

# Low-dimensional lattices. VII

## Coordination sequences

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The coordination sequence  $\{S(n)\}$  of a lattice or net gives the number of nodes that are  $n$  bonds away from a given node.  $S(1)$  is the familiar coordination number. Extending the work of O’Keeffe and others, we give explicit formulae for the coordination sequences of the root lattices  $A_d$ ,  $D_d$ ,  $E_6$ ,  $E_7$ ,  $E_8$  and their duals. Proofs are given for many of the formulae and for the fact that, in every case,  $S(n)$  is a polynomial in  $n$ , although some of the individual formulae are conjectural. In the majority of cases, the set of nodes that are at most  $n$  bonds away from a given node form a polytopal cluster whose shape is the same as that of the contact polytope for the lattice. It is also shown that among all the Barlow packings in three dimensions the hexagonal close packing has the greatest coordination sequence, and the face-centred cubic lattice the smallest, as conjectured by O’Keeffe.

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### 1. Introduction

The *coordination sequence* of an infinite vertex-transitive graph  $\mathcal{G}$  is the sequence  $\{S(0), S(1), S(2), \dots\}$ , where  $S(n)$  is the number of vertices at distance  $n$  from some fixed vertex of  $\mathcal{G}$ . The partial sums  $G(n) = S(0) + S(1) + \dots + S(n)$  are called the *crystal ball* numbers. As in the work of Brunner & Laves (1971), O’Keeffe (1991, 1995), Grosse-Kunstleve (1996) and others, in our examples  $\mathcal{G}$  will usually be the *contact graph* of a  $d$ -dimensional lattice packing (Conway & Sloane 1993) or net (Wells 1977), formed by taking the vertices to be the points of the lattice or net and joining each point to its closest neighbours.

Although we will not study it here, there is another way to construct a graph from a lattice that has some advantages over the contact graph. This is the *Voronoi graph*; again, the vertices represent lattice points, but now two vertices are joined if the corresponding Voronoi cells (Conway & Sloane 1993, p. 33) are adjacent. The contact graph is always a subgraph. The chief advantage of the Voronoi graph is that it is meaningful for any lattice, whereas the contact graph is of little use for general lattices (consider for instance a two-dimensional lattice in which the generating vectors have different lengths). The Voronoi graph may also provide a better model for crystal growth. Consider the body-centred cubic (BCC) lattice  $D_3^*$ , for example, in which the Voronoi cells are truncated octahedra. The vertices within distance  $n$  of a given vertex in the Voronoi graph are the lattice points that can be reached by stacking truncated octahedra to depth  $n$  around a fixed truncated octahedron. These points

form a roughly spherical cluster, whereas as we shall see in § 3 the vertices at distance  $n$  from a given vertex in the contact graph form a cluster with the shape of a cube.

The contact graph has been used by the authors cited above as a way of defining the density of a lattice or net. It is worth mentioning that the theta series (Sloane & Teo 1985; Sloane 1987; Conway & Sloane 1993) may be more appropriate for that purpose, since it exactly gives the numbers of points in ever-increasing spheres about a particular point.

Nevertheless, for lattices and nets that are related to the root lattices  $A_d$ ,  $D_d$ ,  $E_d$ , the contact graphs and the associated coordination sequences are of considerable interest in their own right, and we shall investigate their properties in this paper, extending the work of O’Keeffe (1991, 1995).

Throughout this paper, if  $\mathcal{G}$  is a distance-transitive graph with some fixed choice of origin, and  $u$  is a vertex of  $\mathcal{G}$ , the *height* of  $u$ ,  $ht(u)$ , is the number of edges in the shortest path from  $u$  to the origin. Also, for  $n = 0, 1, 2, \dots$ , we set

$$\begin{aligned} G(n) &= \#\{u \in \mathcal{G} : ht(u) \leq n\}, \\ I(n) &= \#\{u \in \mathcal{G} : ht(u) < n\}, \\ S(n) &= \#\{u \in \mathcal{G} : ht(u) = n\} = G(n) - I(n). \end{aligned}$$

Then  $S(0), S(1), \dots$  is the *coordination sequence* of  $\mathcal{G}$ .

The paper is arranged as follows. In § 2 we study the contact graphs of lattices and introduce the notion of the fractional height of a lattice point  $u$ . This measures by how much the contact polytope of the lattice must be magnified before it contains  $u$ . The fractional height never exceeds the height (theorem 2.2) and differs from it by a bounded amount (theorem 2.3).

A lattice is called *well-coordinated* if the fractional heights are the same as the heights. Well-coordinated lattices have many desirable properties that make them easier to analyse. Although the root lattices  $A_d$  and  $D_d$  are well-coordinated (theorems 3.1 and 3.2), lattices that are not well-coordinated exist in all dimensions above four (theorem 2.6 and subsequent paragraphs). In particular, the lattices  $E_7$ ,  $E_7^*$  and  $E_8$  are not well-coordinated (theorems 3.5–3.7).

An extreme example of a lattice that is not well-coordinated is the 11-dimensional ‘anabasic’ lattice described in § 2; this contains vectors  $u$  with the property that  $ht(2u) < ht(u)$ !

Section 3 studies the coordination sequences of the lattices  $A_d$ ,  $D_d$ ,  $E_6$ ,  $E_7$ ,  $E_8$ , their duals and some related nets. It is worth remarking that in this section we will see graphs in which the crystal ball numbers  $G(n)$  are equinumerous with centred simplices (the sodalite net), centred cubes (the generalized BCC net) and centred orthoplexes ( $\mathbb{Z}^d$ ), representing all the regular polytopes in high dimensions (cf. Coxeter (1971)).

In § 4 it is shown that among all Barlow packings, that is, those formed from layers of the hexagonal lattice, the hexagonal close packing (or h.c.p.) has both the highest coordination sequence and the highest crystal ball sequence, while the face-centred cubic (or FCC) lattice has the lowest. This establishes a conjecture made in O’Keeffe (1995).

The highest crystal ball numbers for packings in dimensions  $d \leq 4$  have a concise description in terms of the function

$$\Delta_k(n) = (n+1)^k - n^k$$

as follows:

$$\left. \begin{aligned} d = 1 : & \text{ for } \mathbb{Z}, G(n) = \Delta_2(n) = 2n + 1, \\ d = 2 : & \text{ for } A_2, G(n) = \Delta_3(n) = 3n^2 + 3n + 1, \\ d = 3 : & \text{ for h.c.p., } G(n) = \frac{7}{8}\Delta_4(n) + (-1)^n \frac{1}{8} = \text{nearest integer to } \frac{7}{8}\Delta_4(n), \\ d = 4 : & \text{ for } D_4, G(n) = \frac{4}{5}\Delta_5(n) + \frac{1}{5} = \text{nearest integer to } \frac{4}{5}\Delta_5(n). \end{aligned} \right\} \quad (1.1)$$

However, in higher dimensions this notation is not especially useful. The formula (3.42) for  $E_8$ , for example, does not simplify when expressed in terms of  $\Delta_k(n)$ .

The following symbols will be used:  $\lfloor \cdot \rfloor$  for integer part or floor,  $\lceil \cdot \rceil$  for ceiling,  $\mathbb{Z}$  for the integers,  $\mathbb{Q}$  for the rationals,  $\mathbb{R}$  for the reals. For undefined terms from lattice theory see Conway & Sloane (1993) and for the definition of less familiar polytopes (see as the ‘ambo-simplex’) see Conway & Sloane (1991). This paper is part of series dealing with the properties of low-dimensional lattices from various points of view, the previous part being Conway & Sloane (1992).

## 2. Contact graphs of lattices

Most of this paper will be concerned with the case when  $\mathcal{G}$  is the contact graph of a  $d$ -dimensional lattice  $\Lambda$  that is spanned by its minimal vectors. Let  $\mathcal{P}$  denote the contact polytope of the lattice; that is, the convex hull of the minimal vectors (Conway & Sloane 1991).

We define the *fractional height* of a vector  $u \in \Lambda$  (or of the corresponding node of  $\mathcal{G}$ ) to be

$$fht(u) = \min_{h \geq 0} \{u \in h\mathcal{P}\},$$

where  $h\mathcal{P} = \{hx : x \in \mathcal{P}\}$ ,  $h \geq 0$ . Let  $G'(h) = \#\{u \in \Lambda : fht(u) \leq h\}$ ,  $I'(h) = \#\{u \in \Lambda : fht(u) < h\}$ , and  $S'(h) = \#\{u \in \Lambda : fht(u) = h\} = G'(h) - I'(h)$ .

In fact, it seems that there are three reasonable ways of measuring height:

- (1) the fractional height,  $fht(u)$ ;
- (2) the fractional height rounded up,  $\lceil fht(u) \rceil$ ; and
- (3) the height,  $ht(u)$ . Obviously we have

$$fht(u) \leq \lceil fht(u) \rceil, \quad (2.1)$$

and we shall prove in a moment that

$$\lceil fht(u) \rceil \leq ht(u), \quad (2.2)$$

and so

$$fht(u) \leq ht(u). \quad (2.3)$$

A lattice for which equality holds in (2.1) is called *well-placed*, because each point appears on the boundary of some  $n\mathcal{P}$ , for  $n \geq 0$ ,  $n \in \mathbb{Z}$ . A lattice for which equality holds in (2.2) is called *well-rounded*, because its heights are obtained just by the appropriate rounding of the fractional heights. Finally, if equality holds in (2.3), or equivalently if equality holds in both (2.1) and (2.2), we call the lattice *well-coordinated*.

**Theorem 2.1.** *A point  $u \in \Lambda$  has fractional height  $h$  if and only if it can be written in the form*

$$u = \sum_{i=1}^d c_i v_i, \quad (2.4)$$

where  $c_i \in \mathbb{Q}$ ,  $c_i \geq 0$ ,  $\sum c_i = h$  and  $v_1, \dots, v_d$  are distinct minimal vectors of  $\Lambda$  belonging to a face of the contact polytope.

*Proof.* If  $fht(u) = h$  then, as we magnify the contact polytope, forming  $a\mathcal{P}$  for increasing  $a$ ,  $u$  first belongs to  $a\mathcal{P}$  when  $a = h$ , at which point  $u$  is on the boundary of  $h\mathcal{P}$ . Since the faces of  $h\mathcal{P}$  are convex  $(d-1)$ -dimensional polytopes, by Carathéodory's theorem (Stoer & Witzgall 1970, theorem 2.2.12), we can write  $u$  as a linear combination of at most  $d$  of the vertices of that face:

$$u = \sum_{i=1}^d \lambda_i(hv_i),$$

with  $\lambda_i \geq 0$ ,  $\sum \lambda_i = 1$ , from which (2.4) follows. The converse is immediate. ■

On the other hand, the points of height  $n$  ( $n \in \mathbb{Z}$ ,  $n \geq 0$ ) are exactly the points that can be written as a linear combination of minimal vectors of  $\Lambda$  with non-negative coefficients that sum to  $n$ . If, instead, we allowed real coefficients with sum at most  $n$ , we would obtain all of  $n\mathcal{P}$ . Of course the vertices of  $n\mathcal{P}$  have height  $n$ . Thus we have established the following.

**Theorem 2.2.** *The points of fractional height at most  $h$  ( $h \geq 0$ ) are all the lattice points in or on  $h\mathcal{P}$ . The points of height at most  $n$  ( $n \in \mathbb{Z}$ ,  $n \geq 0$ ) are a subset—which necessarily includes the vertices—of the lattice points in or on  $n\mathcal{P}$ . Furthermore,  $fht(u) \leq ht(u)$  for all  $u \in \Lambda$ ,  $G'(n) \geq G(n)$  for integers  $n \geq 0$ , and the lattice is well-rounded if and only if  $G'(n) = G(n)$  for all integers  $n \geq 0$ .*

**Theorem 2.3.** *There is a constant  $C$  depending only on the lattice  $\Lambda$  such that*

$$ht(u) - fht(u) \leq C, \quad \text{for all } u \in \Lambda. \tag{2.5}$$

Furthermore,

$$fht(u) = \lim_{n \rightarrow \infty} \frac{ht(nu)}{n}. \tag{2.6}$$

*Proof.* Consider a vector  $u \in \Lambda$  with fractional height  $n$ . From theorem 2.1 we can write  $u = \sum_{i=1}^d c_i v_i$  with  $c_i \geq 0$ ,  $\sum c_i = n$ . If  $u' = \sum [c_i] v_i$ , then  $ht(u') \leq \sum [c_i] \leq fht(u)$ . However,  $u$  and  $u'$  differ only by a lattice vector in  $\mathcal{P}$ , of which there are only finitely many. Equation (2.5) follows, and (2.6) is an immediate consequence. Note that the limit in (2.6) exists, since height is a subadditive function. ■

Theorem 2.3 can be interpreted as saying that, for large  $n$ , the clusters of points of fractional height  $\leq n$  and of height  $\leq n$  look roughly the same, except that the faces of the latter may be somewhat 'pitted'. For well-rounded lattices they are exactly the same.

We shall make frequent use of the following result, which is an immediate consequence of Ehrhart's reciprocity law (Ehrhart 1960, 1967, 1973, 1977; see also Stanley 1980, 1986).

**Theorem 2.4.** *For integral  $n \geq 0$ ,  $G'(n)$  and  $I'(n)$  are respectively given by polynomials  $g'(n)$  and  $i'(n)$  in  $n$  of degree  $d$ , satisfying*

$$g'(-n) = (-1)^d i'(n), \quad n \in \mathbb{Z}. \tag{2.7}$$

Furthermore,  $S'(0) = 1$  while, for  $n > 0$ ,  $S'(n)$  is a polynomial  $s'(n)$  of degree  $d - 1$  satisfying  $s'(0) = 1 - (-1)^d$ .

Since, obviously,  $S(n) = G(n) - G(n - 1)$  for  $n > 0$ , it follows from theorem 2.4 that for well-rounded lattices (for which  $G(n) = G'(n)$ ),  $S(n)$  for  $n > 0$  is also a polynomial  $s(n)$  in  $n$  of degree  $d - 1$ . If this is so then the generating function

$$\mathcal{S}(x) = \sum_{n=0}^{\infty} S(n)x^n$$

can be written as

$$\mathcal{S}(x) = \frac{P_d(x)}{(1-x)^d}, \tag{2.8}$$

for some polynomial  $P_d(x)$ , which we call the *coordinator polynomial*. These polynomials usually provide the most concise specification of the coordination sequences. Equation (2.8) implies that the generating function for the crystal ball numbers  $G(n)$  is

$$\sum_{n=0}^{\infty} G(n)x^n = \frac{\mathcal{S}(x)}{1-x} = \frac{P_d(x)}{(1-x)^{d+1}}.$$

Note that if a lattice  $\Lambda$  is the direct product of lattices  $M$  and  $N$ , then the corresponding generating functions satisfy  $\mathcal{S}_\Lambda(x) = \mathcal{S}_M(x)\mathcal{S}_N(x)$ , and the coordinator polynomial for  $\Lambda$  is the product of those for  $M$  and  $N$ .

It follows from the definition that  $\mathcal{G}$  is well-placed if any one of these three equivalent conditions holds:

$$(a) \text{ } fht(u) \in \mathbb{Z}, \quad \text{for all } u \in \mathcal{G};$$

$$(b) \text{ } I'(n) = G'(n - 1), \quad \text{for } n = 1, 2, \dots; \tag{2.9}$$

$$(c) \text{ } S'(n) = G'(n) - G'(n - 1), \quad \text{for } n = 1, 2, \dots. \tag{2.10}$$

These conditions amount to saying that every point lies on the boundary of  $n\mathcal{P}$ , for some integral  $n \geq 0$ .

The polynomials  $g'(n)$ ,  $s'(n)$  and  $i'(n)$  that give the values of  $G'(n)$ ,  $S'(n)$  and  $I'(n)$  for integral  $n > 0$  are also interesting for negative  $n$ .

**Theorem 2.5.**  $\mathcal{G}$  is well-placed if and only if either

$$(d) \text{ } g'(-n) = (-1)^d g'(n - 1), \quad \text{for all } n \in \mathbb{Z}, \tag{2.11}$$

or

$$(e) \text{ } s'(-n) = (-1)^{d-1} s'(n), \quad \text{for all } n \in \mathbb{Z}, \quad n \neq 0 \tag{2.12}$$

holds.

Equation (2.11) asserts that the values of  $|g'(n)|$  are symmetric about  $n = -\frac{1}{2}$  and (2.12) asserts that  $s'(n)$  is an even polynomial in  $n$  if  $d$  is odd and an odd polynomial in  $n$  if  $d$  is even.

*Proof.* If  $\mathcal{G}$  is well-placed then  $g'(-n) = (-1)^d i'(n)$  (from theorem 2.4) =  $(-1)^d g'(n - 1)$  (from (2.10)). Let  $\sigma(x) = g'(x) - g'(x - 1)$ , so that  $s'(n) = \sigma(n)$  for  $n = 1, 2, \dots$ . Then  $\sigma(x) = (-1)^d \{g'(-x - 1) - g'(-x)\} = (-1)^{d-1} \sigma(-x)$ , so  $s'(-n) = (-1)^{d-1} s'(n)$ ,  $n \neq 0$ . Conversely, if (2.12) holds, there is an even (if  $d$  is odd) or odd (if  $d$  is even) polynomial  $\sigma(x)$  of degree  $d - 1$  such that  $s'(h) = \sigma(h)$  for  $h > 0$ . Then  $g'(h) = \sum_{t < h} \sigma(t)$  is a sum of linear combinations of Bernoulli polynomials of degrees  $d, d - 2, d - 4, \dots$  and (2.11) follows from the symmetry property of

Bernoulli polynomials (Abramowitz & Stegun 1964, equation (23.1.8)). Thus (2.12) implies (2.11), and (2.11) and (2.7) imply (2.10), showing that  $\mathcal{G}$  is well-placed. ■

For example, the FCC lattice is well-coordinated, since  $G'(n) = G(n) = \frac{1}{3}(2n + 1)(5n^2 + 5n + 3)$  satisfies  $G'(-n) = -G'(n - 1)$ : its values at

$$\dots - 3 - 2 - 1 0 1 2 3 \dots$$

being respectively

$$\dots - 55 - 13 - 1 1 13 55 147 \dots$$

Also,

$$S'(n) = G'(n) - G'(n - 1) = 10n^2 + 2 = S(n),$$

for  $n \geq 1$ , an even polynomial.

**Theorem 2.6.** *Every lattice of dimension  $d$  at most 4 is well-coordinated.*

*Proof.* The cases  $d = 1$  and  $2$  are easy. The case  $d = 3$  follows from theorem 2.5 and the fact (cf. O’Keeffe 1995) that for a three-dimensional lattice,  $S(n) = (S(1) - 2)n^2 + 2$ ,  $n > 0$ . The proof for  $d = 4$  is longer and will be given elsewhere. ■

It follows from theorem 2.4 that the coordination sequence for any four-dimensional lattice is given by

$$S(n) = \left(\frac{S(2)}{6} - \frac{S(1)}{3}\right)n^3 - \left(\frac{S(2)}{6} - \frac{4S(1)}{3}\right)n, \tag{2.13}$$

for  $n > 0$  (compare O’Keeffe 1995, p. 906).

On the other hand, the following five-dimensional lattice is not well-placed, and so not well-coordinated. We start from the lattice  $D_5^*$ , generated by the vectors  $v_1 = (1, 0, 0, 0, 0), \dots, v_5 = (0, 0, 0, 0, 1)$  and  $v_6 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ , and ‘squash’ it in the  $v_6$  direction until  $v_1, \dots, v_6$  all have the same length. The resulting lattice has Gram matrix

$$\frac{1}{21} \begin{bmatrix} 20 & -1 & -1 & -1 & 8 \\ -1 & 20 & -1 & -1 & 8 \\ -1 & -1 & 20 & -1 & 8 \\ -1 & -1 & -1 & 20 & 8 \\ 8 & 8 & 8 & 8 & 20 \end{bmatrix},$$

the entries in which are the inner products of the new vectors  $v_1, v_2, v_3, v_4, v_6$ . It is easy to check that  $w = v_6 - v_1 - v_2$  has height 3 but fractional height 2.5 (in fact,  $ht(2w) = 5$ ), showing that the lattice is not well-placed. Further investigation shows that this lattice is well-rounded, with

$$G'(n) = G(n) = \frac{2}{3}n^5 + \frac{5}{4}n^4 + \frac{5}{2}n^3 + \frac{15}{2}n^2 + \frac{23}{6}n + 1,$$

which is indeed not symmetric about  $-\frac{1}{2}$ , and that

$$S'(n) = S(n) = \frac{10}{3}n^4 - \frac{5}{3}n^3 + \frac{20}{3}n^2 + \frac{5}{3}n + 1 \quad (n > 0).$$

Thus lattices that are not well-placed (hence not well-coordinated) exist in all dimensions above four. As we will see, the lattices  $E_7, E_7^*$  and  $E_8$  are also not well-placed.

**Remark 2.7.** Well-coordinated lattices are well-rounded, and it is at first tempting to conjecture that the converse is also true. However, we believe that a counterexample (a well-placed lattice that is not well-rounded) will be found in perhaps as low as five dimensions. The next example shows that, in general, the set of lattice points of height  $\leq n$  need not even be lattice-convex, i.e. need not have the property that every lattice point in the convex hull of the points of height  $n$  has height  $\leq n$ .

**Definition 2.8.** A  $d$ -dimensional lattice  $\Lambda$  is *anabasic* if it has the property that, although it is generated by its minimal vectors, no subset of  $d$  of the minimal vectors generates it. A particular 11-dimensional lattice, which we call ‘the’ anabasic lattice  $B$ , was described in Conway & Sloane (1995a).

The anabasic lattice  $B$  has precisely 24 minimal vectors  $\pm u_1, \dots, \pm u_7, \pm v_1, \dots, \pm v_5$ , satisfying  $2 \sum_{i=1}^7 u_i = 3 \sum_{i=1}^5 v_i = 6w$  (say). Then  $2w = \sum v_i \in B$ ,  $3w = \sum u_i \in B$ , so  $w \in B$ . The heights of the multiples of  $w$  are

vector:	$w$	$2w$	$3w$	$4w$	$5w$	$6w$	$7w$	$8w$	$9w$	$\dots$
height:	12	5	7	10	12	14	17	19	21	$\dots$

and  $fht(w) = \frac{7}{3}$ . The set of points of height  $\leq 5$  is not lattice-convex, since it contains  $2w$  but not  $w$ .

In this example,  $fht(w) = \frac{7}{3}$  while  $ht(w) = 12$ , so the anabasic lattice is neither well-placed nor well-rounded. However, most of the lattices  $\Lambda$  we consider in this paper are well-rounded.

**Theorem 2.9.** If a  $d$ -dimensional lattice  $\Lambda$  is well-rounded, then the set of  $u \in \Lambda$  with  $ht(u) \leq n$  is lattice-convex and the crystal balls are magnified versions of the contact polytope. For integral  $n \geq 0$ ,  $G(n)$  and  $I(n)$  are respectively given by polynomials  $g(n)$  and  $i(n)$  of degree  $d$ , satisfying

$$g(-n) = (-1)^d i(n), \quad n \in \mathbb{Z}.$$

Furthermore,  $S(0) = 1$ , while for  $n > 0$ ,  $S(n)$  is a polynomial  $s(n)$  of degree  $d - 1$  satisfying  $s(0) = 1 - (-1)^d$ .

*Proof.* The hypothesis implies that the set of points of height  $\leq n$  is convex and the other assertions follow from Ehrhart’s reciprocity law (cf. theorem 4). ■

In particular, theorem 2.9 applies if the lattice is well-coordinated.

### 3. Root lattices and their duals

In this section we discuss the coordination sequences of the root lattices, their duals and some related nets.

*The cubic lattice  $\mathbb{Z}^d$*

The contact polytope for  $\mathbb{Z}^d$  is a  $d$ -dimensional cube, and a typical point  $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$  has

$$fht(x) = ht(x) = \sum_{i=1}^d |x_i|. \tag{3.1}$$

The coordination sequence for the 1-dimensional integer lattice  $\mathbb{Z}$  is  $\{1, 2, 2, 2, \dots\}$ ,

with generating function  $\mathcal{S}(x) = (1+x)/(1-x)$ . Therefore, for  $\mathbb{Z}^d$ , the direct product of  $d$  copies of  $\mathbb{Z}$ , we have  $\mathcal{S}(x) = (1+x)^d/(1-x)^d$ ,

$$S(n) = \sum_{k=0}^d \binom{d}{k} \binom{n-k+d-1}{d-1}, \tag{3.2}$$

$$G(n) = \sum_{k=0}^d \binom{d}{k} \binom{n-k+d}{d} \tag{3.3}$$

and  $P_d(x) = (1+x)^d$ . From (3.1), (3.2) we have the identity

$$\sum_{(a_0, a_1, \dots)} d! 2^{d-a_0} / \prod_{i=0}^n a_i! = \sum_{k=0}^d \binom{d}{k} \binom{n-k+d-1}{d-1}, \tag{3.4}$$

the sum being over all  $(a_0, a_1, \dots, a_n) \in \mathbb{Z}^{n+1}$  satisfying  $\sum a_i = d, \sum i a_i = n$ .

The crystal balls are orthoplexes (cf. Conway & Sloane 1991) and the  $G(n)$  are *centred orthoplex numbers*.

The structure of the coordinator polynomials both here and in subsequent examples becomes clearer if the coefficients of the successive polynomials  $P_d(x)$  for  $d = 0, 1, 2, \dots$  are displayed in a triangular array (with coefficients of highest powers on the right). We call this the *coordinator triangle*:

			1		
		1		1	
	1		2		1
	1	3		3	1
1	4		6	4	1
			...		

In this case, of course, the coordinator triangle is simply Pascal’s triangle of binomial coefficients  $\binom{d}{k}$ . O’Keeffe (1991, table 6) gave the coordination sequences for  $d \leq 10$ , but the present description is both simpler and holds for all  $d$ . It follows from (3.2) that the coefficient of  $n^{d-1}$  in  $S(n)$  is  $2^d/(d-1)!$ , as conjectured in O’Keeffe (1991).

*The root lattice  $A_d$*

The contact polytopes of the lattices  $A_n, D_n, E_6, E_7, E_8$  and their duals were described in Conway & Sloane (1991).

We define  $A_d$  to consist of the points  $x = (x_0, x_1, \dots, x_d) \in \mathbb{Z}^{d+1}$  with  $\sum x_i = 0$ . The contact polytope has  $d(d+1)$  vertices of the form  $(1, -1, 0^{d-1})$ . These are at the midpoints of the shorter edges of the diplo-simplex formed by the vectors (from coset [1] of  $A_d$  in  $A_d^*$ )

$$\pm \left( \left( \frac{1}{d+1} \right)^d, \left( \frac{-d}{d+1} \right)^1 \right).$$

The contact polytope was incorrectly described as an ‘ambo-diplo-simplex’ in Conway & Sloane (1991); a better name would be ‘shorter ambo-diplo-simplex’.

A fundamental simplex for the Weyl group  $W(A_n)$  of order  $(n+1)!$  is described in figures 21.1 and 21.6 of Conway & Sloane (1993). It consists of the points satisfying

$$x_0 \geq x_1 \geq \dots \geq x_d, \quad \sum x_i = 0. \tag{3.5}$$

This simplex is an infinite cone which meets the contact polytope in  $d$  faces, one for each non-zero glue vector of  $A_d$  (cf. Conway & Sloane 1993, ch. 4 and 21). The face corresponding to the glue vector

$$[i] = \left( \underbrace{\frac{i}{d+1}, \dots, \frac{i}{d+1}}_{d+1-i}, \underbrace{-\frac{d+1-i}{d+1}, \dots, -\frac{d+1-i}{d+1}}_i \right),$$

for  $1 \leq i \leq d$ , has equation

$$\frac{1}{2}[i] \cdot x = \frac{i}{2(d+1)}(x_0 + \dots + x_{d-i}) - \frac{(d+1-i)}{2(d+1)}(x_{d-i+1} + \dots + x_d) = 1. \quad (3.6)$$

This face contains  $i(d-i)$  vertices of the contact polytope, those with a single  $+1$  in any of the first  $d+1-i$  coordinates and a single  $-1$  in any of the last  $i$  coordinates.

Consider a point  $x \in A_d$  lying in the fundamental simplex, in the cone above the face defined by (3.6). The reflecting planes of the affine (infinite) Weyl group of type  $A_n$  partition the whole space into simplices. The height of  $x$ , and also its fractional height, is given by the number of reflecting planes between  $x$  and the origin, which is  $\frac{1}{2}[i] \cdot x$ .

For an arbitrary point  $x \in A_d$  in the fundamental simplex (3.5), the height is

$$\max_{i=1, \dots, d} \frac{1}{2}[i] \cdot x,$$

which is simply  $\frac{1}{2} \sum |x_i|$ . Thus a point such as  $(7, 3, 0, -5, -5) \in A_4$  can be written as the sum of  $\frac{1}{2} \sum |x_i| = 10$  minimal vectors, and no fewer. From collecting these results and applying theorem 2.9, we obtain the following.

**Theorem 3.1.** *Any point  $x = (x_0, \dots, x_d) \in A_d$  is equivalent under the Weyl group to one with coordinates satisfying (3.5). For such a point we have*

$$fht(x) = ht(x) = \frac{1}{2} \sum |x_i|.$$

The number of points in  $A_d$  equivalent to this point is  $(d+1)! / \prod_{i=-n}^n a_i!$ , where  $a_i$  is the number of components  $x_j$  that are equal to  $i$ , for  $-n \leq i \leq n$ ,  $n = ht(x)$ . The lattice is well-coordinated and  $G(n)$ ,  $S(n)$  ( $n > 0$ ) are polynomials in  $n$  of degrees  $d$ ,  $d-1$  respectively. The crystal balls are shorter ambo-diplo-simplices.

O’Keeffe (1995) empirically determined the coordination sequences for  $A_d$  for  $d \leq 7$ , in each case finding that  $S(n)$  is a polynomial in  $n$  of degree  $d-1$ . The correctness of these expressions is now justified. Using theorem 3.1, we have extended O’Keeffe’s results to  $d = 10$  and find that the coordinator triangle is

			1			
			1	1		
		1	4	1		
	1	9	9	1		
	1	16	36	16	1	
1	25	100	...	100	25	1
			...			

The  $k$ th entry in the  $d$ th row is  $\binom{d}{k}^2$  (for  $k = 0, 1, \dots$ ), so that

$$\mathcal{S}(x) = \sum_{k=0}^d \binom{d}{k}^2 x^k / (1-x)^d \quad (3.7)$$

and hence

$$S(n) = \sum_{k=0}^d \binom{d}{k}^2 \binom{n-k+d-1}{d-1}, \tag{3.8}$$

with a similar expression for  $G(n)$ . The following elegant proof of (3.8) is due to C. L. Mallows. From theorem 3.1, equation (3.8) is equivalent to the identity

$$\sum_a (d+1)! / \prod_{i=-n}^n a_i! = \sum_{k=0}^d \binom{d}{k}^2 \binom{n-k+d-1}{d-1}, \tag{3.9}$$

where the sum on the left extends over all  $a = (a_{-n}, \dots, a_n)$  satisfying

$$\sum_{i=-n}^n a_i = d+1, \quad \sum_{i>0} i a_i = \sum_{i<0} i a_{-i} = n.$$

If we multiply the left-hand side summand of (3.9) by

$$x^{a_0} (xy)^{a_1} (xz)^{a_{-1}} (xy^2)^{a_2} (xz^2)^{a_{-2}} \dots,$$

we see that the left-hand side of (3.9) is equal to the coefficient of  $x^{d+1} y^n z^n$  in

$$(d+1)! \exp \left\{ x + \frac{xy}{1-y} + \frac{xz}{1-z} \right\} = (d+1)! \exp \left\{ x \frac{1-yz}{(1-y)(1-z)} \right\}$$

or, in other words, to

$$\text{coefficient of } y^n z^n \text{ in } \left\{ \frac{1-yz}{(1-y)(1-z)} \right\}^{d+1}. \tag{3.10}$$

On the other hand, the right-hand side of (3.9) is

$$\text{coefficient of } y^n z^n \text{ in } \left\{ \frac{1-yz}{(1-y)(1-z)} \right\}^{-d}. \tag{3.11}$$

Call these two expressions  $c_L(n, d)$  and  $c_R(n, d)$ . Contour integration now shows that

$$\sum_{n=0}^{\infty} \sum_{d=0}^{\infty} c_L(n, d) u^n v^d = \sum_{n=0}^{\infty} \sum_{d=0}^{\infty} c_R(n, d) u^n v^d = \left\{ 1 - 2v \frac{1+u}{1-u} + v^2 \right\}^{-1/2}, \tag{3.12}$$

completing the proof.

It is curious† that equation (3.7) is the expansion of  $L_d((1+x)/(1-x))$  in powers of  $x$ , where  $L_d$  is the  $d$ th order Legendre polynomial (see Pólya & Szegő 1976, p. 86). We are not aware of any other connections between the root system  $A_d$  and the Legendre polynomial  $L_d$ .

*The dual lattice  $A_d^*$*

The contact polytope for  $A_d^*$  is a diplo-simplex (Conway & Sloane 1991, p. 88) with  $2d+2$  vertices  $\pm v_i$ ,  $0 \leq i \leq d$ , where

$$v_i = \left( \left( \frac{1}{d+1} \right)^d, \left( \frac{-d}{d+1} \right)^1 \right),$$

† We are grateful to Herb Wilf for this remark.



Assuming these expressions hold in general, by expanding (2.8) we find that

$$S(n) = \sum_{k=0}^d \binom{n-k+d-1}{d-1} \sum_{i=0}^k \binom{2i}{i} \binom{d-2i}{k-i}. \quad (3.17)$$

This agrees with O'Keefe's empirical results for  $d \leq 7$  and presumably for general  $d$  could be established in a similar manner to equation (3.9).

*The root lattice  $D_n$*

We take  $D_d$  to consist of the points  $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$  with  $\sum x_i$  even. The contact polytope is an 'ambo-orthoplex' (Conway & Sloane 1991, p. 90), having  $2d(d-1)$  vertices, all of the form  $(\pm 1^2, 0^{d-2})$ . The Weyl group  $W(D_d)$  has order  $2^{d-1}d!$  and contains all permutations and all even sign changes of the coordinates.

Any point  $x \in D_d$  is equivalent under this group to one satisfying

$$x_1 \geq x_2 \geq \dots \geq x_{d-1} \geq |x_d|,$$

these inequalities defining the fundamental simplex. As in the case of  $A_d$ , the intersection of this simplex with the contact polytope has a face for each non-zero glue vector of  $D_d$ . There are three faces, defined by

$$\left. \begin{aligned} x_1 &= 1, \\ \frac{1}{2}(x_1 + \dots + x_{d-1} + x_d) &= 1, \\ \frac{1}{2}(x_1 + \dots + x_{d-1} - x_d) &= 1. \end{aligned} \right\} \quad (3.18)$$

The expressions  $x_1$ ,  $\frac{1}{2}(x_1 + \dots + x_d)$ ,  $\frac{1}{2}(x_1 + \dots - x_d)$  then give the fractional heights of points in the cones above these three faces, and the fractional height of a general point in the fundamental simplex is the maximum of these three expressions, which is always an integer. Furthermore, it is easy to show that a point with fractional height  $n$  can actually be written as a sum of  $n$  minimal vectors and so the lattice is well-coordinated.

Finally, the last two faces in (3.18) are equivalent under the full automorphism group of  $D_d$ , since this includes all sign changes of the coordinates.

We collect these results in the following theorem.

**Theorem 3.2.** *Any point  $x = (x_1, \dots, x_d) \in D_d$  is equivalent to one satisfying*

$$x_1 \geq x_2 \geq \dots \geq x_d.$$

For such a point we have

$$fht(x) = ht(x) = \max \left\{ x_1, \frac{1}{2} \sum x_i \right\}.$$

The number of points in  $D_d$  equivalent to  $x$  is

$$2^{d-a_0} d! / \prod_{i=0}^n a_i!, \quad (3.19)$$

where  $a_i$  is the number of components  $x_j$  that are equal to  $i$ , for  $0 \leq i \leq n = ht(x)$ . The lattice is well-coordinated,  $G(n)$  and  $S(n)$  ( $n > 0$ ) are polynomials of degrees  $d$  and  $d-1$ , respectively, and the crystal balls are ambo-orthoplexes.

We had already determined the coordination sequence for  $D_4$  some years ago (see



and

$$P_d(x) = (1+x) \sum_{k=0}^{d-1} \left\langle \begin{matrix} d \\ k \end{matrix} \right\rangle x^k, \quad (3.24)$$

where the  $\left\langle \begin{matrix} d \\ k \end{matrix} \right\rangle$  are Eulerian numbers (Comtet 1974, p. 243; Graham *et al.* 1990, p. 254; Riordan 1958, p. 215). In Comtet's notation (Comtet 1974, p. 244),  $P_d(x) = (x+1)A_n(x)/x$ , where  $A_n(x)$  is an Eulerian polynomial.

*The root lattices  $E_6$ ,  $E_7$ ,  $E_8$  and their duals*

The coordination sequences for  $E_6$  and  $E_6^*$  were found experimentally by O'Keefe (1995) so for these lattices we give only enough information to justify his results.

**$E_6$ .** The contact polytope for the root lattice  $E_6$  is the polytope called  $1_{22}$  in Coxeter's notation (Conway & Sloane 1991, p. 104; Coxeter 1971, p. 201). There are 54 faces, all 5-dimensional hemicubes. If we define  $E_6$  to consist of the points in the standard  $E_8$  (see below) in which the first three coordinates are equal, then the typical face has equation  $\pi \cdot x = 1$ , where  $\pi = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 0, 0, 0, 0, 0)$ . The vertices of  $E_6$  on this face have coordinates  $((\frac{1}{2})^3, (\pm\frac{1}{2})^{5+})$ , the exponent 5+ indicating that only even sign combinations are permitted. The fractional height of a point  $x \in E_6$  in the cone above this face is  $fh(x) = \pi \cdot x$ . This is an integer, so  $E_6$  is well-placed, and it is also easy to see that  $\pi \cdot x = ht(x)$ . Thus we have proved:

**Theorem 3.3.**  *$E_6$  is well-coordinated.*

This establishes the coordination sequence

$$S(n) = \frac{117}{5}n^5 + 36n^3 + \frac{63}{5}n \quad (n > 0) \quad (3.25)$$

found empirically in O'Keefe (1995). Also,

$$G(n) = \frac{39}{10}n^6 + \frac{117}{10}n^5 + \frac{75}{4}n^4 + 18n^3 + \frac{267}{20}n^2 + \frac{63}{10}n + 1. \quad (3.26)$$

**$E_6^*$ .** The contact polytope for the dual lattice  $E_6^*$  is a diplo-Schläfli polytope (Conway & Sloane 1991, p. 104) and coincides with the Voronoi polytope for  $E_6$ . The contact polytope has 72 faces, one for each minimal vector of  $E_6$ . For example, the face defined by the minimal vector  $\pi = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \in E_6$  has equation  $\pi \cdot x = 1$ . This face contains 12 points of  $E_6^*$ , forming a diplo-simplex. All faces are of this type. From this, it is easy to obtain the following.

**Theorem 3.4.**  *$E_6^*$  is well-coordinated.*

This establishes the coordination sequence

$$S(n) = 18n^5 + 30n^3 + 6n \quad (n > 0) \quad (3.27)$$

found empirically in O'Keefe (1995). Also,

$$G(n) = 3n^6 + 9n^5 + 15n^4 + 15n^3 + 9n^2 + 3n + 1. \quad (3.28)$$

The remaining three lattices are not well-placed, although they are well-rounded.

**$E_7$ .** We define  $E_7$  to consist of the points  $x = (x_1, \dots, x_8)$  in the standard  $E_8$ , for which the first two coordinates are equal. The contact polytope is the polytope  $2_{31}$  (Conway & Sloane 1991, p. 100). There are two types of faces: 56 faces that are Schläfli polytopes (also called  $2_{21}$  polytopes) and 576 simplicial faces. We use the extended Coxeter–Dynkin diagram shown in figure 1.

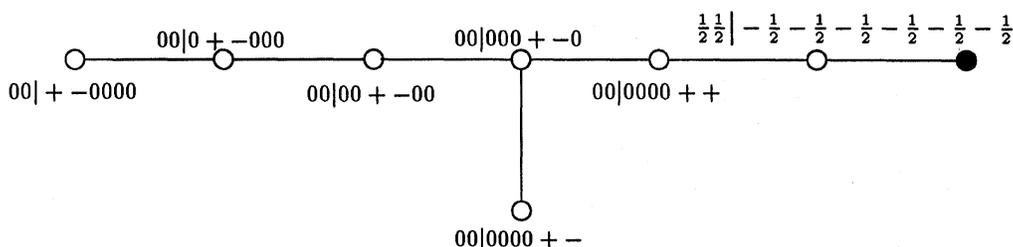


Figure 1. Extended Coxeter–Dynkin diagram for  $E_7$ , labelled to show walls of fundamental simplex.

The extending node in figure 1 is shaded and the other nodes are labelled with the equations that define the walls of the fundamental simplex (compare Conway & Sloane 1993, figure 21.3(b), where slightly different coordinates were used). In figure 1, +1 and -1 have been abbreviated to + and -. The left-most node, for example, defines the wall  $x_3 - x_4 \geq 0$ .

The points in the fundamental simplex therefore satisfy

$$x_1 = x_2, x_3 \geq x_4 \geq x_5 \geq x_6 \geq x_7 \geq |x_8|, \quad x_1 + x_2 \geq x_3 + x_4 + x_5 + x_6 + x_7 + x_8. \quad (3.29)$$

The two faces in which the contact polytope meets this fundamental simplex correspond to the left and bottom nodes of figure 1 and have equations

$$\pi_1 \cdot x = 1, \quad \pi_1 = (\frac{1}{2}, \frac{1}{2}, 1, 0, 0, 0, 0, 0), \quad (3.30)$$

$$\pi_2 \cdot x = 1, \quad \pi_2 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}), \quad (3.31)$$

respectively. The face defined by (3.30) contains 27 points of  $E_7$ ; namely  $110^6$ ,  $001(\pm 1, 0^5)$  and  $\frac{1}{2}\frac{1}{2}\frac{1}{2}(\pm\frac{1}{2})^{5+}$ , forming a Schläfli polytope. These 27 points span  $E_7$ . If  $x$  is a point of  $E_7$  in the cone above this face,

$$fht(x) = \pi_1 \cdot x = x_2 + x_3 \in \mathbb{Z}$$

and  $ht(x) = fht(x)$ . On the other hand, the face defined by (3.31) contains seven points of  $E_7$ ; namely  $110^6$ ,  $(\frac{1}{2})^6$  and  $\frac{1}{2}\frac{1}{2}(-\frac{1}{2}\frac{1}{2}^4) - \frac{1}{2}$ , forming a regular simplex. Now, however, the points on the face only generate a sublattice  $A_7$ , of index 2 in  $E_7$  and, if  $x \in E_7$  is in the cone above this face,

$$fht(x) = \pi_2 \cdot x = x_2 + (x_3 + x_4 + x_5 + x_6 + x_7 - x_8)/4 \in \frac{1}{2}\mathbb{Z}$$

and  $ht(x) = \lceil fht(x) \rceil$ .

For a general point of  $E_7$  satisfying (3.29), we have

$$fht(x) = \max\{x_2 + x_3, x_2 + (x_3 + x_4 + x_5 + x_6 + x_7 - x_8)/4\}, \quad (3.32)$$

and

$$ht(x) = \lceil fht(x) \rceil. \quad (3.33)$$

By applying theorem 2.9, we obtain:

**Theorem 3.5.** *Let  $E_7$  consist of the points  $x = (x_1, \dots, x_8)$  of  $E_8$ , in which the first two coordinates are equal. Any such point of  $E_7$  is equivalent under the Weyl group of  $E_7$  to one satisfying (3.29), for which the fractional height is given by (3.32) and the height by (3.33).  $E_7$  is well-rounded but not well-coordinated.  $G(n)$  and  $S(n)$  ( $n > 0$ ) are polynomials in  $n$  of degrees 7 and 6, respectively.*

Table 1. Numbers of points of fractional height exactly  $h$  ( $S'(h)$ ) and at most  $h$  ( $G'(n)$ ) in  $E_7$  lattice

$h$	$S'(h)$	$G'(h)$
0	1	1
0.5	0	1
1	126	127
1.5	0	127
2	2898	3025
2.5	0	3025
3	25886	28911
3.5	576	29487
4	132930	162417
4.5	4032	166449
5	485982	652431
...	...	...

A computer was now used to determine the numbers of points of fractional height up to 6.5, making use of knowledge of the subgroups of the Weyl group to calculate the number of lattice points equivalent to a given point. The results are partly shown in table 1.

Using theorem 3.5, these computed values suffice to determine  $S(n)$  and  $G(n)$ . From  $S(n) = S'(n) + S'(n - \frac{1}{2})$ , ( $n \geq 1$ ),  $G(n) = G'(n)$  ( $n \geq 0$ ),  $n \in \mathbb{Z}$ , we find

$$S(n) = \frac{148}{5}n^6 - \frac{12}{5}n^5 + 52n^4 + 12n^3 + \frac{212}{5}n^2 - \frac{48}{5}n + 2 \quad (n > 0), \quad (3.34)$$

$$G(n) = \frac{148}{35}n^7 + \frac{72}{5}n^6 + 24n^5 + 28n^4 + \frac{488}{15}n^3 + \frac{98}{5}n^2 + \frac{68}{21}n + 1. \quad (3.35)$$

**$E_7^*$ .** The contact polytope for  $E_7^*$  is a Hesse polytope  $E_{21}$  (Conway & Sloane 1991, p. 101). There are two types of faces: simplices, whose vertices span a sublattice of index 3 in  $E_7^*$  and orthoplexes, whose vertices span a sublattice of index 2. We omit the details and just summarize the result.

**Theorem 3.6.** *The dual lattice  $E_7^*$  is well-rounded but not well-coordinated, the fractional heights are in  $\frac{1}{6}\mathbb{Z}$  and  $G(n)$ ,  $S(n)$  ( $n > 0$ ) are polynomials in  $n$  of degrees 7 and 6, respectively.*

With the aid of a computer we found sufficiently many values to establish that

$$S(n) = \frac{68}{5}n^6 - \frac{216}{5}n^5 + 148n^4 - 192n^3 + \frac{902}{5}n^2 - \frac{264}{5}n + 2 \quad (n > 0), \quad (3.36)$$

$$G(n) = \frac{68}{35}n^7 - \frac{2}{5}n^6 + \frac{74}{5}n^5 + 8n^4 + \frac{56}{5}n^3 + \frac{97}{5}n^2 + \frac{37}{35}n + 1. \quad (3.37)$$

Notice that the formulae (3.34) and (3.36) for  $S(n)$  for these lattices are much more complicated than the corresponding formula (3.25) and (3.27) for  $E_6$  and  $E_6^*$ , reflecting the fact that  $E_7$  and  $E_7^*$  are not well-coordinated.

**$E_8$ .** The 'standard'  $E_8$  consists of the points  $x = (x_1, \dots, x_8)$  whose coordinates are either all integers or all halves of odd integers and whose sum is even. The contact polytope is the Gosset polytope  $4_{21}$  (Conway & Sloane 1991, p. 94). There are two types of faces: 2160 faces that are orthoplexes and 17280 simplicial faces. We use the extended Coxeter–Dynkin diagram shown in figure 2, where we have adopted the same conventions as in figure 1.

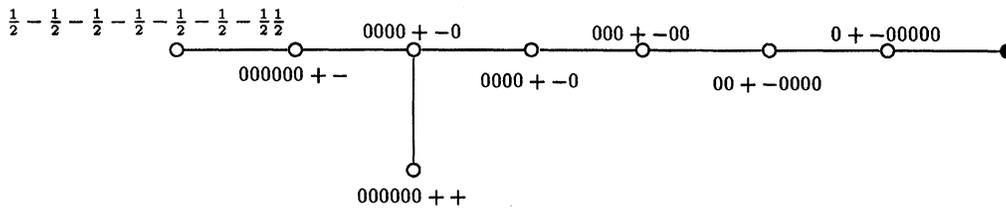


Figure 2. Extended Coxeter–Dynkin diagram for  $E_8$ , labelled to show walls of fundamental simplex.

From the planes defined by the nodes in figure 2, we see that the points in the fundamental simplex satisfy

$$x_2 \geq x_3 \geq x_4 \geq x_5 \geq x_6 \geq x_7 \geq |x_8|, \quad x_1 \geq x_2 + x_3 + x_4 + x_5 + x_6 + x_7 - x_8.$$

However, these together imply  $x_1 \geq x_2$  and so we can conclude that the fundamental simplex consists of the points satisfying

$$x_1 \geq x_2 \geq \dots \geq x_7 \geq |x_8|, \quad x_1 + x_8 \geq x_2 + \dots + x_7. \tag{3.38}$$

The two faces in which the contact polytope meets this fundamental simplex correspond to the left and bottom nodes of figure 2 and have equations:

$$\pi_1 \cdot x = 1, \quad \pi_1 = (1, 0, 0, 0, 0, 0, 0, 0), \tag{3.39}$$

$$\pi_2 \cdot x = 1, \quad \pi_2 = \left(\frac{5}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right), \tag{3.40}$$

respectively. The face defined by (3.39) contains 14 points of  $E_8$ , forming an orthoplex, and spanning a sublattice  $D_8$  of index 2 in  $E_8$ . The face defined by (3.40) contains eight points, forming a regular simplex and spanning a sublattice  $A_3$  of index 3 in  $E_8$ . The fractional heights of points in the cones above these two faces are given by  $\pi_1 \cdot x$  and  $\pi_2 \cdot x$ , respectively. The final result of this analysis is the following.

**Theorem 3.7.** Any point of  $E_8$  is equivalent under the Weyl group to one satisfying (3.38), for which the fractional height is

$$fht(x) = \max\{x_1, (5x_1 + x_2 + x_3 + \dots + x_8)/6\}.$$

$E_8$  is well-rounded but not well-coordinated and  $G(n)$  and  $S(n)$  ( $n > 0$ ) are polynomials of degrees 8 and 7, respectively.

With the help of a computer, we determined the numbers of points of fractional height  $\leq 13$ . In order to do this we precomputed a list of the 256 different types of stabilizers of points satisfying (3.38). The results of the enumeration are partly shown in table 2.

Using theorem 3.7, these values suffice to determine  $S(n)$  and  $G(n)$ . We have  $G(n) = G'(n)$ ,  $n \geq 0$ ,  $n \in \mathbb{Z}$  and  $S(n) = G(n) - G(n - 1)$ ,  $n \geq 1$ ,  $n \in \mathbb{Z}$ , from which it follows that

$$S(n) = \frac{456}{7}n^7 - 120n^6 + 312n^5 - 120n^4 - 48n^3 + 240n^2 - \frac{624}{7}n \quad (n > 0), \tag{3.41}$$

$$G(n) = \frac{57}{7}n^8 + \frac{108}{7}n^7 + 30n^6 + 72n^5 + 39n^4 + 36n^3 + \frac{300}{7}n^2 - \frac{24}{7}n + 1. \tag{3.42}$$

Besides verifying that these polynomials matched the computer results for  $n \leq 13$ , we also checked that  $G(-n) = I(n)$  for  $n \leq 13$  (cf. theorem 2.9). For  $n \geq 4$ , our values for the coordination sequence  $S(n)$  do not agree with those given in O’Keefe (1995);

Table 2. Numbers of points in  $E_8$  lattice by fractional height

( $G'(h)$  is the number with fractional height  $\leq h$ , and  $I'(h)$  is the number with fractional height  $< h$ . The final column gives the coordination sequence.)

$h$	#	$G'(h)$	$I'(h)$	$S(h)$
0	1	1	0	1
between 0 & 1	0			
1	240	241	1	240
between 1 & 2	0			
2	9120	9361	241	9120
between 2 & 3	17 280			
3	657 600	131 041	26 641	121 680
between 3 & 4	1 244 160			
4	9 773 280	996 001	338 401	864 960
between 4 & 5	16 623 360			
5	69 467 520	5 109 841	2 240 161	4 113 840
.	...	...	...	...

we believe the latter are incorrect. Again, the fact that  $E_8$  is not well-coordinated is responsible for the complicated formula in (3.41).

We display the coordinator triangle for  $E_3 = A_1 \oplus A_2$ ,  $E_4 = A_4$ ,  $E_5 = D_5$ ,  $E_6$ ,  $E_7$  and  $E_8$ , followed by the coordinator polynomials for  $E_7^*$  and  $E_6^*$  (the row for  $E_2$  is omitted since that lattice is not generated by its minimal vectors):

				...					
$E_3$ :		1	5	5	1				
$E_4$ :		1	16	36	16	1			
$E_5$ :	1	35	180	180	35	1			
$E_6$ :	1	66	645	1384	645	66	1		
$E_7$ :	1	119	2037	8211	8787	2037	119	1	
$E_8$ :	1	232	7228	55 384	133 510	107 224	24 508	232	1
$E_7^*$ :	1	49	567	2263	3703	2583	625	1	
$E_6^*$ :	1	48	519	1024	519	48	1		

In contrast to the  $A_n$  and  $D_n$  cases, there is no apparent pattern to these coefficients.

Note that the last four lines of this table are not palindromic, displaying again the fact that these lattices are not well-coordinated.

#### Root lattices in general

Looking back over this section, we observe several properties that hold for all root lattices.

**Theorem 3.8.** *Let  $\Lambda$  be one of  $A_d$ ,  $D_d$  or  $E_6$ ,  $E_7$ ,  $E_8$ .*

(i) *Consider the faces in which the contact polytope meets the fundamental simplex. These faces are in one-to-one correspondence with the nodes of the extended Coxeter–Dynkin diagram (Conway & Sloane 1993, figures 21.1 to 21.3) that are not the extending node and whose removal does not make the diagram disconnected.*

(ii) *The fractional height of a lattice point in the cone above such a face is an*

integer if and only if the weight  $c_i$  associated with that node is 1 (see Conway & Sloane 1993, p. 483 and figure 23.1; Coxeter 1971, p. 194).

(iii)  $\Lambda$  is well-placed if and only if the vertices on any face of the contact polytope span  $\Lambda$ .

The explanation for (ii) is that the  $c_i$  give the index of the sublattice spanned by the vertices of the corresponding face.

We suspect that (iii) may hold for all lattices, but do not have a proof.

#### The $d$ -dimensional sodalite net

O’Keeffe (1991) defines the  $d$ -dimensional sodalite net to consist of the holes in the  $A_d^*$  lattice, with each point joined to its  $d + 1$  nearest neighbours. The case  $d = 2$  gives the familiar  $6^3$  hexagonal net. From the coordination sequences of these nets for  $d \leq 6$  given in O’Keeffe (1991), Grosse-Kunstleve (1996) observed that the coordinator polynomial appears to be  $1 + x + x^2 + \dots + x^d$ . If this is true in general it implies

$$S(n) = \binom{n+d}{d} - \binom{n-1}{d}, \tag{3.43}$$

$$G(n) = \binom{n+d+1}{d+1} - \binom{n}{d+1}. \tag{3.44}$$

The expression on the right-hand side of (3.44) is the number of points in a  $d$ -dimensional centred simplex. It should therefore be possible to establish the validity of (3.43) and (3.44) by finding a bijection between the crystal balls in  $d$ -dimensional sodalite and the points of a  $d$ -dimensional centred simplex. This is easy to do for  $d = 2$  but, for higher  $d$ , the expressions (3.43) and (3.44) are at present only conjectures. (Theorems 2.4 and 2.9 do not apply.)

### 4. The Barlow packings

Let  $L$  denote any three-dimensional packing formed by stacking layers of the hexagonal lattice  $A_2$ . As in Conway & Sloane (1995b), we shall refer to these as the *Barlow packings*. Let  $S(N)$ ,  $G(n)$  denote the  $n$ th terms in the coordination and crystal ball sequences with respect to an arbitrary point in any such  $L$ .

**Theorem 4.1.** *For any Barlow packing  $L$ ,*

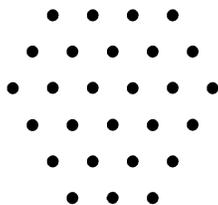
$$10n^2 + 2 \leq S(n) \leq \lfloor 21n^2/2 \rfloor + 2 \quad (n > 0), \tag{4.1}$$

$$\frac{5}{6}\Delta_4(n) + \frac{1}{6}\Delta_2(n) \leq G(n) \leq \frac{7}{8}\Delta_4(n) + (-1)^n \frac{1}{8} \quad (n \geq 0). \tag{4.2}$$

For any  $n > 1$ , the only Barlow packing that achieves either the left-hand value or the right-hand value for all choices of central sphere is the face-centred cubic lattice or hexagonal close-packing, respectively.

**Remark 4.2.** *This interesting result was conjectured by O’Keeffe (1995); it had, in fact, already been established (Conway & Sloane 1993, unpublished notes). The assertion on p. 801 of Hsiang (1993) that any Barlow packing has  $G(2) = 57$  is plainly incorrect: as shown in Conway & Sloane (1995b), there are Barlow packings with  $G(2) = 55, 56$  and  $57$ .*

*Proof.* Let  $H(a, b)$  denote a hexagonal arrangement of points in which the edges of the hexagon contain, respectively,  $a + 1$ ,  $b + 1$ ,  $a + 1$ ,  $b + 1$ ,  $a + 1$ ,  $b + 1$  points. For example,  $H(3, 2)$  is



The number of points in  $H(a, b)$  is  $P(a, b) = T_{a+b+1} + ab$ , where  $T_n = n(n+1)/2$  is a triangular number, and its perimeter is  $3(a+b)$ . The  $n$ th crystal ball with respect to an arbitrary point of  $L$  consists of a stack of  $2n+1$  hexagons  $\{H(a_i, b_i) : -n \leq i \leq n\}$ , for some choice of integers  $a_{-n} < a_{-n+1} < \dots < a_n$ ,  $b_{-n} < b_{-n+1} < \dots < b_n$ . Furthermore,  $a_0 = b_0 = n$ ,  $a_1 + b_1 = a_{-1} + b_{-1} = 2n - 1, \dots, a_n + b_n = a_{-n} + b_{-n} = n$ . At each stage, as we proceed from  $H(a_i, b_i)$  to  $H(a_{i+1}, b_{i+1})$ , for  $0 \leq i < n$ , just one of  $a_i$  and  $b_i$  drops by 1; a similar assertion holds for negative values of  $i$ . The crystal ball for the FCC is obtained if we always reduce  $b$  (say), and that for the h.c.p. if we alternately reduce  $a$  and  $b$ . In any case, we have

$$\begin{aligned} S(n) &= P(a_{-n}, b_{-n}) + 3(a_{-n+1} + b_{-n+1}) + \dots + 3(a_0 + b_0) + \dots \\ &\quad + 3(a_{n-1} + b_{n-1}) + P(a_n, b_n) \\ &= 2T_{n+1} + 6n + 2 \sum_{i=1}^{n-1} 3(2n - i) + a_{-n}b_{-n} + a_nb_n \\ &= S^{\text{fcc}}(n) + a_{-n}b_{-n} + a_nb_n \end{aligned} \tag{4.3}$$

and, similarly,

$$G(n) = G^{\text{fcc}}(n) + \sum_{i=-n}^n a_i b_i. \tag{4.4}$$

The assertions of the theorem follow from (4.3), (4.4) after some elementary algebra which we omit. ■

## 5. Concluding remarks

Several open questions remain. Is there a well-placed lattice that is not well-coordinated? (See remark following theorem 2.6.) Can the reader find a general proof of the formulae for the coordination sequences of  $A_d^*$  (equation (3.17)),  $D_d$  (equation (3.21)) and the sodalite net (equation (3.43))?

The Voronoi graphs (defined at the beginning of §1) should also be investigated. It follows from the work of Rajan & Shende (Rajan & Shende 1997; Conway & Sloane 1993, p. xxviii) that, except for root lattices, the Voronoi graph always properly contains the contact graph. What are the analogues of the coordination sequences for the Voronoi graphs of  $A_d^*$ ,  $D_d^*$ ,  $E_6^*$ ,  $E_7^*$ , for example?

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## References

- Abramowitz, M. & Stegun, I. A. 1964 *Handbook of mathematical functions*. Washington, DC: National Bureau of Standards.
- Brunner, G. O. & Laves, F. 1971 Zum problem der koordinationszahl. *Wiss. Z. Techn. Univ. Dresden* **20**, 387–390.
- Comtet, L. 1974 *Advanced combinatorics*. Dordrecht: Reidel.
- Conway, J. H. & Sloane, N. J. A. 1991 The cell structures of certain lattices. In *Miscellanea mathematica* (ed. P. Hilton, F. Hirzebruch & R. Remmert), pp. 71–107. Berlin: Springer.
- Conway, J. H. & Sloane, N. J. A. 1992 Low-dimensional lattices VI: Voronoi reduction of three-dimensional lattices. *Proc. R. Soc. Lond. A* **436**, 55–68.
- Conway, J. H. & Sloane, N. J. A. 1993 *Sphere packings, lattices and groups*, 2nd edn. Berlin: Springer.
- Conway, J. H. & Sloane, N. J. A. 1995a A lattice without a basis of minimal vectors. *Mathematika* **42**, 175–177.
- Conway, J. H. & Sloane, N. J. A. 1995b What are all the best sphere packings in low dimensions? *Discrete Comput. Geom. (László Fejes Tóth Festschrift)* **13**, 383–403.
- Coxeter, H. S. M. 1971 *Regular polytopes*, 3rd edn. New York: Dover.
- Ehrhart, E. 1960 Sur les polyèdres rationnels et les systèmes diophantiens linéaires. *Comptes Rendus Acad. Sci. Paris* **250**, 959–961.
- Ehrhart, E. 1967 Démonstration de la loi de réciprocité du polyèdre rationnel. *Comptes Rendus Acad. Sci. Paris A* **265**, 91–94.
- Ehrhart, E. 1973 Une extension de la loi de réciprocité des polyèdres rationnels. *Comptes Rendus Acad. Sci. Paris A* **277**, 575–577.
- Ehrhart, E. 1977 *Polyômes arithmétiques et méthode des polyèdres en combinatoire*, Internat. Series Numer. Math., vol. 35. Basel: Birkhäuser.
- Graham, R. L., Knuth, D. E. & Patashnik, O. 1990 *Concrete mathematics*. Reading, MA: Addison-Wesley.
- Grosse-Kunstleve, R. W., Brunner, G. O. & Sloane, N. J. A. 1996 Algebraic description of coordination sequences and exact topological densities for zeolites. *Acta Cryst. A* **52**, 879–889.
- Hsiang, W.-Y. 1993 On the sphere packing problem and the proof of Kepler's conjecture. *Int. J. Math.* **4**, 739–831.
- O'Keefe, M. *N*-dimensional diamond, sodalite and rare sphere packings. *Acta Cryst. A* **47**, 748–753.
- O'Keefe, M. 1995 Coordination sequences for lattices. *Zeit. f. Krist.* **210**, 905–908.
- Pólya, G. & Szegő, G. 1976 *Problems and theorems in analysis*, vol. 2. Berlin: Springer.
- Rajan, D. S. & Shende, A. M. 1997 Minimally and efficiently generated lattices are root lattices. *SIAM J. Discrete Math.* (In the press.)
- Riordan, J. 1958 *An introduction to combinatorial analysis*. New York: Wiley.
- Sloane, N. J. A. 1987 Theta-series and magic numbers for diamond and certain ionic crystal structures. *J. Math. Phys.* **28**, 1653–1657.
- Sloane, N. J. A. 1994 An on-line version of the Encyclopedia of Integer Sequences. *Electronic J. Combinatorics* **1**.
- Sloane, N. J. A. & Plouffe, S. 1995 *The encyclopedia of integer sequences*. New York: Academic.
- Sloane, N. J. A. & Teo, B. K. 1985 Theta series and magic numbers for close-packed spherical clusters. *J. Chem. Phys.* **83**, 6520–6534.
- Stanley, R. P. 1980 Decompositions of rational convex polytopes. *Ann. Discrete Math.* **6**, 333–342.
- Stanley, R. P. 1986 *Enumerative combinatorics*, vol. 1. Monterey, CA: Wadsworth.
- Stoer, J. & Witzgall, C. 1970 *Convexity and optimization in finite dimensions*, vol. I. Berlin: Springer.
- Wells, A. F. 1977 *Three-dimensional nets and polyhedra*. New York: Wiley.

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