

Strategyproof Mechanisms for Purchasing a Shared Resource

[Extended Abstract]

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1. INTRODUCTION

Consider a situation where a group of buyers would like to jointly purchase a particular resource with the intention of sharing it. For example, suppose two individuals who are sharing a living space would like to purchase an object, say an air conditioner or a television, which is available in the market for a certain price. How do they agree upon a division of this price amongst themselves when their utilities for using that object are private? Any scheme that recommends some notion of fair division of this price has to rely on the ability of the scheme to elicit the true utilities of the individuals, which is difficult since each individual wants to minimize his share of the payment. More generally, the resource in question may be congestible, and the utilities may depend on the proportion in which it is shared between the two users. In that case they not only have to decide how to divide the price of the resource but also how it will be shared, and the two decisions would naturally have to go hand in hand. Moreover, it may not be a simple question of paying a given price, but the resource itself may be offered in an auction, in which case the two buyers need to decide how they will jointly bid in the auction, along with the terms of sharing the resource and the division of payment in the event that they win.

Embracing the classical perspective of mechanism design, we can transfer the onus of coming up with a solution to this problem from the buyers to the seller herself. This leads us to consider the converse problem from the perspective of the seller in the market. She intends to sell a resource and several competing groups of buyers are interested in purchasing that resource for their respective groups. Her problem is thus to design a mechanism to allocate the resource to one of the competing groups, along with a proposed division of the resource within the group. The key aspect of this design problem is the kind of incentive properties such a mechanism needs to satisfy. The buyers in a single group are expected to collude in their utility reports and hence such a mechanism needs to be robust to any collusive behavior within a particular group, but perhaps not necessarily across groups.

A practical example, which is our primary motivation for studying this kind of a market, is the market for radio spectrum. Recently there has been a debate concerning the merits and demerits of allocating newly opened blocks of spectrum for free unlicensed use (like WiFi) as opposed to selling them for exclusive licensed use (e.g., to cellular ser-

vice providers) as has been done for the past few decades. It has been argued (see for example [5]) that an open-access unlicensed spectrum can act as an enabler for technological innovations that would increase social welfare and future tax revenue for the government, while at the same time mitigating the inefficiencies of exclusive licensed use. But one can also argue that such a move may cause the government to lose out on the substantial revenue that it generates by sale to licensed users, which could have also been devoted to social benefit. Also, an unlicensed spectrum is prone to the ‘tragedy of the commons’, since firms and their consumers whose quality-of-service requirements are stringent are bound to suffer because of overuse and lack of regulation. Thus it is not immediately evident whether one option is clearly better than the other.

In a proposal in [2] and [3], the authors suggest that auctions may serve as an effective mechanism for allocating the spectrum between licensed and unlicensed users and thus potentially play the role of a fair arbiter between the two paradigms. The broad idea is that groups consisting of smaller content or service providers and equipment vendors who benefit from the additional access to spectrum could jointly submit bids for a shared license, which will compete with bids for exclusive licenses from the bigger firms. A recent analysis of such a market in [9] does not account for the natural possibility of collusive behavior within the participating groups.

It is beneficial to look at an example of what kind of collusive behavior one might expect. One simple mechanism that the seller can use in this case is the classical Vickrey-Clarke-Groves mechanism (VCG), which makes the welfare maximizing allocation and makes each buyer pay the externality he imposes on others by his presence in the optimal allocation. It is well known that VCG is highly vulnerable to collusion (see [1] and [4]). In our case, suppose that a particular group of buyers is competing with other buyers or groups of buyers for procuring a resource being sold using the VCG mechanism. Suppose that the resource is indivisible and each buyer in this group has a particular utility value for having an access to the resource (for example an open access to an unlicensed band). Then each buyer in this group can report an arbitrarily high utility so that even in the absence of any particular buyer, the group would still be allocated the resource under the welfare maximizing allocation. Thus the externality of every buyer in the group is zero and the entire group gets allotted the resource without making any payment!

The main contribution of our work is the design of a

class of collusion-resistant mechanisms that enable a group of buyers to jointly participate in a market for a resource. Specifically, suppose that the resource is being sold in a truthful auction that accepts single bids for the entire resource. Our mechanism then truthfully elicits utility functions from the buyers in the group, prescribes a joint bid, and prescribes a division of the payment and the resource in the event that they win the auction. Moreover, it is collusion-resistant. The notion of collusion-resistance satisfied by the mechanism is the concept of strong group-strategyproofness. A mechanism is strongly group-strategyproof if no coalition of buyers can find a deviation from truthfulness such that no buyer is worse off and at least one buyer is strictly better off, irrespective of the reports of the buyers not in the coalition. From the perspective of the seller, our mechanism gives her a recipe to convert any truthful auction for single buyers into a collusion resistant auction for groups of buyers, which performs an additional task of prescribing a division of the resource to the winning group.

It is important to point out that our design does not strive to achieve any specific notion of fairness in the divisions of the resource or the payment. Rather, we characterize a class of mechanisms which satisfy the key property of group-strategyproofness and which are exactly able to recover the price needed to be paid for the resource. An additional consideration for some notion of fairness of the divisions would require a careful choice of a particular mechanism in this class, and it is an interesting problem that can be looked at separately.

1.1 Related work

The class of mechanisms that we propose are closely related to the carving mechanisms proposed by Moulin and Shenker in [6] and [7] for sharing the cost of a service. In that setup, a set of agents have certain valuations for a service and the cost for the service depends on the set of agents who will be provided the service. The problem is to design a mechanism that decides the set of agents who will be provided the service and their cost shares that will recover the corresponding cost. A carving mechanism sequentially offers the service according to a fixed cost sharing scheme to diminishing subsets of agents until it finds a subset that is able to afford it. As long as this cost sharing scheme satisfies the property of ‘cross-monotonicity’, this mechanism is strongly group-strategyproof.

In our case, the price to be shared between the agents does not depend on the number of agents, but is externally determined by the auction mechanism for the resource and may not be fixed. More importantly, the utilities of the buyers for the resource may depend on their share of the resource. Thus the mechanism has to decide the division of both the resource and the price. The key insight used in our design is that if the utility functions of the agents are concave, and both the resource and price division is done using a single cross-monotonic sharing scheme, then a similar carving-type mechanism which offers the resource to a diminishing subset of agents is strongly group-strategyproof.

2. BUYING A DIVISIBLE RESOURCE

A set L of n agents would like to buy a resource that they intend to share. This resource is being sold in an external market through some auction mechanism. The resource is

assumed to be divisible and each agent i in the group has a utility $U_i(x_i)$ for a fraction x_i of the shared resource. U_i is known only to buyer i and it belongs to the class $\bar{\mathcal{C}}$ of non-negative, concave, non-decreasing utility functions defined on $[0, 1]$ such that $U(0) = 0$ for each $U \in \bar{\mathcal{C}}$. In order to participate in the auction, the group has to submit a single bid for the resource and make the required payment depending on the outcome of the auction.

Our goal is to design a mechanism that accomplishes the following two tasks: 1. Elicit individual utility functions from the agents and then output a group bid to enter into the external auction using an *aggregation procedure* announced a priori and 2. Prescribe a division of the resource and that of the payment needed to be paid in the external auction amongst the buyers, in the event that they win the resource.

Definition: A mechanism is *strongly group-strategyproof* if for any coalition of buyers $S \subseteq L$, fixing any feasible utility function reports of all buyers not in S , for every feasible deviation of the buyers in S from truthful reporting, either all the buyers are indifferent between the original outcome and the new resulting outcome or at least one buyer is strictly worse off.

We define the following class of mechanisms.

Definition: (Cross-monotonic shares aggregation mechanism) For each subset $A \subseteq L$, fix n non-negative numbers $(x_1(A), \dots, x_n(A))$ such that

1. $\sum_{i=1}^n x_i(A) = 1$ and $x_i(A) > 0$ only if $i \in A$.
2. (Cross-monotonicity) If $A \subseteq B$, then $x_i(A) \geq x_i(B)$ for all $i \in A$.

The mechanism elicits utility functions $G_i \in \bar{\mathcal{C}}$ from all the agents and computes a vector of values

$$\bar{\beta} = (\beta_1, \beta_2, \dots, \beta_m)$$

corresponding to diminishing subsets of agents $S_1 \supset S_2 \supset \dots \supset S_m$ as follows.

- Let $S_1 = L$, the set of all agents. For each subset S_j of agents, dynamically define

$$\begin{aligned} \beta_j &= \max\{k \geq 0 : (kx_1(S_j), kx_2(S_j), \dots, kx_n(S_j)) \\ &\preceq (G_1(x_1(S_j)), \dots, G_n(x_n(S_j)))\}. \end{aligned} \quad (1)$$

Here $\preceq$ denotes a componentwise inequality.

- Let $P(S_j)$ be the set of agents who force the inequality in the definition above, i.e. all agents i such that $\beta_j x_i(S_j) = G_i(x_i(S_j))$. Then $S_{j+1} = S_j \setminus P(S_j)$ and m is the smallest integer such that $S_{m+1} = \phi$.
 - Let $\beta^* = \max\{\beta_1, \dots, \beta_m\}$. The mechanism submits the bid β^* to the external auction.
 - Suppose the group has to make a payment p^* in the external auction. Let $r = \min\{i : \beta_i \geq p^*\}$. Then each agent i pays $x_i(S_r)p^*$ and gets a fraction $x_i(S_r)$ if the group is allotted the resource.
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A whole class of mechanisms can be obtained by choosing different fractions $(x_1(A), \dots, x_n(A))$ for the different subsets of S , as long as that they satisfy the required conditions.

Increasing prices implementation: The cross-monotonic shares aggregation mechanism has a natural increasing price implementation which, although not strategically equivalent, is more intuitive. As defined in the mechanism, for each subset $A \subseteq L$, fix n non-negative numbers $(x_1(A), \dots, x_n(A))$ such they satisfy the required conditions. Call this a sharing scheme. First, each buyer i in the set $S_1 = L$ is offered an $x_i(L)$ share of the resource and starting from $k = 0$ all the buyers are offered an increasing set of prices $\{kx_i(L)\}$ for their respective shares. Each buyer can either continue to accept or reject at any point. If the first buyer (or a set of buyers) to reject his price does so at $k = k^*$ then $\beta_1 = k^*$. Thus for the proposed shares of the resource, β_1 is the maximum price such that the entire set of buyers can afford to buy their share of the resource for the corresponding share of the price. The set of buyers who reject their share of the price for $k = \beta_1$ are removed from the set. The mechanism continues with the remaining set S_2 , proposes the set of shares $\{x_i(S_2)\}$ to the buyers according to the sharing scheme, and again finds the largest price such that all the buyers are able to afford to buy their share of the resource for the same share of the price. This price is β_2 . The ‘bottleneck agents’ who drop out at this price are removed and the mechanism continues to find the rest of the vector $\bar{\beta}$ in a similar way till no buyer remains. The largest value in this vector is submitted to the auction. If a payment p^* is to be made in the auction to win the resource, the mechanism looks for the largest subset S_i that can afford to pay the price and both the price and the resource is divided according to the corresponding shares.

Example: Let us consider an example to illustrate the mechanism. Consider a resource A which is being sold in a second price auction. Suppose the group L that intends to buy the resource consists of three buyers 1, 2 and 3 with utility functions $U_1(x) = x$, $U_2(x) = \sqrt{x}$ and $U_3(x) = \ln(1 + x)$. Let the aggregation mechanism prescribe equal shares for every subset of the buyers, i.e. $x_i(A) = \frac{1}{|A|}$ if $i \in A$ and 0 otherwise. These shares clearly satisfy cross-monotonicity. Assume that the buyers truthfully report their utility functions to the mechanism. Set $S_1 = L = \{1, 2, 3\}$. The mechanism thus computes

$$\begin{aligned} \beta_1 &= \max\{k \geq 0 : (\frac{k}{3}, \frac{k}{3}, \frac{k}{3}) \preceq (1/3, \frac{1}{\sqrt{3}}, \ln(1 + \frac{1}{3}))\} \\ &= \min\{1, \sqrt{3}, 3\ln(\frac{4}{3})\} \\ &= 3\ln(\frac{4}{3}) \approx 0.86. \end{aligned}$$

Thus buyer 3 is removed from the group to form $S_2 = \{1, 2\}$. Next, the mechanism computes

$$\begin{aligned} \beta_2 &= \max\{k \geq 0 : (\frac{k}{2}, \frac{k}{2}) \preceq (1/2, \frac{1}{\sqrt{2}})\} \\ &= \min\{1, \sqrt{2}\} = 1. \end{aligned}$$

Thus buyer 1 is removed from S_2 to result in $S_3 = \{2\}$. We thus finally have

$$\beta_3 = \max\{k \geq 0 : k \leq 1\} = 1.$$

Hence the vector $\bar{\beta} = (0.86, 1, 1)$. Thus $\beta^* = 1$ is submitted as a bid in the second price auction. Now assume that there is a single other competing buyer in the second price auction

and suppose that his bid is 0.6. Thus the minimum payment required to win the auction for the group is 0.6, which is feasible. Now $r = \min\{i : \beta_i \geq 0.6\} = 1$. Thus the entire group, i.e. $S_1 = L$ is allotted equal shares of the resource and each buyer pays 0.2 to the seller. Suppose instead that the other buyer in the auction submitted a bid of 0.9. Then in that case $r = \min\{i : \beta_i \geq 0.9\} = 2$. Thus the group $S_2 = \{1, 2\}$ is allotted equal shares of the resource while buyer 3 does not get any share of the resource. Both the winning buyers pay 0.45 to the seller. Thus in short, the resource is shared between the largest subset of buyers who can jointly afford to pay the price, divided according to the prescribed shares.

We can then prove the following main result:

THEOREM 2.1. *Suppose that the resource is being sold in a deterministic dominant strategy truthful auction, in which the payment for not winning the resource is 0. Also, assume that a buyer strictly prefers the outcome where he obtains a non-zero fraction of the resource with a payment equal to his utility for that fraction of the resource, over the outcome where he does not obtain anything and makes no payment. Then any cross-monotonic shares aggregation mechanism is strongly group-strategyproof.*

A deterministic dominant strategy truthful auction is one in which the mapping from the bids to an allocation is deterministic and bidding truthfully is a dominant strategy for all the buyers. An example would be the second price auction with any reserve price. The additional condition that the payment for not winning the resource is 0 is benign and any dominant strategy truthful auction can be modified to satisfy this condition, while maintaining the same allocation rule. For more details, see [8]. Note that the result clearly holds if instead of competing in an auction, there is simply a fixed price for buying the resource.

3. NON-EXCLUDABLE BUYERS

The cross-monotonic shares aggregation mechanism is allowed to exclude buyers from enjoying the resource by assigning them a zero fraction of the resource. But in many cases, the resource is neither divisible nor congestible, but rather each buyer has a particular utility for having an access to the resource. Typically, once that resource is purchased, then a buyer cannot be excluded from enjoying it. An example would be buying a block of spectrum for unlicensed use. Once the block has been assigned for unlicensed use, no user would be restricted from having an access to it. Another example would be buying a centralized air-conditioning system for the house. Once it is bought, a roommate cannot be excluded from enjoying it. In this case how do we design a group strategyproof aggregation mechanism which is not allowed to exclude buyers from enjoying the resource? The solution is obtained from a simple modification of the cross-monotonic shares aggregation mechanism. As before, consider a group L of n buyers. Suppose each buyer i has a utility value V_i for enjoying access to the resource. We define the following class of mechanisms.

Definition: (Non-excludable aggregation mechanism)
Fix n non-negative numbers $(x_1, \dots, x_n)$ such that $\sum_{i=1}^n x_i = 1$. The mechanism elicits bids $(b_1, \dots, b_n)$ from the buyers.

It then submits a bid β^* defined as:

$$\beta^* = \max\{k \geq 0 : (kx_1, kx_2, \dots, kx_n) \preceq (b_1, \dots, b_n)\}. \quad (2)$$

If the group wins the auction and it is supposed to pay a price $p^* \leq \beta^*$, then each buyer j pays $x_j p^*$.

Note the difference between the cross-monotonic shares aggregation mechanism and the non-excludable aggregation mechanism. Since it is not a divisible resource, the fractions only correspond to division of the price to be paid in the auction and not of the resource. Also in a cross-monotonic shares mechanism, from the computed vector $\beta = (\beta_1, \beta_2, \dots, \beta_m)$, the maximum of these values is submitted as the bid, while the non-excludable mechanism simply submits β_1 . Thus the submitted bid is lower. This is because the cross-monotonic shares mechanism computes the best bid it can submit by sequentially excluding low utility ‘bottleneck’ buyers, while the non-excludable aggregation mechanism is not allowed to do so and thus submits the best bid that all the buyers can together afford. The non-excludable aggregation mechanism also has a natural increasing prices implementation. Starting from $k = 0$, each buyer i is offered an increasing price kx_i . Each buyer can continue to accept or reject at any point. If the first buyer to reject his price does so at $k = \beta^*$, then β^* is the submitted bid.

Example: Consider a resource A which is being sold in a second price auction. Suppose the group L that intends to buy access to the resource consists of three buyers 1, 2 and 3. Assume that the shares chosen in the mechanism are $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$. Let utilities of the buyers be $V_1 = 1$, $V_2 = 2$ and $V_3 = 3$ and suppose they truthfully report them to the mechanism. Then the mechanism computes:

$$\beta^* = \max\{k \geq 0 : k(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}) \preceq (1, 2, 3)\} = 2.$$

Thus a bid of 2 will be submitted to the auction. If they win the auction then the payment is divided according to the shares. If the shares were instead chosen to be $(\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$ then we can verify that the submitted bid will be 6, thus improving their chances of winning the resource.

We have the following result:

THEOREM 3.1. *Suppose that the resource is being sold in a dominant strategy truthful auction. Then any non-excludable aggregation mechanism is strongly group-strategyproof.*

Note that in contrast to Theorem 2.1, the auction mechanism for selling the resource is only required to be dominant strategy truthful and not necessarily deterministic. Hence it is a stronger result, although for a simpler mechanism.

4. CONCLUSION AND FUTURE WORK

We designed a class of group-strategyproof mechanisms that enable a group of agents to make purchasing decisions for a shared resource under lack of information about each others’ utilities. The key design requirements were collusion-resistance, which was captured through the notion of group-strategyproofness, and the ability to exactly recover the price required to be paid for the resource in the market. As mentioned in the introduction, no consideration has been

given to the problem of choosing the divisions of the resource and the price that adhere to any particular notion of fairness.

Since the shares in our mechanism are to be decided before the utility functions can be elicited, it is not clear what information could be used to form the basis for some notion of fairness. Here we would like to use the fact that even if the exact utility functions of buyers are not known to each other, there is typically a vague idea of the preferences of the buyers which is commonly known to them. For example, when two roommates buy a television set, they know from behavioral observations in the past that one roommate would like to buy it more than the other. Or if they want to share a cab ride, they know who needs it more than the other and hence should be willing to pay a larger share. A straightforward way to capture this information is to define distributions over utilities of the buyers and assume that these are commonly known between the buyers. In ongoing work we are trying to define a notion of fair allocation of the shares depending on these commonly known distributions of the utilities of the buyers. We can also ask questions like what allocation of shares maximizes the chance that the resource is won by the group? Or what allocation of shares maximizes the expected revenue of the seller? These questions form the basis of our future exploration.

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