_{t=0} p_i + \frac{d}{dt} b_t\big|_{t=0}$$

The vectors $\frac{d}{dt} b_t\big|_{t=0}$ correspond to infinitesimal translations and span a d dimensional space. Thus, in order to compute the dimension of the space spanned by *all* trivial infinitesimal flexes of $G(p)$, it suffices to compute the dimension of the space spanned by the matrices $\frac{d}{dt} A_t\Big|_{t=0}$, which correspond to infinitesimal rotations.

Since A is orthogonal, it follows that

$$0 = \frac{d}{dt} A_t^T(p_i)\Big|_{t=0} A_0 + A_0^T \frac{d}{dt} A_t(p_i)\Big|_{t=0} = \frac{d}{dt} A_t^T(p_i)\Big|_{t=0} + \frac{d}{dt} A_t(p_i)\Big|_{t=0}$$

and thus the matrix $\frac{d}{dt} A_t(p_i)\Big|_{t=0}$ is antisymmetric. Since the space of antisymmetric matrices is $(d^2 - d)/2$-dimensional we have that:

Theorem 2.2.5. *Let* $G(p)$ *be a realization of the abstract framework* $G = (V, E)$ *in* $\mathbb{R}^d$ *such that aff* $\{p_1, \dots, p_{|V|}\} = \mathbb{E}^d$. *Then the dimension of the space of trivial infinitesimal flexes is* $\frac{d(d+1)}{2}$.

Now, let $G(p)$ be a framework in $\mathbb{E}^d$, such that aff $\{p_1, \dots, p_{|V|}\} = \mathbb{E}^d$. Theorem 2.2.5 implies that

$$\dim \text{Ker } \mathcal{R}(p) \geq \frac{d(d+1)}{2}$$

and thus

$$\text{rank } R(p) \leq d|V| - \frac{d(d+1)}{2}$$

Definition 2.2.6. *Let $G(p)$ be a framework in $\mathbb{E}^d$ with aff $\{p_1, \ldots, p_{|V|}\} = \mathbb{E}^d$. Then, $G(p)$ will be called infinitesimally rigid (or first order rigid) in $\mathbb{E}^d$ iff*

$$\textit{rank } R(p) = d|V| - \frac{d(d+1)}{2}$$

A framework $G(p)$ will be called infinitesimally flexible in $\mathbb{E}^d$ iff

$$\textit{rank } R(p) < d|V| - \frac{d(d+1)}{2}$$

Intuitively, this definition says that a framework $G(p)$ is infinitesimally flexible iff there exist non-trivial infinitesimal motions i.e. iff there exists some initial velocity assignment, that does not correspond to that of a rigid motion of the ambient space.

Example 2.2.7. *The framework illustrated in Figure 2.6 is infinitesimally rigid. Indeed, if one goes in the trouble of computing its rigidity matrix, he will end up with the following one:*

$$\mathcal{R}(p) = \begin{pmatrix} -3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 2 & 1 \end{pmatrix}$$

A simple computation shows that the rank of $\mathcal{R}(p)$ is equal to $9 = 2|V| - 3$ and thus the framework $G(p)$ is infinitesimally rigid.

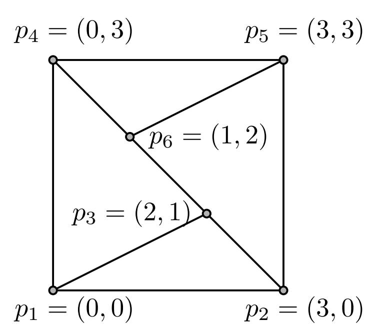

Figure 2.6: An example of an infinitesimally rigid framework in $\mathbb{E}^2$.

By its definition, an infinitesimal motion $(p_1^{'}, \ldots, p_n^{'})$ of a framework $G(p)$ satisfies

$$(p_i - p_j) \cdot (p_i^{'} - p_j^{'}) = 0,\ \forall ij \in E$$

and thus

$$(p_i - p_j) \cdot p_i^{'} = (p_i - p_j) \cdot p_j^{'},\ \forall ij \in E$$

which in turns is equivalent to the following: for every bar ij, the velocities $p_i^{'}, p_j^{'}$ that correspond to its endpoints *have equal projections* in the direction of the bar. This is a very useful observation in constructing small examples of infinitesimally flexible frameworks.

Example 2.2.8. *The flattened triangular prism, illustrated if Figure 2.7, is infinitesimally flexible in $\mathbb{E}^3$. Indeed, there exist an infinitesimal motion which assigns zero velocity to the vertices of the outer triangle and all other vertices are assigned velocities which are perpendicular to the plane of the prism.*

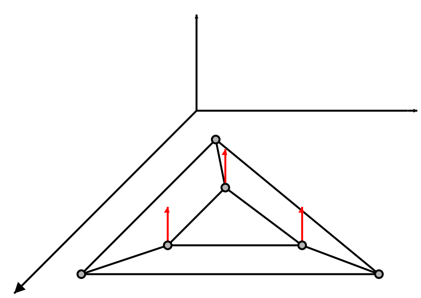

Figure 2.7: An infinitesimal flex of the flattened triangular prism in $\mathbb{E}^3$.

2.3 Statics of frameworks

Consider a framework $G(p)$ in $\mathbb{E}^d$ and suppose that the various bars of the framework are subject to forces of tension or compression, directed along its bars. More precisely, associated with every bar ij is a scalar ω_{ij} such that $\omega_{ij}(p_i - p_j)$ is the force exerted by the bar on vertex i and $\omega_{ij}(p_j - p_i)$ is the force exerted by the bar on vertex j. Intuitively, the scalar ω_{ij} gives the magnitude of the force per unit length.

If $\omega_{ij} < 0$, the force is called a tension on bar ij, otherwise it is called a compression. Notice that bar i exerts forces on vertices i, j that are equal in magnitude but opposite in direction.

Definition 2.3.1. *Let $G = (V, E)$ be an abstract framework. A stress of the framework $G(p)$ is a collection of scalars ω_{ij} such that*

$$\sum_{j \in N(i)} \omega_{ij}(p_i - p_j) = 0, \forall i \in \{1, \ldots, |V|\}$$

Intuitively, a stress of a framework $G(p)$ can be thought of as an assignment of springs to the rods of the framework, each one with spring constant ω_{ij}, such that the net force exerted on each vertex is equal to zero.

The study of stresses of frameworks is important because, roughly speaking, their existence indicates that some edges of the abstract framework are redundant.

Definition 2.3.2. *A stress is called trivial iff $\omega_{ij} = 0$, $\forall ij \in E$. We say a framework $G(p)$ is independent (or stress free) iff it admits only the trivial stress.*

In view of the definitions above, the following Corollary is immediate.

Corollary 2.3.3. *A framework $G(p)$ is independent iff rank $\mathcal{R}(p) = |E|$.*

Definition 2.3.4. *A framework $G(p)$ will be called isostatic iff it infinitesimally rigid and stress free.*

Theorem 2.3.5. *Let $G = (V, E)$ be an abstract framework with $|V| \geq d$ and let $G(p)$ be a framework in $\mathbb{E}^d$. The following are equivalent:*

1. *$G(p)$ is isostatic.*
2. *$|E| = d|V| - d(d+1)/2$ and $G(p)$ is independent.*
3. *$G(p)$ is infinitesimally rigid and removing any rod leaves an infinitesimally flexible framework.*

2.4 Generic behavior of abstract frameworks

We know that an abstract framework $G = (V, E)$ can have rigid, rigid but infinitesimally flexible and flexible realizations. For yet another example, see Figure 2.8.

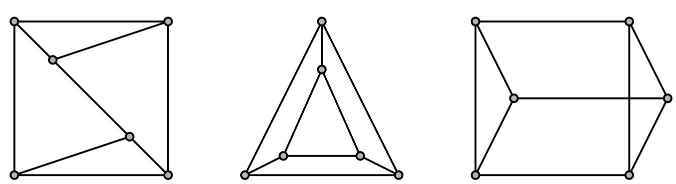

Figure 2.8: An infinitesimally rigid, rigid but infinitesimally flexible and a flexible realization of the triangular prism graph in $\mathbb{E}^2$.

Let's take a closer look at Figure 2.8. The reader may have already noticed that the location of the vertices in the second and third frameworks is rather carefully contrived, while in some sense the first framework represents the typical behavior of a framework of the triangular prism graph in $\mathbb{E}^2$.

This seems to suggest that if we ignore some "singular" realizations, then rigidity (and flexibility) in $\mathbb{E}^d$ can be considered as a property of the abstract framework G, rather than a property of a specific framework.

Definition 2.4.1. *Let $G = (V, E)$ be an abstract framework and let*

$$r_d(G) := max\big\{rank\ \mathcal{R}(p) \mid p \in \mathbb{R}^{d|V|}\big\}$$

Point p will be called a generic point of G iff $rank\ \mathcal{R}(p) = r_d(G)$.

The set of generic points will be denoted by $\mathcal{G}$. We will now establish that *almost all* points are generic, where "almost all" can be interpreted both topologically and measure theoretically.

Definition 2.4.2. *Let $G = (V, E)$ be an abstract framework with $|V| = n$. The d-dimensional indeterminate rigidity matrix of G, denoted by $R(n, d)$ is defined similarly to the rigidity matrix of a realization $G(p)$ for G, where each $(p_i)_j \in \mathbb{R}$ is replaced by the indeterminate $(x_i)_j$.*

For an example of a indeterminate rigidity matrix see Figure 2.3.

$$\begin{bmatrix} . & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & . \\ . & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & . \\ 0 & \cdots & (x_i^1 - x_j^1) & \cdots & (x_i^d - x_j^d) & \cdots & (x_j^1 - x_i^1) & \cdots & (x_j^d - x_i^d) & 0 \\ . & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & . \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & . \end{bmatrix}$$

Table 2.3: The archetypical form of the d-dimensional indeterminate rigidity matrix.

Now, let G be an abstract framework and let $P(x)$ be the polynomial obtained as the sum of the squares of the determinants of all $r_d(G) \times r_d(G)$ submatrices of $R(n, d)$. It is not difficult to verify that:

Lemma 2.4.3. *Point p is a generic point of G iff $P(p) \neq 0$.*

Consequently, the generic points form an *open and dense* subset of $\mathbb{E}^{d|V|}$.

Notice that the preceding discussion implies that our definition of genericity coincides with the analogous concept in algebraic geometry, where a property is said to be generic iff it it holds on the complement of an algebraic variety i.e. the zero-set of a polynomial system.

For $a_1, \ldots, a_n \in \mathbb{R}^d$, let aff$\{a_1, \ldots, a_n\} \subseteq \mathbb{R}^d$ denote the set

$$\text{aff}\{a_1, \ldots, a_n\} = \left\{ \sum \lambda_i a_i \middle| \sum \lambda_i = 1 \right\}$$

i.e. the smallest flat containing $a_1, \ldots a_n$.

Theorem 2.4.4. [4, Section 3] *Let $G = (V, E)$ be an abstract framework and let $p = (p_1, \ldots, p_{|V|}) \in \mathbb{E}^{d|V|}$ be a generic point with $m = \dim \operatorname{aff} \{p_1, \ldots, p_{|V|}\}$. Then*

$$G(p) \text{ is rigid in } \mathbb{E}^d \text{ iff } r_d(G) = d|V| - \frac{(m+1)(2n-m)}{2}$$

and

$$G(p) \text{ is flexible in } \mathbb{E}^d \text{ iff } r_d(G) < d|V| - \frac{(m+1)(2n-m)}{2}$$

Corollary 2.4.5. *Let $G = (V, E)$ be an abstract framework and let $p = (p_1, \ldots, p_{|V|}) \in \mathbb{E}^{d|V|}$ be a generic point with $\operatorname{aff} \{p_1, \ldots, p_{|V|}\} = \mathbb{E}^d$. Then*

$$G(p) \text{ is rigid in } \mathbb{E}^d \text{ iff } r_d(G) = d|V| - \frac{d(d+1)}{2}$$

and

$$G(p) \text{ is flexible in } \mathbb{E}^d \text{ iff } r_d(G) < d|V| - \frac{d(d+1)}{2}$$

Notice that the counts stated in Corollary 2.4.5 are independent of the choice of the specific generic point p. It thus follows that the generic frameworks whose affine span is the entire ambient space, are either all rigid or all flexible.

Since it has been established that the generic points form an open dense subset of $\mathbb{E}^{d|V|}$, it should be clear that Corollary 2.4.5 leads to a notion of *generic* or typical behavior for an abstract framework $G = (V, E)$.

Definition 2.4.6. *An abstract framework G will be called* generically rigid *in $\mathbb{E}^d$ iff $G(p)$ is rigid for one (and thus for all) generic point $p \in \mathbb{E}^{d|V|}$, whose affine span is $\mathbb{E}^d$.*

In other words, given a specific generic framework, its rigidity or flexibility is a property of the structure of the underlying graph, rather than the actual geometry of the embedding. This suggests that there must be some *purely combinatorial way* for determining the generic properties of an abstract framework. This subject is treated in detail in Sections 3.1 and 4.1.

In order to get a feel for the things to come, we will briefly discuss the case $d = 1$. It should be clear that an abstract framework is generically rigid in $\mathbb{E}$ if and only if it is connected. Thus, an an abstract framework G is generically *minimally* rigid in $\mathbb{E}$ iff G is a tree. Moreover, there is a purely combinatorial criterion on the vertices and edges of a connected graph which determine whether it is a tree: the number of edges must equal the number of vertices minus 1.

So for $d = 1$, our suspicion that generic minimal rigidity should be a purely combinatorial property is confirmed.

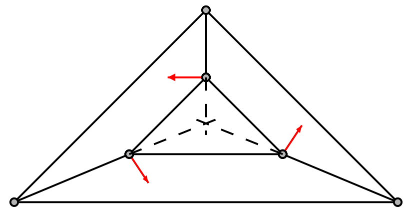

Figure 2.9: A rigid but infinitesimally flexible framework in $\mathbb{E}^2$.

2.5 Equivalence of the notions of rigidity and infinitesimal rigidity

It is natural to ask for the connections (if any) between the notions of rigidity and infinitesimal rigidity. Are they equivalent and if not does one of them imply the other?

It should come as no surprise that if a framework $G(p)$ is infinitesimally rigid then it is also rigid. Indeed, if a framework cannot move even infinitesimally, then it is surely not able to actually move. More formally, suppose for the sake of contradiction that $G(p)$ is infinitesimally rigid but flexible. Then $G(p)$ has a flex χ and the initial velocity assignment of χ to the vertices of $G(p)$ would imply that $G(p)$ is infinitesimally flexible which is absurd.

On the other hand, there exist rigid frameworks that are infinitesimally flexible and thus the two notions are *not equivalent*. One such example is depicted in Figure 2.9. Intuitively, we can think of the infinitesimal motion of the framework in Figure 2.9 as "screwing down" in a counterclockwise direction the inner triangle of the prism while the outer triangle remains fixed.

We should here note that, frameworks which are rigid but infinitesimally flexible are in some sense singular: an infinitesimal flex, while not being realizable in an ideal motion of the framework, will certainly give rise to some swaying or sagging in any physical model of bars and joints. Thus, for engineering applications, infinitesimal rigidity is a superior concept than that of rigidity.

But there is important observation to be made about the framework in Figure 2.9. This framework is clearly *non-generic*, since the three lines defined by adjacent vertices of the inner and outer triangle have a common point of intersection.

It is thus natural to ask for the existence of a *generic* framework which is rigid and in the same time infinitesimally flexible. The answer surprisingly is that this cannot happen. Specifically, we have the following Theorem:

Theorem 2.5.1. [5, Section 3] *Let $G(p)$ be a generic framework in $\mathbb{E}^d$. Then $G(p)$ is rigid iff it is infinitesimally rigid.*

That is, the two notions coincide for generic frameworks.

2.6 Generic global rigidity

Consider the point reconstruction problem: given a set of points in Euclidean 3-space, we wish to determine their relative location (up to rigid motions), where the only information available to us is some *subset* of their pairwise distances. It is clear that in order for this problem to be well defined, the underlying graph of distances has to be generically rigid in $\mathbb{E}^3$. But still, the generic rigidity of the underlying graph is not enough to ensure a *unique* solution to the point reconstruction problem, because as we have seen, a generically rigid graph can have more than one (but in any case finitely many) non-congruent realizations.

If we insist for a *unique solution* to the point reconstruction problem, we have to resort to the the stronger notion of global rigidity. Intuitively, a d-dimensional framework is globally rigid iff it is the *only* framework in $\mathbb{E}^d$ with the same underlying graph and the same edge lengths, up to rigid motions. We note that the problem of recognizing whether an abstract framework is globally rigid is strongly NP-hard [48].

A surprising application of global rigidity to global optimization is highlighted in [35]. Clearly, the point reconstruction problem can be naturally phrased as a non-linear global optimization problem with a cost function that penalizes realizations for unsatisfied distance constraints. A simple example would be the following cost function:

$$f(p_1, \dots, p_n) = \sum_{ij \in E} \left(||p_i - p_j||^2 - d_{ij}^2 \right)^2$$

In [35], B. Henderickson presents an algorithm that follows the divide and conquer paradigm, which enables him to avoid solving the large optimization problem by instead, solving a sequence of smaller ones. This is exactly where global rigidity comes into play. The solutions to the smaller problems can be merged to obtain a solution for the initial optimization problem, because the small optimization problems were defined on globally rigid subcomponents of the underlying graph.

The formal definition of global rigidity follows.

Definition 2.6.1. *Let $G = (V, E)$ be an abstract framework and let $G(p)$ we a realization of G. Then, $G(p)$ is called globally rigid (or uniquely realizable) iff every other realization of G which is equivalent to $G(p)$, is also congruent to it.*

Definition 2.6.2. *An abstract framework $G = (V, E)$ is called generically globally rigid in $\mathbb{E}^d$ iff $G(p)$ is globally rigid for all generic points $p \in \mathbb{E}^{d|V|}$.*

For $d = 1$ it is easy to see that:

Theorem 2.6.3. *An abstract framework is generically globally rigid in $\mathbb{E}$ iff G is 2-vertex connected.*

We continue, with an investigation on conditions that are necessary for generic global rigidity. Clearly, rigidity is a necessary condition for an abstract framework to be globally rigid. On the other hand, there exist rigid graphs which are not globally rigid and thus the two notions are not equivalent. For such an example, see fig. 2.10.

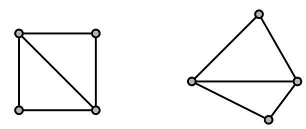

Figure 2.10: A generically rigid abstract framework G in $\mathbb{E}^2$ along with a particular realization. This abstract framework is not generically globally rigid.

This is a simple, yet instructive example that will help us understand why this abstract framework fails to be globally rigid. And the reason is simple enough: given any generic realization of G, one can always reflect the two triangles along their common base, thus obtaining four equivalent but non-congruent embeddings.

Let us now investigate, when can this type of non-uniqueness occur in the general d-dimensional problem. Similarly as before, there must be a few vertices about which a portion of the graph can be reflected and these vertices are said to form a mirror. In order for the reflexion to be possible, there must be no edges between the two parts of the graph separated by the mirror. Clearly, the mirror vertices must lie in a $d-1$ dimensional subspace and moreover notice that in a generic embedding in $\mathbb{E}^d$, at most d vertices can lie on a $d-1$ dimensional subspace. Thus, the size of mirror cannot exceed d. By the definition of a mirror, there can be no edges between the two components separated by the mirror and thus the removal of the vertices of the mirror disconnects the graph.

Consequently, in order to exclude the possibility for the existence of mirrors we need to ensure that there exist no vertex sets of size at most d, whose removal disconnects the graph. Clearly, we have just established the following:

Lemma 2.6.4. [34, Theorem 3.1] *Let $G = (V, E)$ be generically globally rigid in $\mathbb{E}^d$. Then G is $(d+1)$-vertex connected.*

We will now focus our attention for the case $d = 2$. Unfortunately, 3-connectivity is not sufficient to ensure the unique realizability of an abstract framework G in $\mathbb{E}^2$. As an example, consider the Desargues (triangular prism) graph, which is 3-connected but can have as many as 24 embeddings.

Again, let us investigate why the Desargues graph fails to be globally rigid in $\mathbb{E}^2$. The reader is referred to fig. 2.11.

In the leftmost picture of fig. 2.11 we can see a realization of the Desargues framework, where edge af has been removed. Clearly, the quadrilateral $bcde$ is flexible and suppose we start flexing it, where vertices d, f and c follow the orbits illustrated with dashed lines.

This brings us to the middle picture of fig. 2.11, where we have flexed the framework until vertex d is collinear with vertices c and e. We can now start

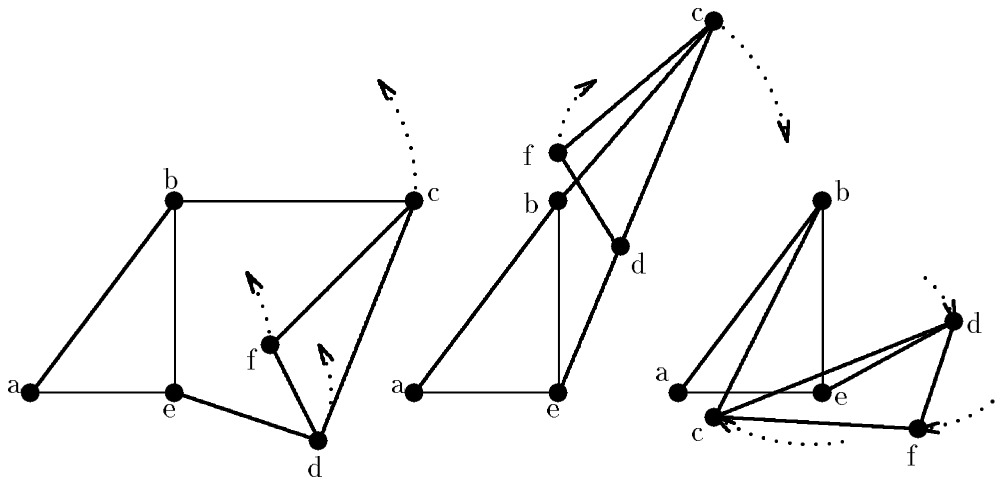

Figure 2.11: Three snapshots of a flex of a Desargues framework, where one of its edges has been removed. The picture is taken from [34].

a new flex of this framework, where again the orbits of vertices f and c are illustrated in dashed lines. Notice that, vertex c can swing all the way around and as it does so, the distance between vertices a and f varies.

Eventually we reach the configuration of the rightmost picture in fig. 2.11, where the distance of vertices a and f is the same as their distance in the leftmost picture of fig. 2.11. This procedure enables us to construct another realization for the Desargues framework, which implies that it is not globally rigid.

Clearly, the problem with the Desargues graph is that once any edge is removed, the resulting graph is generically flexible in $\mathbb{E}^2$. The following definition should come as no surprise.

Definition 2.6.5. *An abstract framework $G = (V, E)$ is called generically redundantly rigid in $\mathbb{E}^d$ iff $G - e$ is generically rigid in $\mathbb{E}^d$, for all $e \in E$.*

Lemma 2.6.6. [34, Theorem 5.9] *Let G be a generically globally rigid graph in $\mathbb{E}^d$ with more than $d+1$ vertices. Then G is redundantly rigid in $\mathbb{E}^d$.*

Combining Lemma's 2.6.4 and 2.6.6 we obtain a necessary condition for a graph to be generically globally rigid in $\mathbb{E}^d$.

Theorem 2.6.7. [34] *Let $G = (V, E)$ be a generically globally rigid graph in $\mathbb{E}^d$. Then, either G is a complete graph on at most $d+1$ vertices, or G is $(d+1)$-connected and redundantly rigid in $\mathbb{E}^d$.*

Hendrickson conjectured that these conditions are also sufficient for the generic global rigidity of an abstract framework in $\mathbb{E}^d$. Since G is generically redundantly rigid in $\mathbb{E}$ iff G is 2-edge connected and since 2-edge connectivity implies 2-vertex connectivity, Theorem 2.6.3 implies that this conjecture is true for $d = 1$. On the other hand, in [15] R. Connelly showed that the conjecture is false for $d = 3$, by constructing a generic realization of the $K_{5,5}$ graph, which is not globally rigid. Similar results hold for larger bipartite graphs in higher dimensions. These appear to be the only known examples of graphs that satisfy Hendrickson's condition and are not globally rigid.

On the other hand, using a combination of arguments due to R. Connelly [15], B. Jackson and T. Jordán [37], it can be shown that Hendrickson's condition is also sufficient, for generic global rigidity in $\mathbb{E}^2$. We will present a rough sketch of the proof of this fact.

The first step in proving the sufficiency of Hendrickson's condition for $d = 2$, is the following Theorem due to B. Jackson and T. Jordán. This Theorem essentially states that the class graphs that are 3-connected and generically redundantly rigid in $\mathbb{E}^2$, have Henneberg constructions, similar to those for Laman graphs. The operation of an edge addition adds one new edge between two non-adjacent vertices.

Theorem 2.6.8. [37, Theorem 6.15] *Let $G = (V, E)$ be an abstract framework which is 3-connected and generically redundantly rigid framework in $\mathbb{E}^2$. Then G can be built up inductively from the K_4 graph by a sequence of edge additions and H_2 steps.*

Now, notice that the K_4 graph is globally rigid in $\mathbb{E}^2$ and moreover the operation of an edge addition *preserves* generic global rigidity in $\mathbb{E}^2$. So, in view of Theorem 2.6.8 it is enough to show that the H_2 operation also preserves generic global rigidity in $\mathbb{E}^2$. This was accomplished by R. Connelly in [15].

Theorem 2.6.9. [15, Theorem 1.5] *Let G be generically globally rigid in $\mathbb{E}^2$ and let $G^{'}$ be obtained from G through a H_2 step. Then, $G^{'}$ is also generically globally rigid in $\mathbb{E}^2$.*

Combining Theorems 2.6.7, 2.6.8 and 2.6.9, we obtain the following characterization for generically globally rigid graphs in $\mathbb{E}^2$.

Corollary 2.6.10. *An abstract framework $G = (V, E)$ is generically globally rigid in $\mathbb{E}^2$ iff either G is a complete graph on at most three vertices or G is 3-vertex connected and redundantly rigid in $\mathbb{E}^2$.*

Notice that Corollary 3.1 implies that global rigidity in $\mathbb{E}^2$ is a generic property.

Only recently, a complete characterization for generic globally rigid graphs in $\mathbb{E}^d, d \geq 3$ was found. Specifically we have the following:

Theorem 2.6.11. [16, 31] *An abstract framework $G = (V, E)$ with $|V| \geq d + 2$ is generically globally rigid in $\mathbb{E}^d$ iff rank $G = d|V| - d(d + 1)$, where G is the so-called "Gauss map" that takes each smooth point of the image of the rigidity map to its tangent space.*

Chapter 3

Generic rigidity in $\mathbb{E}^2$

In this section we deal with the theory of rigid graphs in 2 dimensions. This is a case that has been extensively studied and is fully-understood.

In Section 3.1 we will present a number of combinatorial characterizations of minimally rigid graphs in $\mathbb{R}^2$, some for their historical importance, some for their algorithmic implicatons and some whose generalizations might prove to be useful in obtaining a combinatorial characterization of minimally rigid graphs in $\mathbb{R}^3$.

In Section 3.2 we elaborate on the connections between Laman graphs and pseudo-triangulations. Specifically we will show that the underlying graph of a pointed pseudo-triangulation is a Laman graph and moreover that every planar Laman graph can be embedded in $\mathbb{E}^2$ as a pointed pseudo-triangulation.

Lastly, in Section 3.3, we deal with the Laman decision problem: given an abstract framework G, decide whether it is generically minimally rigid in $\mathbb{E}^2$ or not. We will summarize the various algorithms invented in order to deal with this problem and we will discuss their complexity.

3.1 Combinatorial characterization

Every rigid body constrained to move in the plane has 3 *internal* degrees of freedom that correspond to translations and rotations. On the other hand, $|V|$ vertices have $2|V|$ degrees of freedom to begin with and since each edge removes (at most) one degree of freedom it is clear that a necessary condition for G to be generically rigid in $\mathbb{E}^2$ is that $|E| = 2|V| - 3$.

This condition fails to be sufficient as can be established from the counterexample in Figure 3.1. This is an abstract framework that is generically flexible in $\mathbb{E}^2$, although it has the right number of edges. Upon closer inspection it should be clear that the reason it is flexible, is that its edges are not *well distributed.* Specifically, the abstract framework in Figure 3.1 consists of a flexible square (left component) attached to a redundantly rigid K_4 (right component).

Notice however that if we take one of the edges from the redundantly rigid

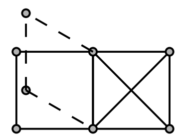

Figure 3.1: A framework that has the right number of edges but is flexible. A flex of the framework is illustrated in dashed lines.

component of the above graph and attach it to its flexible component, the resulting graph (see Figure 3.2) is generically rigid in $\mathbb{E}^2$.

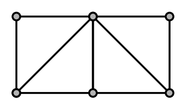

Figure 3.2: A framework with the right number of edges which is rigid, because this time its edges are properly distributed.

Clearly, what we need is $2|V|-3$ *well distributed* edges and this is the basic intuition behind Theorem 3.1.2. Before stating Laman's Theorem we need need the following definition:

Definition 3.1.1. *An abstract framework $G = (V, E)$ is called generically minimally rigid in $\mathbb{E}^d$ iff G becomes generically flexible in $\mathbb{E}^d$ once any edge is removed. Equivalently, an abstract framework $G = (V, E)$ is generically minimally rigid in $\mathbb{E}^d$ iff $G(p)$ is isostatic, for one generic point $p \in \mathbb{E}^{d|V|}$.*

It is worth noticing that in view of the definition above, an abstract framework $G = (V, E)$ is generically rigid in $\mathbb{E}^d$ iff G contains a generically minimally rigid subgraph in $\mathbb{E}^d$. Thus, for the rest of this essay, we will restrict our attention to the class of generically *minimally* rigid frameworks.

We are now ready to state Laman's Theorem. We notice that the necessity of the $2n-3$ counts was already known to J.C. Maxwell. Their sufficiency was proved over 100 years later by G. Laman.

Theorem 3.1.2. [39, 42] *An abstract framework $G = (V, E)$ is generically minimally rigid in $\mathbb{E}^2$ iff*

- $|E| = 2|V| - 3$
- $|E^{'}| \leq 2|V^{'}| - 3$ *for all vertex induced subgraphs* $(V^{'}, E^{'})$*, with* $|V^{'}| \geq 2$

An abstract framework $G = (V, E)$ satisfying the counts above will be said to be Laman or to have the Laman property.

We continue with some characterizations of Laman graphs in terms of cospanning trees. We start with a characterization due to L. Lovasz and Y. Yemini.

Theorem 3.1.3. [40] *The abstract framework $G = (V, E)$ is minimally rigid in $\mathbb{E}^2$ iff for every pair of vertices a, b in V, the multigraph $G^{ab} = (V, E \cup \{a, b\})$ obtained by adding an edge between vertices a, b (even if edge ab was already present) is the edge-disjoint union of two spanning trees.*

A. Recksi has proved a significant refinement of Theorem 3.1.3. Specifically, he shows that it suffices to check the graphs G^{ab}, where *ab is already* an edge of G. In order to prove this, we need to introduce some additional terminology.

Let $G = (V, E)$ an abstract framework and let P be a partition of V. By $E_P(G)$ will denote the set of those edges of G which join vertices belonging to different members of P, and G_P will denote the graph with $V(G_P) = P$, $E(G_P) = E_P(G)$ and an edge of G_P joins in G_P those members of P to which its end-vertices in G belong. We think of G_P as being obtained from G by shrinking each member of P to a single vertex.

The following Theorem was proved in 1961 independently by W.T. Tutte and C. Nash-Williams.

Theorem 3.1.4. [54], [44, Theorem 1] *A graph $G = (V, E)$ has k edge-disjoint spanning trees iff*

$$|E_P(G)| \geq k(|P| - 1)$$

for every partition P of V.

Using Theorem 3.1.4, we can now prove the following:

Theorem 3.1.5. *Let $G = (V, E)$ be an abstract framework. Then G consists of 2 edge-disjoint spanning trees iff the following hold:*

- $|E| = 2|V| - 2$
- $|E^{'}| \leq 2|V^{'}| - 2$ *for all vertex induced subgraphs* $(V^{'}, E^{'})$

Proof. Suppose G is the union of 2 edge-disjoint spanning trees and let $V = \{v_1, \ldots, v_n\}$. Since the two trees are edge-disjoint and spanning it is clear that $|E| = 2|V| - 2$. Let $(V^{'}, E^{'})$ be a vertex induced subgraph and without loss of generality suppose that $V^{'} = \{v_1, \ldots, v_k\}$. By Theorem 3.1.4 it follows that $|E_P(G)| \geq 2(|P| - 1)$ for every partition P of V so in particular this is also the case for the partition $P = \left\{V^{'}, \{v_{k+1}\}, \ldots, \{v_n\}\right\}$. But $|E_P(G)| = |E| - |E^{'}|$ and $|P| = |V| - |V^{'}| + 1$ so the condition $|E_P(G)| \geq 2(|P| - 1)$ implies that $|E^{'}| \leq 2|V^{'}| - 2$.

On the other hand suppose that G satisfies the counts implied by the Theorem. We need to show that $|E_P(G)| \geq 2(|P| - 1)$ for every partition $P = \{P_1, \ldots, P_k\}$ of V. By the hypothesis for each of the induced subgraphs (P_i, E_i) it is true that $|E_i| \leq 2|P_i| - 2$, $\forall i = 1, \ldots, k$. So

$$\sum_{i=1}^{k} |E_i| \leq 2(|P_1| + \ldots |P_k|) - 2|P|$$

and since P is a partition of V

$$\sum_{i=1}^{k} |E_i| \leq 2|V| - 2|P|$$

So it follows that

$$|E_P(G)| = |E| - \sum_{i=1}^{k} |E_i| \geq 2|V| - 2 - 2|V| + 2|P| = 2(|P| - 1)$$

□

Notice that there is a striking similarity between Theorem 3.1.5 and Laman's Theorem, so the following characterization of minimally rigid graphs in $\mathbb{E}^2$ due to A. Recksi should come as no surprise.

Theorem 3.1.6. [47] *An abstract framework $G = (V, E)$ is minimally rigid in $\mathbb{E}^2$ iff for every edge $ij \in E$ the multigraph G^{ij} obtained by doubling edge ij is the union of two edge-disjoint spanning trees.*

Proof. Suppose that G is minimally rigid in $\mathbb{E}^2$ and let $ij \in E$. Consider the multigraph $G^{ij} = (V^{ij}, E^{ij}) = (V, E \cup ij)$, obtained by adding one more copy of edge ij to G. In order to show that G^{ij} is the edge disjoint union of two spanning trees, it suffices to show that it satisfies the counts implied by Theorem 3.1.5. Since G is Laman, it follows that $|E| = 2|V| - 3$ and thus $|E^{ij}| = 2|V^{ij}| - 2$. Now, let $(V^{'}, E^{'})$ be a vertex induced subgraph of G^{ij}. Since G is Laman there are at most $2|V^{'}| - 3$ edges of G incident with the vertices in $V^{'}$, and thus $|E^{ij}| \leq 2|V^{ij}| - 2$ in G^{ij}.

For the other direction, suppose that for every edge $ij \in E$ the multigraph obtained by doubling edge ij is the union of two edge-disjoint spanning trees. So, if we fix an edge $ij \in E$ and double it, the resulting graph G^{ij} will satisfy the counts implied by Theorem 3.1.5. Thus, $|E| = |E^{ij}| - 1 \leq 2|V^{ij}| - 3 = 2|V| - 3$. Now, let $(V^{'}, E^{'})$ be a vertex induced subgraph of G and let $kl \in E^{'}$. By the hypothesis there are at most $2|V^{'}| - 2$ edges of G^{kl} incident with the vertices in $V^{'}$ and thus $|E^{'}| \leq 2|V^{'}| - 3$ in G. □

Before we state and prove the last characterization of Laman graphs due to H. Crapo we need to introduce some new terminology and to prove some preliminary results. An *nTk partition* of a graph G consists of n trees $T_i = (V_i, E_i)$, such that the edge set of G can be expressed as the disjoint union of the trees T_i and every vertex of G belongs to precisely k of them. See Figure 3.3 for an example. A graph G will be called nTk iff it has an nTk partition. An nTk partition of graph G will be called *proper* iff there are no non-trivial (having at least one edge) subtrees of the T_i that have the same underlying vertex set.

Proposition 3.1.7. *Let $G = (V, E)$ be an nTk graph. Then $|E| = k|V| - n$.*

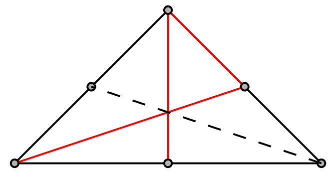

Figure 3.3: A proper 3T2 partition of the $K_{3,3}$ graph.

Proof. Let $T_1 \ldots, T_n$ be the trees of the nTk partition where $T_i = (V_i, E_i)$. By the definition of the nTk partition, each vertex belongs to precisely k of the trees so $\sum |V_i| = k|V|$. Since E is the disjoint union of the trees T_i it follows that

$$|E| = \sum (|V_i| - 1) = k|V| - n$$

□

Proposition 3.1.8. *Let $G = (V, E)$ a graph with the Laman property. Then G is a $3T2$ graph.*

Proof. Since G is Laman, by Theorem 5.1.4 it has a Henneberg-2 construction $G_1, \ldots, G_n$, where G_1 is the single edge $e = uv$. We inductively construct the $3T2$ partition as follows: For the base case $i = 1$ we define

$$T_1 = \{e\},\ T_2 = \{u\},\ T_3 = \{v\}$$

Suppose now that we have constructed a $3T2$ partition for G_i and that G_{i+1} is obtained from G_i through a H_1 step i.e. we add a new vertex n that we connect to vertices a, b of G_i. By the induction, G_i has a $3T2$ partition T_1, T_2, T_3, and by the definition of a $3T2$ partition each vertex belongs to exactly two of the trees. In particular, let $T_a = \{T_{i_1}, T_{i_2}\},\ T_b = \{T_{j_1}, T_{j_2}\}$ be the sets of trees that vertices a, b belong to, respectively. We then add edge na to the tree $T_a \cap T_b$ and edge nb to the tree $T_b - T_a \cap T_b$.
Lastly, suppose that G_{i+1} is obtained from G_i through a H_2 step i.e. we add a new vertex n that we connect to vertices $\{a, b, c\}$ of G_i and additionally we remove edge ab of G_i. By the definition of a $3T2$ partition, every edge of G_i belongs to exactly one tree so we can assume without loss of generality that $ab \in T_1$. Since the removal of edge ab from G_i destroys the connectivity of T_1 it is clear that edges na, nb should be added to T_1. Let T_c be the set of trees to which vertex c belongs. If $T_1 \notin T_c$ then we add edge nc to tree T_1. On the other hand, if $T_1 \in T_c$ then we place edge nc to either T_2 or T_3. □

Theorem 3.1.9. [19, Theorem 1] *Let $G = (V, E)$ be a graph. Then G is minimally rigid in $\mathbb{E}^2$ iff G has a proper $3T2$ partition.*

Proof. Let G be minimally rigid in $\mathbb{E}^2$ and for the sake of contradiction suppose that G has no proper $3T2$ partition. By Theorem 3.1.2 G has the Laman

property and thus by Proposition 3.1.8 G has a $3T2$ partition T_1, T_2, T_3, which cannot be proper by the hypothesis. So there exist $T_i^{'}, T_j^{'}$ non-trivial subtrees of T_i, T_j respectively, that share the same underlying vertex set $V^{'}$. Since the T_j's are edge disjoint it follows that the subgraph of G induced by the vertex set $V^{'}$ has at least $2|V^{'}| - 2$ edges, which is a contradiction since the Laman condition is violated.
On the other hand, suppose that G has a proper $3T2$ partition T_1, T_2, T_3. Proposition 3.1.7 implies that $|E| = 2|V| - 3$. Now, assume that G is not minimally rigid in $\mathbb{E}^2$. By Theorem 3.1.2, G will not have the Laman property so let $(V^{'}, E^{'})$ be a subgraph of G with $|E^{'}| \geq 2|V^{'}| - 2$. Let $T_i^{'} = (V_i^{'}, E_i^{'}),\ i = 1, 2, 3,$ be the subgraphs of T_i induced by the vertex set $V^{'}$. Our goal is to show that two of the $T_i^{'}$ are trees and one of them is empty. Keeping that in mind we define c_i to be the number of connected components of $T_i{'}$. It is then clear that $|E_i^{'}| = |V_i^{'}| - c_i$ and thus

$$|E^{'}| = \sum_{i=1}^{3} |E_i^{'}| = \sum_{i=1}^{3} \left(|V_i^{'}| - c_i\right) = 2|V| - \sum_{i=1}^{3} c_i$$

so

$$\sum_{i=1}^{3} c_i \leq 2$$

Notice that at least two of the c_i's should be strictly positive for otherwise, if two of the c_i's, say c_1, c_2, were zero then $T_1^{'}, T_2^{'}$ would be empty which is absurd since by the definition of a $3T2$ partition every vertex of G belongs to exactly two trees. So exactly two of the c_i's are equal to 1 and one of them equals 0 which in turn means that two of the $T_i^{'}$ are trees and one of them is empty. But since each vertex belongs to two trees it follows that the two non-empty trees have the same underlying set $V^{'}$ which is absurd by the definition of a proper $3T2$ partition. □

3.2 Connections with pseudo-triangulations

Pseudo-triangulations are relatively new objects, initially introduced in the Computational Geometry community for tackling problems such as visibility [46, 45], kinetic data structures [1] and motion planning for robot arms [51]. These exhibit rich combinatorial, rigidity theoretic and polyhedral properties, many of which have only recently have started to be investigated. For example, once any convex hull edge is removed from a pseudo-triangulation, it becomes an expansive 1dof mechanism i.e. as it moves, the distance between any pair of vertices never decreases. Expansive motions were a crucial ingredient in the solution of the Carpenter's rule problem [17]

In this Section we will show that the underlying graph of a pointed pseudo-triangulation is a Laman graph (Corollary 3.2.7) and that any planar Laman graph can be embedded in $\mathbb{E}^2$ as a pointed pseudo-triangulation (Theorem 3.2.8). We begin our study of pseudo-triangulations with some necessary definitions.

Definition 3.2.1. *An angle ω will be called convex, straight or reflex iff it is strictly smaller, equal to or larger than π, respectively.*

Definition 3.2.2. *A vertex v of a polygonal region R will be called convex, straight of reflex iff the angle spanned by its two incident edges that faces the interior of R is convex, straight or reflex respectively.*

We note that, general position for the vertices which we will usually assume, implies *absence* of straight angles.

Definition 3.2.3. *A simple polygon with exactly k convex vertices (and an arbitrary number of reflex vertices) will be called a pseudo-k-gon. The k convex vertices will be called the corners of the pseudo-k-gon.*

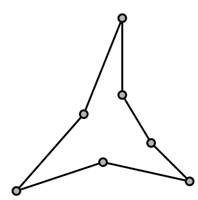

Figure 3.4: A pseudo-triangle.

Notice that triangles are pseudo-triangles but the converse is obviously false.

Definition 3.2.4. *A vertex v will be called pointed iff one of the angles spanned by consecutive edges incident to v is reflex. Otherwise v will be called non-pointed.*

Definition 3.2.5. *A pseudo-triangulation of a finite pointset $P = \{p_1, \ldots, p_n\}$ is a planar subdivision of the convex hull of P into pseudo-triangles.*

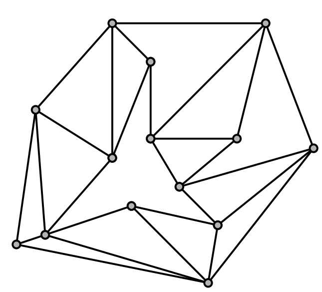

Figure 3.5: A pseudo-triangulation of a pointset.

A pointed pseudo-triangulation is one in which every vertex is pointed.

The following Theorem exhibits the combinatorial properties of pointed pseudo-triangulations which imply useful rigidity theoretic consequences: they are Laman graphs.

Theorem 3.2.6. [52, Theorem 2.3] *Let $G = (V, E)$ be a graph embedded on the pointset $P = \{p_1, \ldots, p_n\}$. The following are equivalent:*

1. *G is a pointed pseudo-triangulation of P.*
2. *G is a pseudo-triangulation of P with the minimum possible number of edges.*
3. *G is a pseudo-triangulation of P with $2n-3$ edges.*
4. *G is planar, pointed and has $2n-3$ edges.*

Proof. Let $|V| = \{v_1, \ldots, v_n\}, |E| = e$ and let f denote the number of interior faces of G. Also, let $d_i = deg\ v_i$ (notice that is also the number of angles incident to v_i) and c_i the number of convex angles incident to v_i. Clearly, if v_i is pointed then $c_i = d_i - 1$ and if v_i is non-pointed then $c_i = d_i$. Let A denote the set of pointed vertices of G and B the set of non-pointed vertices of G, where $b = |B|$ (and thus $|A| = n - b$). It then follows that

$$2e = \sum_{i=1}^{n} d_i = \sum_{u_i \in A} (c_i + 1) + \sum_{v_i \in B} c_i = \left(\sum_{u_i \in A} c_i + \sum_{u_i \in B} c_i \right) + \sum_{u_i \in A} 1 = \sum_{i=1}^{n} c_i + (n - b)$$

Since G is a pseudo-triangulation, every face of its G is a pseudo-triangle and so it has exactly 3 convex corners. It is then clear that $\sum_{i=1}^{n} c_i = 3f$ and thus

$$2e = 3f + (n - b)$$

Applying Euler's formula to G we have that $n - e + (f + 1) = 2$ (notice that we need to add 1 to f, because f is the number of interior faces of G) and using this to eliminate f from the formula above we obtain that

$$e = 2n - 3 + b \tag{3.1}$$

$1 \Leftrightarrow 2 \Leftrightarrow 3$

Since b is non-negative, formula 3.1 implies that the minimum possible number of edges is $2n - 3$ and this achieved iff $b = 0$, which by the definition of b means that G is pointed.

$3 \Rightarrow 4$

Since G is a pseudo-triangulation with $2n - 3$ edges, formula 3.1 implies that $b = 0$ and thus G is pointed. Additionally every pseudo-triangulation is planar by definition, so we are done.

$4 \Rightarrow 3$

Since $e = 2n - 3$, using Euler's formula we obtain that the number of interior faces is $f = n - 2$. Additionally, since G is pointed (thus $B = \emptyset$) the total number of corners of G is equal to

$$c = \sum_{i=1}^{n} c_i = \sum_{i=1}^{n} (d_i - 1) = 2e - n = 3(n - 2)$$

On the other hand, every interior face in a planar graph must have at least 3 inner convex angles. Since G has $n-2$ interior faces it follows that each one of them must have exactly 3 convex angles and thus every interior face of G is a pseudo-triangle. Moreover, G is planar by the hypothesis so it follows that G is a pseudo-triangulation. □

Using Theorem 3.2.6 we can now prove the following:

Corollary 3.2.7. [52, Corollary 2.4] *The underlying graph of a pointed pseudo-triangulation of a pointset $P = \{p_1, \ldots, p_n\}$ is a planar Laman graph.*

Proof. Let $G = (V, E)$ be the underlying graph of a pointed pseudo-triangulation of the pointset P. Theorem 3.2.6 implies that $|E| = 2|V| - 3$ and it remains to show that every vertex induced subgraph of G with k vertices has $\leq 2k-3$ edges. Notice that the properties of planarity of pointedness are hereditary in the sense that, if G is planar and pointed, the same is true for all of its vertex induced subgraphs. Clearly, Theorem 3.2.6 implies that no induced subgraph of G on k vertices can have more than $2k-3$ vertices and thus the claim follows. □

A natural question to ask is if the converse of Corollary 3.2.7 is also true.

Theorem 3.2.8. [33, Theorem 1] *Let $G = (V, E)$ be a planar Laman graph. Then G can be embedded into $\mathbb{E}^2$ as a pointed pseudo-triangulation.*

Before giving the details of the proof we need to introduce the concept of a *combinatorial pseudo-triangulation*, abbreviated as cpt. Intuitively, a cpt can be thought of as an abstract model of a pseudo-triangulation, which carries only the combinatorial information (incidences, convex/reflex angles etc.) of a pseudo-triangulation. A formal definition follows.

Definition 3.2.9. *Let G be a planar 2-connected graph. A combinatorial pseudo-triangulation (cpt) of G is an assignment of labels (reflex and convex) to the angles of G such that:*

- *Every face, excluding the outer one, gets exactly three angles labeled convex.*
- *The angles incident to the outer face receive only reflex labels.*
- *Each vertex is incident to at most one angle labeled big. If a vertex is incident to a big angle then the vertex will be called pointed.*
- *A degree-2 vertex is incident to exactly one angle labeled big.*

The proof of Theorem 3.2.8 goes as follows: we first show that the problem reduces to the case where the outer face is a triangle. Then we establish that every planar Laman graph has a Henneberg construction, with all intermediate graphs being also planar. After that we show that every planar Laman graph admits a cpt assingment (not necessarily unique). Lastly, we show that at least one of the computed cpt's is realizable with straight lines.

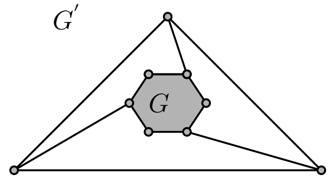

Figure 3.6: Reducing to the case where the outer face is a triangle.

Lemma 3.2.10. [33, Lemma 6] *Embedding a planar Laman graph as a pseudo-triangulation, reduces to the case when the outer face is a triangle.*

Proof. Let $G = (V, E)$ be a planar Laman graph, whose outer face has more than 3 vertices. We construct a new Laman graph $G^{'}$ on $|V| + 3$ vertices as follows: we add 3 new vertices, which form a triangle, to the outer face of G so that the original graph is contained in the triangle's interior. We then join each triangle vertex with arbitrary vertex on the boundary of G. The reader is referred to Figure 3.2 for an example of this procedure.

We claim that if $G^{'}$ is realized as a pseudo-triangulation with the new triangle as its outer face, then G is also realized as a pseudo-triangulation. Clearly, it suffices to show that in the realization of $G^{'}$, the outer face of G is in convex position. The way to do that is by showing that all angles of G incident to its outer face are reflex. But, the three new interior edges of $G^{'}$ provide two convex angles at their endpoint incident to the outer face and at least one convex angle at their other endpoint. Since the three pseudo-triangles that surround G have in total 9 convex angles, the claim follows. □

Lemma 3.2.11. [33, Lemma 8] *Every planar Laman graph admits a combinatorial pseudo-triangulation assignment.*

Proof. The proof goes by induction on the number of vertices of a planar Laman graph. In what follows, we will depict a face of a cpt as a circle with its 3 convex angles marked with black dots. Unmarked angles are reflex.

The base case is a triangle and it has a unique cpt labelling.

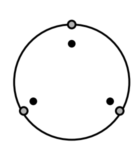

Figure 3.7: The unique cpt of a triangle.

Suppose now the claim is true for planar Laman graphs on n vertices. We know that a planar Laman graph on $n + 1$ vertices is obtained by a planar Laman graph on n vertices through a H_1 or H_2 step. Thus, it suffices to show

how to extend a cpt labelling in order to facilitate the addition of a new vertex through a H_1 or a H_2 step. These two cases are treated separately.

In a H_1 step, a new vertex v is inserted in a face F (which by induction already has a cpt labelling) and joined to two old vertices v_1, v_2. The new edges vv_1, vv_2 partition the face F into two new ones. We need to consider the following cases:

If neither of v_1, v_2 is among the convex vertices of F, the convex vertices of F can be split between the two new faces either as 2+1 or 3+0 (see Figure 3.8).

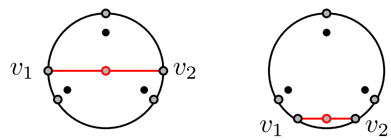

Figure 3.8: Neither of v_1, v_2 is among the convex vertices of F.

If one of them (say v_1) is among the convex vertices of F then the other two convex vertices are either both in the same new face or they are separated (see Figure 3.9).

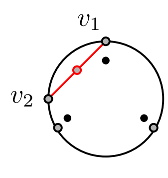

Figure 3.9: Only v_1 is a convex vertex of F.

Last case to consider is when both of v_1, v_2 are convex vertices of F. The remaining vertex is in one of the newly created faces (see Figure 3.10).

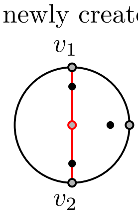

Figure 3.10: Both of v_1, v_2 are convex vertices of F.

In either one of these cases, the assignment of convex and reflex labels is what one would expect: a convex vertex is split in two convex ones, a reflex vertex is split in a reflex one and a convex one and the new point gets exactly one reflex angle. A representative case is illustrated in Figure 3.11.

In a H_2 step, edge v_1v_2 is first removed, merging two faces of the existing cpt into a new face F. This process forces some pairs of angles to merge into one. The rules for assigning labels to merged angles are natural: if one of the angles

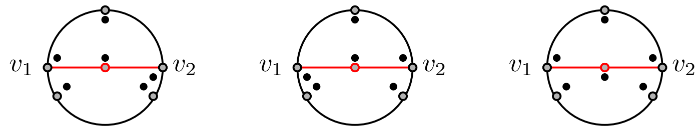

Figure 3.11: A representative case of the labeling process. These are the three possible cpt extensions of the first cpt in Figure 3.8.

that merged together was reflex than the resulting angle is reflex, otherwise it is convex. See Figure 3.12 for a representative case.

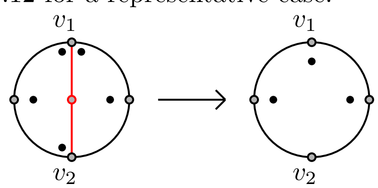

Figure 3.12: A representative case of the merging process.

Therefore face F has *exactly* four convex angles. The boundary of F is separated by v_1 and V_2 into two chains, each of which contains at least one convex vertex. Depending on whether v_1, v_2 are convex or reflex, there are 4 cases to consider (see Figure 3.13): both of them are convex and the other two convex vertices are distributed as 1-1 to the two chains; one of them is convex and the other three convex vertices are distributed as 1-2 on the chains; both of them are reflex and the four convex vertices are distributed as either 2-2 or 1-3 to the chains.

For the last part of the H_2 step, a new vertex v is inserted into face F and gets connected to v_1, v_2 and some other arbitrary vertex v_3. The new edges vv_1, vv_2, vv_3 partition F into three parts and a tedious but straightforward case analysis shows that in each one of the four possible cases mentioned above, a cpt labeling can be constructed. See Figure 3.14 for a representative case. $\square$

Finally, combining Lemma's 3.2.10 and 3.2.11 we obtain that:

Theorem 3.2.12. *Every planar Laman graph can be embedded as a pseudo-triangulation.*

The proof follows the same analysis as that of Lemma 3.2.11 and is omitted. The details of the proof can be found in [33].

3.3 Algorithms for generic 2-rigidity

There is a number of questions of algorithmic nature, concerning generically rigid graphs in $\mathbb{E}^2$, the most important being the following decision problem: given a graph G determine whether it is Laman i.e. generically minimally rigid

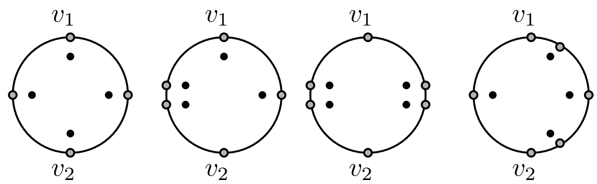

Figure 3.13: The four possible cases after the removal of edge v_1v_2.

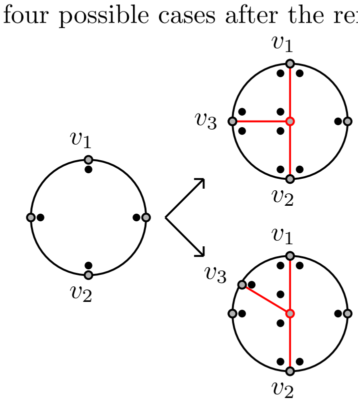

Figure 3.14: The two possible cpt labelings, following a H_2 step.

in $\mathbb{E}^2$. Other problems of interest include the construction of a Henneberg 2-sequence for a given Laman graph G and additionally the identification of the rigid and flexible subcomponents of a generically flexible graph G. We will focus our attention on the Laman decision problem. It should be clear by now that Laman graphs is an extensively studied and well understood class of graphs, so the existence of efficient polynomial time algorithms for all these problems should come as no surprise.

We have seen that there is a number of characterizations of Laman graphs and each of them leads to an associated algorithm for verifying generic minimal rigidity in the plane. Clearly, some of them are better suited for algorithmic verification than others. For purposes of comparison we compile a list of them here, in the form of a Theorem.

Theorem 3.3.1. *For a graph $G = (V, E)$ with $|V| \geq 2$ the following are equivalent:*

1. *G is generically minimally rigid in $\mathbb{E}^2$;*
2. *there exists a Henneberg 2-sequence for G;*
3. *$|E| = 2|V| - 3$, and for each subgraph $G^{'} = (V^{'}, E^{'})$ with $|V^{'}| \geq 2, |E^{'}| \leq 2|V^{'}| - 3$;*
4. *for every pair of vertices a, b in V, the multigraph $G^{ab} = (V, E \cup ab)$ obtained by adding an edge between vertices a, b is the edge-disjoint union of two spanning trees;*

5. *for every edge $ij \in E$ the multigraph G^{ij} obtained by doubling edge ij is the union of two edge-disjoint spanning trees;*

6. *for each edge ab, the graph $G^* = (V^*, E^*)$ obtained by adding two loops at vertex a and one loop at vertex b satisfies $|E^*| = 2|V^*|$ and $|E^{''}| \leq 2|V^{''}|$ for all subgraphs $G^{''} = (V^{''}, E^{''})$;*

7. *for each edge ab the graph $G^* = (V^*, E^*)$ obtained by adding two loops at vertex a and one loop at vertex b contains two edge disjoint matchings of edges to vertices;*

8. *G has a proper $3T2$ partition;*

9. *G admits a red-black hierarchy (RBH);*

10. *$|E| = 2|V| - 3$ and for each $e \in E$ the multigraph G^{4e} obtained by quadrupling (add three additional copies) of e has no induced subgraph $G^{'}$ with $|E^{'}| > 2|V^{'}|$;*

Clearly, characterization (3) is not suited for algorithmic verification of Laman graphs, because checking whether a graph satisfies the Laman counts leads to a poor algorithm since it involves counting the edges in every subgraph, of which there is an exponential number.

Characterizations (4) and (5), are given in terms of cospanning trees. These characterizations are better suited for algorithmic verification, thanks to the existence of polynomial time algorithms for decomposing a graph into two spanning trees [23, 29]. This approach leads to the best known algorithm for the Laman decision problem, which runs in time $O(n\sqrt{nlogn})$.

Characterizations (6) and (7) were employed by K. Sugihara [53] and led to the first polynomial time algorithm for determining the independence of a set of edges in 2 dimensions.

Characterization (9) is due to S. Bereg. The RBH for a graph G is a hierarchical decomposition of the graph into trees which is a certificate for generic rigidity in the plane [6, Corollary 4]. Moreover, an RBH can be constructed in $O(n^2)$ time [6, Theorem 3] and a hierarchy can be verified to be RBH in $O(n)$ time [6, Lemma 5]. Having an RBH of a Laman graph G, enables us to compute its Henneberg 2-sequence in $O(n^2)$ time. Additionally, recall that Theorem 3.2.8 implies that every planar Laman graph G can be embedded in the plane as a pointed pseudotriangulation. By [6, Theorem 8] it follows that using a RBH for G, such an embedding can be computed in $O(n^2)$ time, speeding up a recent algorithm [33, Section 3.1] by a factor of $O(n)$. O. Daescu and A. Kurdia extend these results and obtain an algorithm for verifying Laman graphs which runs in $O(T_{st}(n) + nlogn)$ time, where $T_{st}(n)$ is the best time to extract two edge disjoint spanning trees from G or decide that no such trees exist [20, Theorem 4.3]. Moreover, they speed up the construction of an RBH to $O(nlgn)$.

Characterization (10) leads to a very simple and elegant algorithm first proposed by B. Hendrickson and J. Jacobs [36] and generalized by I. Streinu, A. Lee and L. Tehran in a number of papers. The basic idea behind the algorithm is to grow a maximal set of independent edges $\hat{E}$ one at a time. A new edge will be added to $\hat{E}$ iff it is independent of the existing set. If $2|V|-3$ independent edges are found then by Laman's theorem G will be rigid. So the key to an efficient algorithm is to be able to determine easily whether or not a new edge is independent of $\hat{E}$. Characterization (10) implies that we can do that by quadrupling each edge of $\bar{G}=(V,\hat{E}\cup e)$ and then checking that no subgraph has too many edges. But in fact we can do much better than that. The following Lemma shows that it is enough to quadruple only edge e.

Lemma 3.3.2. *A new edge e is independent of $\hat{E}$ iff the graph G^{4e} has no induced subgraph with $|E^{'}|>2|V^{'}|$.*

Proof. For the first direction, let us assume that G^{4e} has no induced subgraph with $|E^{'}|>2|V^{'}|$ and for the sake of contradiction suppose that e is not independent of $\hat{E}$. It then follows from characterization (10) that there exists an edge $e^{'}\in\hat{E}\cup e$ whose quadrupling creates a subgraph of $G^{4e^{'}}$ denoted by $G^{'}=(V^{'},E^{'})$, with $|E^{'}|>2|V^{'}|$. Since the edges in $\hat{E}$ are independent, characterization (10) implies that edge e belongs to $E^{'}$. We now consider the following two cases: edge $e^{'}$ either belongs to $E^{'}$ or not.

- If $e^{'}\in E^{'}$ then since $G^{'}$ is an induced subgraph, all 4 copies of $e^{'}$ belong to $E^{'}$. Consider the graph $G^{''}=(V^{'},\{E^{'}-3e^{'}\}\cup 3e)$ obtained from $G^{'}$ by substituting 3 copies of edge $e^{'}$ with 3 copies of edge e. But clearly $G^{''}$ is a subgraph of G^{4e} with $|E^{''}|=|E^{'}|>2|V^{'}|=2|V^{''}|$ which is a contradiction by the hypothesis.

- On the other hand if $e^{'}\notin E^{'}$ then the graph $G^{'''}=(V^{'},\{E^{'}-\{e_i,e_j,e_k\}\}\cup 3e)$ obtained by removing any three edges $e_i,e_j,e_k\in E^{'}-e$ and substituting them with 3 copies of edge e is an induced sugraph of G^{4e} with $|E^{'''}|=|E^{'}|>2|V^{'}|=2|V^{'''}|$ which is a contradiction by the hypothesis.

For the other direction, suppose that edge e is independent of $\hat{E}$. The claim follows immediately from characterization (10). □

The preceding Lemma, reduces the complexity of testing whether a new edge is independent, to that of counting edges in subgraphs once the new edge is quadrupled. In order to do this efficiently, we will make use of the *pebble game* algorithm described below.

Each vertex is assigned two pebbles which will be used to cover any two of its incident edges. An edge is said to be covered iff ithas an assigned pebble to one of its incident vertices. A *pebble covering* is an assignment of the pebbles such that all edges of the graph are covered. The goal of the algorithm is to compute a pebble covering or to establish that no such cover exists.

The pebble game algorithm

Assume that we have a set of edges that are covered with pebbles. For each one of the four copies of the new edge do:

- If there is a free pebble at either one of vertices the new edge is incident to, use that pebble to cover the new edge. Afterwards direct this edge such that it points away from the vertex that donated the pebble.

- Otherwise perform a DFS in the directed graph of existing edges. If a free pebble is found then perform the appropriate sequence of swaps such that the new edge is covered. If no free pebbles are found, return "no cover".

The total running time of the pebble game Algorithm is $O(|V||E|)$. Indeed, the Algorithm is called $|E|$ times and each DFS takes $O(|V|)$ time. The connection between independence testing and pebble coverings is made explicit in the following Theorem.

Theorem 3.3.3. *A new edge e is independent of $\hat{E}$ iff there exists a pebble covering for G^{4e}.*

Proof. First suppose that there exists a pebble cover for G^{4e} and let $G^{'} = (V^{'}, E^{'})$ be a vertex induced subgraph of G^{4e}. Then it must be the case that $|E^{'}| \leq 2|V^{'}|$ since the edges in $E^{'}$ can only be covered using pebbles from the vertices in $V^{'}$ and there is only $2|V^{'}|$ of them. Thus by characterization (10) it follows that edge e is independent of $\hat{E}$.

On the other hand, suppose that that e is independent of $\hat{E}$ and for the sake of contradiction let us assume that no pebble covering exists for G^{4e} i.e. the pebble game Algorithm returned "no cover". Let $|V^{'}|$ be the number of vertices encountered during that call of the pebble game Algorithm and let $G^{'} = (V^{'}, E^{'})$ be the subgraph of G^{4e} induced by $V^{'}$. It is then clear that $|E^{'}| > 2|V^{'}|$, for otherwise a free pebble would have been found. But that is a contraction because of the hypothesis combined with characterization (10). $\square$

Chapter 4

Generic rigidity in $\mathbb{E}^3$

One of the major open problems in rigidity theory is that of the combinatorial characterization of generically minimally rigid graphs in 3-space.

In Section 4.1 we will establish that the 3-dimensional analogue of the Laman condition, fails to be a sufficient condition for generic minimal rigidity in $\mathbb{E}^3$. Consequently, it should come as no surprise that a lot of effort has been spent in trying to determine appropriate refinements to the Laman condition, so that it yields a sufficient condition for generic rigidity in $\mathbb{E}^3$. Some of these are presented in the end of this Section.

In Section 4.2 we consider the class of graphs that corresponds to the 1-skeleta (edge graphs) of convex simplicial polyhedra in $\mathbb{E}^3$ and we show that these graphs are generically minimally rigid in $\mathbb{E}^3$.

4.1 Search for a combinatorial characterization

Each point in 3-space has 3 degrees of freedom. Additionally, every body in 3-space has 6 "internal" degrees of freedom that correspond to translations and rotations (rigid motions) of $\mathbb{R}^3$. Since the addition of an edge destroys at most one degree of freedom it is clear that if we expect a graph on n vertices to be rigid in $\mathbb{R}^3$ then it should have at least $3n-6$ edges (in order to destroy all "external" degrees of freedom).

The following "necessary counts Theorem" is the natural generalization of the Maxwell-Laman condition to 3-space.

Theorem 4.1.1. *Let $G = (V, E)$ a generically minimally rigid graph in $\mathbb{E}^3$. Then the following hold:*

- $|E| = 3|V| - 6$
- $|E^{'}| \leq 3|V^{'}| - 6$ *for all vertex induced subgraphs* $(V^{'}, E^{'})$, *with* $|V^{'}| \geq 2$

Similar to the planar case, we say that graph G has the Laman property in $\mathbb{R}^3$ iff it satisfies the counts above.

Although the conditions stated above are necessary for a graph to be generically rigid in $\mathbb{R}^3$, they are not sufficient as can be easily established by the counterexample of the "double banana" due to W. Whiteley, illustrated in Figure 4.1. The two "bananas" can rotate relative to one another along their implied hinge (dotted line).

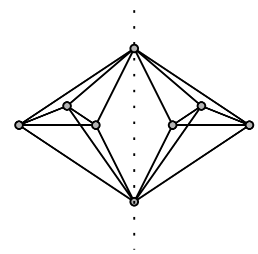

Figure 4.1: The double banana.

Since the Laman property ceases to be a sufficient condition for generic rigidity in $\mathbb{E}^3$, it should come as no surprise, that a lot of effort has been put in trying to determine appropriate refinements, that would yield a combinatorial characterization of generic rigidity in $\mathbb{E}^3$. We now present one of these.

Definition 4.1.2. *A graph G is said to be k-vertex connected iff there does not exist a set of $k-1$ vertices whose removal disconnects the graph. The vertex connectivity of a graph G is the largest k for which the graph is k-vertex-connected.*

As an example consider the graph of the double banana. It is clear that the graph of the double banana is 2-connected, since there is no vertex whose removal disconnects the graph. On the other hand, the removal of the two vertices on the implied hinge disconnects the graph and thus the graph of the double banana is not 3-connected. It then follows that the graph of the double banana has vertex connectivity 2.

One can naturally wonder what would happen if we demand vertex connectivity strictly greater than 2. Does that make the Laman property a sufficient condition for generic rigidity in $\mathbb{E}^3$? The answer is negative as was established in [41]. The case $k = 3$ appears also in [57, Figure 60.1.3].

The proof goes as follows: we first show that a graph G with the Laman property can have vertex connectivity at most 5; then we present three graphs that have the Laman property and have vertex connectivity 3,4,5, respectively, but are flexible. These graphs are slight modifications of the double banana that are obtained by adding mechanisms (spiders) which increase the connectivity but in the same time preserve the flexibility.

Lemma 4.1.3. [41, Theorem 1] *Let $G = (V, E)$ be a graph that has the Laman property. Then G has vertex connectivity at most 5.*

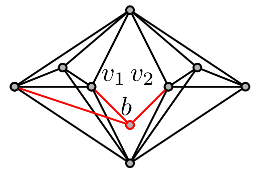

Figure 4.2: The double banana with a 3-spider attached to it.

Proof. Suppose that $G = (V, E)$ has vertex connectivity k. Then G is k-vertex connected and by definition there does not exist a set of $k-1$ vertices whose removal disconnects the graph. This implies that there exist at least k-vertex disjoint paths between any pair of vertices of G and thus $\deg(v) \geq k, \forall v \in V$. Then

$$|E| = \frac{1}{2}\sum_{v \in V} \deg(v) \geq \frac{1}{2}k|V|$$

and since $|E| = 3|V| - 6$ it follows that

$$k \leq 6 - \frac{12}{|V|}$$

Thus $k \leq 5$, as claimed. □

Theorem 4.1.4. [41, Lemma's 2,3,4] *There exists graphs that have the Laman property and have vertex connectivity 3,4,5, respectively.*

Proof. We will only deal with the case $k = 3$. The other cases are treated similarly and the reader is referred to [41] for the details.

The Figure below illustrates the simplest spider that converts the double banana to a flexible graph with connectivity 3. This spider consists of the single vertex b (spider body) connected by three edges (spider legs) to the two bananas. Notice that the legs are not connected to the implied hinge vertices. The intuition of why this graph should be flexible is the following: as the two bananas rotate along their implied hinge, vertices v_1 and v_2 move closer of farther apart, causing vertex b to swing up or down.

It is routine to check that the resulting graph has the Laman property. Clearly the graph has connectivity 3. To verify that tis graph remains flexible we compute the dimension of the space of infinitesimal motions. Consider first the double banana together with the spider body only (without the spider legs). The space of motions of this graph is at least 10 since we have 3 dof for the spider body, 6 dof for the double banana that correspond to rigid motions or $\mathbb{R}^3$ and 1 dof for the implied hinge. Since each edge reduces the dimension of the space of motions by 1, adding the three spider legs results in a graph with at least 7 dof. Since the *internal* dof are 6 it follows that the graph is flexible. □

4.2 Frameworks given by convex polyhedra in $\mathbb{E}^3$

The origins of the mathematical study of the rigidity of polyhedra can be traced back at least to L. Euler, who in 1766 conjectured that *every* polyhedron is rigid in the following sense: consider a polyhedron $P \subseteq \mathbb{E}^3$, viewed as a "panel and hinge" structure, meaning that the facets are 2-dimensional panels and the edges 1-dimensional hinges. Now imagine that, the panels are free to move *continuously* in $\mathbb{E}^3$, subject to the following constraints: their shape and the adjacencies between panels is preserved throughout the motion and the relative motion between adjacent pairs of panels is rotation about their common hinge. The polyhedron P will be called rigid iff every such motion results in a framework which is congruent to P. The answer to Euler's conjecture turned out to be negative and we will now briefly summarize the events that led to this discovery.

The first major breakthrough occurred in 1813, when A.L. Cauchy verified the conjecture for the case when P is convex [13]. Specifically, Cauchy proved the following, now known as the Cauchy Rigidity Theorem:

Theorem 4.2.1. *If two 3-dimensional polyhedra convex P and $P^{'}$ are combinatorially equivalent with corresponding facets being congruent, then P is congruent to $P^{'}$.*

For a proof of Cauchy's Rigidity Theorem the reader is referred to [2]. We should here note that the convexity assumption is essential for Cauchy's rigidity Theorem. For a counterexample where one of the polyhedra is non-convex see Figure 4.3. Notice however, that the example of Figure 4.3 does not constitute a counterexample to Euler's conjecture, since the reflexion that transforms the one polyhedron to the other is not a continuous motion.

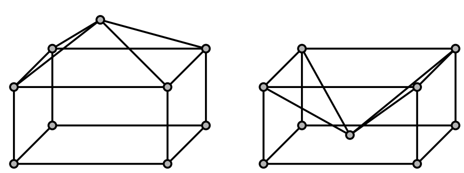

Figure 4.3: Two combinatorially equivalent polytopes with corresponding facets congruent that are not themselves congruent. The reason is that the second one is non-convex.

Moreover, in 1975, H. Gluck established the truth of Euler's conjecture when the coordinates of the vertices of P are generic [30]. Of course, Gluck's proof that almost all polyhedra are rigid, does not imply that flexible polyhedra cannot exist. It does however mean that, if they do exist, they are extremely rare.

This small window of opportunity attracted the attention of B. Connelly, who in 1978, provided a counterexample i.e. a flexible polyhedron consisting of 18 triangular faces, thus establishing that Euler's conjecture was false [14]. Illustrated in Figure 4.4 is a "cut-out" paper model of a flexible polyhedron due to K. Steffen. The bold edges are to be folded as "mountains" whereas the dashed edges as "valleys".

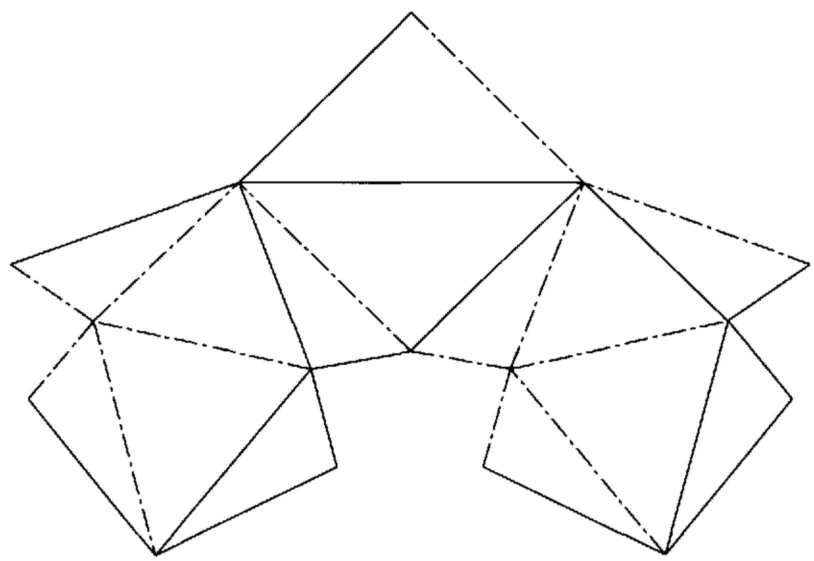

Figure 4.4: A "cut-out" paper model of a flexible polyhedron.

Consider now a polyhedron $P \subseteq \mathbb{E}^3$, only this time, view it as a bar and joint framework, rather than a panel and hinge structure as we did before. It is natural to ask, what are the rigidity theoretic properties of a framework obtained in such a manner. We will prove that if $G(p)$ is a framework obtained by a convex polyhedron $P \subseteq \mathbb{E}^3$, then $G(p)$ is rigid in $\mathbb{E}^3$ iff every face of P is a triangle (P is simplicial).

We start by proving the well-known Cauchy Index Lemma, that Cauchy used in proving his Rigidity Theorem which is regarded as one of the most impressive arguments in geometry. We will state it here in a somewhat more graph theoretical form as opposed to its original statement.

Lemma 4.2.2. [13] *Let $G = (V, E)$ be a planar graph, without loops or multiple edges and suppose that each one of its edges is labeled with a plus or minus sign. As a result, sign changes may occur as we circle around some vertex of G, say in a counterclockwise direction. The claim is that, there exists either a vertex with no sign changes or a vertex with exactly two sign changes.*

In fact, we will prove a somewhat sharper assertion from which Lemma 4.2.2 follows. But first we need the following definitions.

Definition 4.2.3. *Let $G = (V, E)$ be a planar graph, without loops or multiple edges and suppose that each one of its edges is labeled with a plus or minus sign.*

The index of vertex $v \in V$, denoted by $I(v)$, is the number of sign changes encountered as v is circled in the counterclockwise direction.

The index I of G, is the sum of the indices of all the vertices of G i.e.

$$I = \sum_{v \in V} I(v)$$

See Figure 4.5 for an example of a vertex with index 4.

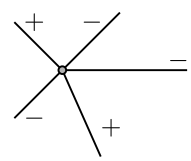

Figure 4.5: A vertex with index 4.

Notice that by its definition, the index of a vertex v will always be even. We will now state and prove a Lemma that implies the Cauchy index Lemma.

Lemma 4.2.4. *Let $G = (V, E)$ be a planar graph, without loops or multiple edges and suppose that each one of its edges is labeled with a plus or minus sign. Then, its index satisfies $I \leq 4|V| - 8$.*

Proof. We start with a definition, that at first sight will definitely strike the reader as strange unnatural but is nonetheless useful, for reasons that will become apparent during the proof. Given a (bounded) face f of G, then the number of edges of f *will not be* what one would expect it to be i.e. the number of the boundary edges of f. Instead, the number of edges of f is computed as follows: we count *twice* each edge that does not separate f from another *bounded* region and once all the other ones. For an example see Figure 4.6.

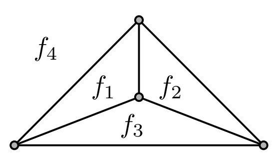

Figure 4.6: A graph G with four faces (regions). One of them (f_4) is unbounded and each one of the other three has four edges according to or definition.

Let F be the set of faces of G and let F_i be the number of faces with exactly i edges. Since G contains no loops or multiple edges it follows that $F_1 = F_2 = 0$ and thus

$$|F| = \sum_{i \geq 3} F_i \tag{4.1}$$

Additionally, by the way we defined the number of edges of a face f it should be clear that

$$2|E| = \sum_{i\geq 3} iF_i \tag{4.2}$$

Another way to compute the index I of G is by circling regions of G. Specifically, we start by orienting G and then we count sign changes that occur as one moves around each of the faces of G. Since two edges are adjacent iff they are adjacent in moving around the region to whose boundary the belong, the total number of sign changes encountered while circling the faces of G will be the index I of G. For more details see [3], Section 2.2.

Now, since the number of sign changes as one traverses the boundary of a face with n edges is at most n, it follows that

$$I \leq 2F_3 + 4F_4 + 4F_5 + 6F_6 + 6F_7 + \dots \tag{4.3}$$

At this point we will make use of the generalized Euler formula for disconnected graphs (reference). In particular, if $G = (V, E)$ is a planar graph with k connected components then the generalized Euler formula implies that $|V| - |E| + |F| = 1 + k$, and thus

$$|V| - |E| + |F| \geq 2 \tag{4.4}$$

Consequently, $4|V| - 8 \geq 4|E| - 4|F|$ and combining this with Formulas (4.1) and (4.2) we obtain that

$$4|V| - 8 \geq \sum_{i\geq 3}(2i-4)F_i = 2F_3 + 4F_4 + 6F_5 + \dots \tag{4.5}$$

Clearly, the right hand side of (4.5) dominates the right hand side of (4.3) and thus it follows that

$$I \leq 4|V| - 8$$

$\square$

Theorem 4.2.5. *Let $G(p)$ be a framework in $\mathbb{E}^3$ given by a convex polyhedron P. Then*

$$\textit{rank}\ \mathcal{R}(p) = |E|$$

Proof. Recall that $\mathcal{R}(p) \in \mathbb{R}^{|E|\times 3|V|}$ and thus $\mathcal{R}(p)^T \in \mathbb{R}^{3|V|\times|E|}$. Using some elementary Linear Algebra we have that

$$|E| = \dim \ker \mathcal{R}(p)^T + \dim \operatorname{Im} \mathcal{R}(p)^T = \dim \operatorname{Ker} \mathcal{R}(p)^T + \dim \operatorname{Im} \mathcal{R}(p)$$

and by the definition of the rank of a matrix it follows that

$$|E| = \dim \operatorname{Ker} \mathcal{R}(p)^T + \operatorname{rank} \mathcal{R}(p)$$

It is thus enough to show that $\dim \operatorname{Ker} \mathcal{R}(p)^T = 0$ and this is exactly what we will try to do.

Let $\omega = (\omega_{ij})_{ij \in E} \in \mathrm{Ker}\mathcal{R}(p)^T$ and suppose for the sake of contradiction that ω is not the zero vector. By the definition of ω we have that

$$\sum_{j \in N(i)} \omega_{ij}(p_i - p_j) = 0, \forall i \in \{1, \ldots, |V|\} \tag{4.6}$$

Now, we will use the signs of the coefficients ω_{ij} in order to label the edges of G with plus and minus signs. Specifically, if $\omega_{ij} > 0$, then edge ij is labeled with a plus sign while if $\omega_{ij} < 0$ then edge ij is labeled with a minus sign. The edge ij is left unmarked if $\omega_{ij} = 0$.

Consider the graph $G^{'}$ induced by the *marked edges* of G, meaning that the edges of $G^{'}$ are the edges of G that are marked with a plus or minus sign and the vertices of $G^{'}$ are those vertices of G that are incident with at least one marked edge. We will reach a contradiction by showing that the index of every vertex of $G^{'}$ is ≥ 4 which is absurd by Lemma 4.2.2. Since the index of a vertex is always an even number it is enough to show that the index of a vertex cannot be zero or two.

Fixing some arbitrary $i \in V$, Formula 4.6 implies that

$$\sum_{j \in N(i)} \omega_{ij}(p_i - p_j) = 0$$

and since if j is not a vertex of $G^{'}$ $\omega_{ij} = 0$, it follows that

$$\sum_{j \in N'(i)} \omega_{ij}(p_i - p_j) = 0 \tag{4.7}$$

where $N^{'}(i)$ denotes the set of neighbors of vertex i in $G^{'}$.

Suppose now that $I(i) = 0$. By the definition of the index of a vertex, this means that the signs of the scalars ω_{ij} for $j \in N^{'}(i)$ are either all positive or negative. Since P is convex, let $\eta \cdot (p_i - x) = 0$ be a supporting hyperplane of P at vertex i, where η is its normal vector. By the definiton of the supporting hyperplane, all other vertices lie on one side of the plane and thus $\eta \cdot (p_i - p_j)$ is either positive or negative, for all $j \in N^{'}(i)$. Therefore,

$$\sum_{j \in N'(i)} \omega_{ij} \left[\eta \cdot (p_i - p_j)\right] \neq 0$$

that implies

$$\sum_{j \in N'(i)} \omega_{ij}(p_i - p_j) \neq 0$$

which contradicts Formula (4.7).

On the other hand, suppose that $I(i) = 2$. Then in the circular order around vertex i, there exists a set of edges marked with plus followed by a set of edges marked with minus. Again, since P is supposed to be convex, there exists a supporting hyperplane of P at vertex i, with all the edges marked with a plus sign on one of its sides and those marked with a minus sign on the other. Then, a similar argument as before concludes the proof. $\square$

Now, let $G(p)$ be a framework in $\mathbb{E}^3$ given by a convex polyhedron and suppose that we wish to determine whether it is infinitesimally rigid or not. In view of Definition 2.2.6 and Theorem 4.2.5 it suffices to check whether $|E| = 3|V| - 6$.

Lemma 4.2.6. *Let P be a polyhedron and let V, E, F be its set of vertices, edges and facets respectively. Then $|E| \leq 3|V| - 6$ and equality holds iff every face of P is a triangle.*

Proof. Let F_i be the number of facets of P which have exactly i boundary edges. Clearly, $F_1 = F_2 = 0$ and $|F| = \sum_{i \geq 3} F_i$. Since each edge of P belongs to exactly two facets, it follows that

$$\sum_{i \geq 3} iF_i = 2|E|$$

and thus

$$3|F| = 3\sum_{i \geq 3} F_i \leq \sum_{i \geq 3} iF_i = 2|E|$$

Also notice that equality holds iff $F_i = 0, \forall i \geq 4$ which in turns means that $|F| = F_3$ i.e. every face of P is a triangle.

Now, Euler's formula applied to P implies that $|V| - 2 = |E| - |F|$ and thus

$$3|V| - 6 = 3(|V| - 2) = 3(|E| - |F|) = |E| + (2|E| - 3|F|)$$

Since we established above that $2|E| = 3|F|$ iff every face of P is a triangle, the claim follows. $\square$

We have finally achieved our goal, namely we are now in the position to prove the following:

Theorem 4.2.7. *Let $G(p)$ be a framework obtained by a convex polyhedron $P \subseteq \mathbb{E}^3$. Then $G(p)$ is rigid in $\mathbb{E}^3$ iff every face of P is a triangle (P is simplicial).*

Proof. By definition 2.2.6, $G(p)$ is infinitesimally rigid iff rank $\mathcal{R}(p) = 3|V| - 6$. Since $G(p)$ is obtained from a convex polyhedron, Theorem 4.2.5 implies that rank $\mathcal{R}(p) = |E|$. Thus $G(p)$ is infintesimally rigid iff $|E| = 3|V| - 6$ which in view of Lemma 4.2.7 happens iff every face of P is a triangle. $\square$

At this point it is natural to ask whether Theorem 4.2.7 can be generalized to arbitrary dimension. The answer is affirmative, and in fact a stronger assertion due to A. Fogelsanger is true. Specifically, we have the following:

Theorem 4.2.8. [28] *For $d \geq 2$, the graph of a triangulated d-pseudo-manifold is generically $d+1$ rigid.*

Notice that Theorem 4.2.7 is a special case of Fogelsangre's Theorem, when applied to the sphere.

Chapter 5

Inductive constructions for generically minimally rigid graphs

This chapter addresses the following problem: is there some kind of systematic way to generate *all* generically minimally rigid graphs in $\mathbb{E}^d$?

In Section 5.1 we deal with the case $d = 2$, where the problem is extensively studied and fully understood. Specifically, we will show that an abstract framework G is generically minimally in $\mathbb{E}^2$ iff it can be constructed inductively using the so-called Henneberg operations.

In Section 5.2 we deal with the general case, where the plot thickens. We begin by generalizing the Henneberg operations to d-space and we prove that every graph obtained by a Henneberg d-sequence is generically minimally rigid in $\mathbb{E}^d$. On the other hand, we will show that the Henneberg d-sequences *are not sufficient* to generate all generically minimally rigid graphs in $\mathbb{E}^d$.

The case $d = 3$ deserves some extra attention, due its importance in applications and it is dealt with separately in Section 5.3. Here we analyze one of the most important conjectures concerning generic rigidity in $\mathbb{E}^3$: G is generically minimally rigid graph in $\mathbb{E}^3$ iff G has an "extended" Henneberg d-construction, in the sense that along with the two Henneberg steps, we are allowed to use two more operations, the so-called X and V replacement.

Lastly, since we have established in Section 4.2 that the edge graphs (or 1-skeleta) of convex simplicial polyhedra are generically minimally rigid in $\mathbb{E}^3$, it is natural to ask whether these graphs posses a property analogous to that of generically minimally rigid graphs in $\mathbb{E}^2$ i.e. whether they can be inductively constructed using some kind of Henneberg operations. The answer is affirmative, as we will establish in Section 5.4. Moreover we will exploit the fact that this class of graphs can be constructed inductively, in order to prove some properties about them. Specifically, we will prove that convex simplicial polyhedra always have an even number of facets and additionally that the the Hirsch conjecture is true for them.

5.1 Henneberg constructions in the plane

We start with a simple but important Lemma.

Lemma 5.1.1. *Let $G = (V, E)$ be a generically minimally rigid graph in $\mathbb{E}^2$. Then G has at least one vertex of degree 2 or 3.*

Proof. Let us assume for the sake of contradiction that deg $v \geq 4, \forall v \in V$. By Laman's Theorem we have that $|E| = 2|V| - 3$ and thus

$$2|E| = 2(2|V| - 3) = \sum_{v \in V} \deg v \ \geq 4|V|$$

which is absurd. □

We will now introduce what will be our most important tool throughout this Section.

Definition 5.1.2. *Let $G = (V, E)$ be an abstract framework. The Henneberg-I (H_1) step (or vertex addition) applied to G, inserts one new vertex that gets connected to 2 existing ones.*

The Henneberg-II (H_1) step (or edge split) applied to G, replaces an edge by a new vertex that gets connected to its endpoints and to one more arbitrary vertex.

For an example of a H_1 and a H_2 step see Figure 5.1.

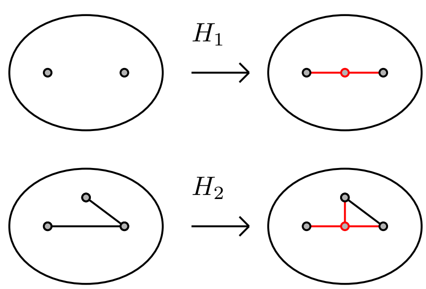

Figure 5.1: A H_1 step and a representative case of a H_2 step. The newly added vertex and the new edges are depicted in red.

Henneberg 2-sequences are a systematic way of generating minimally rigid graphs in $\mathbb{E}^2$ based on the H_1 and H_2 operations. The formal definition follows:

Definition 5.1.3. *A Henneberg 2-sequence for a graph G is a sequence of graphs $G_1, \ldots, G_n$ with the following properties :*

- $G_1 = K_3$
- $G_n = G$
- *G_{i+1} is obtained from G_i, through a H_1 or a H_2 step,* $\forall i \in \{2, \ldots, n-1\}$.

See Figure 5.2 for an example of a Henneberg 2-sequence for the $K_{3,3}$ graph. It is important to notice that the Henneberg 2-sequence for an abstract frame-

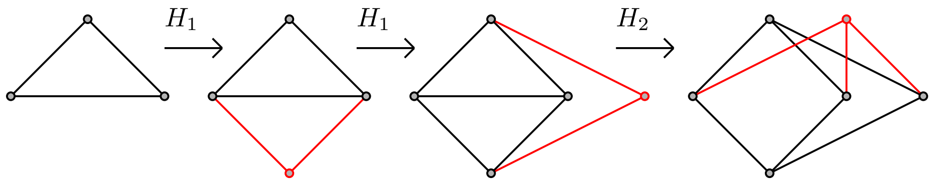

Figure 5.2: A Henneberg 2-sequence for the $K_{3,3}$ graph. At each step, the added vertex and the new edges are depicted in red.

work G *may not* be unique i.e. an abstract framework G can have many Henneberg 2-sequences.

The following Theorem by T.S. Tay and W. Whiteley, fully justifies our interest in Henneberg 2-sequences.

Theorem 5.1.4. *An abstract framework $G = (V, E)$ is generically minimally rigid in $\mathbb{E}^2$ iff it has a Henneberg 2-sequence.*

Proof. Let G be an abstract framework that is generically minimally rigid in $\mathbb{E}^2$. By Lemma 5.1.1 there exists at least one vertex of degree 2 or 3. We consider these two cases seperately:

If there is a vertex of degree 2, remove that vertex and its two adjacent edges i.e. perform a H_1 step in reverse. The resulting graph clearly has the Laman property.

On the other hand, suppose there is a vertex v of degree 3 and let a, b, c be its 3 neighbours. These cannot form a triangle because the Laman property would be violated on the subgraph induced a, b, c, v. So we can perform a H_2 step in reverse i.e. we remove vertex v and add the missing edge between vetrices a, b, c. The resulting graph has the Laman property so we can proceed inductively until the remaining graph is a triangle.

For the other direction we proceed by induction. Suppose that all graphs on n vertices constructed by Henneberg 2-sequences have the Laman property. We can proceed by performing either a H_1 or a H_2 step and we will consider these two cases separately. Performing either a H_1 or a H_2 step we obtain an abstract framework $G = (V, E)$ with $|E| = 2|V| - 3$, so we only need to check the second condition implied by Laman's Theorem. Suppose $V^{'} \subseteq V$ and let v be the new vertex.

Suppose v was added through a H_1 step. If $v \notin V^{'}$ then by the induction hypothesis it follows that $|E^{'}| \leq 2|V^{'}| - 3$. On the other hand if $v \in V^{'}$

then by the hypothesis the induced subgraph $(V^{'} - \{v\}, E^{'})$ will have at most $2(|V^{'}| - 1) - 3$ edges and thus $(V^{'}, E^{'})$ will have at most $2(|V^{'}| - 1) - 3 + 2 = 2|V^{'}| - 3$ edges.

Suppose v was added through a H_2 step. Again if $v \notin V^{'}$ we are done by the induction hypothesis. So suppose $v \in V^{'}$ and let a, b, c be its three neighbours. Now, if one of the vertices a, b, c does not belong to $V^{'}$ then by the hypothesis the induced subgraph $(V^{'} - \{v\}, E^{'})$ will have at most $2(|V^{'}| - 1) - 3$ edges and thus $(V^{'}, E^{'})$ will have at most $2(|V^{'}| - 1) - 3 + 2 = 2|V^{'}| - 3$ edges.

On the other hand, if all of the vertices a, b, c are included in $V^{'}$ it follows that the induced subgraph $(V^{'} - \{v\}, E^{'})$ will have at most $2(|V^{'}| - 1) - 4$ edges and thus $(V^{'}, E^{'})$ will have at most $2(|V^{'}| - 1) - 4 + 3 = 2|V^{'}| - 3$ edges. □

5.2 Henneberg constructions in higher dimensions

We start by generalizing the definitions of the planar Henneberg steps, to arbitrary dimension.

Definition 5.2.1. *Let $G = (V, E)$ be an abstract framework. The H_1^d step (or vertex addition) applied to G, inserts one new vertex that gets connected to d existing ones.*

The H_2^d step (or edge split) applied to G, replaces an edge by a new vertex that gets connected to its endpoints and additionally to $d - 1$ other vertices.

See Figure 5.3 for an example when $d = 3$. Notice that the H_1 and H_2 steps that were introduced in Section 5.1 coincide with the H_1^2 and H_2^2 steps.

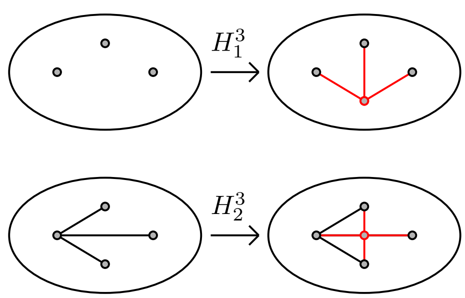

Figure 5.3: A H_1^3 step and a representative case of a H_2^3 step. The newly inserted vertex and the new edges are depicted in red.

The generalization of the definition of the Henneberg 2-sequence to arbitrary dimension is what one would expect it to be.

Definition 5.2.2. *A Henneberg d-sequence for a graph G is a sequence of graphs $G_1, \ldots, G_n$ with the following properties :*

- $G_1 = K_{d+1}$
- $G_n = G$
- *G_{i+1} is obtained from G_i, through a H_1^d or a H_2^d step, $\forall i \in \{2, \dots, n-1\}$.*

We now continue by proving that if G has a Henneberg d-sequence, then G is generically minimally rigid in $\mathbb{E}^d$.

Theorem 5.2.3. [58, Propositions 5.1, 5.2] *Let $G = (V, E)$ be generically minimally rigid in $\mathbb{E}^d$. Every abstract framework obtained by G through a H_1^d or H_2^d step, is also generically minimally rigid in $\mathbb{E}^d$.*

Since the K_{d+1} graph is generically minimally rigid in $\mathbb{E}^d$ it follows that:

Corollary 5.2.4. *Every abstract framework G obtained by a Henneberg d-construction is generically minimally rigid in $\mathbb{E}^d$.*

In order to get an idea of the techniques involved in the proof of Theorem 5.2.3, we will prove it only for the special case $d = 2$ (which is of course Theorem 5.1.4). The details of the proof for the general case are omitted.

We note that the approach taken in this proof, is radically different than that of Theorem 5.1.4. This should come as no surprise, since the proof of Theorem 5.1.4 relied heavily on the combinatorial characterization of generically minimally rigid graphs in $\mathbb{E}^2$.

We begin by stating a Lemma due to W. Whiteley, which provides us with a broad substitution principle.

Lemma 5.2.5. [56, Theorem 2.6] *Let $G(p)$ be an isostatic framework in $\mathbb{E}^d$ and let $G^{'}(p)$ be a sub-framework on k vertices, which is infinitesimally rigid in the affine space of its joints. If $G^{'}(p)$ is replaced by another sub-framework on k vertices which is isostatic in the affine space of its joints, then the resulting framework is isostatic.*

We are now ready to prove Theorem 5.2.3 for the special case $d = 2$.

Proof. Suppose $G = (V, E)$ is generically minimally rigid in $\mathbb{E}^2$ and let $G^* = (V^*, E^*)$ be an abstract framework obtained by G through a H_1 or a H_2 step. These two cases will be treated seperately.

Adding a vertex through a H_1 step: Let v be the added vertex and a, b the vertices to which it gets attached. In view of Definition 3.1.1, in order to show that G^* is generically minimally rigid in $\mathbb{E}^2$, it suffices to find a generic point $q \in \mathbb{E}^{d(|V|+1)}$ such that $G^*(q)$ is isostatic.

Let $G(p)$ be an independent realization of G. Place the new vertex v at point p_v, so that it is not collinear with a and b, and let $q = (p, p_v)$ be the resulting generic point. We first prove that $G^*(q)$ is independent.

Suppose that $G^*(q)$ admits a stress (see Figure 5.4). It suffices to show that it is trivial.

By the definition of a stress, the forces exerted on vertex v must cancel out. But since v was placed such that it is not collinear with a and b, this can only

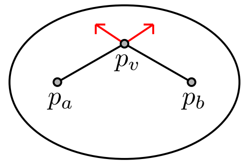

Figure 5.4: A stress of $G^*(q)$. The forces exerted on vertex v are depicted in red.

happen if the forces exerted on v are zero. Now, if we ignore vertex v, the remaining framework is just $G(p)$, which by hypothesis only admits the trivial stress. So the stress is trivial and consequently $G^*(q)$ is independent.

On the other hand, it is clear that $|E^*| = 2|V^*| - 3$ and thus, in view of Theorem 2.3.5, $G^*(q)$ is isostatic.

Adding a vertex through a H_2 step: Let v be the added vertex, $\{a, b\}$ the splitted edge and c the third vertex that v gets connected to. Also, let $G(p)$ be an independent realization of G, where points p_a, p_b and p_c are not collinear. Notice that we can find one such, since the independent realizations of G form an open and dense subset of $\mathbb{E}^{d|V|}$.

Let $G^*(q)$ be a realization of G^*, where $q = (p, (p_a + p_b)/2)$. Geometrically, this means that the new vertex v is placed at the midpoint of segment p_a, p_b. Also, let $G^{'}$ be the abstract framework obtained by G^* by removing edge $\{v, b\}$ and adding edge $\{a, p\}$. By the first case of the Theorem, $G^{'}$ is generically minimally rigid in $\mathbb{E}^2$ and let $G^{'}(q)$ the corresponding framework for $G^{'}$.

Since q_a, q_v, q_c are not collinear, it follows from the proof of the first part of the Theorem that the realization $G^{'}(q)$ is isostatic. Because of the collinearity of q_a, q_v, q_b, the segments $[q_a, q_v]$ and $[q_v, q_b]$ form an infinitesimally rigid sub-framework on the affine space of their endpoints (the line defined by p_a and p_b). By Lemma 5.2.5, this can be substituted with the isostatic sub-framework formed by segments $[p_a, p_v]$ and $[p_v, p_b]$, thus obtaining an isostatic framework for G^*. Again, we are done by the same argument as in the first case of the Theorem.

$\square$

A natural question to ask, is if the converse of Theorem 5.2.3 is also true i.e. does every generically minimally rigid graph in $\mathbb{E}^d$ have a Henneberg d-sequence? It is not difficult to see that the answer to this question is negative. Specifically, notice that every abstract framework that is constructed through a Henneberg d-sequence, has *at least one* vertex of degree d or $d+1$ (created by the last Henneberg step in the sequence). So, it suffices to find a graph which is generically minimally rigid in $\mathbb{E}^d$, but all of its vertices have degree strictly greater than $d+1$. For such an example when $d = 3$, see Figure 5.5.

On the other hand, it is known that given a graph which is generically minimally rigid graph in $\mathbb{E}^d$, if there exist vertices of degree d or $d+1$, these can be removed and the resulting graph will be again generically minimally rigid

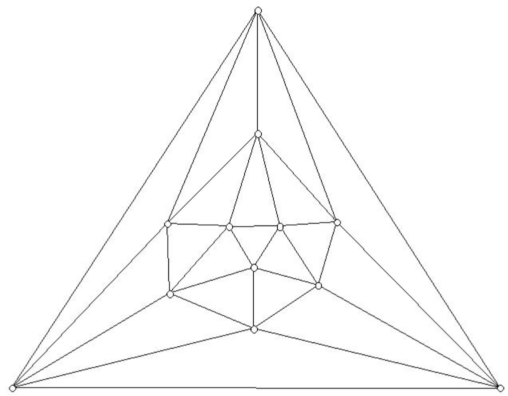

Figure 5.5: The edge graph of the icosahedron is generically minimally rigid in $\mathbb{E}^3$ but all its vertices have degree 5.

in $\mathbb{E}^d$. Specifically, we have the following Theorem concerning the removal of vertices of degree d and $d+1$.

Theorem 5.2.6. [58, Propositions 5.1, 5.3] *Let $G = (V, E)$ be generically minimally rigid in $\mathbb{E}^d$.*

If G has a vertex v of degree d, then the abstract framework obtained by removing v and all its incident edges is generically minimally rigid in $\mathbb{E}^d$.

If G has a vertex v of valence $d+1$, then there is a choice of vertices among the neighbors of v (say u and w) such that, the graph obtained by removing v and all its incident vertices and adding an edge uw is generically minimally rigid in $\mathbb{E}^d$.

5.3 Spatial Henneberg constructions

For the remainder of this section we will focus on the case where $d = 3$, which undoubtedly is the most important one, when it comes to applications. Let us first summarize what we know so far.

By Theorem 5.2.3 it follows that all graphs obtained by K_4 through a sequence of H_1^3 and H_2^3 steps, are generically minimally rigid in $\mathbb{E}^3$. Moreover, by Theorem 5.2.6 we know that we can remove vertices of degree 3 and 4 by performing inverse Henneberg steps while preserving the property of being generically minimally rigid in $\mathbb{E}^3$. On the other hand, the counterexample of Figure 5.5 implies that the H_1^3 and H_2^3 generate a proper subclass of the generically minimally rigid graphs in $\mathbb{E}^3$.

Notice that the problem with the graph of the icosahedron is that all its vertices have degree strictly greater than 4, so they cannot be created using the

H_1^3 and H_2^3 steps. A natural idea, that will enable us to cope with this "degeneracy" would be to introduce new Henneberg steps that will insert vertices of degree 5 and higher. But there is a catch here. How many new Henneberg steps do we need? Our strategy seems to imply that we need to introduce one such step for every possible degree, so infinitely many? Fortunately not, as the following Lemma implies.

Lemma 5.3.1. *Let $G = (V, E)$ be generically minimally rigid in $\mathbb{E}^3$. Then G has at least one vertex of degree 3,4, or 5.*

The proof of the Lemma is similar to that of Lemma 5.1.1 and is omitted. Lemma 5.3.1 clearly implies that in order to find some way to generate all generically minimally rigid graphs in $\mathbb{E}^3$ we need to develop additional methods, that will enable us to cope with degree 5 vertices. We continue with a Theorem that provides us with a way of removing degree 5 vertices, while preserving generic rigidity.

Theorem 5.3.2. [58, Proposition 4.5] *Let $G = (V, E)$ be generically minimally rigid in $\mathbb{E}^3$ and let $v \in V$, a degree-5 vertex, with edges $\{v_i, v\}$, $i = 1, \ldots, 5$. Then, one of the following is true:*

- *for some choice of non-incident edges e and $e^{'}$ among vertices v_i (X configuration), the graph obtained by removing vertex v and adding edges e and $e^{'}$ is generically minimally rigid in $\mathbb{E}^3$. For an example see Figure 5.6.*
- *for two choices of incident pairs of edges $e, e^{'}$ and $d, d^{'}$ (V configuration), which are not incident to the same vertex, the two graphs formed by removing vertex v and adding these pairs are both generically minimally rigid in $\mathbb{E}^3$. For an example see Figure 5.7.*

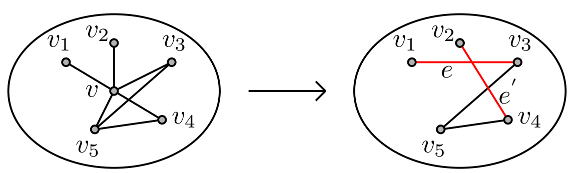

Figure 5.6: Replacement of a 5-valent vertex v with an X configuration.

Now what about adding degree 5 vertices? Unfortunately there is no simple procedure for adding degree 5 vertices, although there are some partial results (for example [58, Proposition 4.10]). However, it is conjectured that the converse of Theorem 5.3.2 is also valid and this is usually referred to as the $3D$ replacement conjecture. We note that no analogue conjecture exists for $d \geq 4$.

Conjecture 5.3.3. [57, Conjecture 60.1.16]

X ***replacement****: Let $G = G^{'} \cup ab \cup cd$ be a generically minimally rigid graph in $\mathbb{E}^3$, where the a, b, c, d are distinct. Then, the graph G^{*} obtained by*

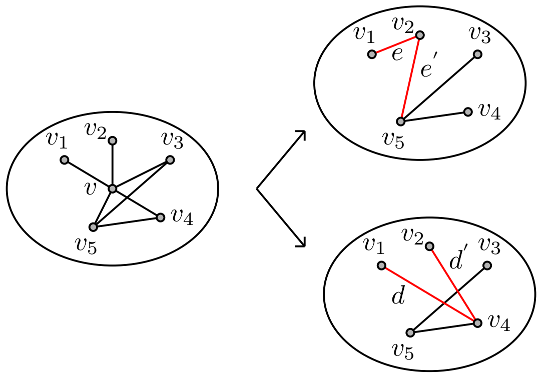

Figure 5.7: Replacement of a 5-valent vertex v with a V configuration. Notice that according to Theorem 5.3.2, there are always two possible V configurations, that we can use to replace vertex v.

G' through the insertion of a new degree 5 vertex, that gets attached to vertices a, b, c, d and an arbitrary vertex e, is generically minimally rigid in $\mathbb{E}^3$. See Figure 5.8 for an example.

Double V replacement: *Let $G_1 = G' \cup ab \cup bc$ and $G_2 = G' \cup a'b' \cup b'c'$ be two generically minimally rigid graphs in $\mathbb{E}^3$, with $b \neq b'$. Then, the graph G^* obtained by G' through the insertion of a new degree 5 vertex, that gets attached to 5 vertices from the set $\{a, a', b, b', c, c'\}$ is also generically minimally rigid in $\mathbb{E}^3$. See Figure 5.9 for an example.*

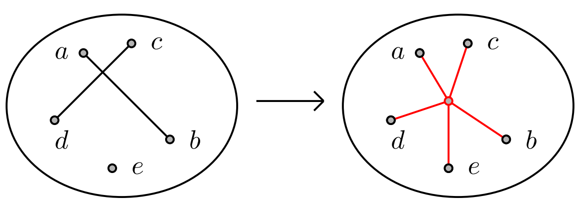

Figure 5.8: The X replacement.

There is important observation to be made here. Notice that for $d = 2$, the Henneberg sequence for a graph G is a *certificate* for generic rigidity in $\mathbb{E}^2$. However, in view of Conjecture 5.3.3, it seems reasonable to assume that we cannot expect a simple Henneberg sequence to verify that a graph G is generically minimally rigid in $\mathbb{E}^3$. The natural pattern to consider here is *a tree of graphs*.

Intuitively, imagine a directed rooted tree with G as it root, where all the nodes of the tree are indexed by graphs. Specifically, the leaf nodes of the tree correspond to the K_4 graph and every path from a leaf node to the root

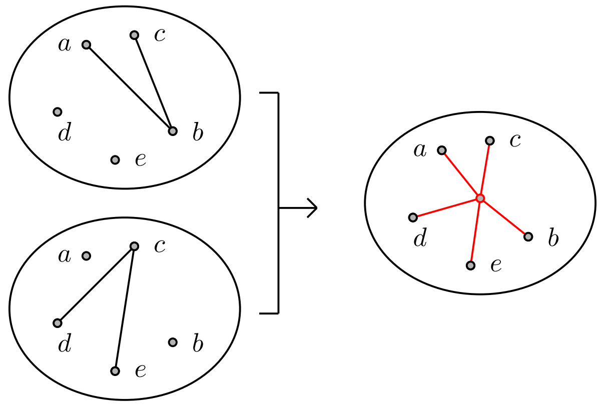

Figure 5.9: The double V replacement.

corresponds to a Henneberg 3-construction for G. If a vertex H has one incoming edge $H^{'} \rightarrow H$, this means that graph H was obtained by $H^{'}$ either through a H_1^3 or a H_2^3 or a X replacement step. If a vertex H has two incoming edges $H^{'} \rightarrow H$ and $H^{''} \rightarrow H$, this means that graph H was obtained by graphs $H^{'}$ and $H^{''}$ through a V replacement step. A formal definition follows.

Let $G = (V, E)$ be a graph and $v \in V$. By $G - v$ we will denote the graph obtained by G through the removal of v and of its incident edges.

Definition 5.3.4. *A Henneberg 3-tree for a graph G is a directed rooted tree whose vertices are indexed by graphs, such that:*

- *G is the root*
- *the direction of every edge is towards the root*
- *any vertex of the tree has at most two entering edges*
- *if a vertex has no entering edges then it corresponds to the K_4 graph*
- *for each edge $H^{'} \rightarrow H$ we have $V(H^{'}) = V(H) - v$, for some $v \in V(H)$*
- *if a vertex H has one entering edge $H^{'} \rightarrow H$ then one of the following is true:*
  - *$deg_H v$=3 and $E(H^{'}) = E(G - v)$*
  - *$deg_H v = 4$ and $E(H^{'}) = E(G-v) \cup ab$, for some vertices a, b adjacent to v in H*
  - *$deg_H v = 5$ and $E(H^{'}) = E(G-v) \cup ab \cup cd$ for some distinct neighbors a, b, c, d of v in H*

- *if a vertex H has two entering edges $H^{'} \to H$ and $H^{''} \to H$ then $deg_H v = 5$ and $E(H^{'}) = E(G-v) \cup a^{'}b^{'} \cup b^{'}c^{'}$ and $E(H^{''}) = E(G-v) \cup a^{''}b^{''} \cup b^{''}c^{''}$, where $b^{'} \neq b^{''}, a^{'} \neq c^{'}, a^{''} \neq c^{''}$ and the 5 neighbors of v in H are among the set $\{a^{'}, a^{''}, b^{'}, b^{''}, c^{'}, c^{''}\}$*

In view of Definition 5.3.4, the contents of this Section can be summarized in the following:

Theorem 5.3.5. *If a graph is generically minimally rigid in $\mathbb{E}^3$ then there exists a Henneberg 3-tree for G.*

Conjecture 5.3.6. *If there exists a Henneberg 3-tree for G then G is generically minimally rigid in $\mathbb{E}^3$.*

5.4 Constructions of the 1-skeleta of simplicial polyhedra

We start with a fundamental Theorem due to E. Steinitz, which characterizes the graphs that correspond to 1-skeleta of 3-dimensional convex polyhedra.

Theorem 5.4.1. *A graph G is the 1-skeleton of a convex polyhedron in $\mathbb{E}^3$ iff G is simple, planar and 3-vertex connected.*

For a proof of this Theorem the reader is referred to [32, Section 13.1]. We continue with a useful Lemma.

Lemma 5.4.2. [22, Corollary 4.4.7] *Every triangulation with at least four vertices is 3-vertex connected.*

Now, combining Theorem 5.4.1 and Lemma 5.4.2 we have that:

Corollary 5.4.3. *A graph $G = (V, E)$ is the 1-skeleton of a convex simplicial polyhedron iff G is a triangulation with at least four vertices.*

We now introduce the Henneberg steps that we will use in order to construct all 1-skeleta of convex simplicial polyhedra. We will call these H_1, H_2 and H_3 and they are illustrated in Figure 5.10. Notice that the H_1 and H_2 steps, coincide with the H_1^3 and H_2^3 steps.

We continue with a Lemma similar in flavor to Lemma 5.1.1.

Lemma 5.4.4. *Let $G = (V, E)$ be the 1-skeleton of a convex simplicial polyhedron in $\mathbb{E}^3$. Then G contains at least one vertex of degree 3,4 or 5.*

Proof. Suppose on the contrary that deg $v \geq 6, \forall v \in V$. Then

$$\sum_{v \in V} \deg v \geq 6|V| \Leftrightarrow 2|E| \geq 6|V| \Leftrightarrow 3|V| - 6 \geq 3|V|$$

which is absurd. Finally by Theorem 5.4.1, since G is 3-vertex connected, every vertex has degree at least 3, thus establishing the claim. □

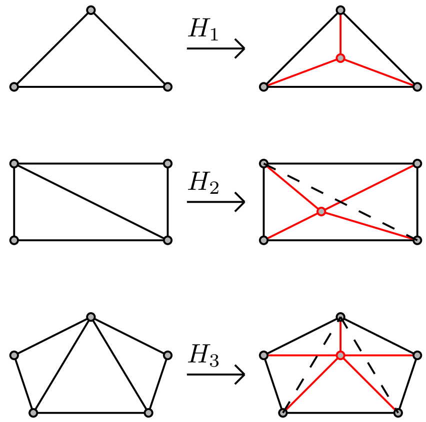

Figure 5.10: The Henneberg steps that will enable us to construct all 1-skeleta of convex simplicial polyhedra.

We are now in a position where we can state and prove our main Theorem.

Proposition 5.4.5. [10] *Graph G is the 1-skeleton of a convex simplicial polyhedron in $\mathbb{E}^3$ iff it has a construction that begins with the 1-skeleton of the 3-simplex followed by a sequence of H_1, H_2, H_3 steps.*

Proof. Let G be the 1-skeleton of a simplicial polyhedron, so by Corollary 5.4.3, G is a triangulation. By Lemma 5.4.4 there exists at least one vertex of degree 3, 4 or 5, and we choose one with minimum degree. Suppose without loss of generality that the minimum degree is 5 (similar considerations hold for the other cases). The reader is referred to Figure 5.11. Since the graph is planar

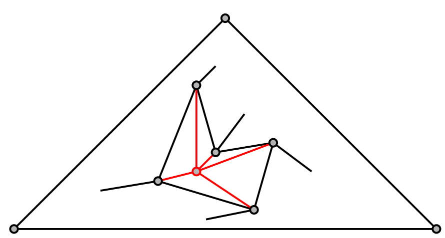

Figure 5.11: A representative case of the proof, where a degree-5 vertex is removed.

none of the diagonals of the pentagon can present. Thus we do a H_3 step in reverse, by removing the red vertex and adding two diagonals in such a way, so that the pentagon becomes triangulated. The resulting graph G' is a triangulation, so in view of Corollary 5.4.3 it is the 1-skeleton of a simplicial polyhedron. If G' has exactly 4 vertices, then it is the 3-simplex and we reached the base case. Otherwise, an inductive argument completes the proof.

For the other direction, every graph produced by a H_1, H_2, H_3 sequence that begins with a 3-simplex is clearly a triangulation with at least 4 vertices and thus by Corollary 5.4.3, it is the 1-skeleton of a simplicial polyhedron. □

Since we have established the existence of an inductive procedure that enables us to construct all 1-skeleta of 3-dimensional convex simplicial polyhedra, one is immediately tempted to explore the possibility of exploiting these constructions, in order to prove facts about this important class of graphs. We will now offer two such examples, but first some definitions are in order.

Definition 5.4.6. *The diameter of a graph $G = (V, E)$ denoted by $\delta(G)$, is the longest path between any two vertices of the graph i.e.*

$$\delta(G) = max\{d(u, v)|u, v \in V\}$$

where $d(u, v)$ is the number of edges in a shortest path connecting u and v.

Definition 5.4.7. *For $f > d \geq 2$, let $\Delta(d, f)$ denote the maximal diameter of all 1-skeleta of d-dimensional polyhedra, with at most f facets.*

As an example, it is easy to verify that $\Delta(2, f) = \lfloor \frac{n}{2} \rfloor$. In 1957, in a letter to G.B. Dantzig, W.M. Hirsch conjectured that $\Delta(d, f) \leq f - d$. This conjecture is known to be true for $d < 4$ and for various special cases but the general status of the problem is open [21, 43, 38].

The value of $\Delta(d, f)$ is a lower bound on the number of iterations for the simplex algorithm with *any* pivot rule and thus determining the behavior of the $\Delta(d, f)$ function is closely related to the question of whether there is some pivot rule for which the simplex algorithm is strongly polynomial. For more details concerning the Hirsch conjecture, the reader is referred to [59, Section 3.3].

Keeping our promise, we present the first Lemma concerning convex simplicial polytopes, whose proof relies on the fact that this class of graphs can be constructed inductively using the Henneberg steps.

Lemma 5.4.8. *Every convex simplicial polytope $\mathcal{P}$ has an even number of facets.*

Proof. By Theorem 5.4.5 we know that every 1-skeleton of a convex simplicial polytope has a Hennenberg construction that starts with the 1-skeleton of the 3-simplex and continues with the H_1, H_2 and H_3 steps. It is clear that each Henneberg step increases the number of triangles in the triangulation by two and thus the number of facets of the corresponding polytope will also be increased by the same number. Since every Hennenberg construction begins with the 3-simplex which has 4 facets the claim is established. □

We continue with our second result.

Corollary 5.4.9. *The Hirsch conjecture is true for the 1-skeleta of convex simplicial polyhedra in $\mathbb{E}^3$.*

Proof. The proof goes by induction on the number $2k$ of facets of a convex simplicial polyhedron.

For the base case $k = 2$, since the only convex simplicial polyhedron with 4 facets is the 3-simplex it is clear that $\Delta(3, 4) \leq 1$.

Suppose now that the theorem is true for all polyhedra with up to $2k$ facets. We need to show that $\Delta(3, 2k+2) \leq 2k-1$ so let $\mathcal{P}$ be a polyhedron with $2k+2$ facets and let $S_3, \mathcal{P}_1, \ldots, \mathcal{P}_{k-2}, \mathcal{P}_{k-1} = \mathcal{P}$ be its Hennenberg sequence, where S denotes the 1-skeleton of the 3-simplex. By Lemma 5.4.8 the polytope $\mathcal{P}_{k-2}$ has $2k$ facets and thus by the induction hypothesis we obtain that $\delta(\mathcal{P}_{k-2}) \leq 2k-3$. Let n be the new vertex, u, v two old vertices and a one of $n's$ neighbour. Now there are three cases to consider:

- $\mathcal{P}_{k-1}$ is obtained by $\mathcal{P}_{k-2}$ through a H_1 step (see Figure 5.12).

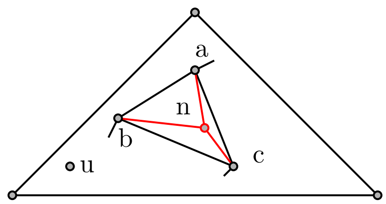

Figure 5.12: $\mathcal{P}_{k-1}$ is obtained by $\mathcal{P}_{k-2}$ through a H_1 step.

Since a H_1 step removes no edges it follows the maximum distance between all the old vertices is less than $2k - 3 < 2k - 1$. It remains to show that the distance between any old vertex u and the new vertex n is at most $2k - 1$. But $d(u, n) = d(u, a) + 1 \leq 2k - 3 + 1 = 2k - 2 < 2k - 1$, thus the claim is established.

- $\mathcal{P}_{k-1}$ is obtained by $\mathcal{P}_{k-2}$ through a H_2 step (see Figure 5.13).

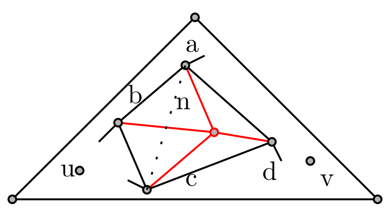

Figure 5.13: $\mathcal{P}_{k-1}$ is obtained by $\mathcal{P}_{k-2}$ through a H_2 step.

Let ac be the edge that was removed. We first show that the distance between any pair of old vertices u, v in $\mathcal{P}_{k-1}$ is at most $2k - 2$.

By the induction hypothesis we know that there existed a path P between vertices u, v in $\mathcal{P}_{k-2}$, with length at most $2k-3$. If $ac \notin \mathcal{P}$ then $d(u, v) \leq$

$2k-3 < 2k-1$ in $\mathcal{P}_{k-1}$. On the other hand, if $ac \in P$, we construct a new path $P^{'}$ between vertices u and v, where $P^{'} = P - \{ac\} \cup \{cn\} \cup \{na\}$ and $\text{length}(P^{'}) = \text{length}(P) - 1 + 2 \leq 2k-3-1+2 = 2k-2 < 2k-1$. Thus we have established that $d(u,v) \leq 2k-2$ for every pair of old vertices u, v in $\mathcal{P}_{k-1}$.

It remains to show that $d(u,n) \leq 2k-1$ for every old vertex u in $\mathcal{P}_{k-1}$. Picking any neighbor of n, for example vertex a, we have that $d(u,n) = d(u,a) + 1 \leq 2n-1$.

- $\mathcal{P}_{k-1}$ is obtained by $\mathcal{P}_{k-2}$ through a H_3 step (see Figure 5.14).

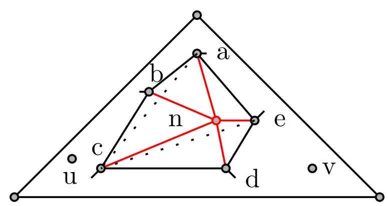

Figure 5.14: $\mathcal{P}_{k-1}$ is obtained by $\mathcal{P}_{k-2}$ through a H_3 step.

Let ac and ec be the edges that were removed. We first show that the distance between any pair of old vertices u, v in $\mathcal{P}_{k-1}$ is at most $2k-2$.

By the induction hypothesis we know that there existed a path P between vertices u and v in $\mathcal{P}_{k-2}$, whose length was at most $2k-3$. If path P does not involve any of the edges that were removed the claim is trivially true. So suppose that path P involves at least one of the edges ac, ec.

If path P contains exactly one of the egdes ac, ec then the claim follows by the same argument that was used in the previous case. On the other hand, if path P contains both edges, then P contains a loop L based on vertex c. Since L contains at least three vertices it follows that the path $P^{'} = P - L$ connects vertices u and v and moreover $\text{length}(P^{'}) \leq \text{length}(P) - 3 \leq (2k-3) - 3 < 2k-2$.

Finally we need to show that $d(u,n) \leq 2n-1$, for every old vertex u in $\mathcal{P}_{n-1}$. Picking any neighbor of n, say a for example, we have that $d(u,n) = d(u,a) + 1 \leq 2n-2+1 = 2n-1$, thus establishing the claim.

$\square$

Chapter 6

Counting the number of embeddings of minimally rigid graphs

In this chapter our goal will be to compute tight bounds on the number of distinct planar and spatial Euclidean embeddings of generically minimally rigid graphs, up to rigid motions, as a function of the number of vertices. In order to accomplish this, we define a square polynomial system, obtained the edge length constraints, whose real solutions correspond precisely to the different embeddings.

An example for $d = 3$ can be seen below. Here (x_i, y_i, z_i) are the coordinates of the i-th vertex, and 3 vertices (which define a facet) are fixed to discard translations and rotations:

$$\begin{cases} x_i = a_i,\ y_i = b_i,\ z_i = c_i, & i = 1, 2, 3, \quad a_i, b_i, c_i \in \mathbb{R}, \\ (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = l_{ij}^2, & ij \in E - \{12, 13, 23\} \end{cases}$$

Notice that all nontrivial equations are quadratic; there are $2n - 4$ for Laman graphs, and $3n - 9$ for 1-skeleta of simplicial polyhedra, where n is the number of vertices. The classical Bézout bound on the number of roots of a polynomial system is equal to the product of the polynomials' degrees, and yields 4^{n-2} and 8^{n-3}, respectively. It is indicative of the hardness of the problem that efforts to substantially improve these bounds have failed.

Specifically, for the planar case the best known upper bound is $\binom{2n-4}{n-2} \approx 4^{n-2}/\sqrt{\pi(n-2)}$ and for the spatial case $\frac{2^{n-3}}{n-2}\binom{2n-6}{n-3} \approx 8^{n-3}/\big((n-2)\sqrt{\pi(n-3)}\big)$, when restricted to the class of graphs that correspond to 1-skeleta of convex simplicial polyhedra. Both of these bounds were obtained in [9] using complex algebraic geometry.

Mixed volume (or BKK bound) exploits the sparseness of the equations to bound the number of common roots, it is always bounded by Bézout's bound and typically much tighter. Nonetheless, mixed volume yields an upper bound of 4^{n-2} in $\mathbb{E}^2$ [50] and an upper bound of 8^{n-1} in $\mathbb{E}^3$ [49], when restricted to the class of graphs that correspond to 1-skeleta of convex simplicial polyhedra. Notice again that both of these bounds fail to improve on the Bézout bound.

Passing to lower bounds, the best known ones in $\mathbb{E}^2$ are $24^{\lfloor (n-1)/2 \rfloor} \simeq 2.21^n$ and $2 \cdot 12^{\lfloor (n-3)/3 \rfloor} \simeq 2.29^n/6$ [1], obtained using a "caterpillar" and a "fan"construction, respectively [9]. Both of them are based on the Desargues (or triangular prism) graph (Figure 6.4), which admits 24 embeddings [9].

In applications, it is crucial to know the number of embeddings for specific (small) values of n. The most important results in this direction are that the Desargues graph admits 24 embeddings in $\mathbb{E}^2$ [9] and that the $K_{3,3}$ graph admits 16 [55] and 32 [26] embeddings in $\mathbb{E}^2$. Additionally it is also known that the cyclohexane graph admits 16 embeddings in $\mathbb{E}^3$ [25].

Our main contribution is twofold: first, we derive an improved lower bound in $\mathbb{E}^2$ and the first non-trivial lower bound in $\mathbb{E}^3$:

$$32^{\lfloor (n-2)/4 \rfloor} \simeq 2.37^n,\ n \geq 10, \quad \text{and} \quad 16^{\lfloor (n-3)/3 \rfloor} \simeq 2.52^n,\ n \geq 9,$$

by designing a $K_{3,3}$ caterpillar and a cyclohexane caterpillar, respectively. The way these bounds are derived is simple enough to allow for improvements.

Second, we give tight bounds for $n = 7, 8$ in $\mathbb{E}^2$ and $n = 6, 7$ in $\mathbb{E}^3$; these are important for applications, and may lead to tighter lower bounds. We also reduce the existing gap for $n = 9, 10$ in $\mathbb{E}^2$, and $n = 8, 9, 10$ in $\mathbb{E}^3$, see Tables 6.2 and 6.3. For this, we have reformulated the corresponding polynomial system to remove spurious solutions, since we prove that (6.1) cannot yield tight bounds. Further, we have implemented specialized software that generates all rigid graphs, for small n, up to isomorphism, and computes the respective mixed volumes. Our results indicate that mixed volume can be of general interest in enumeration problems.

Some of these results appeared in [27] in preliminary form. A more detailed account of our results can be found in [26].

6.1 Mixed volume basics

In this section we introduce our main algebraic tool, of which we will make extensive use throughout the rest of this chapter. Specifically, we will study how the geometry of polytopes can be used to predict the number of solutions of a square polynomial system. For background see [7, 12, 18] and references therein. We start with some necessary definitions.

Definition 6.1.1. *Given a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$, its support is defined to be the set of exponents that corresponding to monomials of f.*

For example, the support of the polynomial $f(x, y) = x^2 + y^2 + xy + 1$ is the set $\{(2, 0), (0, 2), (1, 1), (0, 0)\}$.

[1]This corrects the exponent of the original statement.

Definition 6.1.2. *The Newton polytope of the polynomial* $f \in \mathbb{C}[x_1, \ldots, x_n]$*, denoted by NP(f), is a polytope in* $\mathbb{R}^n$ *defined as the convex hull of the points in the support of* f*.*

For two examples see Figure. 6.1.

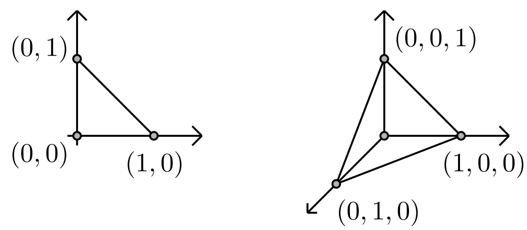

Figure 6.1: The Newton polytopes of the polynomials $f(x, y) = x + y + 1$ and $f(x, y, z) = x + y + z + 1$, respectively.

We proceed by describing two operations induced the vector space structure of $\mathbb{R}^n$, that will be used to form new polytopes from old ones.

Definition 6.1.3. *Let* P, Q *be polytopes in* $\mathbb{R}^n$ *and let* $\lambda \geq 0$ *be a real number. The Minkowski sum of* P *and* Q*, denoted by* $P + Q$*, is defined as*

$$P + Q = \{p + q \mid p \in P \, and \; q \in Q\}$$

where $p + q$ *denotes the usual vector sum in* $\mathbb{R}^n$*.*
Additionally, the polytope λP *is defined as*

$$\lambda P = \{\lambda p \mid p \in P\}$$

where λp *is the usual scalar multiplication in* $\mathbb{R}^n$*.*

See Figure 6.2 for an example of the Minkowski sum of two polytopes.

Now, let $P_1, \ldots, P_n$ be a collection of polytopes in $\mathbb{R}^n$ and let $\lambda_i \in \mathbb{R}, \lambda_i \geq 0$, for $i = 1, \ldots, n$. Consider the Minkowski sum of the scaled polytopes $\lambda_1 P_1 + \cdots + \lambda_n P_n \in \mathbb{R}^n$. It is a known fact that the n-th dimensional Euclidean volume $vol_n(\lambda_1 P_1 + \cdots + \lambda_n P_n)$ is a homogeneous polynomial of degree n in the λ_i.

Definition 6.1.4. *The coefficient of the monomial* $\lambda_1 \lambda_2 \cdots \lambda_n$ *in* $vol_n(\lambda_1 P_1 + \cdots + \lambda_n P_n)$ *is defined to be the mixed volume of the polytopes* $P_1, \ldots, P_n$ *and is denoted by* $MV_n(P_1, \ldots, P_n)$*.*

Among other things, the mixed volume is linear in each argument i.e.

$$MV_n(\ldots, aP_i + \beta P_i^{'}, \ldots) = aMV_n(\ldots, P_i, \ldots) + \beta MV_n(\ldots, P_i^{'}, \ldots)$$

and it generalizes the usual volume in the sense that if $P_1 = \cdots = P_n = P$, then

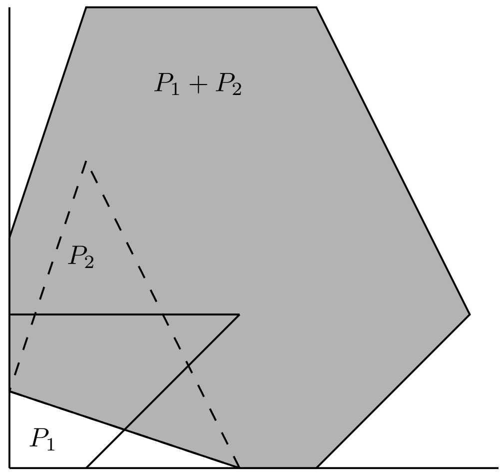

Figure 6.2: The Minkowski sum (shaded) of the two polygons P_1 (solid line) and P_2 (dashed line).

$$MV_n(P, \ldots, P) = n! vol_n(P)$$

For an extensive study of the properties of the mixed volume, the reader is referred to [18, Chapter 7].

In what follows, we focus on the topological *torus* $\mathbb{C}^* = \mathbb{C} - \{0\}$. We now state a fundamental Theorem due to D.N. Bernstein.

Theorem 6.1.5. [7] *Let $f_1, \ldots f_n \in \mathbb{C}[x_1, \ldots, x_n]$ and consider the square polynomial system $f_1 = \ldots = f_n = 0$. Then, the number of isolated solutions in $(\mathbb{C}^*)^n$ is bounded above by the* mixed volume *of the Newton polytopes of the f_i. Moreover, this bound is tight for a generic choice of coefficients of the f_i's.*

In addition to Bernstein's original paper, there are also closely related papers by Kushnirenko and Khovanskii. For this reason, the mixed volume bound on the number of solutions given by Theorem 6.1.5 is sometimes referred to as the BKK bound.

In the same paper Bernstein obtains an explicit condition, now known as Bernstein's Second Theorem, that describes when a choice of coefficients is generic. Before we state the Theorem we need the following definition:

Definition 6.1.6. *Given $v \in \mathbb{R}^n - \{0\}$ and a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$, we denote by $\partial_v f$ the polynomial obtained by keeping only those terms of f, whose exponents minimize the inner product with v.*

Notice that, the Newton polytope of the polynomial $\partial_v f$ is just the face of the $NP(f)$, which supported by v. We are now ready to state Bernstein's second Theorem.

Theorem 6.1.7. [7] *Let $f_1, \dots f_n, \in \mathbb{C}[x_1, \dots, x_n]$ and consider the square polynomial system $f_1 = \dots = f_n = 0$. If for all $v \in \mathbb{R}^n - \{0\}$ the face system $\partial_v f_1 = \dots = \partial_v f_n = 0$ has no solutions in $(\mathbb{C}^*)^n$, then the mixed volume of the f_i is exactly equal to the number of solutions in $(\mathbb{C}^*)^n$, and all solutions are isolated. Otherwise, the mixed volume is a strict upper bound on the number of isolated solutions.*

6.2 Algebraic formulation of the problem

Let $G = (V, E)$ with $|V| = n$, be a generically minimally rigid graph in $\mathbb{E}^d$ and let $G(p)$ be a specific realization of G in the plane. Moreover, let $l_{ij} = ||p_i - p_j||$, $ij \in E$, be the corresponding lengths that are induced to the edges of G by the framework $G(p)$.

A crucial observation is to made here: the problem of computing the number of realizations of G, which are equivalent but non-congruent to $G(p)$ can be formulated as an algebraic one.

Specifically, we can construct a polynomial system, whose real solutions correspond to all possible realizations of G which are non-congruent to $G(p)$ but induce the same edge lengths l_{ij}. For a representative case when $d = 2$, see system (6.1) below:

$$\begin{cases} x_i = a_i,\ y_i = b_i, & i = 1, 2, \quad a_i, b_i \in \mathbb{R}, \\ (x_i - x_j)^2 + (y_i - y_j)^2 = l_{ij}^2, & ij \in E - \{12\} \end{cases} \tag{6.1}$$

This is a square $2n \times 2n$ polynomial system in the unknowns $x_1, y_1, \dots, x_n, y_n$, where (x_i, y_i) corresponds to the coordinates of the vertex i in the embedding.

Notice that, the equations of this system can be divided in two distinct groups (the first and second line of system (6.1)). The equations of the second group merely express our interest in frameworks which are equivalent to $G(p)$. So, what does the first group of equations stand for? Thinking geometrically, we see that the effect of the first group of equations is to fix the edge defined by vertices v_1 and v_2 (where we have assumed without loss of generality that there exists one between them). This is necessary in order to discard the solutions of system (6.1) which correspond to rigid motions (translations and rotations) of $\mathbb{E}^2$, for otherwise the system would have an infinite number of real solutions. This is why these equations are usually referred to as the "pin down" equations.

Clearly, in order to bound the number of embeddings of a minimally rigid graph G, it is enough to bound the number of real solutions of system (6.1). Since this task is overwhelmingly difficult, we restrict ourselves with bounds on the number of complex roots of our system.

Perhaps the most famous bound on the number of roots of a polynomial system is the classical Bézout bound, which is just the product of the degrees of the polynomials. Thus in our case, the Bézout bound is equal to 4^{n-2} and 8^{n-3} for $d = 2, 3$, respectively.

Since system (6.1) is quite sparse, it is reasonable to assume that the BKK bound introduced in Section 6.1 will yield better results then the Bézout bound.

However, this is not the case (at least for $d = 2$) as was established in [50]. Specifically, we have the following:

Theorem 6.2.1. [50, Theorem 8] *For any Laman graph on n vertices, the mixed volume of system (6.1) is exactly* 4^{n-2}.

We should here note that, it is indicative of the hardness of the problem that all efforts to substantially improve the Bézout bound have essentially failed. Along with the 4^{n-2} bound of [50], the best known upper bounds are $\binom{2n-4}{n-2} \approx 4^{n-2}/\sqrt{\pi(n-2)}$ and $\frac{2^{n-3}}{n-2}\binom{2n-6}{n-3} \approx 8^{n-3}/\left((n-2)\sqrt{\pi(n-3)}\right)$, for the planar and spatial case, respectively [8, 9]. These bounds were obtained using complex algebraic geometry.

In trying to explain why the mixed volume of system (6.1) fails to give something better than the Bézout bound, a first observation is that the formulation system (6.1), does not satisfy Bernstein's second Theorem and consequently, the computed mixed volume is not a tight bound an the number of solutions in $(\mathbb{C}^*)^{2n}$. This was observed in [50] for the case $d = 2$. In, [27] we extend this to the case when $d = 3$. We now go briefly through the proof of this fact.

For $d = 3$, the corresponding formulation to that of system (6.1) is the following $3n \times 3n$ square polynomial system:

$$\begin{cases} x_i = a_i,\ y_i = b_i,\ z_i = c_i, & i = 1,2,3, \quad a_i, b_i, c_i \in \mathbb{R}, \\ (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = l_{ij}^2, & ij \in E - \{12, 13, 23\} \end{cases} \tag{6.2}$$

We have assumed here without loss of generality that edges $12, 13, 23$ define a facet. Choosing direction $v = (0,0,0,0,0,0,0,0,0,-1,\ldots,-1) \in \mathbb{R}^{3n}$, the corresponding face system is:

$$\begin{cases} x_i = a_i, y_i = b_i, z_i = c_i, & i = 1,2,3, \quad a_i, b_i, c_i \in \mathbb{R}, \\ (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = 0, & ij \in E,\ i,j \notin \{1,2,3\}, \\ x_i^2 + y_i^2 + z_i^2 = 0, & ij \in E:\ i \notin \{1,2,3\},\ j \in \{1,2,3\}. \end{cases}$$

This system has

$$(\underbrace{a_1, b_1, c_1}_{v_1}, \ldots, \underbrace{a_3, b_3, c_3}_{v_3}, \underbrace{1, 1, \gamma\sqrt{2}}_{v_4}, 1, 1, \gamma\sqrt{2}, \ldots, \underbrace{1, 1, \gamma\sqrt{2}}_{v_n}) \in (\mathbb{C}^*)^{3n}$$

as a solution, where $\gamma = \pm\sqrt{-1}$. Consequently, according to Theorem 6.1.7, the mixed volume is not a tight bound on the number of solutions in $(\mathbb{C}^*)^{3n}$.

To remove this degeneracy we will apply an idea of Ioannis Z. Emiris proposed to the authors of [50] at EuroCG'08. Specifically, we will introduce new variables for common subexpressions that appear in the polynomials of the system. Surprisingly enough, the introduction of new variables which increases the Bézout bound, can nontheless decrease the BKK bound [24].

In order to remove spurious solutions at toric infinity, we introduce new variables $s_i = x_i^2 + y_i^2 + z_i^2$, for $i = 1, \ldots, n$. This yields the $4n \times 4n$ polynomial system seen below:

$$\begin{cases} x_i = a_i, y_i = b_i, z_i = c_i, & i = 1, 2, 3, \\ s_i = x_i^2 + y_i^2 + z_i^2, & i = 1, \ldots, n, \\ s_i + s_j - 2x_i x_j - 2y_i y_j - 2z_i z_j = l_{ij}^2, & ij \in E - \{12, 13, 23\} \end{cases} \tag{6.3}$$

This system is equivalent to system (6.2) and this is the formulation we will be using from now on, although we were not able to show that it satisfies Bernstein's second Theorem. However, in practice, it yields better results (i.e. lower mixed volumes) than formulation (6.2).

For the rest of this capter, we will focus our attention on the class of Laman graphs and of the 1-skeleta of convex simplicial polyhedra. In order to bound the number of embeddings of of both of these graph classes, we have developed specialized software that constructs all Laman graphs for $n \leq 9$ and all 1-skeleta of simplicial polyhedra in $\mathbb{E}^3$ with $n \leq 10$. The reason we are able to do this is because both of these graph classes admit inductive Henneberg constructions, as explained in Section 5.

Our computational platform is SAGE [2] and the Henneberg steps are implemented, using SAGE's interpreter, in Python. After we construct all the graphs, we then classify them up to isomorhism using SAGE's interface for N.I.C.E., an open-source isomorphism check engine, keeping for each graph the Henneberg sequence with largest number of H_1 steps.

For each one of these graphs we then set up its corresponding polynomial system and for each system we bound the number of its (complex) solutions by computing its mixed volume, using [12]. Notice that for every Laman graph, in order to discard translations and rotations, we assume that one edge is of unit length, aligned with an axis, with one of its vertices at the origin. In $\mathbb{E}^3$, a third vertex also needs to be fixed so as to belong to a coordinate plane. The corresponding coordinates are given specific values and are no longer unknowns.

We should note here that, the business of fixing an edge in $\mathbb{E}^2$ (a facet, in $\mathbb{E}^3$) can have a somewhat unexpected consequence : depeding on the choice of the edge (facet) to fix, we obtain systems that might have different mixed volumes. But, since all of these systems bound the actual number of embeddings, we use the minimum of the computed mixed volumes as an upper bound to the number of embeddings.

For our computations, we used an Intel Core2, at 2.4GHz, with 2GB of RAM. We tested more that 20 000 graphs and computed the mixed volume of more than 40 000 polynomial systems. The total time of experiments was about 2 days. Tables 6.2 and 6.3 summarize our results.

6.3 Planar embeddings of Laman graphs

Recall that the class of Laman graphs coincides the class of generically minimally rigid graphs in $\mathbb{E}^2$ and moreover they can be constructed by a *Henneberg*

[2] `http://www.sagemath.org/`

sequence, which starts with a triangle, followed by a sequence of Henneberg-1 (or H_1) and Henneneerg-2 steps (or H_2). We represent each Laman graph by $\triangle s_4 \ldots, s_n$, where $s_i \in \{1, 2\}$, and $s_4, \ldots, s_n$ is its Henneberg sequence. A Laman graph G is called of type H_1 if it can be constructed using only H_1 steps, and of type H_2 if a H_2 step is necessary in order to construct G.

We start with a simple Lemma, which establishes the effect of a H_1 step on the number of embeddings of an abstract framework $G = (V, E)$.

Lemma 6.3.1. *A H_1 step exactly doubles the number of embeddings of a generically minimally rigid graph $G = (V, E)$ in $\mathbb{E}^2$.*

Proof. Let $G = (V, E)$ be a generically minimally rigid graph and let k be the maximum number of non-congruent embeddings of G in $\mathbb{E}^2$, over all generic admissible edge length assignments. We will first show that a H_1 step *at least* doubles the number of embedings. Let $G^{'}$ be a graph obtained by G through a H_1 step. The claim is that each embedding of G induces exactly two embeddings for $G^{'}$.

So, let $G(p)$ be an embedding of G. In order to obtain an embedding of $G^{'}$ we need to insert one new vertex through a H_1 step. By the the definition of the H_1 step, it should be clear that the new point lies on the intersection of two circles, centered at the two vertices the H_1 step is applied to, see Figure 6.3. So, if we choose the radii of the two circles to be sufficiently large, then the two circles are guaranteed to have two points of intersection in each of the k embeddings of G and thus we get two points where the new vertex can be placed. For each one of these two points we obtain one embedding for $G^{'}$ and the claim follows.

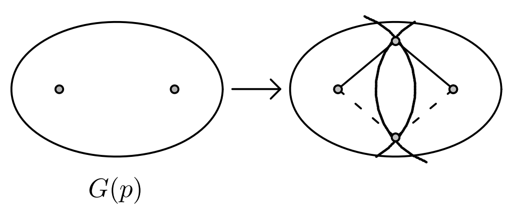

Figure 6.3: A realization of G and the two points (intersection points of the two circles) where the new vertex can be placed at.

On the other hand, given an embedding of $G^{'}$, if one removes the new vertex and its incident edges the resulting framework is an embedding of G. Notice that the same thing happens for the symmetric embedding $G^{'}$, where by symmetric we mean the embedding where the new vertex is placed in its symmetric position. Since G has at most k embeddings it follows that $G^{'}$ cannot have more than $2k$ embeddings. $\square$

Corollary 6.3.2. [9, Proposition 5.2] *The number of embeddings in $\mathbb{E}^2$ of a H_1 graph on n vertices, is at most 2^{n-2} and this is tight.*

Proof. A H_1 graph on n vertices has a Laman sequence that starts with a triangle and continues with $n-3$, H_1 steps. Since a triangle has two embeddings and since a H_1 step exactly doubles the number of embeddings, the claim follows. □

At this point it is worth noticing that for the special class of H_1 graphs, the mixed volume of system (6.3) is 2^{n-2} and thus, in this case, it is a tight bound on the number of embeddings [50, Lemma 6]. The same thing holds for $d = 3$ and a detailed proof can be found in the next Section. We now continue with some lower bounds.

Theorem 6.3.3. [9, Lemma 5.3] *There exist edge lengths for which the Deasrgues graph can be embedded in $\mathbb{E}^2$ in at least 24 ways.*

Proof. Recall that the Desargues framework has a Henneberg construction and thus it is generically minimally rigid in the plane. Since it is minimally rigid, if we remove *any edge*, the resulting framework will be generically flexible in $\mathbb{E}^2$ (see Figure 6.4)

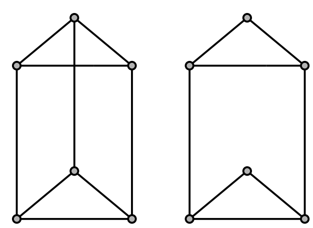

Figure 6.4: The Desargues framework and a generically flexible framework in $\mathbb{E}^2$ obtained by it, through the removal of an edge.

Now, consider some generic realization of the generically flexible framework we constructed above. See Figure 6.5 for an example.

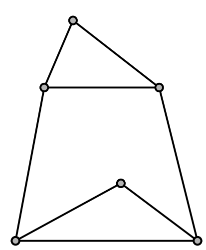

Figure 6.5: A generic realization of the generically flexible framework, constructed above.

Let us discard (for now) the lower triangle of the generic framework depicted in Figure 6.5. Our goal now, is to compute appropriate lengths for the three erased edges, such that the corresponding framework can be embedded in at least 24 ways in the plane.

After the removal of the lower triangle, the resulting framework is the 4-bar mechanism illustrated in Figure 6.6.

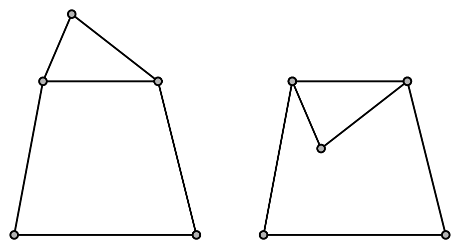

Figure 6.6: The 4-bar mechanism obtained by the framework of Figure 6.5, together with a 4-bar mechanism obtained by it, by reflecting the upper triangle.

Clearly, once an edge of this framework is pinned down, the other vertices trace algebraic curves. Moreover, it is a known fact, that as this mechanism moves, the top vertex traces a curve of degree 6 reference (see Figure 6.7) and the same thing is true for the 4-bar mechanism, where the upper triangle is reflected (the second framework in Figure 6.6).

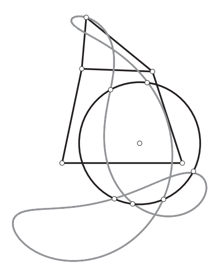

Figure 6.7: The bottom bar is pinned down to the plane and as the mechanism moves, the top vertex traces an algebraic curve of degree 6, illustrated in light gray.

Now, place a circle in such a way, so that it intersects *each one* of the two degree 6 curves at six points, that are distinct. Such a circle is illustrated in Figure 6.8.

Now, we claim that if the two deleted triangle edges are assigned edge lengths equal to the distance between the bottom vertices and the center of the circle, respectively, and the other deleted edge, is assigned length equal to the radius of the circle, then the corresponding framework has 24 embeddings.

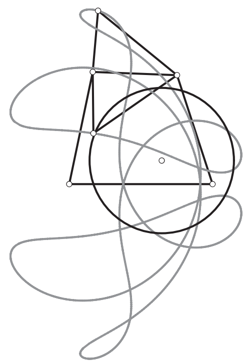

Figure 6.8: A circle that intersects each one of the two curves at 6 points.

Indeed, suppose we place the deleted vertex at the center of the circle and let $p_1, \ldots, p_{12}$, be an arbitrary labeling of the 12 intersection points of the two curves with the circle. Then, each p_i induces an embedding of the Desargues framework with the edge lengths prescribed above.

To see why this is true, fix some $i \in \{1, \ldots 12\}$ and suppose p_i lies on the curve traced by the top vertex of the 4-bar mechanism in Figure 6.7. Since p_i lies on the coupler curve, this means that the 4-bar mechanism can be flexed so that the location of the top vertex coincides with that of p_i. Then, if all the removed edges are added, we obtain one embedding of the Desargues framework with the prescribed edge lengths. Since we have 12 distinct intersection points, it follows that the same procedure allows us to construct 12 distinct embeddings. An example, taken from the Phd thesis of R. Steffens [49] can be seen in Figure 6.9. Moreover, by reflecting each one of these 12 embeddings about the bottom bar, we obtain twelve more and thus obtain a total of 24 embeddings.

□

Based on the existence of edge lengths, for which the Desargues graph has (at least) 24 embeddings, we can construct graphs which will provide us with general lower bounds. The first such example, is the so-called Desargues "caterpillar" framework, introduced in [9].

Given a number of Desargues frameworks, we can "glue" them together along an edge, in a "caterpillar" fashion. For an example with 3 copies, see Figure 6.10

One can use induction on the number of Desargues copies in the caterpillar, in order to show that the resulting graph has the Laman property. A straightforward calculation gives the following:

Lemma 6.3.4. [9, Lemma 5.4] *There exist edge lengths for which the Desargues caterpillar framework has* $24^{\lfloor (n-2)/4 \rfloor} \approx 2.21^n$ *embeddings.*

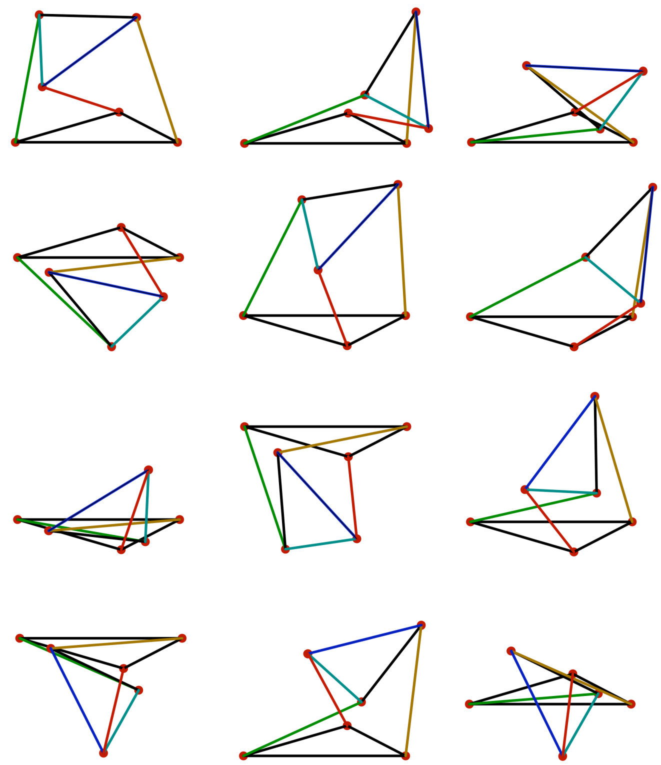

Figure 6.9: Twelve embeddings of the Desargues framework taken from [49].

In order to get a better lower bound we will use another iterative construction, known as the Desargues "fan" construction, which was also introduced in [9].

This time, we fix a bar in the plane and place on it a 4-bar mechanism. We also fix the positions of the symmetric triangles and the corresponding circles that give the 24 embeddings. Then, we perturb the 4-bar mechanism (but not the base triangle) several times to obtain a fan-like gluing of Desargues congurations. For an example with 3 copies, see Figure 6.11.

Again, using induction on the number of copies of the Desargues framework, it is easy to show that the resulting framework has the Laman property and moreover:

Lemma 6.3.5. [9, Proposition 5.6] *There exist edge lengths for which the Desargues "fan" framework has* $2 \cdot 12^{\lfloor (n-3)/3 \rfloor} \approx 2.29^n/6$ *embeddings.*

We continue with another lower bound that has been extensively used (cf. [50]) although there seems to be no rigorous proof of this fact. In [55, Section

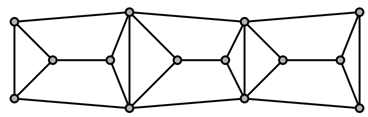

Figure 6.10: A Desargues caterpillar framework, with 3 copies.

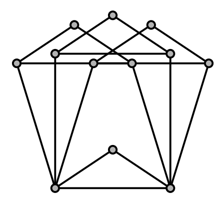

Figure 6.11: A Desargues fan framework, with 3 copies. The edges connecting the triangles were removed, for the sake of clarity.

4], D. Walter and M. Husty show that there exist edge lengths for which the $K_{3,3}$ graph has 16 embeddings. It turns out that more than that is true:

Theorem 6.3.6. *The $K_{3,3}$ graph has 32 embeddings for the edge lengths prescribed in* [55, Section 4].

The proof of this fact is straightforward i.e. if one solves the corresponding polynomial system using the edge lengths prescribed in [55, Section 4] the result is 32 real solutions. This computation was performed by E.P. Tsigaridas and the results are illustrated in Table 6.1.

Now, we establish a new lower bound for general n by constructing a $K_{3,3}$ caterpillar. The way to do that is by "gluing" together copies of the $K_{3,3}$ graph so that each new copy has exactly one edge in common with the previous one. For an example with 3 copies see Figure 6.12.

Using induction on the number of $K_{3,3}$ copies, it is easy to show that the resulting graph is Laman.

Theorem 6.3.7. *There exist edge lengths for which the $K_{3,3}$ caterpillar construction has $32^{\lfloor (n-2)/4 \rfloor} \simeq 2.37^n$ embeddings, for $n \geq 10$.*

Proof. Notice that each copy of the $K_{3,3}$ adds 4 vertices, except the first one which adds 6 vertices. So, given a $K_{3,3}$ caterpillar on n vertices, it follows that $n = 6 + 4k$, where k is the number of $K_{3,3,}$ copies excluding the first one. Thus $k + 1 = (n - 2)/4$ and since there exist edge lengths for which the $K_{3,3}$ has 32 embeddings, the claim follows. □

Now, one can easily verify that every △2 graph is isomorphic to a △1 graph and that every △12 graph is isomorphic to a △11 graph. Consequently, all

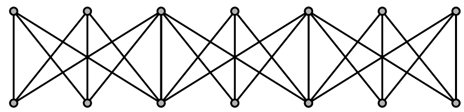

Figure 6.12: A $K_{3,3}$ caterpillar with three copies.

Laman graphs on 4 and 5 vertices are of type H_1, so in view of Corollary 6.3.2 it follows that:

Lemma 6.3.8. *The maximum number of Euclidean embeddings for Laman graphs on 4 and 5 vertices is equal to 4 and 8, respectively. Moreover, these bounds are tight.*

A simple case analysis shows that the first time a H_2 step is necessary in order to construct all Laman graphs is for $n = 6$. Moreover, up to isomorphism, there are only two H_2 graphs on 6 vertices. Specifically, we have that:

Theorem 6.3.9. *The only H_2 graphs on 6 vertices are the $K_{3,3}$ and the Desargues graph.*

Lemma 6.3.10. *The maximum number of Euclidean embeddings for Laman graphs on $n = 6, 7$ and 8 vertices is 32,64 and 128, respectively. Moreover, these bounds are tight.*

Proof. Using our software (see Section 6.1), we constructed all Laman graphs on $n = 6, 7$ and 8 vertices, and computed the mixed volumes of the respective polynomial systems, thus establishing the upper bounds. On the other hand, we obtain matching lower bounds starting with the $K_{3,3}$, which has 32 embeddings, and applying H_1 steps, which exactly double the number of embeddings. $\square$

We end this section with Table 6.2 which summarizes our results for $n \leq 10$. For $n = 9, 10$, the upper bounds are obtained by our software, whereas the lower bounds follow from the Desargues fan [9] and Theorem 6.3.7, respectively.

6.4 Spatial embeddings of the 1-skeleta of simplicial polyhedra

This section extends the previous results to 1-skeleta of convex simplicial polyhedra. We have already seen that these graphs are generically minimally rigid in $\mathbb{E}^3$ (Theorem 4.2.7) and moreover they can be constructed using the appropriate Henneberg steps (Proposition 5.4.5). For more details, see Section 4.2.

Again, we start with a Lemma which determines the effect a spatial H_1 step has on the number of embeddings of an abstract framework G. Notice that this Lemma applies to *any* abstract framework and not just to those that correspond to 1-skeleta of simplicial polyhedra.

Lemma 6.4.1. *A spatial H_1 step exactly doubles the number of embeddings of a generically minimally rigid graph in $\mathbb{E}^3$.*

Proof. Since 3 spheres intersect generically in two points, a spatial H_1 at most doubles the number of spatial embeddings. Moreover, since we can choose the edge lengths to be appropriately large, we are guaranteed that the three spheres will intersect and thus a spatial H_1 step exactly doubles the number of embeddings. □

Corollary 6.4.2. *The number of embeddings in $\mathbb{E}^3$ of a H_1 graph on n vertices, is at most 2^{n-2} and this is tight.*

We will continue by showing that for the special case of H_1 graphs, the mixed volume of the polynomial system using formulation (6.3) is exactly 2^{n-2} and thus a tight bound on the number of embeddings. The proof for the case $d = 2$ can be found in [50, Lemma 6]. We start with a key Lemma, which will enable us to decouple the mixed volume calculation in smaller pieces.

Lemma 6.4.3. [11] *Let $P_1, \ldots, P_k$ be polytopes in $\mathbb{R}^{m+k}$ and $Q_1, \ldots, Q_m$ polytopes in $\mathbb{R}^m \subseteq \mathbb{R}^{m+k}$. Then*

$$MV_{m+k}(Q_1, \ldots, Q_m, P_1, \ldots, P_k) = MV_m(Q_1, \ldots, Q_m) \cdot MV_k(\pi(P_1), \ldots, \pi(P_k))$$

where $\pi : \mathbb{R}^{m+k} \mapsto \mathbb{R}^k$ denotes the projection on the last k coordinates.

Let G be an abstract framework on n vertices and suppose that we perform a spatial H_1 step where we add vertex v_{n+1} that gets connected to vertices v_k, v_l and v_m. Using formulation (6.3), our system gets four new equations, namely:

$$\begin{aligned}
s_{n+1} - x_{n+1}^2 - y_{n+1}^2 - z_{n+1}^2 &= 0 & (6.4)\\
s_{n+1} + s_k - 2x_{n+1}x_k - 2y_{n+1}y_k - 2z_{n+1}z_k - l_{n+1,k}^2 &= 0 & (6.5)\\
s_{n+1} + s_l - 2x_{n+1}x_l - 2y_{n+1}y_l - 2z_{n+1}z_l - l_{n+1,l}^2 &= 0 & (6.6)\\
s_{n+1} + s_m - 2x_{n+1}x_m - 2y_{n+1}y_m - 2z_{n+1}z_m - l_{n+1,m}^2 &= 0 & (6.7)
\end{aligned}$$

Let $Q_1, \ldots, Q_{4n}$ be the Newton polytopes of the $4n$ old equations and let $P_1, \ldots, P_4$ be the Newton polytopes of the four new equations (6.4)-(6.7). Since the variables $s_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}$ appear only in equations (6.4)-(6.7) it follows that $Q_i \subseteq \mathbb{R}^{4n}, i = 1, \ldots, 4n$ and $P_i \subseteq \mathbb{R}^{4n+4}, i = 1 \ldots 4$ so in view of Lemma 6.4.3 we obtain that

$$MV_{4n+4}(Q_1, \ldots, Q_{4n}, P_1, \ldots, P_4) = MV_{4n}(Q_1, \ldots, Q_{4n}) \cdot MV_4(\pi(P_1), \ldots, \pi(P_4))$$

On the other hand, since

$$\pi(P_1) = \operatorname{conv}\{(2,0,0,0), (0,2,0,0), (0,0,2,0), (0,0,0,1)\}$$

and

$$\pi(P_2) = \pi(P_3) = \pi(P_4) = \mathrm{conv}\{(1,0,0,0),(0,1,0,0),(0,0,0,1),(0,0,0,0)\}$$

a straightforward calculation using for example the PHC pack[3] shows that

$$MV_4(\pi(P_1),\ldots,\pi(P_4)) = 2$$

and thus it follows that

$$MV_{4n+4}(Q_1,\ldots,Q_{4n},P_1,\ldots,P_4) = 2\cdot MV_{4n}(Q_1,\ldots,Q_{4n})$$

Lemma 6.4.4. *The mixed volume of the polynomial system that corresponds to a H_1 graph on n vertices, using formulation (6.3), is exactly 2^{n-2}.*

Proof. The proof goes by induction on the number of vertices of a H_1 graph. The clam is true for the base case i.e. the 1-skeleton of the 3-simplex as one can easily establish using for example the PHC pack. Suppose now, that the claim is true for H_1 graphs up to n vertices. Since a H_1 graph on $n+1$ vertices is obtained from a H_1 graph on n vertices through a H_1 step and since a H_1 step exactly doubles the mixed volume of the polynomial system, the claim follows. □

From now on, we will focus our attention on graphs that correspond to 1-skeleta of convex simplicial polyhedra. Notice that for $n = 4$, the only convex simplicial polytope is the 3-simplex. Thus it follows that:

Lemma 6.4.5. *The 1-skeleton of a convex simplicial polyhedron on 4 vertices has at most 2 embeddings and this bound is tight.*

For $n = 5$, it is known that all 1-skeleta of convex simplicial polyhedra are isomorphic to the graph illustrated in Figure 6.13 [10].

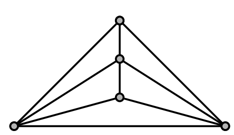

Figure 6.13: The only 1 skeleton of a convex simplicial polytope on 5 vertices.

Clearly, this graph is obtained from the graph of the 3-simplex after a H_1 step, so in view of Theorem 6.4.1 we have that:

Lemma 6.4.6. *The 1-skeleton of a convex simplicial polyhedron on 5 vertices has at most 4 embeddings and this bound is tight.*

The first interesting case is when $n = 6$. Again, it is known that there exist only two non-isomorphic graphs G_1, G_2 for $n = 6$ [10]. These are illustrated in Figure 6.14.

[3] http://www.math.uic.edu/ jan/

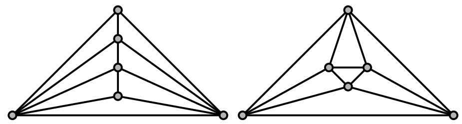

Figure 6.14: All 1-skeleta of convex simplicial polyhedra on 6 vertices.

Theorem 6.4.7. *The 1-skeleton of a simplicial polyhedron on 6 vertices has at most 16 embeddings and this bound is tight.*

Proof. We start with the upper bound. Using our software, we computed that the mixed volume of the polynomial system that corresponds to graph G_1 is equal to 8. On the other hand, since all facets of G_2 are symmetric, we arbitrarily fix one of them and compute the mixed volume of the corresponding polynomial system. In this case the mixed volume turns out to be 16, so the upper bound for $n = 6$ is 16.

We now turn to the lower bound. The graph of the cyclohexane molecule, is essentially a 6-cycle. For the rest of the conversation the reader is referred to Figure 6.15. In [25] it is shown that for equal bond lengths (edge lengths) $L_1, \ldots, L_6$ and equal bond angles $\phi_1, \ldots, \phi_6$, there exist 16 different realizations of the cyclohexane graph. See Figures 6.16 and 6.17 for two of them.

Now, notice that for constant bond lengths (edge lengths) $L_1, \ldots, L_6$ and constant bond angles ϕ_1, ϕ_2, ϕ_3, each of the triangles $T_1 = \triangle(p_1, p_2, p_6), T_2 = \triangle(p_2, p_3, p_4), T_3 = \triangle(p_4, p_5, p_6)$ is fixed and thus the edges p_2p_6, p_2p_4, p_4p_6 have constant length. Similarly, the edges p_1p_3, p_3p_5, p_5p_1 also have constant length. Consequently, the 16 computed realizations of the cyclohexane molecule can be viewed as realizations of graph G_2 and thus the lower bound follows.

□

We now pass to general n and establish a new lower bound using the cyclohexane "caterpillar" construction. Specifically, we "glue" together copies of cyclohexanes so that they share a common triangle. For an example with 2 copies see Figure 6.18.

Theorem 6.4.8. *There exist edge lengths for which the cyclohexane "caterpillar" construction has $16^{\lfloor (n-3)/3 \rfloor} \simeq 2.52^n$ embeddings, for $n \geq 9$.*

Proof. Each new copy of the cyclohexane adds 3 new vertices except the first one, which adds 6 vertices. So, given a cyclohexane caterpillar on n vertices it follows that $n = 6 + 3k$, where k is the number of cyclohexanes in the caterpillar excluding the first one. Thus $k + 1 = (n-3)/3$ and since there exist edge lengths for which the cyclohexane has 16 embeddings, the claim follows.

□

We conclude this Section with Table 6.3, which summarizes our results.

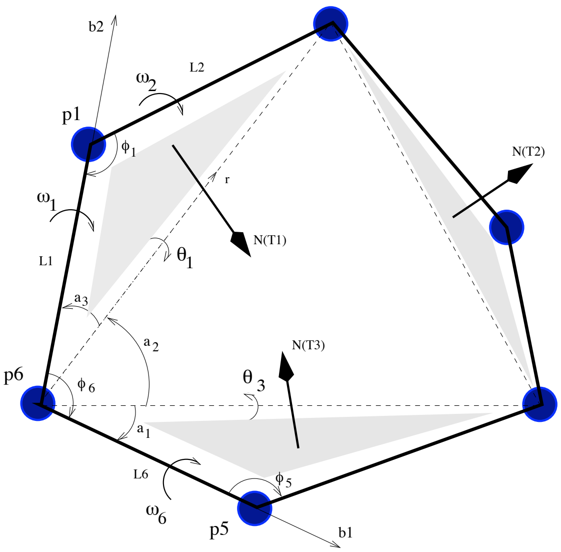

Figure 6.15: A picture of the cyclohexane molecule taken from [25].

Upper bounds for $n = 7, \ldots, 10$ are computed by our software. The lower bound for $n = 9$ follows from Theorem 6.4.8. All other lower bounds are obtained by applying H_1 to a graph with one vertex less.

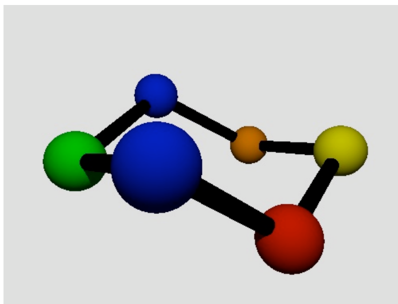

Figure 6.16: A "chair" configuration of the cyclohexane graph.

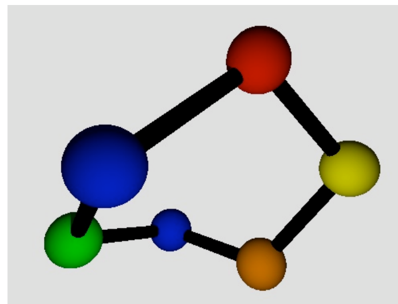

Figure 6.17: A "boat" configuration of the cyclohexane graph.

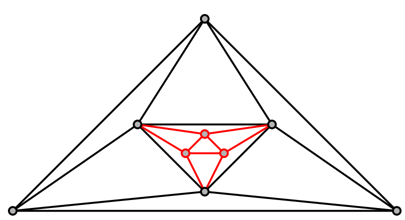

Figure 6.18: A Cyclohexane "caterpillar" with 2 copies.

x_3	x_4	x_5	x_6	y_2	y_3	y_4	y_5	y_6
−6.304252150	1.583912965	−2.520487030	9.180990118	−6.076155659	1.071360653	8.802909730	2.444769823	2.170119919
−6.304252150	1.583912965	−2.520487030	9.180990118	6.076155659	−1.071360653	−8.802909730	−2.444769823	−2.170119919
−6.003839304	−1.659203675	4.329211977	8.638705901	6.076155659	−1.365976019	8.789029705	14.67361735	3.791142355
−6.003839304	−1.659203675	4.329211977	8.638705901	−6.076155659	1.365976019	−8.789029705	−14.67361735	−3.791142355
−5.955967222	−1.699158152	−1.716531329	9.382794358	6.076155659	−1.410740154	8.781392918	−2.734091644	0.9814122655
−5.955967222	−1.699158152	−1.716531329	9.382794358	−6.076155659	1.410740154	−8.781392918	2.734091644	−0.9814122655
−4.823943592	−8.045062818	−0.3839806083	7.753633764	6.076155659	14.47382021	3.908575731	−3.038919638	5.374119784
−4.823943592	−8.045062818	−0.3839806083	7.753633764	−6.076155659	−14.47382021	−3.908575731	3.038919638	−5.374119784
−3.977923994	4.390587084	1.721687429	−9.422598320	6.076155659	15.00136499	7.792480032	−3.115099275	0.4632935405
−3.977923994	4.390587084	1.721687429	−9.422598320	−6.076155659	−15.00136499	−7.792480032	3.115099275	−0.4632935405
−0.4700410465	−6.003497690	0.9765766589	−8.892938529	6.076155659	16.18941766	6.630084124	−3.143359044	3.148911608
−0.4700410465	−6.003497690	0.9765766589	−8.892938529	−6.076155659	−16.18941766	−6.630084124	3.143359044	−3.148911608
0.3645140994	5.903147116	−1.094769062	8.847494855	6.076155659	16.27592264	6.719587348	−2.902259712	3.274421290
0.3645140994	5.903147116	−1.094769062	8.847494855	−6.076155659	−16.27592264	−6.719587348	2.902259712	−3.274421290
4.937406465	7.561392207	−0.1476386693	−7.308507033	6.076155659	15.50673991	4.777588083	15.22399286	5.965377183
4.937406465	7.561392207	−0.1476386693	−7.308507033	−6.076155659	−15.50673991	−4.777588083	−15.22399286	−5.965377183
4.943802466	−3.403993825	−1.804567816	9.412454201	6.076155659	15.50406693	8.271204631	−2.706463504	0.6369504796
4.943802466	−3.403993825	−1.804567816	9.412454201	−6.076155659	−15.50406693	−8.271204631	2.706463504	−0.6369504796
5.159054355	4.442038120	−2.587586308	−8.353858671	6.076155659	−3.258797784	7.763265894	14.56904825	4.383268793
5.159054355	4.442038120	−2.587586308	−8.353858671	−6.076155659	3.258797784	−7.763265894	−14.56904825	−4.383268793
5.289353813	7.688459341	0.04244002377	−7.185915300	6.076155659	15.35195928	4.570294625	−3.093526936	6.112497141
5.289353813	7.688459341	0.04244002377	−7.185915300	−6.076155659	−15.35195928	−4.570294625	3.093526936	−6.112497141
5.700566089	4.128088088	1.705788110	−9.419621275	6.076155659	−2.998191003	7.934663744	−3.116333838	0.5203220464
5.700566089	4.128088088	1.705788110	−9.419621275	−6.076155659	2.998191003	−7.934663744	3.116333838	−0.5203220464
6.185509902	−3.358600186	−7.368365280	−9.297429100	−6.076155659	2.730068047	8.289740936	−9.945323857	1.599316145
6.185509902	−3.358600186	−7.368365280	−9.297429100	6.076155659	−2.730068047	−8.289740936	9.945323857	−1.599316145
6.554081152	−2.704474015	2.740741564	−8.963900709	−6.076155659	2.502379849	8.525597944	2.977562848	2.940830510
6.554081152	−2.704474015	2.740741564	−8.963900709	6.076155659	−2.502379849	−8.525597944	−2.977562848	−2.940830510
7.328562126	−1.294824919	−8.211811506	−7.841451708	−6.076155659	1.948082019	8.850052454	−6.453685499	5.245153487
7.328562126	−1.294824919	−8.211811506	−7.841451708	6.076155659	−1.948082019	−8.850052454	6.453685499	−5.245153487
8.024653295	8.596997995	2.364212648	−6.145482803	6.076155659	13.49864412	2.468121852	15.19421049	7.157725974
8.024653295	8.596997995	2.364212648	−6.145482803	−6.076155659	−13.49864412	−2.468121852	−15.19421049	−7.157725974

Table 6.1: 32 realizations for the $K_{3,3}$ obtained using the edge lengths prescribed in [55, Section 4] and $x_1 = y_1 = 0$ and $x_2 = 1$.

$n =$	3	4	5	6	7	8	9	10
lower	2	4	8	32	64	128	288	1024
upper	2	4	8	32	64	128	512	2048
H_1	$\triangle$	$\triangle 1$	$\triangle 11$	$\triangle 111$	$\triangle 1^4$	$\triangle 1^5$	$\triangle 1^6$	
H_2				$\boldsymbol{\triangle 112}$	$\boldsymbol{\triangle 1^3 2}$	$\boldsymbol{\triangle 1^4 2}$	$\triangle 1^5 2$ $\triangle 1^4 21$ $\boldsymbol{\triangle 1^4 22}$	

Table 6.2: Bounds and Henneberg sequences for Laman graphs for $n \leq 10$. Bold text indicates the graph yielding the upper bound.

$n =$	4	5	6	7	8	9	10
lower	2	4	16	32	64	256	512
upper	2	4	16	32	160	640	2560
H_1	$\triangle$	$\triangle 1$	$\triangle 11$	$\triangle 111$	$\triangle 1^4$	$\triangle 1^5$	$\triangle 1^6$
H_2			$\boldsymbol{\triangle 12}$	$\boldsymbol{\triangle 1^2 2}$	$\triangle 1^3 2$ $\boldsymbol{\triangle 1^2 2^2}$ $\triangle 1^2 21$	$\triangle 1^4 2$ $\triangle 1^3 2^2$ $\triangle 1^3 21$ $\boldsymbol{\triangle 1^2 2^3}$	$\triangle 1^5 2$ $\triangle 1^4 2^2$ $\triangle 1^4 21$ $\triangle 1^3 2^3$ $\triangle 1^3 21^2$ $\triangle 1^3 212$ $\triangle 1^3 2^2 1$ $\boldsymbol{\triangle 1^2 2^4}$
H_3							

Table 6.3: Bounds and Henneberg sequences for 1-skeleta of simplicial polyhedra for $n \leq 10$, where $\triangle$ denotes the 1-skeleton of the 3-simplex. Notice that up to $n = 10$ there is no need to apply a H_3 step. In fact, the first time a H_3 step is necessary is for $n = 13$ [10]. Bold text indicates the Henneberg sequence of the graph that gives the upper bound.

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