Proc. R. Soc. Lond. A **419**, 29–68 (1988)
Printed in Great Britain

Low-dimensional lattices. II. Subgroups of $GL(n, \mathbb{Z})$

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(*Received* 28 *January* 1988)

The maximal finite irreducible groups of $n \times n$ integers for $n = 4, 5, \ldots, 9, 11, 13, 17, 19, 23$ were determined by Dade, Ryskov, Bülow, Plesken & Pohst and Plesken, as the automorphism groups of certain quadratic forms. This paper presents a geometric description of the corresponding n-dimensional lattices, and gives coordinates which display their symmetries and minimal vectors. Some very interesting lattices appear.

1. INTRODUCTION

For a given value of n there are only finitely many non-isomorphic finite groups of $n \times n$ integer matrices. This theorem has a long history and is associated with the names Jordan, Minkowski, Bieberbach and Zassenhaus (see Milnor 1976; Brown *et al.* 1978). For $n = 2$ and 3 these groups were classified in the past century, because they are needed in crystallography. The maximal finite subgroups of $GL(4, \mathbb{Z})$ were given by Dade (1965), and the complete list of finite subgroups of $GL(4, \mathbb{Z})$ by Bülow *et al.* (1971), Neubüser *et al.* (1971), Wondratschek *et al.* (1971) and Brown *et al.* (1972*a*, *b*, 1973, 1978). The maximal irreducible finite subgroups of $GL(5, \mathbb{Z})$ were found independently by Ryskov (1972*a*, *b*) and Bülow (1973). That work was greatly extended by Plesken & Pohst (1977*a*, *b*, 1980*a*–*c*), who determined the maximal irreducible subgroups of $GL(n, \mathbb{Z})$ for $n = 6, 7, 8, 9$, and by Plesken (1985), who dealt with $n = 11, 13, 17, 19, 23$.

In these papers the subgroups are usually specified as the automorphism groups of certain quadratic forms. In this paper we shall give a geometric description of the maximal irreducible subgroups of $GL(n, \mathbb{Z})$ for $n = 1, \ldots, 9, 11, 13, 17, 19, 23$, by exhibiting lattices corresponding to the quadratic forms; the automorphism groups of the lattices are the desired groups. By giving natural coordinates for these lattices and determining their minimal vectors, we are able to make their symmetry groups clearly visible. There are 176 lattices, many of which have not been studied before (although they are implicit in the above references and in Conway *et al.* (1985)) and are very beautiful.

Our goal in this series of papers, as we stated in part I (Conway & Sloane 1988*b*), is to simplify and systematize the work of others, rather than to present new material. We do not rederive the enumerations of Dade, Ryskov, Bülow, Plesken & Pohst and Plesken, but take their lists as our starting point. We feel, however, that this geometric approach does throw additional light on these groups.

The relations between different lattices are also clarified. For example, Plesken & Pohst's nine-dimensional lattice F_8 belongs to a set of four lattices $F_{10} = A_3 \otimes A_3$, $F_9 = (A_3 \otimes A_3)^{+2}$, $F_8 = (A_3 \otimes A_3)^{+2048}$, $F_{11} = (A_3 \otimes A_3)^{+4096}$ whose interrelations are displayed in figure 1.

The geometric approach also makes it possible to fill in a gap in Plesken's enumeration, by determining the minimal vectors of the seventeen 23-dimensional lattices associated with the Leech lattice (see §11).

The lattices and groups are summarized in table 1, and figure 1 displays their interrelations. The individual lattices are described in §§3–11. To conserve space we have not repeated Plesken and Pohst's Gram matrices. Nor have we attempted to give theta series, although these are easy to write down when the lattice is obtained by applying constructions A or B to a code or by gluing up root lattices (Conway & Sloane 1988*a*, chaps 4, 7). Section 2 contains some background material.

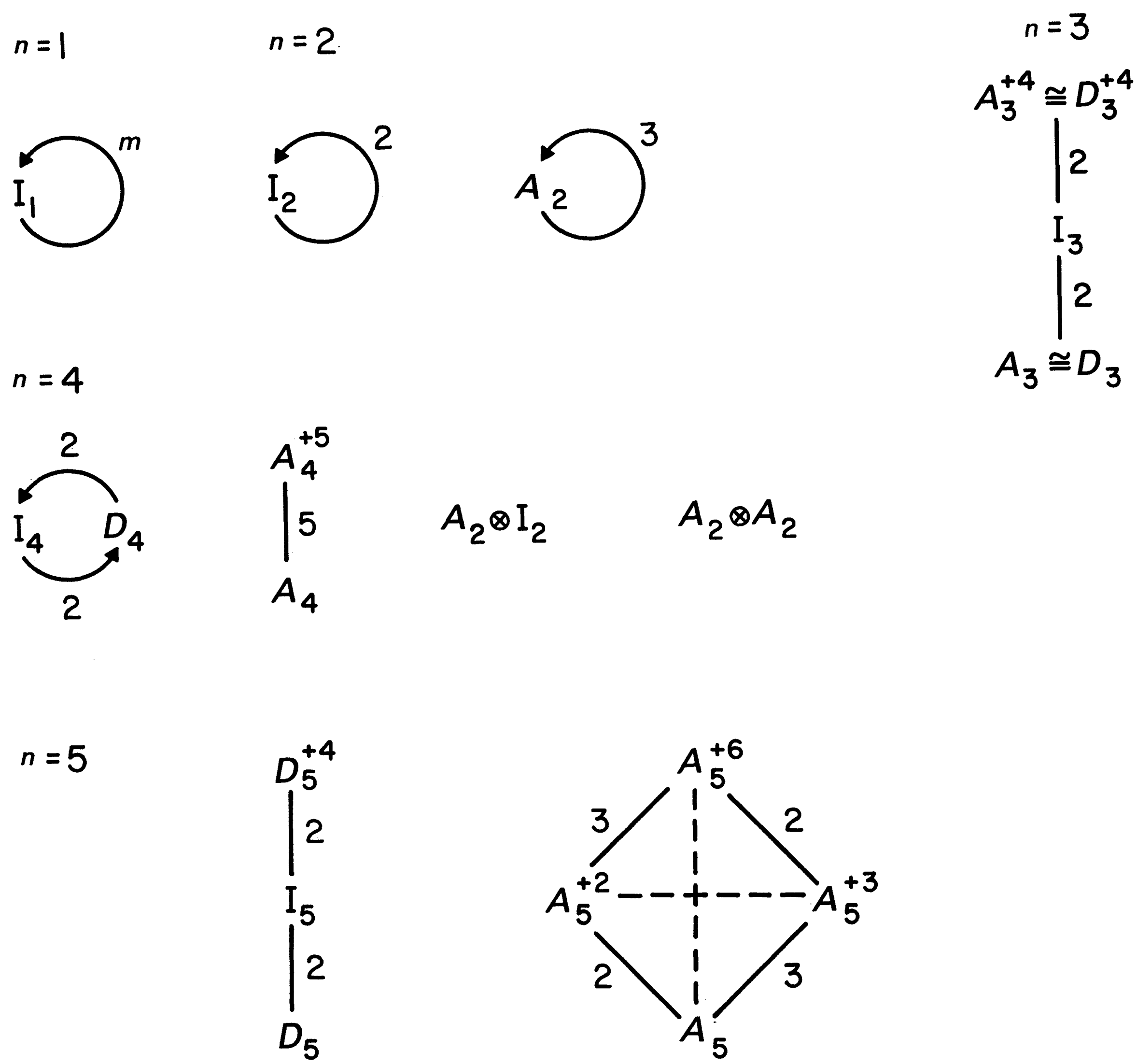

FIGURE 1. For description see page 35.

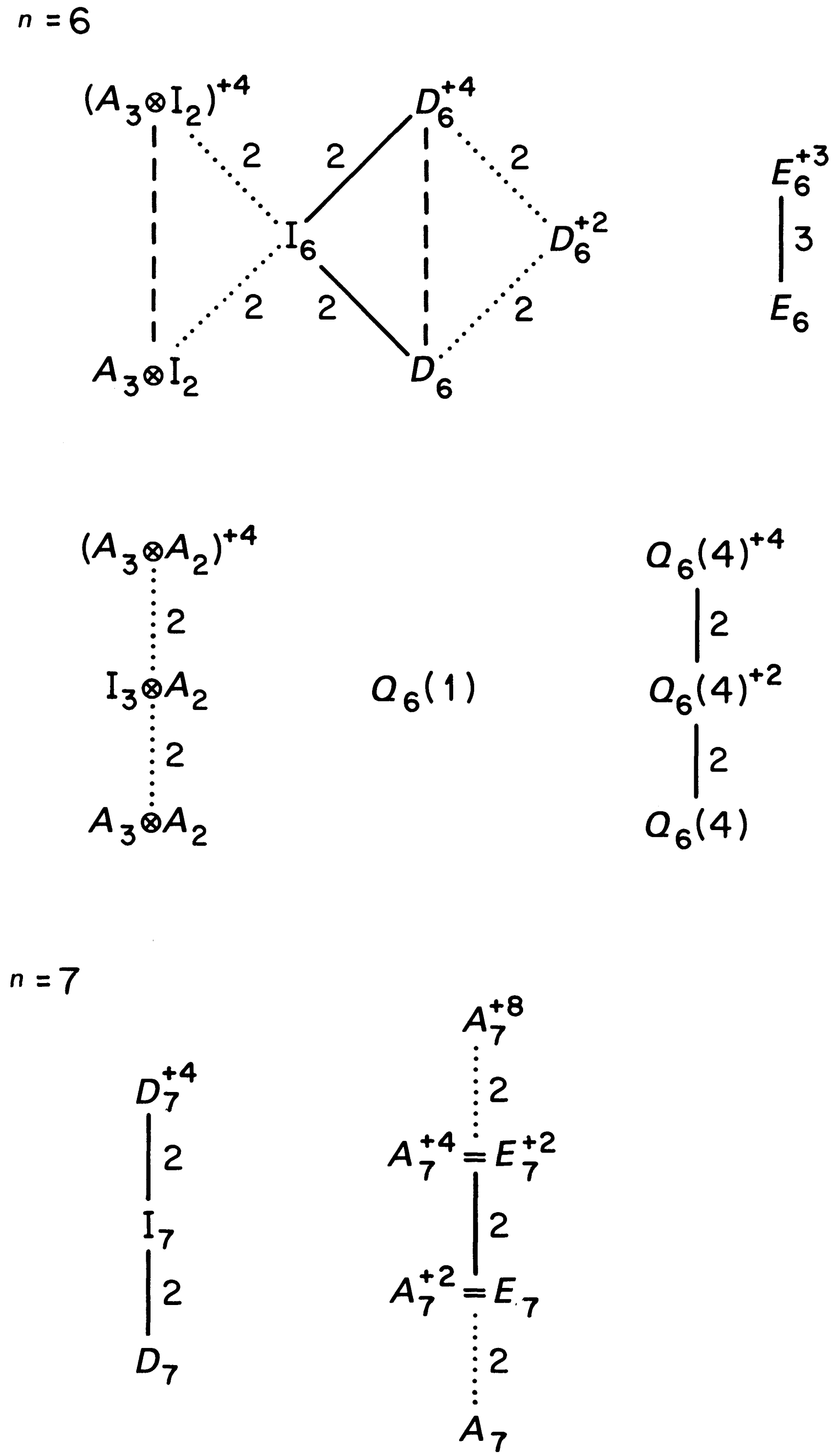

FIGURE 1. For description see page 35.

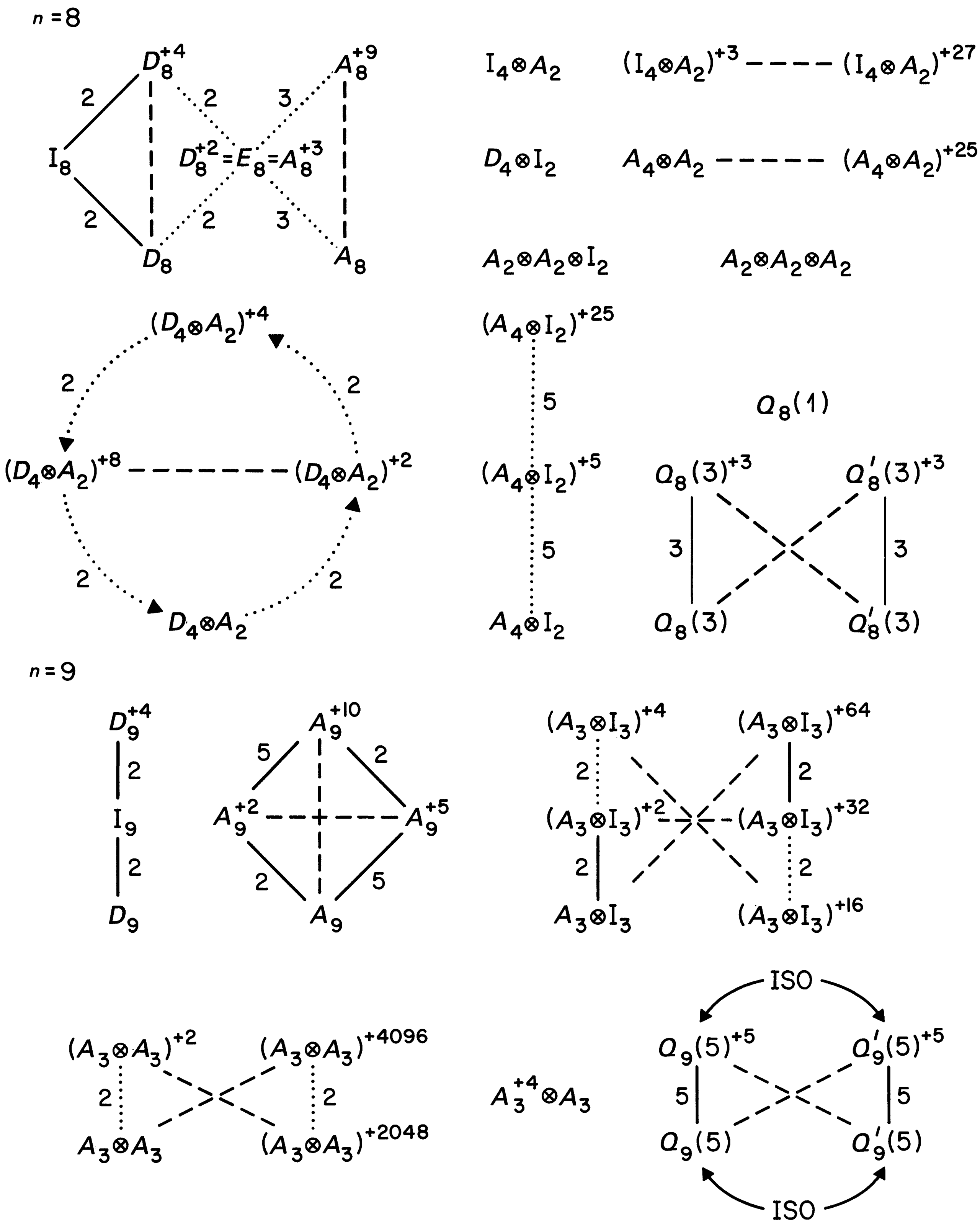

FIGURE 1. For description see page 35.

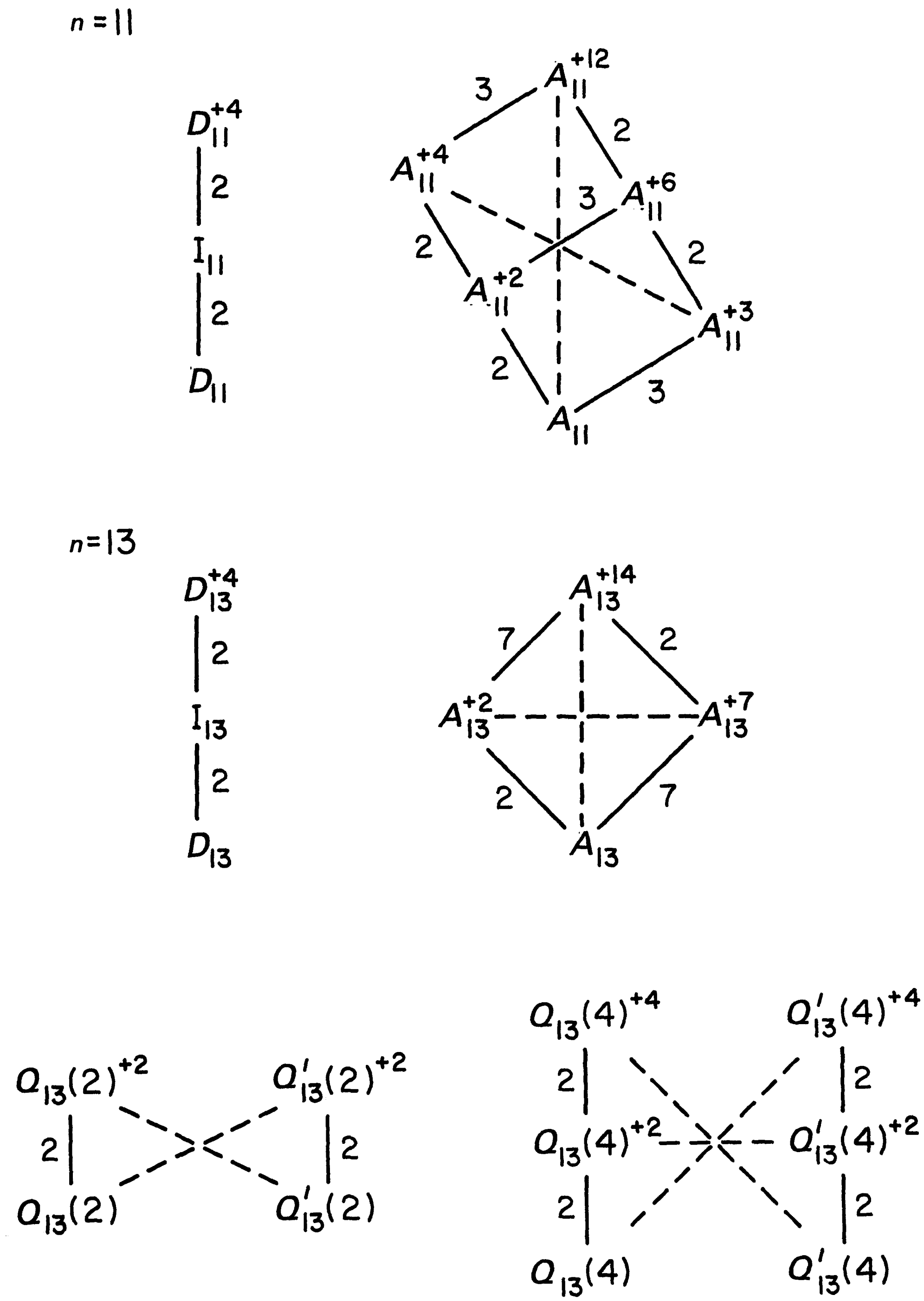

FIGURE 1. For description see page 35.

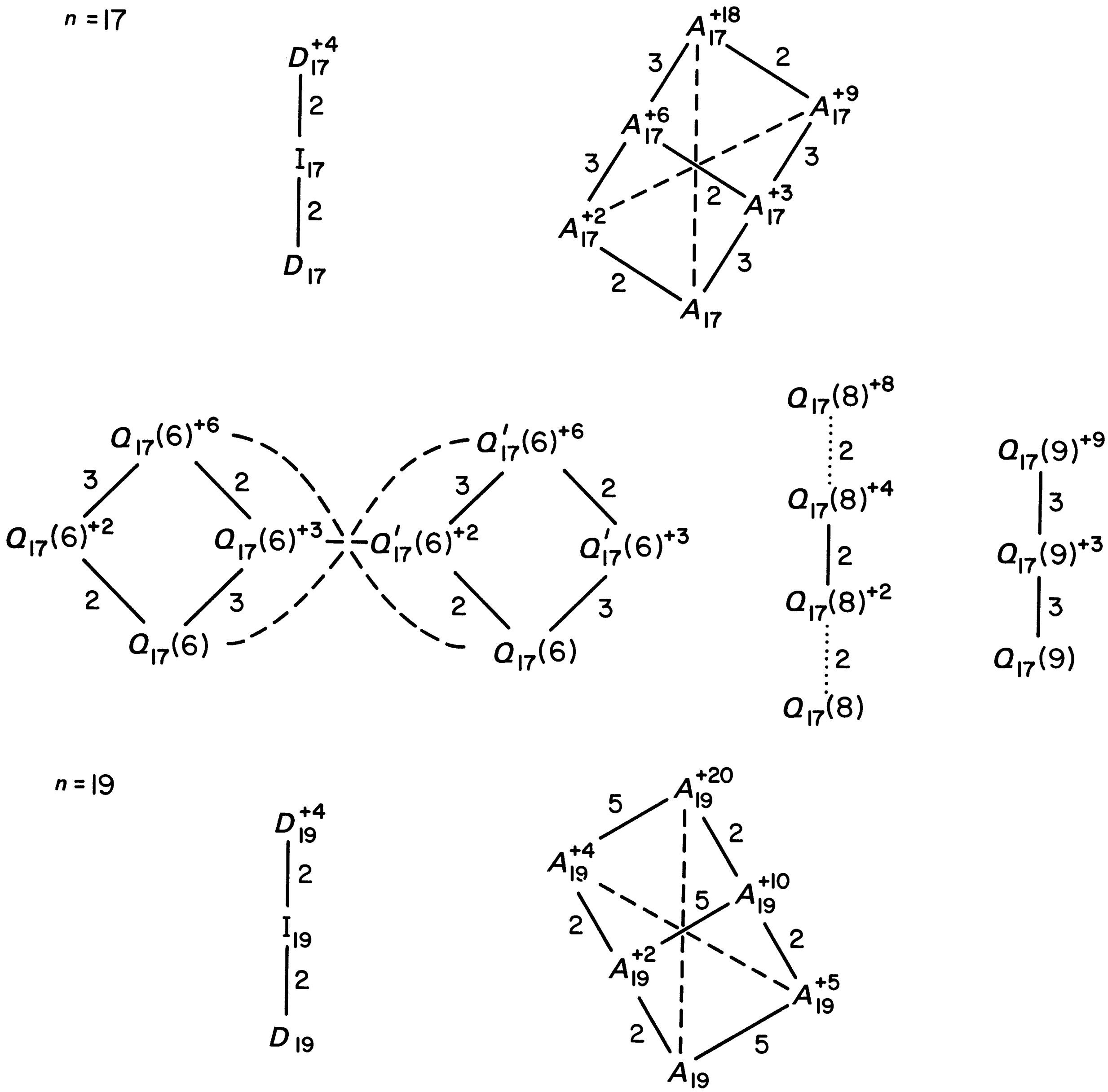

FIGURE 1. For description see opposite.

Study of the lattices in dimensions 13 and 17 has made it clear that there are interesting 'parent' lattices in dimensions 14 and 18, analogous to the Leech lattice Λ_{24} in dimension 24, although no single lattice dominates everything in the way that Λ_{24} does. In dimension 18, for example, there appear to be four principal lattices, two associated with the extended quadratic residue code of length 18 and two with the symplectic group $S_4(4)$. We plan to discuss these lattices elsewhere.

We shall refer to Conway & Sloane (1988*b*) as part I, Conway & Sloane (1988*c*) as part III, Conway & Sloane (1988*a*) as SLG, and Conway *et al.* (1985) as the ATLAS. Notation from Part I will be used without comment. Isomorphism between lattices is denoted by $\cong$, and $+$ stands for $+1$, and $-$ for -1. Unless

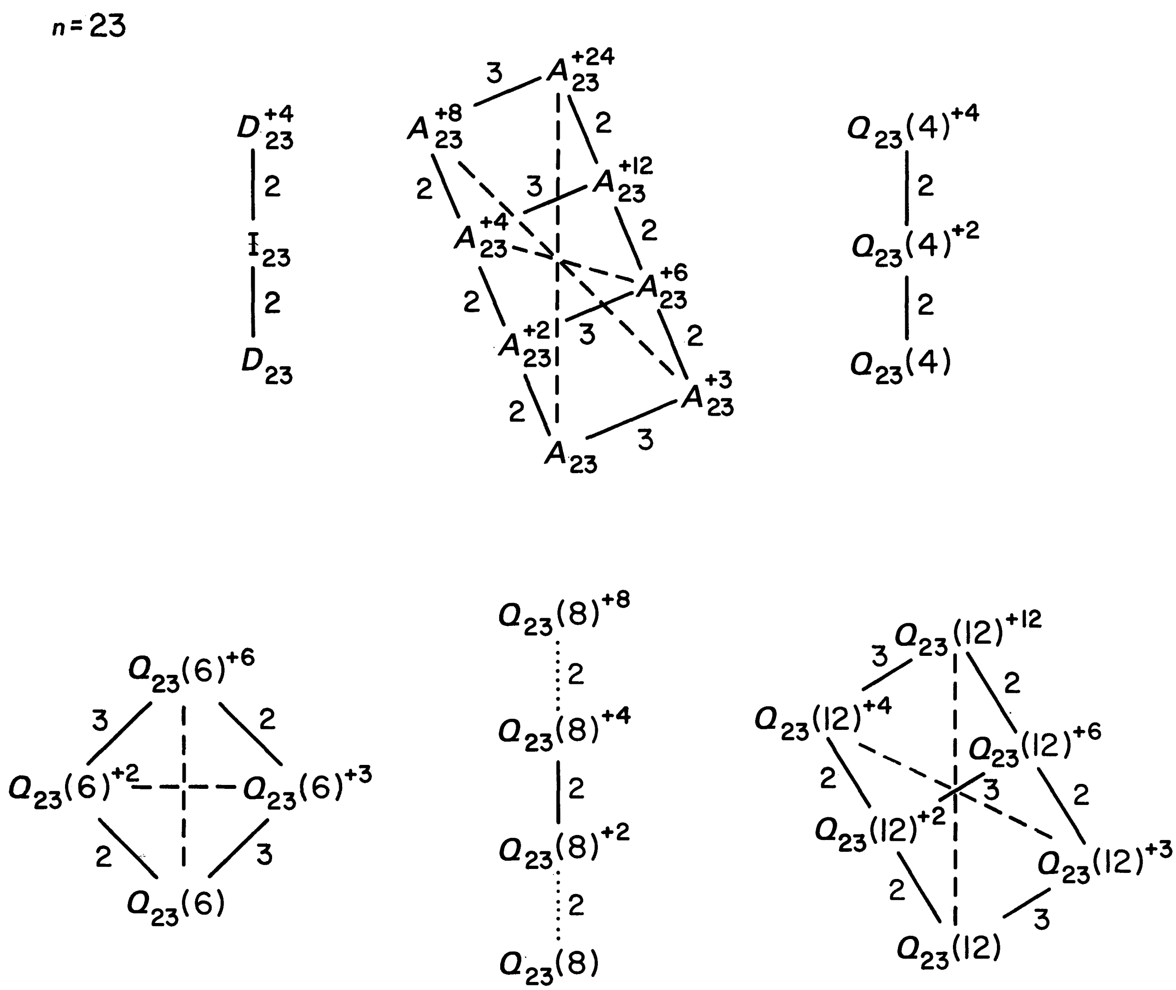

FIGURE 1. The lattices and their relations. A solid line marked i between two lattices indicates that they are adjacent n-dimensional lattices with the same group, and that the upper contains the lower to index i. Here 'upper' usually means higher on the page, otherwise arrowheads are inserted. Dotted lines similarly indicate adjacent lattices with different groups. Dashed lines indicate duality but are omitted when they would fall on other lines.

specified otherwise we use the 'standard' inner product $(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = \sum x_i y_i$.

We remind the reader that Plesken & Pohst (1980*c*) contains errata for Plesken & Pohst (1977*b*). Also, in Plesken & Pohst (1980*c*, p. 298) the order of $\mathrm{Aut}_{\mathbb{Z}}(F_{15})$ is twice as large as is stated. In Plesken (1985, p. 305, line 9) $a = 52$ should be $a = 104$, and in line 13, $\det F = 5^8 \cdot 10^8$ should be $\det F = 5^8 \cdot 10$.

Some closely related papers are Ryskov & Lomakina (1980), Plesken (1981), Plesken & Hanrath (1984) and Gudivok *et al.* (1982, 1986). We should also mention the parallel classification of maximal finite subgroups of $GL(n, \mathbb{C})$ which has been done for $n \leqslant 10$ by Blichfeldt, Brauer, Lindsay, Wales, Huffman & Wales and Feit (see Feit 1976).

TABLE 1. LATTICES DEFINING MAXIMAL IRREDUCIBLE SUBGROUPS OF $GL_n(\mathbb{Z})$

(The entry for a typical n-dimensional lattice describes the lattice, gives Plesken and Pohst's or Plesken's symbol (in the column headed P), the elementary divisors (d_i), determinant (d), minimal norm (μ), number of pairs of minimal vectors (s) and group order (g). To make the lattice integral, multiply it by $\sqrt{c}$, where c is the least common multiple of the denominators of the d_i. The lattices are divided by horizontal lines into blocks; dualizing reverses the order within a block. An asterisk (*) indicates that there is more than one orbit of minimal vectors. Several lattices may have the same automorphism group: this occurs just when the lattices are in the same block and have the same value of g (and is indicated by a brace if the lattices are adjacent in the table).)

description	P.	d_i	d	μ	s	g
dimension 1						
I_1	F_1	1^1	1	1	1	2
dimension 2						
I_2	F_1	1^2	1	1	2	8
A_2	F_2	$1^1 3^1$	3	2	3	12
dimension 3						
$A_3 \cong D_3$	F_3	$1^2 4^1$	4	2	6	
I_3	F_2	1^3	1	1	3	48
$A_3^{+4} \cong D_3^{+4}$	F_1	$\frac{1}{4}^1 1^2$	$\frac{1}{4}$	$\frac{3}{4}$	4	
dimension 4						
I_4	F_1	1^4	1	1	4	384
D_4	F_2	$1^2 2^2$	4	2	12	1152
A_4	F_5	$1^3 5^1$	5	2	10	240
A_4^{+5}	F_6	$\frac{1}{5}^1 1^3$	$\frac{1}{5}$	$\frac{4}{5}$	5	
$A_2 \otimes \mathrm{I}_2$	F_3	$1^2 3^2$	9	2	6	288
$A_2 \otimes A_2$	F_4	$1^1 3^2 9^1$	81	4	9	144
dimension 5						
D_5	F_2	$1^4 4^1$	4	2	20	
I_5	F_1	1^5	1	1	5	3840
D_5^{+4}	F_3	$\frac{1}{4}^1 1^4$	$\frac{1}{4}$	1	5	
A_5	F_5	$1^4 6^1$	6	2	15	
A_5^{+2}	F_7	$\frac{1}{2}^1 1^3 3^1$	$\frac{3}{2}$	$\frac{3}{2}$	10	1440
A_5^{+3}	F_6	$\frac{1}{3}^1 1^3 2^1$	$\frac{2}{3}$	$\frac{4}{3}$	15	
A_5^{+6}	F_4	$\frac{1}{6}^1 1^4$	$\frac{1}{6}$	$\frac{5}{6}$	6	

TABLE 1 (*cont.*)

description	$P.$	d_i	d	μ	s	g
dimension 6						
D_6	F_2	$1^4 2^2$	4	2	30	
I_6	F_1	1^6	1	1	6	$2^6 \cdot 6!$
D_6^{+4}	F_3	$\frac{1}{2}^{2} 1^4$	$\frac{1}{4}$	1	6	
D_6^{+2}	F_6	$\frac{1}{2}^{1} 1^4 2^1$	1	$\frac{3}{2}$	16	23040
E_6	F_8	$1^5 3^1$	3	2	36	$2\lvert W(E_6)\rvert$
E_6^{+3}	F_9	$\frac{1}{3}^{1} 1^5$	$\frac{1}{3}$	$\frac{4}{3}$	27	
A_6	F_{13}	$1^5 7^1$	7	2	21	$2 \cdot 7!$
A_6^{+7}	F_{12}	$\frac{1}{7}^{1} 1^5$	$\frac{1}{7}$	$\frac{6}{7}$	7	
$A_3 \otimes \mathrm{I}_2$	F_4	$1^4 4^2$	16	2	12	4608
$(A_3 \otimes \mathrm{I}_2)^{+16}$	F_5	$\frac{1}{4}^{2} 1^4$	$\frac{1}{16}$	$\frac{3}{4}$	8	
$\mathrm{I}_3 \otimes A_2$	F_7	$1^3 3^3$	27	2	9	10368
$A_3 \otimes A_2$	F_{10}	$1^3 3^1 12^2$	432	4	18	288
$(A_3 \otimes A_2)^{+16}$	F_{11}	$\frac{1}{4}^{2} 1^1 3^3$	27/16	$\frac{3}{2}$	12	
$Q_6(1)$	F_{14}	$1^3 7^3$	343	4	21	$4\lvert L_2(7)\rvert$
$Q_6(4)$	F_{16}	$1^3 5^1 10^2$	500	4	15	
$Q_6(4)^{+2}$	F_{15}	$1^3 5^3$	125	3	10	$4\lvert A_5\rvert$
$Q_6(4)^{+4}$	F_{17}	$\frac{1}{2}^{2} 1^1 5^3$	$\frac{125}{4}$	$\frac{5}{2}$	12	
dimension 7						
D_7	F_2	$1^6 4^1$	4	2	42	
I_7	F_1	1^7	1	1	7	$2^7 \cdot 7!$
D_7^{+4}	F_3	$\frac{1}{4}^{1} 1^6$	$\frac{1}{4}$	1	7	
E_7	F_6	$1^6 2^1$	2	2	63	$\lvert W(E_7)\rvert$
E_7^{+2}	F_7	$\frac{1}{2}^{1} 1^6$	$\frac{1}{2}$	$\frac{3}{2}$	28	
A_7	F_5	$1^6 8^1$	8	2	28	$2 \cdot 8!$
A_7^{+8}	F_4	$\frac{1}{8}^{1} 1^6$	$\frac{1}{8}$	$\frac{7}{8}$	8	
dimension 8						
D_8	F_2	$1^6 2^2$	4	2	56	
I_8	F_1	1^8	1	1	8	$2^8 \cdot 8!$
D_8^{+4}	F_4	$\frac{1}{2}^{2} 1^6$	$\frac{1}{4}$	1	8	
E_8	F_5	1^8	1	2	120	$\lvert W(E_8)\rvert$
A_8	F_{10}	$1^7 9^1$	9	2	36	$2 \cdot 9!$
A_8^{+9}	F_{13}	$\frac{1}{9}^{1} 1^7$	$\frac{1}{9}$	$\frac{8}{9}$	9	

TABLE 1 (*cont.*)

description	*P*.	d_i	d	μ	s	g
dimension 8						
$D_4 \otimes \mathrm{I}_2$	F_3	$1^4 2^4$	16	2	24	2654208
$\mathrm{I}_4 \otimes A_2$	F_7	$1^4 3^4$	81	2	12	497664
$(\mathrm{I}_4 \otimes A_2)^{+3}$	F_{11}	${\frac{1}{3}}^1 1^4 3^3$	9	2	12 }	62208
$(\mathrm{I}_4 \otimes A_2)^{+27}$	F_8	${\frac{1}{3}}^3 1^4 3^1$	$\frac{1}{9}$	$\frac{4}{3}$	54 }	
$D_4 \otimes A_2$	F_6	$1^4 6^4$	1296	4	36	6912
$(D_4 \otimes A_2)^{+2}$	F_{21}	$1^4 3^2 6^2$	324	3	16	1152
$(D_4 \otimes A_2)^{+4}$	F_{22}	${\frac{1}{2}}^2 1^2 3^2 6^2$	81	3	48	3456
$(D_4 \otimes A_2)^{+8}$	F_{20}	${\frac{1}{2}}^2 1^2 3^4$	$\frac{81}{4}$	2	12	1152
$A_4 \otimes \mathrm{I}_2$	F_{14}	$1^6 5^2$	25	2	20	115200
$(A_4 \otimes \mathrm{I}_2)^{+5}$	F_{17}	${\frac{1}{5}}^1 1^6 5^1$	1	$\frac{8}{5}$	25	57600
$(A_4 \otimes \mathrm{I}_2)^{+25}$	F_{16}	${\frac{1}{5}}^2 1^6$	$\frac{1}{25}$	$\frac{4}{5}$	10	115200
$A_4 \otimes A_2$	F_{19}	$1^4 3^2 15^2$	$3^4 5^2$	4	30 }	1440
$(A_4 \otimes A_2)^{+25}$	F_{18}	${\frac{1}{5}}^2 1^2 3^4$	$3^4 5^{-2}$	$\frac{8}{5}$	15 }	
$A_2 \otimes A_2 \otimes \mathrm{I}_2$	F_9	$1^2 3^4 9^2$	6561	4	18	41472
$A_2 \otimes A_2 \otimes A_2$	F_{12}	$1^1 3^3 9^3 27^1$	3^{12}	8	27	2592
$Q_8(1)$	F_{15}	$1^4 5^4$	625	4	60	$2\|[3, 3, 5]\|$
$Q_8(3)$	F_{25}	$1^5 7^2 21^1$	$343 \cdot 3$	4	42 }	$4\|L_2(7)\|$
$Q_8(3)^{+3}$	F_{23}	${\frac{1}{3}}^1 1^4 7^3$	$343/3$	$\frac{8}{3}$	21 }	
$Q_8'(3)$	F_{24}	$1^3 7^4 21^1$	$7^5 \cdot 3$	6	28 }	
$Q_8'(3)^{+3}$	F_{26}	${\frac{1}{3}}^1 1^2 7^5$	$7^5/3$	$\frac{14}{3}$	24 }	
dimension 9						
D_9	F_2	$1^8 4^1$	4	2	72 }	$2^9 \cdot 9!$
I_9	F_1	1^9	1	1	9 }	
D_9^{+4}	F_3	${\frac{1}{4}}^1 1^8$	$\frac{1}{4}$	1	9 }	
A_9	F_{16}	$1^8 10^1$	10	2	45 }	$2 \cdot 10!$
A_9^{+2}	F_{18}	${\frac{1}{2}}^1 1^7 5^1$	$\frac{5}{2}$	2	45 }	
A_9^{+5}	F_{17}	${\frac{1}{5}}^1 1^7 2^1$	$\frac{2}{5}$	$\frac{8}{5}$	45 }	
A_9^{+10}	F_{15}	${\frac{1}{10}}^1 1^8$	$\frac{1}{10}$	$\frac{9}{10}$	10 }	
$A_3 \otimes \mathrm{I}_3$	F_4	$1^6 4^3$	64	2	18 }	663552
$(A_3 \otimes \mathrm{I}_3)^{+2}$	F_6	$1^7 4^2$	16	2	18 }	
$(A_3 \otimes \mathrm{I}_3)^{+4}$	F_{13}	${\frac{1}{4}}^1 1^6 4^2$	4	2	18 }	165888
$(A_3 \otimes \mathrm{I}_3)^{+16}$	F_{12}	${\frac{1}{4}}^2 1^6 4^1$	$\frac{1}{4}$	$\frac{3}{2}$	48 }	
$(A_3 \otimes \mathrm{I}_3)^{+32}$	F_7	${\frac{1}{4}}^2 1^7$	$\frac{1}{16}$	1	9 }	663552
$(A_3 \otimes \mathrm{I}_3)^{+64}$	F_5	${\frac{1}{4}}^3 1^6$	$\frac{1}{64}$	$\frac{3}{4}$	12 }	

TABLE 1 (*cont.*)

description	$P.$	d_i	d	μ	s	g
$A_3 \otimes A_3$	F_{10}	$1^4 4^4 16^1$	4096	4	36	2304
$(A_3 \otimes A_3)^{+2}$	F_9	$1^4 4^5$	1024	4	81*	36864
$(A_3 \otimes A_3)^{+2048}$	F_8	$\frac{1}{4}^5 1^4$	1/1024	$\frac{3}{4}$	24	
$(A_3 \otimes A_3)^{+4096}$	F_{11}	$\frac{1}{16}^1 \frac{1}{4}^1 1^4$	1/4096	$\frac{9}{16}$	16	2304
$A_3^{+4} \otimes A_3$	F_{14}	$\frac{1}{4}^2 1^5 4^2$	1	$\frac{3}{2}$	24	1152
$Q_9(5)$	F_{20}	$1^4 2^1 4^3 20^1$	2560	4	45	$2\|S_6\|$
$Q_9(5)^{+5}$	F_{19}	$\frac{1}{5}^1 1^3 2^1 4^4$	512/5	$\frac{12}{5}$	15	

dimension 11

description	$P.$	d_i	d	μ	s	g
D_{11}	F_2(i)	$1^{10} 4^1$	4	2	110	
I_{11}	F_1(i)	1^{11}	1	1	11	$2^{10} \cdot 10!$
D_{11}^{+4}	F_3(i)	$\frac{1}{4}^1 1^{10}$	$\frac{1}{4}$	1	11	
A_{11}	F_6(ii)	$1^{10} 12^1$	12	2	66	
A_{11}^{+2}	F_5(ii)	$1^{10} 3^1$	3	2	66	
A_{11}^{+3}	F_4(ii)	$\frac{1}{3}^1 1^9 4^1$	$\frac{4}{3}$	2	66	$2 \cdot 12!$
A_{11}^{+4}	F_3(ii)	$\frac{1}{4}^1 1^9 3^1$	$\frac{3}{4}$	2	66	
A_{11}^{+6}	F_2(ii)	$\frac{1}{3}^1 1^{10}$	$\frac{1}{3}$	$\frac{5}{3}$	66	
A_{11}^{+12}	F_1(ii)	$\frac{1}{12}^1 1^{10}$	$\frac{1}{12}$	$\frac{11}{12}$	12	

dimension 13

description	$P.$	d_i	d	μ	s	g
D_{13}	F_2(i)	$1^{12} 4^1$	4	2	156	
I_{13}	F_1(i)	1^{13}	1	1	13	$2^{13} \cdot 13!$
D_{13}^{+4}	F_3(i)	$\frac{1}{4}^1 1^{12}$	$\frac{1}{4}$	1	13	
A_{13}	F_2(ii)	$1^{12} 14^1$	14	2	91	
A_{13}^{+2}	F_4(ii)	$\frac{1}{2}^1 1^{11} 7^1$	$\frac{7}{2}$	2	91	$2 \cdot 14!$
A_{13}^{+7}	F_3(ii)	$\frac{1}{7}^1 1^{11} 2^1$	$\frac{2}{7}$	$\frac{12}{7}$	91	
A_{13}^{+14}	F_1(ii)	$\frac{1}{14}^1 1^{12}$	$\frac{1}{14}$	$\frac{13}{14}$	14	
$Q_{13}(2)$	F_1(iv)	$1^9 5^3 10^1$	1250	4	390*	
$Q_{13}(2)^{+2}$	F_4(iv)	$\frac{1}{2}^1 1^8 5^4$	625/2	$\frac{5}{2}$	26	$4\|L_2(25)\|$
$Q_{13}'(2)$	F_3(iv)	$1^4 5^8 10^1$	$5^9 \cdot 2$	6	65	
$Q_{13}'(2)^{+2}$	F_2(iv)	$\frac{1}{2}^1 1^3 5^9$	$5^9/2$	6	65	
$Q_{13}(4)$	F_3(iii)	$1^7 3^5 12^1$	2916	4	234	
$Q_{13}(4)^{+2}$	F_1(iii)	$1^7 3^6$	729	3	52	
$Q_{13}(4)^{+4}$	F_6(iii)	$\frac{1}{4}^1 1^6 3^6$	729/4	3	52	$4\|L_3(3)\|$
$Q_{13}'(4)$	F_4(iii)	$1^6 3^6 12^1$	8748	4	117	
$Q_{13}'(4)^{+2}$	F_2(iii)	$1^6 3^7$	2187	3	26	
$Q_{13}'(4)^{+4}$	F_5(iii)	$\frac{1}{4}^1 1^5 3^7$	2187/4	3	26	

TABLE 1 (*cont.*)

description	$P.$	d_i	d	μ	s	g
dimension 17						
D_{17}	F_2(i)	$1^{16}4^1$	4	2	272	
I_{17}	F_1(i)	1^{17}	1	1	17	$2^{17}\cdot 17!$
D_{17}^{+4}	F_3(i)	$\frac{1}{4}^1 1^{16}$	$\frac{1}{4}$	1	17	
A_{17}	F_2(ii)	$1^{16}18^1$	18	2	153	
A_{17}^{+2}	F_4(ii)	$\frac{1}{2}^1 1^{15}9^1$	$\frac{9}{2}$	2	153	
A_{17}^{+3}	F_5(ii)	$1^{16}2^1$	2	2	153	$2\cdot 18!$
A_{17}^{+6}	F_6(ii)	$\frac{1}{2}^1 1^{16}$	$\frac{1}{2}$	2	153	
A_{17}^{+9}	F_3(ii)	$\frac{1}{9}^1 1^{15}2^1$	$\frac{2}{9}$	$\frac{16}{9}$	153	
A_{17}^{+18}	F_1(ii)	$\frac{1}{18}^1 1^{16}$	$\frac{1}{18}$	$\frac{17}{18}$	18	
$Q_{17}(6)$	F_3(vi)	$1^8 2^8 6^1$	1536	4	1020	
$Q_{17}(6)^{+2}$	F_4(vi)	$\frac{1}{2}^1 1^8 2^7 6^1$	384	$\frac{7}{2}$	408	
$Q_{17}(6)^{+3}$	F_5(vi)	$\frac{1}{3}^1 1^7 2^9$	512/3	$\frac{10}{3}$	510	
$Q_{17}(6)^{+6}$	F_6(vi)	$\frac{1}{6}^1 1^8 2^8$	128/3	$\frac{17}{6}$	120	$8\lvert L_2(16)\rvert$
$Q_{17}'(6)$	F_2(vi)	$1^8 2^8 12^1$	3072	4	663*	
$Q_{17}'(6)^{+2}$	F_1(iv)	$1^9 2^7 6^1$	768	3	68	
$Q_{17}'(6)^{+3}$	F_8(vi)	$\frac{1}{3}^1 1^7 2^8 4^1$	1024/3	$\frac{8}{3}$	51	
$Q_{17}'(6)^{+6}$	F_7(vi)	$\frac{1}{3}^1 1^8 2^8$	256/3	$\frac{8}{3}$	51	
$Q_{17}(8)$	F_1(iv)	$1^8 4^8 16^1$	2^{20}	6	1088	$2^9\cdot 17\cdot 8$
$Q_{17}(8)^{+2}$	F_2(iii)	$1^8 4^9$	2^{18}	4	17	$2^{17}\cdot 17\cdot 8$
$Q_{17}(8)^{+4}$	F_1(iii)	$1^9 4^8$	2^{16}	4	17	
$Q_{17}(8)^{+8}$	F_2(iv)	$\frac{1}{4}^1 1^8 4^8$	2^{14}	4	17	$2^9\cdot 17\cdot 8$
$Q_{17}(9)$	F_2(v)	$1^8 2^1 4^7 36^1$	$2^{17}\cdot 9$	6	1020*	
$Q_{17}(9)^{+3}$	F_1(v)	$1^8 2^1 4^8$	2^{17}	4	102	$2\lvert L_2(17)\rvert$
$Q_{17}(9)^{+9}$	F_3(v)	$\frac{1}{9}^1 1^7 2^1 4^8$	$2^{17}/9$	$\frac{34}{9}$	18	
dimension 19						
D_{19}	—	$1^{18}4^1$	4	2	342	
I_{19}	—	1^{19}	1	1	19	$2^{19}\cdot 19!$
D_{19}^{+4}	—	$\frac{1}{4}^1 1^{18}$	$\frac{1}{4}$	1	19	
A_{19}	—	$1^{18}20^1$	20	2	190	
A_{19}^{+2}	—	$1^{18}5^1$	5	2	190	
A_{19}^{+4}	—	$\frac{1}{4}^1 1^{17}5^1$	$\frac{5}{4}$	2	190	$2\cdot 20!$
A_{19}^{+5}	—	$\frac{1}{5}^1 1^{17}4^1$	$\frac{4}{5}$	2	190	
A_{19}^{+10}	—	$\frac{1}{5}^1 1^{18}$	$\frac{1}{5}$	$\frac{9}{5}$	190	
A_{19}^{+20}	—	$\frac{1}{20}^1 1^{18}$	$\frac{1}{20}$	$\frac{19}{20}$	20	
dimension 23						
D_{23}	F_2(ii)	$1^{22}4^1$	4	2	506	
I_{23}	F_1(ii)	1^{23}	1	1	23	$2^{23}\cdot 23!$
D_{23}^{+4}	F_3(ii)	$\frac{1}{4}^1 1^{22}$	$\frac{1}{4}$	1	23	

TABLE 1 (*cont.*)

description	*P.*	d_i	d	μ	s	g
A_{23}	F_1(i)	$1^{22}24^1$	24	2	276	
A_{23}^{+2}	F_3(i)	$1^{22}6^1$	6	2	276	
A_{23}^{+3}	F_8(i)	$\frac{1}{3}^1 1^{21} 8^1$	$\frac{8}{3}$	2	276	
A_{23}^{+4}	F_5(i)	$\frac{1}{2}^1 1^{21} 3^1$	$\frac{3}{2}$	2	276	
A_{23}^{+6}	F_6(i)	$\frac{1}{3}^1 1^{21} 2^1$	$\frac{2}{3}$	2	276	$2\cdot 24!$
A_{23}^{+8}	F_7(i)	$\frac{1}{8}^1 1^{21} 3^1$	$\frac{3}{8}$	2	276	
A_{23}^{+12}	F_4(i)	$\frac{1}{6}^1 1^{22}$	$\frac{1}{6}$	$\frac{11}{6}$	276	
A_{23}^{+24}	F_2(i)	$\frac{1}{24}^1 1^{22}$	$\frac{1}{24}$	$\frac{23}{24}$	24	
$Q_{23}(4)$	F_2(vi)	$1^{22}4^1$	4	4	46575	
$Q_{23}(4)^{+4}$	F_1(vi)	1^{23}	1	3	2300	$2\|Co_2\|$
$Q_{23}(4)^{+4}$	F_3(vi)	$\frac{1}{4}^1 1^{22}$	$\frac{1}{4}$	3	2300	
$Q_{23}(6)$	F_1(vii)	$1^{22}6^1$	6	4	37950	
$Q_{23}(6)^{+2}$	F_3(vii)	$\frac{1}{2}^1 1^{21} 3^1$	$\frac{3}{2}$	$\frac{5}{2}$	276	$2\|Co_3\|$
$Q_{23}(6)^{+3}$	F_4(vii)	$\frac{1}{3}^1 1^{21} 2^1$	$\frac{2}{3}$	$\frac{10}{3}$	11178	
$Q_{23}(6)^{+6}$	F_2(vii)	$\frac{1}{6}^1 1^{22}$	$\frac{1}{6}$	$\frac{5}{2}$	276	
$Q_{23}(8)$	F_1(iv)	$1^{22}8^1$	8	4	32890*	$2^{12}\|M_{23}\|$
$Q_{23}(8)^{+2}$	F_1(iii)	$1^{22}2^1$	2	2	23	$2^{23}\|M_{23}\|$
$Q_{23}(8)^{+4}$	F_2(iii)	$\frac{1}{2}^1 1^{22}$	$\frac{1}{2}$	2	23	
$Q_{23}(8)^{+8}$	F_2(iv)	$\frac{1}{8}^1 1^{22}$	$\frac{1}{8}$	2	23	$2^{12}\|M_{23}\|$
$Q_{23}(12)$	F_3(v)	$1^{22}12^1$	12	4	26841*	
$Q_{23}(12)^{+2}$	F_1(v)	$1^{22}3^1$	3	3	186760*	
$Q_{23}(12)^{+3}$	F_6(v)	$\frac{1}{3}^1 1^{21} 4^1$	$\frac{4}{3}$	$\frac{8}{3}$	759	
$Q_{23}(12)^{+4}$	F_5(v)	$\frac{1}{4}^1 1^{21} 3^1$	$\frac{3}{4}$	3	186760*	$2\|M_{24}\|$
$Q_{23}(12)^{+6}$	F_2(v)	$\frac{1}{3}^1 1^{22}$	$\frac{1}{3}$	$\frac{8}{3}$	759	
$Q_{23}(12)^{+12}$	F_4(v)	$\frac{1}{12}^1 1^{22}$	$\frac{1}{12}$	$\frac{23}{12}$	24	

2. Properties of lattices

Automorphism groups. The automorphism group Aut (L) of a lattice L is the group of all euclidean congruences of that lattice which fix the origin. This group can be represented as a group of matrices in various ways. With respect to an integral basis for L, Aut (L) is represented by a group of integral matrices, and the lattices we are considering are precisely those for which this group is a maximal finite irreducible subgroup of $GL(n, \mathbb{Z})$. Two such groups are conjugate in $GL(n, \mathbb{Z})$ if and only if they represent automorphism groups of the same lattice with respect to possibly different integral bases.

Alternatively we could describe Aut (L) with respect to an orthonormal basis for the real euclidean space $\mathbb{R}L$ containing L. This represents Aut (L) as a finite group of real orthogonal matrices.

It is possible for two subgroups of $GL(n, \mathbb{Z})$ to be conjugate in $GL(n, \mathbb{Q})$ but not in $GL(n, \mathbb{Z})$. This happens when the corresponding lattices L and M are commensurable (i.e. their intersection has finite index in each of them), as in many of our d-families. In this case we can choose an orthonormal basis so that their groups are represented by the same group of real orthogonal matrices.

From the geometric viewpoint espoused in this paper, it is natural to describe this by saying that L and M have the same group. From an arithmetic viewpoint one would instead speak of rationally equivalent groups of integral matrices.

Duality. The dual lattice L^* of a lattice L was defined in part I. The Gram matrix for L^* is the inverse of that for L, and $\mathrm{Aut}(L^*) = \mathrm{Aut}(L)$.

We shall often loosely describe two lattices as duals when one is congruent or even merely similar to the dual of the other. When it is necessary to be more precise, we say that L and M are λ-duals if $M = \lambda^{\frac{1}{2}}L^*$. This relation is symmetric, and implies that any inner product $u \cdot v (u \in L, v \in M)$ is a multiple of $\lambda^{\frac{1}{2}}$.

Direct sums. If L, M are lattices of dimensions l, m with Gram matrices A, B respectively, their direct sum $A \oplus B$ is the $(l+m)$-dimensional lattice with Gram matrix $A \oplus B$. So $\det L \oplus M = \det L \det M$, and the dual lattice $(L \oplus M)^* \cong L^* \oplus M^*$. If M_1, M_2 are generator matrices for L, M, then $M_1 \oplus M_2$ is a generator matrix for $L \oplus M$.

The automorphism group of a direct sum of indecomposable lattices, of which a are isomorphic to L, b are isomorphic to M, ..., is

$$(\mathrm{Aut}(L) \,\mathrm{wr}\, \mathsf{S}_a) \times (\mathrm{Aut}(M) \,\mathrm{wr}\, \mathsf{S}_b) \times \dots$$

where wr denotes a wreath product and S_n is the symmetric group of order $n!$

Tensor products. Again, if L, M are lattices of dimensions l, m with Gram matrices A, B respectively, their tensor product $L \otimes M$ is the lm-dimensional lattice with Gram matrix $A \otimes B$. So $\det(L \otimes M) = (\det L)^m (\det M)^l$, and the dual lattice $(L \otimes M)^* \cong L^* \otimes M^*$. If M_1, M_2 are generator matrices for L, M, then $M_1 \otimes M_2$ is a generator matrix for $L \otimes M$.

One cannot say in general what the automorphism group of $L \otimes M$ is, because accidental symmetries may always appear. Let $G_L = \mathrm{Aut}(L)$, $G_M = \mathrm{Aut}(M)$. In all cases $\mathrm{Aut}(L \otimes M)$ contains the central product $(G_L \times G_M)/(-1, -1)$, of order $\frac{1}{2}|G_L|\,|G_M|$, and usually this is all of $\mathrm{Aut}(L \otimes M)$. In particular cases the group may be larger. For instance $\mathrm{Aut}(L \otimes L)$ contains $(G_L \,\mathrm{wr}\, \mathsf{S}_2)/(-1, -1)$, of order $|G_L|^2$. Again the automorphism group of $L \otimes \mathrm{I}_m = L \oplus \dots \oplus L$ (m times) contains a group isomorphic to $G_L \,\mathrm{wr}\, \mathsf{S}_m$, of order $|G_L|^m m!$.

Elementary divisors. The elementary divisors $d_1, \dots, d_n$ of an n-dimensional lattice L are the elementary divisors of a Gram matrix for L, and satisfy $d_i | d_{i+1}$, $\det L = \prod d_i$ (cf. Newman 1972). These numbers have a geometric interpretation: there exists a basis $u_1, \dots, u_n$ for L such that $u_1/d_1, \dots, u_n/d_n$ is a basis for the dual lattice L^*. So L is integral (i.e. $L \subseteq L^*$) if and only if the d_i are integers. In this case the dual quotient L^*/L is a direct product $C_{d_1} \times \dots \times C_{d_n}$ of cyclic groups. The elementary divisors of the λ-dual of L are $\lambda^{\frac{1}{2}}/d_n, \dots, \lambda^{\frac{1}{2}}/d_1$.

3. The individual lattices

We now describe the individual lattices listed in table 1, sometimes in rather telegraphic language.

The root lattices. The root lattices A_n, D_n, E_6, E_7 and E_8 were defined in part I, together with their glue vectors $[i]$ and automorphism groups.

The dual of D_n is the lattice D_n^{+4}, obtained by adjoining the glue vectors

$$[1] = (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, \tfrac{1}{2}),$$
$$[2] = (0, 0, \ldots, 0, 1),$$
$$[3] = (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, -\tfrac{1}{2}),$$

each with n coordinates. If we simply adjoin [2] we get a lattice which might have been called D_n^{+2}, but for which we prefer the name I_n, because it is just the n-dimensional simple cubic lattice consisting of all points with integer coordinates. Instead we reserve the name D_n^{+2} (when n is even) for the lattice obtained by adjoining [1] to D_n.

The lattice $A_n^{+r} = \langle A_n, sv\rangle$, where $rs = n+1$ and

$$v = [1] = \left(\frac{1}{n+1}, \frac{1}{n+1}, \ldots, \frac{1}{n+1}, \frac{-n}{n+1}\right),$$

with $n+1$ coordinates. The glue vector

$$[s] = \left(\frac{s}{n+1}, \ldots, \frac{s}{n+1}, \frac{-r}{n+1}, \ldots, \frac{-r}{n+1}\right)$$

(with r coordinates $s/(n+1)$ and s coordinates $-r/(n+1)$) is a minimal vector in the coset $A_n + sv$, and has norm $rs/(n+1)$.

Apart from certain notorious exceptions in low dimensions (see table 1), D_n, I_n and D_n^{+4} have the same automorphism group, of shape $2^n . n!$, consisting of all permutations and sign changes of the n coordinates. The group $2^{n-1}. n!$ of D_n^{+2} is only half as large, only the sign changes of an even number of coordinates being permitted. The automorphism group of A_n^{+r} is $2 \times \mathsf{S}_{n+1}$ (again with some exceptions), and is generated by negation and all permutations of the $n+1$ coordinates.

d-Families of lattices. Let us say that a lattice L is *adjacent* to any lattice of the form $\langle L, v\rangle$ in the same dimension, and that two lattices are in the same *family* if they are connected by a chain of adjacencies.

Then in the range covered by our table most of the families have a fairly uniform structure. The smallest (or coarsest) lattice of a family, L say, is contained in all the others, which can therefore be written as L^{+r} for various values of r. The largest (or finest) member of the family, L^{+d} say, contains all the others, so that r divides d. In this case we call the family a *d-family*. There is often a one-to-one correspondence between the divisors r of d and the lattices L^{+r} in the family.

Furthermore, there is usually a large group of automorphisms common to all lattices in a given *d-family*, although in some cases individual family members may have additional automorphisms.

The dual of a d-family $\{L^{+r}\}$ is another (often the same) d-family $\{M^{+s}\}$, the lattices L^{+r} and M^{+s} being duals just when $rs = d$.

The exceptional lattices $Q_n(d)^{+r}$. In the dimensions n covered by table 1 we find that most of the lattices are easily expressible in terms of the root lattices A_m, D_m and E_m. The *exceptional* lattices that remain fall into d-families in such a way that for any given d there is either a unique d-family $\{Q_n(d)^{+r}\}$ or two mutually dual d-families $\{Q_n(d)^{+r}\}$ and $\{Q_n'(d)^{+r}\}$. The dual of a particular lattice $Q_n(d)^{+r}$ is therefore (up to a scale factor) the lattice $Q_n'(d)^{+d/r}$, except that the prime is omitted for a self-dual family.

When $n = 17$, for example, the exceptional lattices form two dual 6-families

$$Q_{17}(6),\ Q_{17}(6)^{+2},\ Q_{17}(6)^{+3},\ Q_{17}(6)^{+6},$$

$$Q_{17}'(6),\ Q_{17}'(6)^{+2},\ Q_{17}'(6)^{+3},\ Q_{17}'(6)^{+6},$$

a self-dual 8-family

$$Q_{17}(8),\ Q_{17}(8)^{+2},\ Q_{17}(8)^{+4},\ Q_{17}(8)^{+8},$$

and a self-dual 9-family

$$Q_{17}(9),\ Q_{17}(9)^{+3},\ Q_{17}(9)^{+9}$$

(cf. figure 1).

Figure 1 displays all the lattices and most of the adjacencies. The notation $L \xrightarrow{i} M$ means that L and M have the same automorphism group and $M = \langle L, v\rangle$ contains L to index i. A dotted arrow indicates adjacent lattices with different groups. We omit the heads from upward arrows. Duality is indicated by dashed lines, except when these would fall onto other lines.

The same lattices are also described in table 1. The entry for a typical lattice describes the lattice, gives Plesken & Pohst's or Plesken's symbol for it, when available (in the column headed P), and then gives the elementary divisors $\{d_i\}$, the determinant $d = \prod d_i$, the minimal norm μ, the number s of pairs of minimal vectors, and the order g of the automorphism group. The elementary divisors are derived from Plesken & Pohst and Plesken.

Duality up to scale, rotation, and possibly reflection is indicated by horizontal lines dividing table 1 into blocks; duality simply reverses the order of lattices in a block. The lattices in a block either belong to two dual d-families or to a single self-dual d-family.

4. Dimensions 1 to 5

Dimension 2. The lattices I_2 and A_2 are the familiar square and hexagonal planar lattices (see, for example, SLG, figure 1.3)). The lattices $D_2 \cong D_2^*$ and A_2^* are scaled copies of I_2 and A_2 respectively.

Dimension 3. The lattice $A_3 \cong D_3$ is isomorphic to the face-centred cubic lattice, and its dual $A_3^* \cong D_3^*$ is the body-centred cubic lattice. Although we normally use the names A_3 and A_3^*, the reader should remember that these lattices are often best understood as D_3 and D_3^*.

Dimension 4. The lattice D_4 has an exceptional automorphism group which transitively permutes the three cosets [1], [2], [3] (see part I, appendix). So $D_4^{+2} \cong I_4$. Also D_4^{+4} is a scaled copy of D_4.

The lattice $A_2 \otimes I_2$ is the direct sum $A_2 \oplus A_2$ of two A_2 lattices. Its automorphism group is the direct product of their automorphism groups extended by the symmetry that interchanges them, or in other words is $\mathrm{Aut}(A_2)\ \mathrm{wr}\ S_2$.

The typical vector of $A_2 \otimes A_2$ is a 3×3 array of integers with each row and column adding to 0. The group is generated by all permutations of rows, all permutations of column, transposition, and negation, and has order 144.

Dimension 5. The lattices D_5, I_5, D_5^{+4} share the same group $2^5 . \mathsf{S}_5$, while A_5, A_5^{+2}, A_5^{+3}, A_5^{+6} share the group $2 \times \mathsf{S}_6$.

5. Dimension 6

Besides D_6, I_6, D_6^{+4} we now have the lattice D_6^{+2}, whose group is only half as large (see §3).

The lattices

$$E_6 \text{ and } E_6^{+3} = E_6[1] = E_6^*$$

share the same group $2 \times W(E_6)$, of order $2^8 \cdot 3^4 \cdot 5 = 103680$, generated by negation and the Weyl group $W(E_6)$ of type E_6. Similarly

$$A_6 \text{ and } A_6^{+7} = A_6[1] = A_6^*$$

share the group $2 \times \mathsf{S}_7$.

The lattice $A_3 \otimes \mathrm{I}_2$ is the direct sum $A_3 \oplus A_3 \cong D_3 \oplus D_3$, with group $\mathrm{Aut}\,(A_3)$ wr S_2, of order $48^2 \cdot 2 = 4608$. The dual lattice is $(A_3 \otimes \mathrm{I}_2)^{+16} = A_3^* \oplus A_3^* \cong D_3^* \oplus D_3^*$.

Similarly $\mathrm{I}_3 \otimes A_2 = A_2 \oplus A_2 \oplus A_2$ has group $\mathrm{Aut}\,(A_2)$ wr S_3, of order $12^3 \cdot 6 = 10368$.

The typical vector of $A_3 \otimes A_2 \cong D_3 \otimes A_2$ is a 4×3 array of integers with each row and column adding to zero. The group has order 288 and structure $\mathsf{S}_4 \times \mathsf{S}_3 \times 2$, generated by permutations of rows, permutations of columns, and negation. This lattice may alternatively be described as the set of 3×3 arrays of integers with row-sums zero and column-sums even. The dual lattice

$$(A_3 \otimes A_2)^{+16} = (A_3 \otimes A_2)^* \cong A_3^* \otimes A_2 \cong D_3^* \otimes A_2$$

is most simply described as the set of 3×3 arrays of integers with row-sums zero and all entries in any column having the same parity.

The six-dimensional lattice related to PGL_2 (7). The exceptional lattice $Q_6(1)$ can be described by three complex coordinates, using $[x, y, z]$ as an abbreviation for $(x, y, z, \bar{x}, \bar{y}, \bar{z})$, where the bar denotes complex conjugation, and a suitable inner product is

$$\begin{aligned}[x_1, y_1, z_1] \cdot [x_2, y_2, z_2] &= \mathrm{Re}\,(x_1 \bar{x}_2 + y_1 \bar{y}_2 + z_1 \bar{z}_2) \\ &= \tfrac{1}{2}(x_1 \bar{x}_2 + y_1 \bar{y}_2 + z_1 \bar{z}_2 + \bar{x}_1 x_2 + \bar{y}_1 y_2 + \bar{z}_1 z_2). \qquad (1)\end{aligned}$$

Then $Q_6(1)$ is generated by the 42 norm 4 vectors consisting of all cyclic permutations of

$$[\pm 2, 0, 0], \quad [\pm\lambda, \pm 1, \pm 1] \quad \text{and} \quad [0, \pm\mu, \pm\mu],$$

where $\lambda = (-1 + \sqrt{-7})/2$, $\mu = \bar{\lambda} = (-1 - \sqrt{-7})/2$ are the roots of $X^2 + X + 2 = 0$. These 42 vectors are all the minimal vectors. The group has structure $2 \times PGL_2(7)$ and order 672, and is generated by the unitary reflections in these vectors together with complex conjugation.

This lattice $Q_6(1)$ has appeared in several different guises in the literature. It was called P_6 or ϕ_6 by Barnes (1957*a*, *b*) and others, and is Plesken and Pohst's F_{14}. It is the real form of the complex laminated lattice $\Lambda_3(\lambda)$ (Conway & Sloane 1983; ATLAS, p. 3). It is also the lattice $A_6^{(2)}$ in the notation of part III, §5. In this form the minimal vectors consist of all cyclic permutations of

$$\begin{array}{lllllll} \pm(0, & 0, & 1, & -1, & -1, & 1, & 0), \\ \pm(0, & -1, & +1, & 0, & 0, & +1, & -1), \\ \pm(0, & -1, & 0, & +1, & +1, & 0, & -1), \end{array}$$

and the group is described in these coordinates in §6 of part III.

Lattices related to the icosahedron. The exceptional lattices $Q_6(4)$, $Q_6(4)^{+2}$, $Q_6(4)^{+4}$ all have the same group G, which arises from the three-dimensional group of symmetries of an icosahedron by adjoining elements that change the sign of $\sqrt{5}$. We also describe these lattices in a three-dimensional notation, using the triple $[x, y, z]$ as an abbreviation for (x, y, z, x', y', z'), where $(a+b\sqrt{5})' = a-b\sqrt{5}$, for $a, b \in \mathbb{Q}$, and define the inner product by

$$[x_1, y_1, z_1]\cdot[x_2, y_2, z_2] = \tfrac{1}{2}(x_1x_2+y_1y_2+z_1z_2+x_1'x_2'+y_1'y_2'+z_1'z_2'). \tag{2}$$

Then the group G consists of the 120 symmetries of the icosahedron, extended by the conjugating map

$$[x, y, z] \to [x', z', y'], \tag{3}$$

and has order 240.

We label the 12 vertices of the icosahedron in two ways: with the triples $\pm v_\infty$, $\pm v_0, \ldots, \pm v_4$, which are the cyclic permutations of $[0, \pm\sigma, \pm 1]$ (see figure 2), and also with the triples $\pm w_\infty, \pm w_0, \ldots, \pm w_4$, which are the cyclic permutations of $[0, \pm 1, \pm\tau]$, where $\tau = \tfrac{1}{2}(1+\sqrt{5})$, $\sigma = -\tau' = \tfrac{1}{2}(-1+\sqrt{5})$. Then v_i and $w_i = \tau v_i$ represent the same vertex, for $i \in \{\infty, 0, \ldots, 4\}$: although as three-dimensional vectors their lengths are different, they rationalize to six-dimensional vectors of the same length. For example $v_\infty = [\sigma, 1, 0]$ and $w_\infty = [1, \tau, 0]$ rationalize to

$$(\sigma, 1, 0, -\tau, 1, 0) \quad \text{and} \quad (1, \tau, 0, 1, -\sigma, 0),$$

respectively, both of norm $\tfrac{5}{2}$.

Then the lattice $Q_6(4)^{+4}$ is generated by the 24 such triples $\pm v_i, \pm w_i$ representing the vertices; these norm $\tfrac{5}{2}$ vectors are also the minimal vectors of this lattice.

We label the midpoints of the edges of the icosahedron by the 30 triples consisting of the cyclic permutations of $[\pm 2, 0, 0]$ and $[\pm 1, \pm\sigma, \pm\tau]$, of norm 4 (using the sums $v_\infty + v_0, \ldots$ of the v-coordinates of the end-points of an edge, rather than the w-coordinates, which produce vectors of greater norm). The lattice $Q_6(4)$ is generated by these 30 triples, which are also the minimal vectors. These 30 triples are root vectors in the sense that the reflections in them generate the symmetries of the icosahedron. Furthermore, $Q_6(4) = \langle \pm v_i \pm v_j, \pm w_i \pm w_j \rangle$, and has index 4 in $Q_6(4)^{+4}$; as non-zero glue vectors we may take v_∞, w_∞ and $v_\infty + w_\infty$.

Similarly we label the face-centres of the icosahedron by the 20 triples consisting of the cyclic permutations of $[\pm 1, \pm 1, \pm 1]$ and $[0, \pm\tau, \pm\sigma]$, of norm 3 (these are the differences $w_\infty - v_0, \ldots$, between the w-coordinates of one end of an edge and the v-coordinates of the other end; the difference labels the face that is pointed to). The lattice $Q_6(4)^{+2}$ is generated by these 20 triples, which are also the minimal

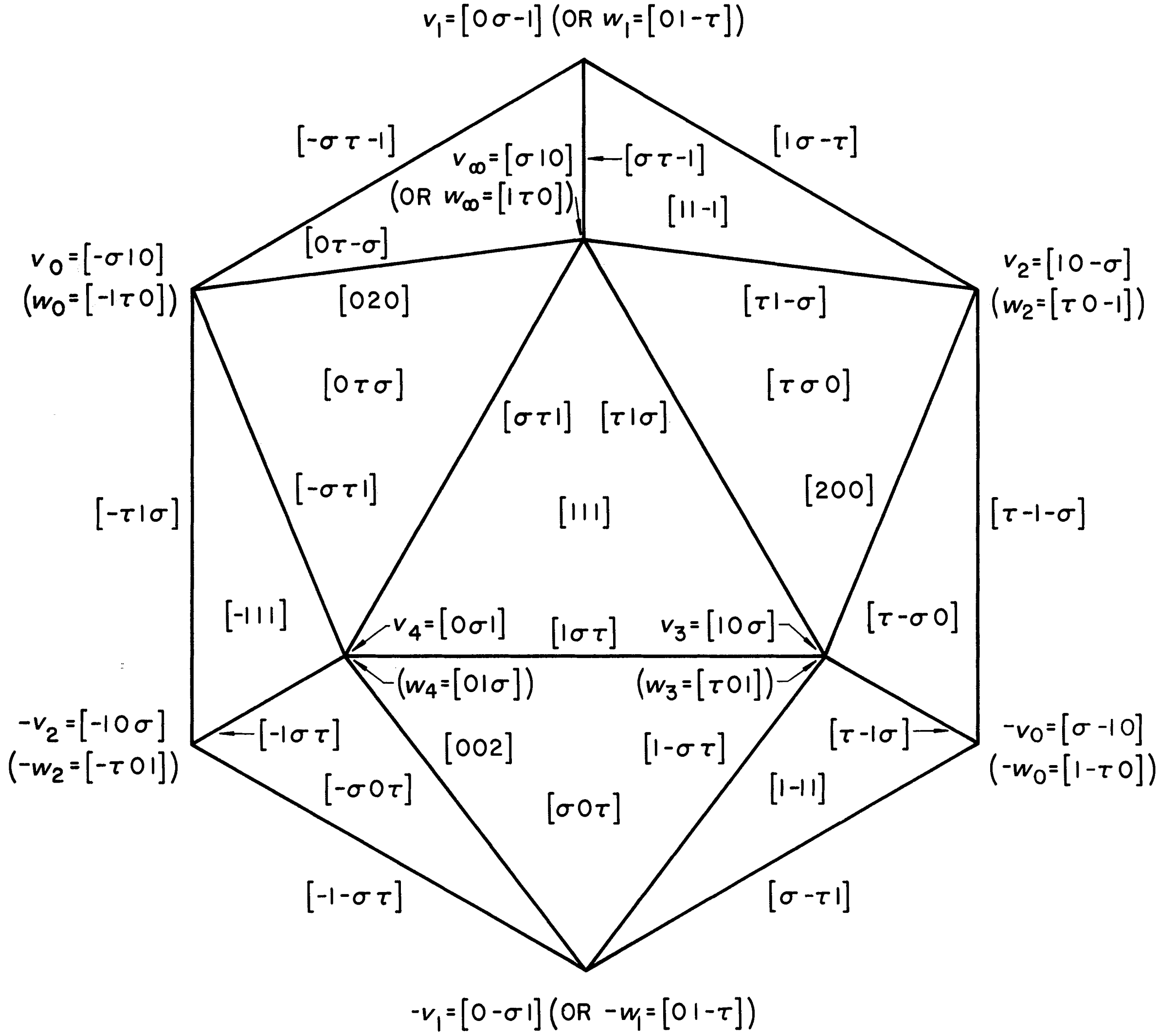

FIGURE 2. Icosahedron, labelled with coordinates for vertices (in two alternative forms), mid-points of edges, and centres of faces; $\tau = \frac{1}{2}(1+\sqrt{5})$, $\sigma = \frac{1}{2}(-1+\sqrt{5})$.

vectors. Furthermore $Q_6(4)^{+2} = \langle Q_6(4),\ v_\infty + w_\infty \rangle$; $w_\infty - v_0$ is a minimal vector in the coset $v_\infty + w_\infty + Q_6(4)$.

The linear relations among the vectors $\pm v_i$, $\pm w_j$ are as follows. In either coordinates, the *icosahedral rule* says that the sum of the neighbours of a vertex is $\sqrt{5}$ times the vertex; for example

$$v_0 + v_1 + v_2 + v_3 + v_4 = \sqrt{5}\, v_\infty.$$

Hence the w-coordinates for a vertex are half the sum of the v-coordinates for that vertex and its neighbours; for example

$$w_\infty = \tfrac{1}{2}(v_\infty + v_0 + v_1 + v_2 + v_3 + v_4),$$

$$w_0 = \tfrac{1}{2}(v_0 + v_1 + v_\infty + v_4 - v_2 - v_3).$$

Finally, the sum of the v-coordinates for two *adjacent* vertices equals the sum of the w-coordinates for the two *adjoining* vertices; for example

$$v_\infty + v_0 = w_1 + w_4, \quad v_4 - v_2 = w_3 - w_2.$$

The triple at any point (vertex, or midpoint of edge or face) consists of the last three coordinates of some unit icosian u (see §7), and the map $q \to u^{-1}qu$ is then a rotation about that point (where $q = xi + yj + zk$). The rotation group of the icosahedron is the set of all such rotations, together with the identity. This group acts on the six diameters $\pm v_\infty, \pm v_0, \ldots, \pm v_4$ in the way that $L_2(5)$ acts on $\infty, 0, \ldots, 4$. For example, the unit icosians for v_∞ are $[\pm\tau, \sigma, 1, 0]$ and $[\pm\sigma, 1, \tau, 0]$, and represent the four rotations $(0\ 1\ 2\ 3\ 4)^n$ $(n = 1, \ldots, 4)$ about that vertex. Some other rotations of the icosahedron are also simply expressible in these coordinates; for example, the maps taking (x, y, z) to $(-x, y, z)$, $(x, -y, z)$, $(x, y, -z)$ and (y, z, x) become respectively $(\infty\ 0)\ (2\ 3)$, $(\infty\ 0)\ (1\ 4)$, $(1\ 4)\ (2\ 3)$, $(\infty\ 3\ 4)$.

6. Dimension 7

We do not provide individual discussions of the lattices D_n^{+r}, I_n, A_n^{+r} in dimensions $n \geqslant 7$.

The automorphism group of the lattices

$$E_7 = A_7^{+2} \quad \text{and} \quad E_7^{+2} = E_7^* = A_7^{+4}$$

is the Weyl group of type E_7, of order $2^{10}\cdot 3^4 \cdot 5 \cdot 7 = 2903040$.

7. Dimension 8

The automorphism group of

$$E_8 = D_8^{+2} = A_8^{+3} = E_8^*$$

is the Weyl group of type E_8, of order $2^{14}\cdot 3^5\cdot 5^2 \cdot 7 = 696729600$.

The lattice $D_4 \otimes \mathrm{I}_2 = D_4 \oplus D_4$ is self-explanatory. Because $D_4^* \cong D_4$ this is also isomorphic to its dual, as is the next lattice.

The lattice $\mathrm{I}_4 \otimes A_2$ is the direct sum $A_2^4 = A_2 \oplus A_2 \oplus A_2 \oplus A_2$, with group $\mathrm{Aut}(A_2) \operatorname{wr} \mathbf{S}_4$, of order $12^4 \cdot 4! = 497664$.

The lattice $L = (\mathrm{I}_4 \otimes A_2)^{+3}$ may be described (in the notation of part I) as $A_2^4[1\ 1\ 1\ 1]$, or in other words it is obtained by 'gluing up' the lattice A_2^4 using the glue code generated by $[1\ 1\ 1\ 1]$. Because A_2^*/A_2 is a cyclic group of order 3, this code contains three codewords $[0\ 0\ 0\ 0]$, $[1\ 1\ 1\ 1]$ and $[2\ 2\ 2\ 2]$, and

$$L = A_2^4 \cup (A_2[1])^4 \cup (A_2[2])^4.$$

As usual, gluing theory makes it easy to determine the order g of the automorphism group of L. In the notation of part I, we find $g_0(L) = g_0(A_2)^4 = 6^4$, $g_1(L) = 2$ (corresponding to negation), and $g_2(L) = 4!$. Thus $g = 6^4 \cdot 2 \cdot 4! = 62208$.

The dual lattice $(\mathrm{I}_4 \otimes A_2)^{+27}$ is $A_2^4[(0\ 0\ 1\ 2)]$, obtained from A_2^4 by using the glue code generated by all cyclic shifts of [0 0 1 2]. (This is the 'zero-sum' ternary code of length 4, dimension 3 and minimal distance 2, i.e. a [4, 3, 2] code.)

The typical vector of $D_4 \otimes A_2$ is a 3×4 array of integers with row-sums even and column-sums zero. There are $3 \times 24 = 72$ minimal vectors, of norm 4, namely the arrays whose rows are $\{v, -v, 0\}$ in some order, where v is a minimal vector of D_4. The group (see §2) has order $\frac{1}{2} \cdot 1152 \cdot 12 = 6912$.

The lattice $(D_4 \otimes A_2)^{+2}$ is the union of $D_4 \otimes A_2$ and its translate by the array whose rows are y_1, y_2, y_3, where these are the non-trivial glue vectors for D_4 and $y_1 + y_2 + y_3 = 0$. By a slight abuse of our usual notation we describe this as the lattice $D_4 \otimes A_2$ [1 2 3]. There are 32 minimal vectors, of norm 3 (like the above one), all lying in the translate. The group is abstractly isomorphic to the automorphism group of D_4, and has order 1152. (An automorphism of D_4 first acts in each row, and then bodily permutes the rows in the way it permutes the three cosets $D_4 + y_1$, $D_4 + y_2$, $D_4 + y_3$.)

Similarly the dual lattice $(D_4 \otimes A_2)^{+8}$ may be described as $D_4 \otimes A_2$ glued up by the code generated by [1 1 0], [2 0 2] and [0 3 3]. There are eight glue words. The minimal norm is 2, and there are 24 minimal vectors, e.g. the arrays with rows $0, y_1, -y_1$.

The final lattice in this group is $(D_4 \otimes A_2)^{+4}$, which is $D_4 \otimes A_2$ [(1 2 3)]. The four glue words are [0 0 0], [1 2 3], [2 3 1], [3 1 2]. This contains $(D_4 \otimes A_2)^{+2}$ as a sublattice of index 2, has three times as many minimal vectors as that lattice, and its group is correspondingly three times as large, there being a new automorphism that cyclically permutes the three rows.

The lattice $A_4 \otimes I_2 = A_4^2$ has group $\mathrm{Aut}(A_4) \operatorname{wr} S_2$, of order $(2 \cdot 5!)^2 \cdot 2 = 115200$. The dual lattice $A_4^* \otimes I_2 = (A_4 \otimes I_2)^{+25} = A_4^2[10, 01]$ has minimal norm $\frac{4}{5}$, a typical minimal vector being $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, -\frac{4}{5}, \frac{1}{5}; 0, 0, 0, 0, 0)$.

The lattice $(A_4 \otimes I_2)^{+5} = A_4^2[11]$ has minimal norm $\frac{8}{5}$, and $g_0 = (5!)^2$, $g_1 = 2$, $g_2 = 2$, so the group has order $g = (5!)^2 \cdot 2 \cdot 2 = 57600$.

The next four lattices, $A_4 \otimes A_2$, $(A_4 \otimes A_2)^{+25} = (A_4 \otimes A_2)^* \cong A_4^* \otimes A_2$, $A_2 \otimes A_2 \otimes I_2 = (A_2 \otimes A_2) \oplus (A_2 \otimes A_2)$, and $A_2 \otimes A_2 \otimes A_2$, need no further explanation.

The lattice related to the polytope $\{3, 3, 5\}$. The exceptional lattice $Q_8(1)$ can be described in terms of the symmetries of the four-dimensional polytope $\{3, 3, 5\}$, just as we described the lattices $Q_6(4)^{+r}$ in §5 in terms of the symmetries of the three-dimensional icosahedron $\{3, 5\}$. We use a four-dimensional notation, where $[w, x, y, z]$ means $(w, x, y, z, w', x', y', z')$, the prime has the same meaning as in §5, and the inner product is given by

$$[w_1, x_1, y_1, z_1] \cdot [w_2, x_2, y_2, z_2] = \tfrac{1}{2}(w_1 w_2 + x_1 x_2 + y_1 y_2 + z_1 z_2 + w_1' w_2' + x_1' x_2' + y_1' y_2' + z_1' z_2') \quad (4)$$

Then the automorphism group of $Q_8(1)$ consists of the 14400 symmetries of $\{3, 3, 5\}$, extended by the conjugating map

$$s : [w, x, y, z] \to [w', x', z', y'], \quad (5)$$

and has order 28800.

The 120 minimal vectors of $Q_8(1)$, of norm 4, are obtained from

$$[\pm 2, 0, 0, 0], \quad [\pm 1, \pm 1, \pm 1, \pm 1], \quad [0, \pm 1, \pm \sigma, \pm \tau]$$

by even permutations of the coordinates. These vectors generate the lattice, and the reflections in them generate the symmetries of $\{3, 3, 5\}$.

If we associate the quaternion $\frac{1}{2}(w+xi+yj+zk)$ with $[w, x, y, z]$, then the quaternions associated with all the lattice vectors form the *icosian ring*, and those corresponding to the 120 minimal vectors are the *unit icosians* (see SLG, chap. 8, §2; Du Val 1964; ATLAS, p. 2; Wilson 1986). The unit icosians form a multiplicative group which is isomorphic to the perfect double cover $2.A_5$, the binary icosahedral group.

The automorphism group of $Q_8(1)$ can be described in quaternionic language as follows. The maps $q \to uq$ and $q \to qv$ (u, v unit icosians) form two subgroups L and R isomorphic to $2.A_5$. The rotation group $[3, 3, 5]^+$ of $\{3, 3, 5\}$ is $L \circ R = L \times R/\{\pm 1\}$, of structure $2.A_5^2$. The map $t: q \to \bar{q}$ is a wreathing involution (taking $q \to uq$ to $q \to q\bar{u}$), and extends this to the full group $[3, 3, 5]$ of $\{3, 3, 5\}$. This group has structure $2(A_5 \text{ wr } S_2)$. Finally the map s (5) extends each A_5 to S_5, and the full group of $Q_8(1)$ has structure

$$2.\tfrac{1}{2}(S_5 \times S_5).2$$

corresponding to

$$\{\pm 1\} \cdot \begin{matrix}\text{even perms} \\ \text{of } S_5 \times S_5\end{matrix} \cdot t,$$

and has order 28800 (Plesken & Pohst omit a factor of two).

This lattice is closely related to E_8. Let q_r correspond to $[w_r, x_r, y_r, z_r]$, for $r = 1, 2$, and let $\operatorname{Re}\{q_1 \bar{q}_2) = \alpha + \beta\sqrt{5}$, for $\alpha, \beta \in \mathbb{Q}$. Then the inner product (4) is proportional to α. If instead we use $\alpha + \beta$, the icosian ring becomes a copy of the E_8 lattice.

The eight-dimensional lattices related to $PGL_2(7)$. The exceptional lattices $Q_8(3)$, $Q_8(3)^{+3}$, $Q_8'(3)$, $Q_8'(3)^{+3}$ share the same group G of order 672. Although G is of course real, we prefer to use eight-dimensional complex coordinates involving $\omega = e^{\frac{2}{3}\pi i}$ to describe these lattices, because then we can represent G by (complex) monomial matrices. In fact G is generated by negation and the four elements

$$\alpha = (e_\infty)\,(e_0, e_1, e_2, e_3, e_4, e_5, e_6),$$

$$\beta = (e_\infty, \bar{\omega}e_\infty, \bar{\omega}e_\infty)\,(e_0, \omega e_0, \bar{\omega}e_0)\,(e_1, \omega e_2, \bar{\omega}e_4)\,(e_3, \omega e_6, \bar{\omega}e_5),$$

$$\gamma = (e_\infty, e_0)\,(e_1, e_6)\,(e_2, \omega e_3)\,(e_4, \bar{\omega}e_5),$$

$$\delta = (e_\infty)\,(e_0)\,(e_1, e_6)\,(e_2, e_5)\,(e_3, e_4),$$

where $e_\infty, e_0, e_1, \ldots, e_6$ are an orthonormal coordinate frame. The elements α, β, γ generate the simple group $PSL_2(7)$ of order 168, and δ extends this to $PGL_2(7)$ of order 336.

(To avoid any possibility of confusion, we mention that these four lattices are not $\mathbb{Z}[\omega]$-modules; ω is used here just as an abbreviation.)

For the lattice $Q_8(3)^{+3}$ we define 21 vectors

$$\begin{array}{lcccccccc}
 & e_\infty & e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
0_1 = \theta^{-1}\times & (1 & \bar\omega & \omega & 1 & 1 & 1 & 1 & \omega) \\
1_1 = \theta^{-1}\times & (1 & \omega & \bar\omega & \omega & 1 & 1 & 1 & 1) \\
\ldots & & & & \cdot & \cdot & \cdot & & \\
0_2 = \theta^{-1}\times & (\omega & \omega & \bar\omega & \bar\omega & 1 & 1 & \bar\omega & \bar\omega) \\
1_2 = \theta^{-1}\times & (\omega & \bar\omega & \omega & \bar\omega & \bar\omega & 1 & 1 & \bar\omega) \\
\ldots & & & & \cdot & \cdot & \cdot & & \\
0_4 = \theta^{-1}\times & (\bar\omega & 1 & \omega & \bar\omega & \omega & \omega & \bar\omega & \omega) \\
1_4 = \theta^{-1}\times & (\bar\omega & \omega & 1 & \omega & \bar\omega & \omega & \omega & \bar\omega) \\
\ldots & & & & \cdot & \cdot & \cdot & &
\end{array}$$

where n_t is obtained by permuting the last *seven* coordinates of 0_t cyclically by n places to the right, for $0 \leqslant n \leqslant 6$, $t = 1$, 2 or 4, and $\theta = \omega - \bar\omega = \sqrt{-3}$. We use the inner product

$$\textstyle\sum a_r e_r \cdot \sum b_s e_s = \sum a_r \bar b_s, \tag{6}$$

which is real-valued for these vectors.

In fact those inner products are given by

$$n_t \cdot n_t' = \begin{cases} -\frac{1}{3} & \text{if } \ dt \equiv \pm 3 \\ \frac{8}{3} & \text{if } \ d \equiv 0 \\ \frac{2}{3} & \text{otherwise,} \end{cases} \tag{7}$$

$$n_t \cdot n'_{t'} = \begin{cases} -\frac{1}{3} & \text{if } \ \pm d \equiv tt' \text{ or } 0 \\ \frac{2}{3} & \text{if } \ \pm 2d \equiv tt' \\ -\frac{4}{3} & \text{if } \ \pm 4d \equiv tt', \end{cases} \tag{8}$$

where the congruences are modulo 7, $d \equiv n - n'$, and in (8) we have $t \neq t'$. Then $Q_8(3)^{+3}$ is generated by these 21 vectors; also these vectors and their negatives are the minimal vectors of $Q_8(3)^{+3}$.

These vectors satisfy many linear relations, of which the simplest are

$$1_4 - 6_4 = 4_2 - 3_2,$$

$$2_2 - 5_2 = 1_1 - 6_1,$$

$$4_1 - 3_1 = 2_4 - 5_4,$$

and their images under α.

The lattice $Q_8(3)$ is generated by the differences of these 21 vectors. There are just 84 distinct non-zero differences of the minimal norm (which is 4); they are obtained from the following six vectors by cyclic permutations of the last seven coordinates and/or negation:

$$\begin{array}{lcccccccc}
 & e_\infty & e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
4_1 - 3_1 = & (0 & 0 & 0 & \bar\omega & 1 & -1 & -\bar\omega & 0) \\
6_4 - 1_4 = & (0 & 0 & \omega & 0 & 1 & -1 & 0 & -\omega) \\
1_1 - 6_1 = & (0 & 0 & \omega & -\bar\omega & 0 & 0 & \bar\omega & -\omega) \\
1_1 - 2_2 = & (\bar\omega & 1 & 0 & 0 & -\omega & -\omega & 0 & 0) \\
5_4 - 3_1 = & (\omega & \omega & -\bar\omega & 0 & 0 & 0 & 0 & -\bar\omega) \\
4_2 - 1_4 = & (1 & \bar\omega & 0 & -1 & 0 & 0 & -1 & 0).
\end{array}$$

These are the 84 minimal vectors of $Q_8(3)$.

For the dual lattice $Q_8'(3)^{+3} = Q_8(3)^*$ we similarly define 24 vectors of norm $\frac{14}{3}$:

$$\begin{array}{llllllllll}
 & & e_\infty & e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
\infty_3 = & \theta^{-1}\times & (2-\omega & 1 & 1 & 1 & 1 & 1 & 1 & 1) \\
\infty_5 = & \theta^{-1}\times & (2\bar\omega-1 & \omega & \omega & \omega & \omega & \omega & \omega & \omega) \\
\infty_6 = & \theta^{-1}\times & (2\omega-\bar\omega & \bar\omega & \bar\omega & \bar\omega & \bar\omega & \bar\omega & \bar\omega & \bar\omega) \\
0_3 = & \theta^{-1}\times & (1 & 2-\omega & 1 & \bar\omega & \omega & \omega & \bar\omega & 1) \\
0_5 = & \theta^{-1}\times & (\bar\omega & 2\omega-\bar\omega & \bar\omega & \omega & 1 & 1 & \omega & \bar\omega) \\
0_6 = & \theta^{-1}\times & (\omega & 2\bar\omega-1 & \omega & 1 & \bar\omega & \bar\omega & 1 & \omega), \\
 & & \ldots & & & & & & &
\end{array}$$

where n_t is obtained by permuting the last seven coordinates of 0_t cyclically by n places to the right, for $0 \leqslant n \leqslant 6$, $t = 3$, 5 or 6. The inner products (again defined by (9)) are given by

$$n_t \cdot n_{t'} = \begin{cases} \frac{14}{3} & \text{if} \quad t \equiv t' \\ -\frac{7}{3} & \text{otherwise}, \end{cases} \tag{9}$$

$$n_t \cdot n'_{t'} = \begin{cases} -\frac{1}{3} & \text{if} \quad \pm d \equiv tt' \\ -\frac{4}{3} & \text{if} \quad \pm 4d \equiv tt' \\ \frac{5}{3} & \text{if} \quad \pm 5d \equiv tt', \end{cases} \tag{10}$$

where $d \equiv n-n'$, except that $d \equiv 1$ if n or n' is ∞, and in (10) we have $n \neq n'$.

Then $Q_8'(3)^{+3}$ is generated by these 24 vectors, and they and their negatives are the minimal vectors of this lattice. These vectors also satisfy many linear relations, such as

$$n_3+n_5+n_6 = 0 \quad (n = \infty, 0, 1, \ldots, 6),$$

$$1_5-6_5 = 2_6-5_6 = 4_3-3_3,$$

$$\infty_3+0_6 = 2_3+5_3,$$

$$\infty_6+0_5 = 1_6+6_6,$$

$$\infty_5+0_3 = 4_5+3_5,$$

$$1_5+3_3 = 4_3+6_5,$$

$$2_6+6_5 = 1_5+5_6,$$

$$4_3+5_6 = 2_6+3_3.$$

The lattice $Q_8'(3)$ is generated by the differences of these 24 vectors. There are just 56 distinct non-zero differences of the minimal norm (which is 6): they are obtained from the following four vectors by cyclic permutations of the last seven coordinates and/or negation:

$$\begin{array}{lllllllll}
 & e_\infty & e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
1_5-6_5 = & (0 & 0 & 1 & \omega & -\bar\omega & \bar\omega & -\omega & -1) \\
\infty_6-0_6 = & (1 & -\omega & -1 & \omega & 0 & 0 & \omega & -1) \\
\infty_5-0_5 = & (\omega & -1 & 1 & 0 & -\bar\omega & -\bar\omega & 0 & 1) \\
\infty_3-0_3 = & (\bar\omega & -\bar\omega & 0 & -\omega & \bar\omega & \bar\omega & -\omega & 0).
\end{array}$$

These are the 56 minimal vectors of $Q_8'(3)$.

8. Dimension 9

The lattice $A_3 \otimes I_3 = A_3^3$ has group Aut (A_3) wr S_3, of order $48^3 \cdot 3! = 2^{13} \cdot 3^4 = 663552$. Its dual is $(A_3 \otimes I_3)^{+64} = (A_3^3)^* = (A_3^*)^3$.

For the lattice $(A_3 \otimes I_3)^{+2} = A_3^3[2\ 2\ 2]$ we have $g_0 = 24^3$, $g_1 = 2^3$, $g_2 = 3!$, and the group is the same as that of $A_3 \otimes I_3$. The dual lattice is $(A_3 \otimes I_3)^{+32} = A_3^3[(0\ 1\ 1), (2\ 0\ 0)]$.

For the lattice $(A_3 \otimes I_3)^{+4} = A_3^3\,[1\ 1\ 1]$ we have $g_0 = 24^3$, $g_1 = 2$, $g_2 = 3!$, so the group has order $2^{11} \cdot 3^4 = 165888$. The dual is $(A_3 \otimes I_3)^{+16} = A_3^3[(0\ 1\ 3)]$.

The next five lattices are best described in terms of $D_3 \otimes D_3$ $(\cong A_3 \otimes A_3)$, arranging the coordinates in a 3×3 array or *board*.

The first lattice, $A_3 \otimes A_3$, is $D_3 \otimes D_3$ itself. The 72 minimal vectors have shape $(\pm 1^4, 0^5)$, where there are evenly many minus signs and the four non-zero coordinates form a 'square'; for example

1	−1	0
0	0	0
−1	1	0

In other words this lattice is obtained by applying construction B (Leech & Sloane 1971; SLG) to the binary [9, 4, 4] code C spanned by the corresponding 'square' codewords of shape $(1^4, 0^5)$. From §2, the group is isomorphic to $2 \times (S_4 \text{ wr } S_2)$, of order 2304.

By adjoining any one of the 18 vectors $(\pm 2, 0^8)$ to $D_3 \otimes D_3$ we obtain $(D_3 \otimes D_3)^{+2} \cong (A_3 \otimes A_3)^{+2}$, whose minimal vectors are those 18 vectors together with *all* the 144 vectors $(\pm 1^4, 0^5)$ whose support is a 'square' as above. This contains the previous lattice to index 2. In other words $(D_3 \otimes D_3)^{+2}$ is obtained by applying construction A (Leech & Sloane 1971; SLG) to the code C. Also Aut (C) is generated by bodily permutations of the rows or columns of the board, and by transposition, and the group of the lattice has structure $2^9 \times (S_3 \text{ wr } S_2)$ and order 36864.

The lattice $(D_3 \otimes D_3)^{+2048} \cong (A_3 \otimes A_3)^{+2048}$ is dual to $(D_3 \otimes D_3)^{+2}$, and has 48 minimal vectors $(\pm \tfrac{1}{2}^3, 0^6)$, supported on any row or column of the board. In other words this lattice is obtained by applying construction A to the [9, 5, 3] dual code C^*, spanned by the corresponding binary vectors of shape $(1^3, 0^6)$. (C^* is a 3×3 'lightbulb code' in the notation of Graham & Sloane (1985), Fishburn & Sloane 1988).

By adjoining the vector $(\tfrac{1}{4}^9)$ to $(D_3 \otimes D_3)^{+2048}$ we obtain the lattice

$$(D_3 \otimes D_3)^{+4096} = (D_3 \otimes D_3)^* = D_3^* \otimes D_3^* \cong (A_3 \otimes A_3)^*.$$

Its 32 minimal vectors have the shape $((\pm \tfrac{1}{4})^9)$, where the minus signs correspond to a codeword of C^*; for example

$-\tfrac{1}{4}$	$\tfrac{1}{4}$	$-\tfrac{1}{4}$
$\tfrac{1}{4}$	$-\tfrac{1}{4}$	$\tfrac{1}{4}$
$\tfrac{1}{4}$	$-\tfrac{1}{4}$	$\tfrac{1}{4}$

The lattice $A_3^{+4} \otimes A_3 = A_3^* \otimes A_3 \cong D_3^* \otimes D_3$ has 48 minimal vectors, which are the tensor products $a \otimes b$ of minimal vectors $a \in D_3^*$, $b \in D_3$; for example

$$\begin{array}{|ccc|} \tfrac{1}{2} & \tfrac{1}{2} & 0 \\ \tfrac{1}{2} & \tfrac{1}{2} & 0 \\ -\tfrac{1}{2} & -\tfrac{1}{2} & 0 \end{array} \; .$$

From §2, the group is $2 \times S_4 \times S_4$, of order 1152.

Lattices related to duads and synthemes. The exceptional lattices $Q_9(5)$, $Q_9(5)^{+5}$ have a natural description in terms of the 'duads' and 'synthemes' of S_6 (cf. Sylvester 1844). The situation is parallel to that for the lattices $Q_8(3)^{+r}$. Again there are really two dual families ($\{Q_9(5), Q_9(5)^{+5}\}$ and $\{Q_9'(5), Q_9'(5)^{+5}\}$), and two of the lattices are generated by the differences of the other two ($Q_9(5)$ by the differences of $Q_9(5)^{+5}$ and $Q_9'(5)$ by the differences of $Q_9'(5)^{+5}$). However, we obtain just two isomorphism classes of lattices, because the outer automorphism of S_6 implies that $Q_9(5) \cong Q_9'(5)$ and $Q_9(5)^{+5} \cong Q_9'(5)^{+5}$.

For these lattices we use ten coordinates adding to zero. The ten coordinates are labelled with the symbols $abc|def$, $abd|cef$, ..., $aef|bcd$, corresponding to the partitions of six letters $\{a, b, c, d, e, f\}$ into two triples. (To save space we shorten the labels to abc, ..., aef.) For each *duad* or pair of distinct letters xy, there is a *duad vector* V_{xy}, whose abc coordinate is $\frac{3}{5}$ if the duad lies inside either abc or def, and is $-\frac{2}{5}$ otherwise. There are $\binom{6}{2} = 15$ duad vectors. For example,

	abc	abd	abe	abf	acd	ace	acf	ade	adf	aef
$V_{ab} =$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$-\frac{2}{5}$	$-\frac{2}{5}$	$-\frac{2}{5}$	$-\frac{2}{5}$	$-\frac{2}{5}$	$-\frac{2}{5}$.

The inner products of the duad vectors are given by

$$V_{uv} \cdot V_{xy} = \begin{cases} \frac{12}{5} & \text{if } |\{u, v\} \cap \{x, y\}| = 2 \\ -\frac{3}{5} & \text{if } |\{u, v\} \cap \{x, y\}| = 1 \\ \frac{2}{5} & \text{if } |\{u, v\} \cap \{x, y\}| = 0. \end{cases}$$

The lattice $Q_9(5)^{+5}$ is generated by these fifteen duad vectors; they and their negatives are the minimal vectors of $Q_9(5)^{+5}$. The group of all four of these lattices is $2 \times S_6$, of order 1440.

The lattice $Q_9(5)$ is generated by the differences of the duad vectors. The minimal norm is then 4, and there are 90 minimal vectors. In fact for each pair of letters, for example ef, there are six minimal vectors, for example

	abc	abd	abe	abf	acd	ace	acf	ade	adf	aef
$V_{ab} - V_{cd} =$	1	1	0	0	−1	0	0	0	0	−1
$V_{ac} - V_{bd} =$	1	−1	0	0	1	0	0	0	0	−1
$V_{ad} - V_{bc} =$	−1	1	0	0	1	0	0	0	0	−1

and their negatives. These are supported on the four coordinates $ghi|jkl$ for which the pair ef lies inside either triple.

Similarly, for each *syntheme* or partition of the six letters into three pairs

$uv.wx.yz$, there is a *syntheme vector* $S_{uv.wx.yz}$, whose abc coordinate is $\frac{3}{5}$ if a, b, c lie in distinct pairs uv, wx, yz, and is $-\frac{2}{5}$ otherwise. There are $5\cdot 3\cdot 1 = 15$ syntheme vectors. For example

	abc	abd	abe	abf	acd	ace	acf	ade	adf	aef
$S_{ab.cd.ef} =$	$-\frac{2}{5}$	$-\frac{2}{5}$	$-\frac{2}{5}$	$-\frac{2}{5}$	$-\frac{2}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$-\frac{2}{5}$.

The lattice $Q_9'(5)^{+5}$ is generated by these fifteen syntheme vectors; the outer automorphism of S_6 shows that $Q_9'(5)^{+5}$ and $Q_9(5)^{+5}$ are isomorphic (cf. SLG, chapter 10, Section 1.3); and (up to a scale factor) $Q_9'(5)^{+5}$ is dual to $Q_9(5)$.

The lattice $Q_9'(5)$ is generated by the differences of the syntheme vectors. This is isomorphic to $Q_9(5)$ and (again up to a scale factor) is dual to $Q_9(5)^{+5}$. In fact the vector

$$\Delta_{gh.ij} = S_{gi.hj.kl} - S_{gj.hi.kl}$$

has inner product ± 2 with V_{gi}, V_{gj}, V_{hi}, V_{hj} and inner product 0 with the other V_{xy}. Thus $Q_9(5)^{+r}$ and $Q_9'(5)^{+5/r}$ are 4-duals in the terminology of §2.

9. Dimension 13

Lattices related to $L_2(25)$. The exceptional lattices $Q_{13}(2)^{+r}$, $Q_{13}'(2)^{+r}$ ($r = 1, 2$) all have the same group G, isomorphic to $2\times L_2(25).2$, and of order 31200, where $L_2(25).2 = P\Sigma L_2(25)$ is the extension of $L_2(25)$ by the field automorphism $\sigma: x \to x^5$ of $\mathbb{F}_{25}$. We take the field $\mathbb{F}_{25}$ to consist of the elements $\{a+b\theta : a, b\in\mathbb{F}_5\}$, where $\theta^2 = -3$, and describe these lattices using 26 coordinates $\{e_\infty, e_{a+b\theta}: a, b\in \mathbb{F}_5\}$, arranged in an array as shown in figure $3a$.

The group G is generated by negation and the maps $\alpha = \alpha_1$, $\alpha' = \alpha_\theta$, $\gamma = \gamma_{-1}$ and δ, where

$$\alpha_t : e_r \to e_{t+r},$$
$$\beta_t : e_r \to e_{t.r},$$
$$\gamma_t : e_r \to \pm e_{t/r},$$
$$\delta : e_r \to e_{r^5},$$

for $t\in\mathbb{F}_{25}$. The sign in $\gamma: e_r \to \pm e_{-1/r}$ is plus if r is ∞ or a square in $\mathbb{F}_{25}$, minus otherwise (this also determines the signs in γ_t). The actions of γ and δ are displayed in figures $3b$, c; a small circle indicates that the coordinate should be negated. Note that α represents a cyclic shift to the right in figure $3a$, and α' a shift upwards.

The lattice $Q_{13}(2)^{+2}$ is generated by the particular vector u_∞ shown in figure $3d$ and its images under the group G. We set $u_0 = (u_\infty)^\gamma$ (shown in figure $3e$), and $u_t = (u_0)^{\alpha_t}$ for $t\in\mathbb{F}_{25}$. Then the 52 vectors $\pm u_\infty$, $\pm u_t$, of norm $\frac{5}{2}$, are the minimal vectors of $Q_{13}(2)^{+2}$. It may be verified that these vectors lie in a 13-dimensional space. For example the vectors u_t for $t\in\{0, \pm\theta, \pm 2\theta, 1, 1\pm\theta, 1+2\theta, 2, 2\pm\theta, 2+2\theta\}$ form a basis for $Q_{13}(2)^{+2}$. The + entries in u_0 occur at ∞ and on three straight lines (at the non-zero points $a+b\theta$ for which b/a is 0 or ± 1).

The lattice $Q_{13}(2)$ is generated by the differences of the minimal vectors of

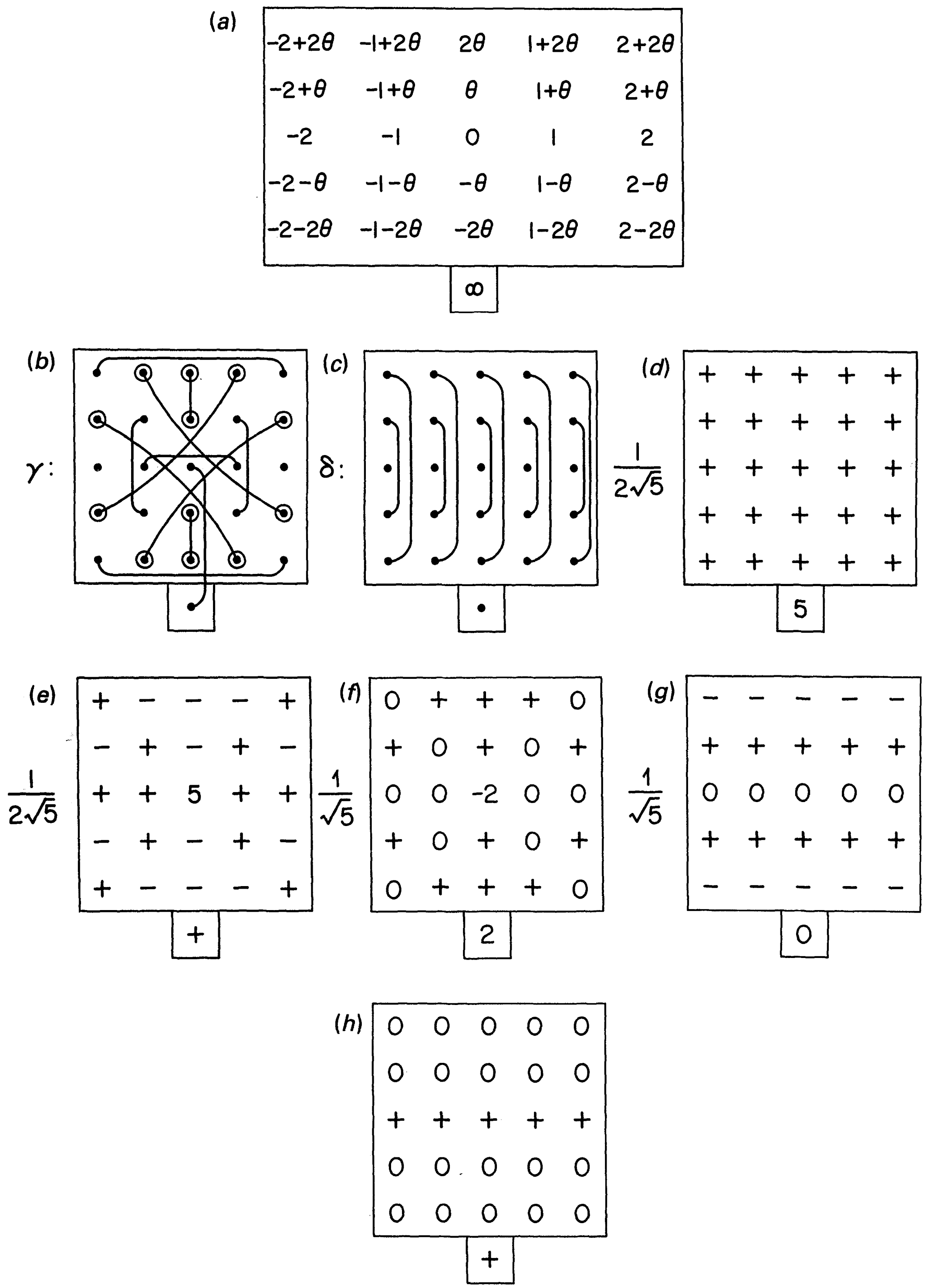

FIGURE 3. (*a*) Labels for 26 coordinates for the lattices $Q_{13}(2)^{+r}$, $Q_{13}'(2)^{+r}$. (*b*) (*c*) Action of group elements γ, δ. (*d*)–(*h*) The vectors u_∞, u_0, w, y, z.

$Q_{13}(2)^{+2}$, and has two orbits of minimal vectors. There are $26 \cdot 25 = 650$ minimal vectors of shape $5^{-\frac{1}{2}}(\pm 2^2, \pm 1^{12}, 0^{12})$, and norm 4, such as the vector $w = 2^{-\frac{1}{2}}(u_\infty - u_0)$ shown in figure 3*f*. There are also 130 minimal vectors of shape $5^{-\frac{1}{2}}(\pm 1^{20}, 0^6)$, namely the vector y shown in figure 3*g* and its images under $G . y$ is stabilized by the subgroup $H \cong PGL_2(5) . 2$ of order 240 generated by α, β_2, γ and δ, and therefore has $31200/240 = 130$ images; the total number of minimal vectors in $Q_{13}(2)$ is $650 + 130 = 780$.

The lattice $Q_{13}'(2)$ is spanned by the images under G of the vector z shown in figure 3*h*, which is its typical minimal vector. Because z is also stabilized by H, $G_{13}'(2)$ has just 130 minimal vectors.

The fourth lattice $Q_{13}'(2)^{+2}$ is then obtained by adjoining $\sqrt{5}\, u_\infty$ to the previous lattice, and has the same minimal vectors as that lattice. The lattices $Q_{13}(2)^{+r}$ and $Q_{13}'(2)^{2/r}$ are 5-duals.

Lattices related to the projective plane of order 3. The exceptional lattices $Q_{13}(4)^{+r}$ and $Q_{13}'(4)^{+r}$ $(r = 1, 2, 4)$ share a group G isomorphic to $2 \times L_3(3) . 2$, of order 22464. These lattices are related both to the projective plane Π of order 3 and to the ternary Hamming code C of length 13.

The points of Π are called $P_0, \ldots, P_{12}$, and correspond to our coordinate vectors (multiplied by $\sqrt{3}$), for which we use the same names:

$$\begin{array}{rccccccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ P_0 = & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_1 = & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_2 = & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & \cdots & & & & & & \end{array} \tag{11}$$

The lines of Π are called $L_0, \ldots, L_{12}$ and correspond to a second orthogonal set of vectors:

$$\begin{array}{rccccccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ L_0 = 1/\sqrt{3} & (0 & 0 & + & 0 & + & + & + & - & - & 0 & + & - & +) \\ L_1 = 1/\sqrt{3} & (0 & + & 0 & + & + & + & - & - & 0 & + & - & + & 0) \\ L_2 = 1/\sqrt{3} & (+ & 0 & + & + & + & - & - & 0 & + & - & + & 0 & 0) \\ & & & & & & & \cdots & & & & & & \end{array} \tag{12}$$

(L_{i+1} is a cyclic permutation to the left of L_i.) A point P_i and line L_j are incident in the plane (see figure 4) just when the corresponding vectors are orthogonal (this happens if and only if $i + j \equiv 0, 1, 3$ or $9 \pmod{13}$).

The group G is generated by negation together with $\alpha, \beta, \gamma, \delta$, where α, β, γ (generators for the subgroup $L_3(3)$) are monomial matrices with the following actions:

$$\alpha = (P_0, P_1, P_2, \ldots, P_{12})\,(L_0, L_{12}, L_{11}, \ldots, L_1),$$

$$\begin{aligned} \beta = {} & (P_1, P_3, P_9)\,(P_2, P_6, P_5)\,(P_4, P_{12}, P_{10})\,(P_7, P_8, P_{11}) \cdot \\ & (L_1, L_3, L_9)\,(L_2, L_6, L_5)\,(L_4, L_{12}, L_{10})\,(L_7, L_8, L_{11}), \end{aligned}$$

$$\begin{aligned} \gamma = {} & (P_5, -P_5)\,(P_2, P_{11})\,(P_4, P_7)\,(P_6, P_8)\,(P_{10}, -P_{12}) \cdot \\ & (L_0, -L_0)\,(L_1, L_3)\,(L_2, L_{12})\,(L_5, -L_7)\,(L_6, L_{10}), \end{aligned}$$

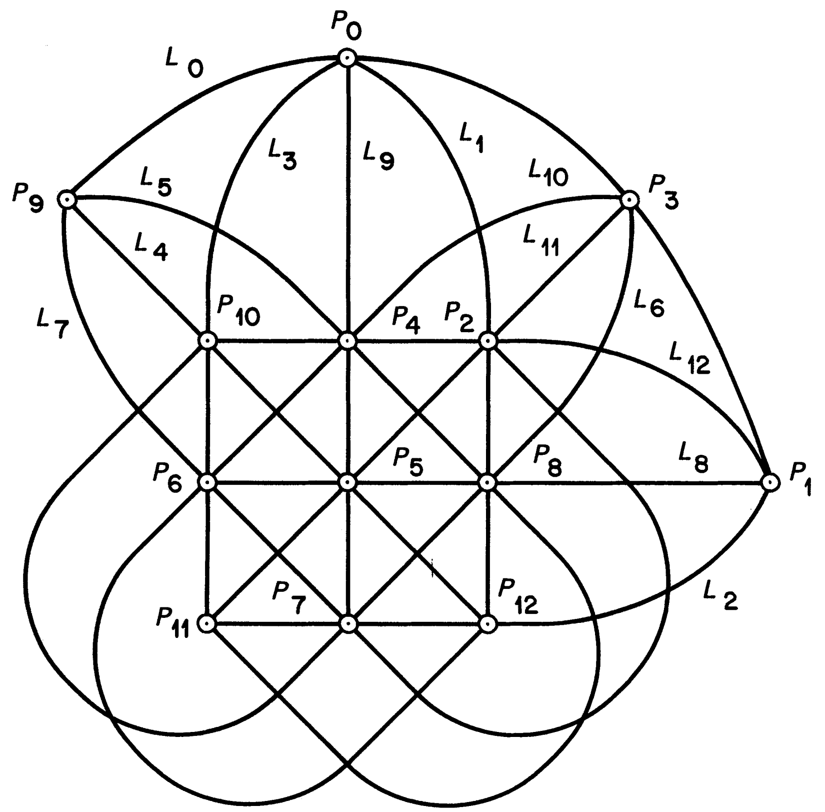

FIGURE 4. Projective plane Π of order 3. A point P_i is incident with a line L_j if and only if $i+j \equiv 0, 1, 3$ or $9 \pmod{13}$.

and δ is the orthogonal matrix

$$\frac{1}{3}\begin{bmatrix}
0 & 0 & + & 0 & + & + & + & - & - & 0 & + & - & + \\
0 & + & 0 & + & + & + & - & - & 0 & + & - & + & 0 \\
+ & 0 & + & + & + & - & - & 0 & + & - & + & 0 & 0 \\
0 & + & + & + & - & - & 0 & + & - & + & 0 & 0 & + \\
+ & + & + & - & - & 0 & + & - & + & 0 & 0 & + & 0 \\
+ & + & - & - & 0 & + & - & + & 0 & 0 & + & 0 & + \\
+ & - & - & 0 & + & - & + & 0 & 0 & + & 0 & + & + \\
- & - & 0 & + & - & + & 0 & 0 & + & 0 & + & + & + \\
- & 0 & + & - & + & 0 & 0 & + & 0 & + & + & + & - \\
0 & + & - & + & 0 & 0 & + & 0 & + & + & + & - & - \\
+ & - & + & 0 & 0 & + & 0 & + & + & + & - & - & 0 \\
- & + & 0 & 0 & + & 0 & + & + & + & - & - & 0 & + \\
+ & 0 & 0 & + & 0 & + & + & + & - & - & 0 & + & -
\end{bmatrix}. \tag{13}$$

Note that α sends P_i to P_{i+1} and L_i to L_{i-1} (with subscripts modulo 13), β sends P_i to P_{3i} and L_i to L_{3i}, γ is a signed version of the involutory homology that fixes L_0 and P_5, and δ is the polarity that interchanges each P_i with L_i.

The 52 norm 3 vectors $\pm P_i$, $\pm L_i$ are the minimal vectors of $Q_{13}'(4)^{+2}$, and generate that lattice.

To each incidence $I_{i,j}$ between a point P_i and a line L_j in Π there corresponds a pair $\pm I_{i,j}$ of minimal vectors of $Q_{13}(4)^{+2}$. For example

$$\begin{array}{rccccccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ I_{0,0} = & 0 & + & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot\ , \\ I_{1,0} = & + & 0 & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & - & \cdot & \cdot & \cdot\ , \\ I_{3,0} = & + & - & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot\ , \\ I_{9,0} = & + & + & \cdot & - & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot\ , \end{array} \qquad (14)$$

when the dots indicate zero entries. These $2 \times 13 \times 4 = 104$ norm 3 vectors generate the lattice $Q_{13}(4)^{+2}$.

The lattices $Q_{13}(4)^{+2}$ and $Q_{13}'(4)^{+2}$ are 3-duals. In fact, suppose P_i and L_j are incident, let P_i meet lines $L_j, L_{j_1}, L_{j_2}, L_{j_3}$, and let L_j contain points $P_i, P_{i_1}, P_{i_2}, P_{i_3}$. Then the vector $I_{i,j}$ has inner product $\pm\sqrt{3}$ with $P_{i_1}, P_{i_2}, P_{i_3}, L_{j_1}, L_{j_2}, L_{j_3}$, and is orthogonal to the other P_k and L_k.

Alternatively, we may define the [13, 10, 3] perfect Hamming code C over $\mathbb{F}_3$ to be the cyclic code generated by the vector $I_{9,0}$ in (14). The lattice $Q_{13}(4)^{+2}$ is equivalently obtained by applying construction A_3 (SLG, chapter 5) to C. The vectors $\pm I_{i,j}$ are the 104 minimal weight codewords in C. The automorphism group of C is the monomial portion $2 \times L_3(3)$ of G, generated by negation, α, β and γ.

The dual lattice $Q_{13}'(4)^{+2}$ is similarly obtained by applying construction A to the dual code $C^{\perp}$, a [13, 3, 9] ternary code whose 26 non-zero codewords are the vectors $\pm L_i$ (with the factor $1/\sqrt{3}$ omitted).

The lattice $Q_{13}(4)$ is generated by the differences of the minimal vectors of $Q_{13}(4)^{+2}$, or equivalently is obtained by applying construction B_3 (SLG, chapter 5) to C. Two minimal vectors, of norm 4, are supported on each quadrilateral of Π, for a total of $2 \cdot 13 \cdot 12 \cdot 9 \cdot 4/4! = 468$ minimal vectors.

The lattice $Q_{13}(4)^{+4}$ is obtained by adjoining the vector $(\frac{1}{2}^{13})$ to $Q_{13}(4)^{+2}$, and has the same 52 minimal vectors.

Similarly $Q_{13}'(4)$ is generated by the differences of the minimal vectors of $Q_{13}'(4)^{+2}$, or equivalently is obtained by applying construction B_3 to $C^{\perp}$. The minimal vectors are of the form $\pm P_i \pm L_j$, where P_i and L_j are not incident. There are $2 \times 13 \times 9 = 234$ such vectors, of norm 4.

Finally $Q_{13}'(4)^{+4}$ is obtained by adjoining the vector $((\sqrt{3}/2)^{13})$ to $Q_{13}'(4)^{+2}$. The minimal vectors are unchanged.

We note that $Q_{13}(4)^{+r}$ and $Q_{13}'(4)^{+4/r}$ are 3-duals, and $\sqrt{3}Q_{13}(4)^{+r} \subseteq Q_{13}'(4)^{+r}$, $\sqrt{3}Q_{13}'(4)^{+r} \subseteq Q_{13}(4)^{+r}$ for $r = 1, 2, 4$.

10. Dimension 17

Lattices associated with $L_2(16)$. The exceptional lattices $Q_{17}(6)^{+r}$, $Q_{17}'(6)^{+r}$ ($r = 1, 2, 3$) share the same group G, isomorphic to $2 \times L_2(16).4$ and of order 32640, where $L_2(16).4$ is the extension of $L_2(16)$ by the field automorphism of order 4. Although G is real, we use 17-dimensional complex coordinates involving $\omega = e^{\frac{2}{3}\pi i}$ to represent these lattices (compare the treatment of $Q_8(3)^{+r}$ in §7).

We construct the field $\mathbb{F}_{16}$ by adjoining to $\mathbb{F}_2$ an element ϵ satisfying $\epsilon^4+\epsilon^3+\epsilon^2+\epsilon+1=0$ (a non-primitive polynomial!). If we define $\omega=\epsilon+\epsilon^4$ then $\omega^2=\bar{\omega}=\epsilon^2+\epsilon^3$, $\omega^2+\omega+1=0$, and $\epsilon\omega$ is a primitive element of $\mathbb{F}_{16}=\{0, \epsilon^i\omega^j: 0\leqslant i\leqslant 4, 0\leqslant j\leqslant 2\}$: We shall describe these lattices using 17 coordinates $\{e_\infty, e_t: t\in\mathbb{F}_{16}\}$, arranged in an array as shown in figure 5*a*.

The group G is generated by negation and the maps α_t, β_t ($t\in\mathbb{F}_{16}$), γ and δ, where

$$\alpha_t: e_r\rightarrow e_{r+t},$$

$$\beta_t: e_\infty\rightarrow\overline{\chi(t)}\,e_\infty,\ e_r\rightarrow\chi(t)\,e_{rt}\quad (r\neq\infty),$$

$$\gamma: e_r\rightarrow\chi(r)\,e_{1/r},$$

$$\delta: e_r\rightarrow e_{r^2},$$

$\chi: \mathbb{F}_{16}\cup\{\infty\}\rightarrow\mathbb{C}$ is given by

$$\chi(\infty)=\chi(0)=1,\ \chi(\epsilon^i\omega^j)=\omega^j,$$

except that when applying δ we must also replace the coefficients of all e_k by their complex conjugates. The actions of α_ϵ, β_ω, β_ϵ, γ and δ are displayed in figure 5(*b*)–(*f*). The element $\rho=\alpha_\epsilon\gamma$ (first α_ϵ then γ) has order 17 and is displayed in figure 5*g*.

The following paragraphs briefly describe the eight lattices $Q_{17}(6)^{+r}$, $Q_{17}'(6)^{+r}$. Because the relations between them are more complicated than those for the other families in this paper, we shall give a *generating vector* z for each lattice, with the property that its 17 images $\{\rho^{ij}(z): 0\leqslant i\leqslant 16\}$ (for some j) have (up to a scale factor) the Gram matrix given by Plesken (1985). We also give one or two representative minimal vectors (shown in figure 6) for each lattice, and we specify, for such a minimal vector u, the subgroup $H(u)$ of $L_2(16)$ that fixes u up to sign. If $H(u)$ has order h there are $2|L_2(16)|/h=2\cdot 15\cdot 16\cdot 17/h$ minimal vectors in the orbit of u under $2\times L_2(16)$. The properties of these eight lattices are summarized in table 1 and figure 1.

$Q_{17}(6)$: two orbits of minimal vectors under $L_2(16)$ with representatives u_1, u_1', (see figure 6); these orbits fuse under δ. The stabilizer $H(u_1)=\langle\alpha_t: t$ in top half of array$\rangle$ has order $h=8$; there are $1020+1020=2040$ minimal vectors. Generator $z=u_1$.

$Q_{17}(6)^{+2}$: one orbit; u_2; $H(u_2)=\langle\beta_\epsilon,\gamma\rangle$; $h=10$; 816 minimal vectors; $z=u_2$.

$Q_{17}(6)^{+3}$: one orbit; u_3; $H(u_3)\cong H(u_1)$; $h=8$; 510 minimal vectors; $z=u_3$.

$Q_{17}(6)^{+6}$: one orbit; u_6; $H(u_6)=\langle\alpha_\epsilon,\rho\rangle$; $h=34$; 240 minimal vectors; $z=(u_6)^{\alpha_1}$.

$Q_{17}'(6)$: two orbits of minimal vectors under both $L_2(16)$ and G, with representatives v_1, v_1'; $H(v_1)=\langle\alpha_t: t\in\mathbb{F}_{16}\rangle$; $h=16$; $H(v_1')=\langle\beta_\epsilon,\gamma\rangle$; $h=10$; $510+816=1326$ minimal vectors; $z=v_1$.

$Q_{17}'(6)^{+2}$: one orbit; v_2; $H(v_2)=\langle\alpha_1,\beta_\omega,\gamma\rangle\cong \mathsf{A}_5$; $h=60$; 136 minimal vectors; $z=v_2$.

$Q_{17}'(6)^{+3}$: one orbit; v_3; $H(v_3)=\langle\alpha_t: t\in F_{16},\beta_\epsilon\rangle$; $h=80$; 51 minimal vectors; $z=(v_3)^{\beta_\omega}$.

$Q_{17}'(6)^{+6}$: same minimal vectors as previous lattice; $z=w_3$.

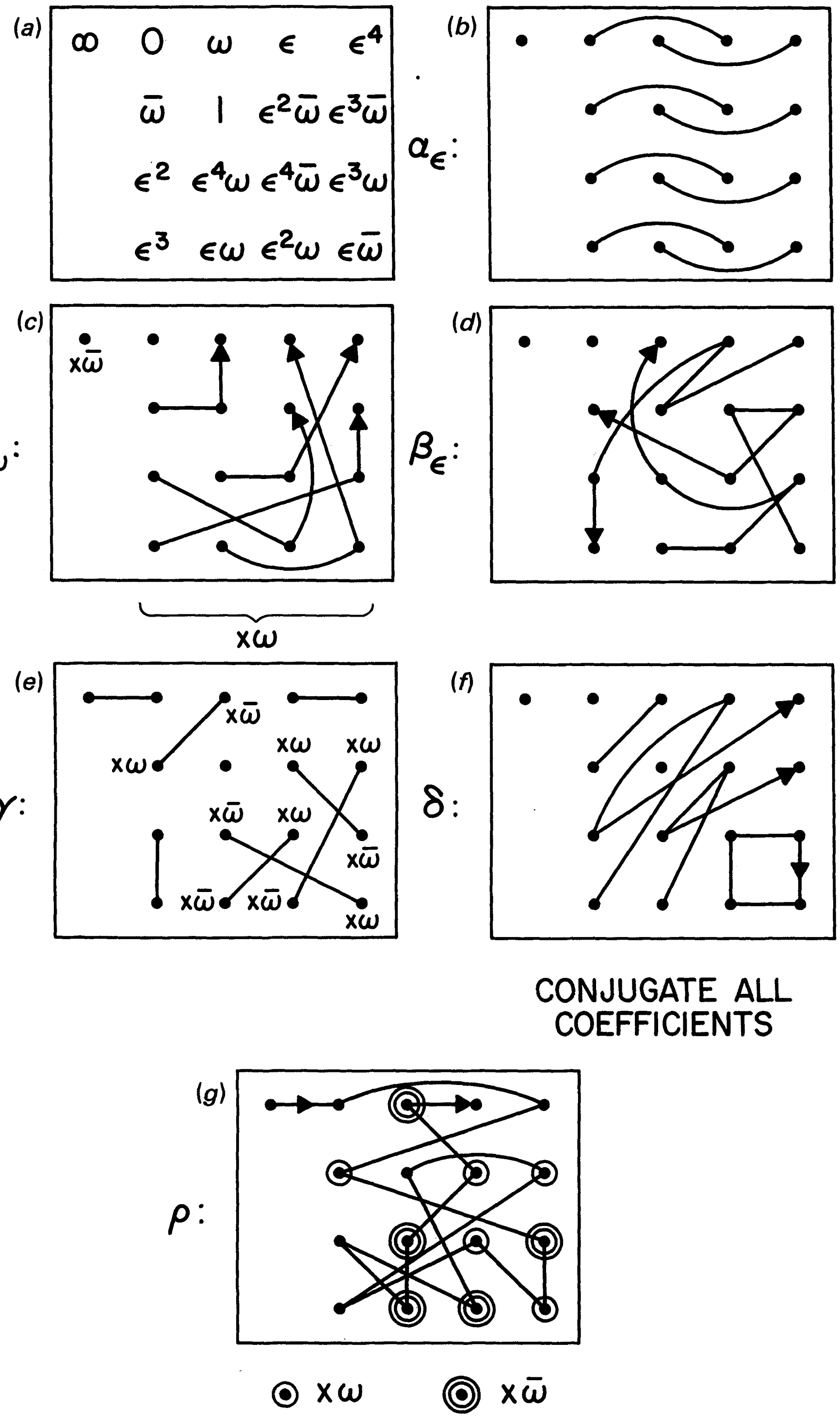

FIGURE 5. (*a*) Labels for 17 coordinates for the lattices $Q_{17}(6)^{+r}$, $Q_{17}'(6)^{+r}$. (*b*)–(*g*) Action of group elements α_ϵ, β_ω, β_ϵ, γ, δ, ρ. After the permutations in (*c*), (*e*), (*f*), (*g*) the indicated actions are to be performed: thus in (*c*) the leading coordinate is to be multiplied by $\bar{\omega}$ and the other 16 coordinates by ω, while in (*f*) all coordinates must be conjugated.

$$u_1:\ \frac{1}{\sqrt{2}}\ \begin{array}{|ccccc|}\hline 0 & \bar\omega & \bar\omega & \bar\omega & \bar\omega \\ & -\bar\omega & -\bar\omega & -\bar\omega & -\bar\omega \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ \hline\end{array} \qquad u_1':\ \frac{1}{\sqrt{2}}\ \begin{array}{|ccccc|}\hline 0 & \omega & -\omega & 0 & 0 \\ & \omega & -\omega & 0 & 0 \\ & \omega & -\omega & 0 & 0 \\ & \omega & -\omega & 0 & 0 \\ \hline\end{array}$$

$$u_2:\ \frac{1}{\sqrt{2}}\ \begin{array}{|ccccc|}\hline -1 & -1 & 0 & 1 & 1 \\ & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ \hline\end{array} \qquad u_3:\ \frac{1}{\theta\sqrt{2}}\ \begin{array}{|ccccc|}\hline -2\omega & \omega & \omega & \omega & \omega \\ & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & \omega & \omega & \omega & \omega \\ \hline\end{array}$$

$$u_6:\ \frac{1}{\theta\sqrt{2}}\ \begin{array}{|ccccc|}\hline 1 & 1 & 1 & 1 & 1 \\ & \omega & 1 & \omega & 1 \\ & \bar\omega & 1 & \bar\omega & 1 \\ & \bar\omega & \omega & \bar\omega & \omega \\ \hline\end{array} \qquad v_1:\ \frac{\omega}{2}\ \begin{array}{|ccccc|}\hline 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & -1 & -1 & -1 & -1 \\ & -1 & -1 & -1 & -1 \\ \hline\end{array}$$

$$v_1':\ \frac{1}{2}\ \begin{array}{|ccccc|}\hline -\theta & \theta & -\omega & 0 & 0 \\ & \bar\omega & 0 & \bar\omega & \bar\omega \\ & 0 & -\omega & \bar\omega & -\omega \\ & 0 & -\omega & -\omega & \bar\omega \\ \hline\end{array} \qquad v_2:\ \frac{1}{2}\ \begin{array}{|ccccc|}\hline 0 & 0 & 0 & \omega & \omega \\ & 0 & 0 & \omega & \omega \\ & \bar\omega & \bar\omega & 1 & 1 \\ & \bar\omega & \bar\omega & 1 & 1 \\ \hline\end{array}$$

$$v_3:\ \frac{1}{2\theta}\ \begin{array}{|ccccc|}\hline 4 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ \hline\end{array} \qquad w_6:\ \frac{1}{2\theta}\ \begin{array}{|ccccc|}\hline -1 & -\bar\omega & 2\bar\omega & 2\bar\omega & 2\omega \\ & -1 & -\omega & -\omega-3 & -\bar\omega \\ & -1 & -\bar\omega & 2\omega & -1 \\ & 1-2\omega & -\bar\omega & -1 & 2 \\ \hline\end{array}$$

FIGURE 6. Representative vectors occurring in the lattices $Q_{17}(6)^{+r}$, $Q_{17}'(6)^{+r}$.

We remark that $\sqrt{2}u_6 = v_6$ (say) $\in Q_{17}'(6)^{+6}$, and, for $r = 2, 3, 6$,

$$Q_{17}(6)^{+r} = \langle Q_{17}(6), 6u_6/r \rangle.$$

$$Q_{17}'(6)^{+r} = \langle Q_{17}'(6), 6v_6/r \rangle.$$

Also $Q_{17}(6)^{+r}$ and $Q_{17}'(6)^{+6/r}$ are 2-duals, and

$$\sqrt{2}Q_{17}(6)^{+r} \subseteq Q_{17}'(6)^{+r},$$

$$\sqrt{2}Q_{17}'(6)^{+r} \subseteq Q_{17}(6)^{+r},$$

for $r = 1, 2, 3, 6$.

Lattices related to the quadratic residue codes of length 17. The next two families of exceptional lattices, $Q_{17}(8)^{+r}$ $(r = 1, 2, 4, 8)$ and $Q_{17}(9)^{+r}$ $(r = 1, 3, 9)$ are connected with 17- and 18-dimensional representations of $L_2(17)$. In contrast to 23 dimensions, however, when all the exceptional lattices can be obtained from the Leech lattice, here there is no single 18-dimensional lattice which yields all the exceptional lattices, and it seems best to treat the two families separately.

The exceptional lattices $Q_{17}(8)^{+r}$ $(r = 1, 2, 4, 8)$ are related to the [17, 8, 6] binary quadratic residue code C and its dual the [17, 9, 5] code $C^{\perp} = C \cup \{1^{17} + C\}$. The weight distributions of these codes may be found in (Berlekamp 1968, p. 432).

The lattices $Q_{17}(8)^{+2}$, $Q_{17}(8)^{+4}$, $Q_{17}(8)$ are respectively obtained by applying construction A to the code C, construction A to $C^{\perp}$, and construction B to C, while $Q_{17}(8)^{+8} = \langle Q_{17}(8)^{+4}, (\frac{1}{2}^{17}) \rangle$. The minimal vectors in $Q_{17}(8)$ have norm 6 and shape $(\pm 1^6, 0^{11})$, and are obtained from the 68 minimal weight words in C by changing any odd number of 1s to -1s. There are $2^5 \times 68 = 2176$ minimal vectors. In each of the other three lattices there are just 34 minimal vectors, of shape $(\pm 2^1, 0^{16})$.

The automorphism group of the [18, 9, 6] extended quadratic residue code is $L_2(17)$, (MacWilliams & Sloane 1977, p. 492), so Aut(C) has order $8 \cdot 17 = 136$. Then $Q_{17}(8)^{+2}$ and $Q_{17}(8)^{+4}$ have group $2^{17}.\text{Aut}(C)$, while $Q_{17}(8)$, $Q_{17}(8)^{+8}$ have group $2^9.\text{Aut}(C)$. (Compare the lattices $Q_{23}(8)^{+r}$ in the following section, which have a similar structure.)

The A_{17}*-type lattices related to* $L_2(17)$. The second set of exceptional lattices related to $L_2(17)$, the lattices $Q_{17}(9)^{+r}$ $(r = 1, 3, 9)$, have structure similar to A_{17}. (In particular, $Q_{17}(9)^{+r} \subseteq A_{17}^{+r}$.)

The lattices $Q_{17}(9)^{+r}$ share a group G isomorphic to $2 \times L_2(17)$ and of order 4896. We use 18 coordinates adding to 0, labelled $\{\infty, 0, 1, \ldots, 16\}$. Then G is generated by negation, $\alpha: x \to x+1$, $\beta: x \to 2x$ and $\gamma: x \to -1/x$ (mod 17).

The norm 4 vector

	∞	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$w =$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$

is fixed or negated by $\beta^2, \gamma, \delta: x \to -x$ (mod 17) and $\alpha\gamma\alpha^{-8}\beta$, which generate a subgroup isomorphic to S_4. Thus w has 204 images under G, and these vectors are the minimal vectors of $Q_{17}(9)^{+3}$ and generate that lattice. Let us call a minimal vector *positive* if its coordinates are congruent to $\frac{1}{3}$ modulo 1. Then the differences of the positive minimal vectors of $Q_{17}(9)^{+3}$ generate $Q_{17}(9)$.

The minimal vectors of $Q_{17}(9)$ have shape $(+1^3, -1^3, 0^{12})$, supported on one of the 102 hexads where a minimal vector of $Q_{17}(9)^{+3}$ is $\pm\frac{2}{3}$. There are $102\binom{6}{3} = 2040$ minimal vectors of $Q_{17}(9)$, falling into two orbits under G, of sizes 816 and 1224.

By adjoining any one of the 36 norm 34/9 vectors $\pm((\frac{1}{9})^{17}, -(\frac{17}{9})^1)$ to $Q_{17}(9)^{+3}$ we obtain $Q_{17}(9)^{+9}$, whose minimal vectors are these 36 vectors.

11. Dimension 23

Lattices related to the Leech lattice. The exceptional 23-dimensional lattices $Q_{23}(d)^{+r}$ ($d = 4, 6, 8, 12, r|d$) are all closely related to the Leech lattice Λ_{24}. We use the standard coordinates for Λ_{24} and suppose some familiarity with that lattice (see SLG or ATLAS). For $d = 4, 6, 8$ or 12 we take a vector $v = v_d \in \Lambda_{24}$ of norm d. If $d = 4, 6$ or 8 all choices for v_d are equivalent, and for $d = 12$ we choose $v_{12} = (2^{24})$. Then $Q_{23}(d)^{+r}$ for $r|d$ consists of the projections onto $v_d^{\perp}$ of all Leech lattice vectors whose inner products with v_d are divisible by $s = d/r$.

For $d = 4$ we obtain three lattices. $Q_{23}(4)$ is $v_4^{\perp}$, i.e. consists of the Leech lattice vectors that are orthogonal to a minimal vector. This is the laminated lattice Λ_{23} of Leech & Sloane (1971), Conway & Sloane (1982), and gives the densest sphere-packing known in 23 dimensions. $Q_{13}(4)^{+2}$ is the projection onto $v_4^{\perp}$ of the vectors w having even inner product with it. This is an integral unimodular lattice which we called the shorter Leech lattice O_{23} in SLG. Its 4600 minimal vectors (of norm 3) are obtained from the minimal vectors w of Λ_{24} at angle 60° with v_4. (Later in this section we enumerate the minimal vectors in all these lattices.) Finally $Q_{23}(4)^{+4} = \Lambda_{23}^*$ is the projection of all of Λ_{24} onto $v_4^{\perp}$, and gives the thinnest known covering of 23-dimensional space by spheres. The group of these three lattices is isomorphic to the group $2 \times Co_2$ of automorphisms of the Leech lattice that fix or negate v_4.

Similarly the four lattices $Q_{23}(6)^{+r}$ ($r = 1, 2, 3, 6$) have group isomorphic to $2 \times Co_3$, fixing or negating v_6. The bottom lattice $Q_{23}(6)$ consists of the Leech lattice vectors orthogonal to v_6, and the top one is the projection of Λ_{24} onto $v_6^{\perp}$.

The 552 minimal vectors of $Q_{23}(6)^{+2}$ (which span that lattice) are especially interesting, since they lie on 276 equiangular lines (cf. Delsarte *et al.* 1975). They are also the minimal vectors of $Q_{23}(6)^{+6}$.

If we take $v_8 = (0^{23}, 8)$ we see that $Q_{23}(8)$ consists of the Leech lattice vectors whose last coordinate is 0. The other coordinates are therefore even. This is the lattice obtained by applying construction B to the [23, 11, 8] binary Golay code. Its automorphism group, of structure $2^{12}.M_{23}$, consists of permutations of M_{23} and sign changes on $\mathscr{C}$-sets.

The lattice $Q_{23}(8)^{+2}$ is obtained by adjoining the vector $(4, 0, 0, \ldots)$, the projection of $(4, 0, 0, \ldots; 4)$. This is the lattice obtained by applying construction A to the same code; its automorphism group $2^{23}.M_{23}$ contains *all* sign changes.

The lattice $Q_{23}(8)^{+4}$ has the same group; it is obtained by adjoining $(2^7, 0, 0, \ldots)$, the projection of $(2^7, 0, 0, \ldots; 2)$. Equivalently it is obtained by applying construction A to the [23, 12, 7] perfect Golay code.

Finally $Q_{23}(8)^{+8}$ is the projection of Λ_{24} onto $v_8^{\perp}$, and has the same group as $Q_{23}(8)$, to which it is dual.

The symmetries of Λ_{24} that fix or negate $v_{12} = (2, 2, \ldots, 2)$ form the group $2 \times M_{24}$, generated by negation and the permutations of the 24 coordinates that fix the [24, 12, 8] extended Golay code $\mathscr{C}$. This group is the automorphism group of the lattices $Q_{23}(12)^{+r}$ $(r = 1, 2, 3, 4, 6, 12)$.

The lattice $Q_{23}(12)$ is $v_{12}^{\perp}$, and its dual is the projection of Λ_{24} onto $v_{12}^{\perp}$. The 2×759 minimal vectors of $Q_{23}(12)^{+3}$, also of $Q_{23}(12)^{+6}$, are the projections of the octad vectors $(2^8, 0^{16})$. The 2×24 minimal vectors of $Q_{23}(12)^{+12}$ are the projections of the coordinate vectors $(\pm 8, 0^{23})$.

Minimal vectors of the exceptional 23-*dimensional lattices.* Our description of the lattices $Q_{23}(d)^{+r}$ in terms of the Leech lattice enables us to enumerate their minimal vectors. We note that the table on p. 181 of the ATLAS is useful in finding the number of vectors w of given norm that have a given inner product with a fixed vector v.

The minimal vectors of $Q_{23}(d)^{+r}$ are the projections $\bar{w}$ onto $v^{\perp}$ of certain vectors $w \in \Lambda_{24}$. We find them by supposing that w is as small as possible. Let $w \cdot w = m$, $w \cdot v = i$; also $v \cdot v = d$. Then because (possibly at the cost of negating w) we can replace w by $nv \pm w$, we may suppose that $0 \leqslant i \leqslant \frac{1}{2}d$. If $i < \frac{1}{2}d$, w is unique. But if $i = \frac{1}{2}d$, the two equally short vectors w and $w - v$ yield the same $\bar{w}$. We find

$$N(\bar{w}) = m - \frac{i^2}{d}, \qquad \frac{d}{r} \mid i,$$

and

$$i \leqslant \tfrac{1}{2}(d + m - 4),$$

because $N(v - w) \geqslant 4$ in the Leech lattice.

We discuss $Q_{23}(12)^{+4}$ to show the method. Our conditions yield $3 \mid i$, $i \leqslant \frac{1}{2}d = 6$, and we wish to minimize $N(\bar{w}) = m - \frac{1}{12}i^2$. We try $m = 4$ first. There is no minimal vector of Λ_{24} having inner product $i = 6$ with v, but there are minimal vectors with $i = 3$. These would give $N(\bar{w}) = 4 - 3^2/12 = 3\frac{1}{4}$. However, by using $m = 6$ we can do better. There are two orbits of norm 6 vectors with $i = 6$, namely

	shape	number
$\pm w =$	$(2^{12}, 0^{12})$	$2 \cdot 2576 = 5152$,
$\pm w =$	$(4, 2^{11}, -2, 0^{11})$	$2 \cdot 2576 \cdot 12^2 = 741\,888$.

The projections $\bar{w}$ of these vectors have norm $6 - 6^2/12 = 3$. Because vectors of norm $m > 6$ have projections of norm at least $m - 6^2/12 > 3$, the above $\bar{w}$ are the minimal vectors of $Q_{23}(12)^{+4}$. However, the number of them is only $\frac{1}{2}(5152 + 741\,888) = 373\,520$, because in this case $i = \frac{1}{2}d$. We notice that all these vectors have $i = 6$, and so also belong to the sublattice $Q_{23}(12)^{+2}$, which therefore has the same minimal vectors as $Q_{23}(12)^{+4}$.

Similarly we find that the minimal vectors of $Q_{23}(d)^{\pm r}$ are the projections $\bar{w}$ onto $v_d^{\perp}$ of all the Leech lattice vectors w that have norm m and inner product i with v_d, where m and i are given in table 2.

TABLE 2. MINIMAL VECTORS OF $Q_{23}(d)^{+r}$ ARE THE PROJECTIONS ONTO $v_d^{\perp}$ OF ALL $w \in \Lambda_{24}$ WITH $w \cdot w = m$, $w \cdot v_d = i$

$d = 4$	$r = 1$	2	4				
	$m = 4$	4	4				
	$i = 0$	2	2				
$d = 6$	$r = 1$	2	3	6			
	$m = 4$	4	4	4			
	$i = 0$	3	2	3			
$d = 8$	$r = 1$	2	4	8			
	$m = 4$	4	4	4			
	$i = 0$	4	4	4			
$d = 12$	$r = 1$	2	3	4	6	12	
	$m = 4$	6	4	6	4	4	
	$i = 0$	6	4	6	4	5	

In most of these cases the vectors $\bar{w}$ are in one orbit. The exceptions are:

lattice	shape of $\pm w$	number
$Q_{23}(8)$	$((\pm 4)^2 0^{21}; 0)$	$2^2\binom{23}{2} = 1012$
	$((\pm 2)^8 0^{15}; 0)$	$2^7 \cdot 506 = 64768$
$Q_{23}(12)$	$(4, -4, 0^{22})$	552
	$(2^4, -2^4, 0^{16})$	$759 \cdot 70 = 53130$
$Q_{23}(12)^{+r}$	$(2^{12}, 0^{12})$	$2 \cdot 2576 = 5152$
$(r = 2, 4)$	$(4, 2^{11}, -2, 0^{11})$	$2 \cdot 2576 \cdot 12^2 = 741888$

These are indicated by asterisks in table 1.

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