Error functions

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1 Error function $\operatorname{erf} x$ and complementary error function $\operatorname{erfc} x$

(*Gauss*) **error function** is

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \tag{1}$$

and has properties

$$\operatorname{erf}(-\infty) = -1, \qquad \operatorname{erf}(+\infty) = 1$$
$$\operatorname{erf}(-x) = -\operatorname{erf}(x), \qquad \operatorname{erf}(x^*) = [\operatorname{erf}(x)]^*$$

where the asterisk denotes complex conjugation. **Complementary error function** is defined as

$$\operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt = 1 - \operatorname{erf} x \tag{2}$$

Note also that

$$\frac{2}{\sqrt{\pi}} \int_{-\infty}^x e^{-t^2}\, dt = 1 + \operatorname{erf} x$$

Another useful formula:

$$\int_0^x e^{-\frac{t^2}{2\sigma^2}}\, dt = \sqrt{\frac{\pi}{2}}\sigma \operatorname{erf}\left[\frac{x}{\sqrt{2}\sigma}\right]$$

Some Russian authors (e.g., *Mikhailovskiy*, 1975; *Bogdanov et al.*, 1976) call $\operatorname{erf} x$ a *Cramp function.*

2 Faddeeva function $w(x)$

Faddeeva (or *Fadeeva*) **function** $w(x)$ (*Fadeeva and Terent'ev*, 1954; *Poppe and Wijers*, 1990) does not have a name in *Abramowitz and Stegun* (1965, ch. 7). It is also called *complex error function* (or *probability integral*) (*Weideman*, 1994; *Baumjohann and Treumann*, 1997, p. 310) or *plasma dispersion function* (*Weideman*, 1994). To avoid confusion, we will reserve the last name for $Z(x)$, see below. Some Russian authors (e.g., *Mikhailovskiy*, 1975; *Bogdanov et al.*, 1976) call it a *(complex) Cramp function* and denote as $W(x)$. Faddeeva function is defined as

$$w(x) = e^{-x^2}\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^x e^{t^2}\,dt\right) = e^{-x^2}[1 + \mathrm{erf}\,(ix)] = e^{-x^2}\mathrm{erfc}\,(-ix) \quad (3)$$

Integral representations:

$$w(x) = \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{-t^2}\,dt}{x-t} = \frac{2ix}{\pi}\int_0^{\infty}\frac{e^{-t^2}\,dt}{x^2-t^2} \quad (4)$$

where $\Im x > 0$. These integral representations can be converted to (3) using

$$\frac{1}{x+i\Delta-t} = -2i\int_0^{\infty} e^{2i(x+i\Delta-t)u}\,du \quad (5)$$

3 Plasma dispersion function $Z(x)$

Plasma dispersion function $Z(x)$ (*Fried and Conte*, 1961) is also called *Fried-Conte function* (*Baumjohann and Treumann*, 1997, p. 268). In the book by *Mikhailovskiy* (1975), notation is $Z_{\mathrm{Mikh}}(x) \equiv xZ(x)$, which may be a source of confusion. Plasma dispersion function is defined as:

$$Z(x) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{e^{-t^2}}{t-x}\,dt \quad (6)$$

for $\Im x > 0$, and its analytic continuation to the rest of the complex x plane (i.e, to $\Im x \leq 0$).

Note that $Z(x)$ is just a scaled $w(x)$, i.e.

$$Z(x) \equiv i\sqrt{\pi}w(x)$$

We see that

$$Z(x) = 2ie^{-x^2} \int_{-\infty}^{ix} e^{-t^2}\,dt = i\sqrt{\pi}e^{-x^2}[1 + \mathrm{erf}\,(ix)] = i\sqrt{\pi}e^{-x^2}\mathrm{erfc}\,(-ix) \quad (7)$$

One can define $\bar{Z}$ which is given by the same equation (6), but for $\Im x < 0$, and its analytic continuation to $\Im x \geq 0$. It is related to $Z(x)$ as

$$\bar{Z}(x) = Z^*(x^*) = Z(x) - 2i\sqrt{\pi}e^{-x^2} = -Z(-x)$$

4 (Jackson) function $G(x)$

Another function useful in plasma physics was introduced by *Jackson* (1960) and does not (yet) have a name (to my knowledge):

$$G(x) = 1 + i\sqrt{\pi}xw(x) = 1 + xZ(x) = -Z'(x)/2 \quad (8)$$

Integral representation:

$$G(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{te^{-t^2}\,dt}{t - x} \quad (9)$$

where again $\Im x > 0$.

5 Other definitions

5.1 Dawson integral

Dawson integral $F(x)$ (*Abramowitz and Stegun*, 1965, eq. 7.1.17), also denoted as $S(x)$ by *Stix* (1962, p. 178) and as $\mathrm{daw}\,x$ by *Weideman* (1994), is defined as:

$$F(z) = e^{-z^2} \int_0^z e^{t^2}\,dt = (-Z(x) + i\sqrt{\pi}e^{-x^2})/2 = xY(x)$$

(see also the definition of $Y(x)$ below).

5.2 Fresnel functions

Fresnel functions (*Abramowitz and Stegun*, 1965, ch. 7) $C(x)$, $S(x)$ are defined by

$$C(x) + iS(x) = \int_0^x e^{\pi i t^2/2}\,dt$$

5.3 (Sitenko) function $\varphi(x)$

Another function (*Sitenko*, 1982, p. 24) is $\varphi(x)$, defined only for real arguments:

$$\varphi(x) = 2xe^{-x^2} \int_0^x e^{t^2}\, dt \tag{10}$$

so that

$$G(x) = 1 - \varphi(x) + i\sqrt{\pi} x e^{-x^2} \tag{11}$$

5.4 Function $Y(x)$ of *Fried and Conte* (1961)

Fried and Conte (1961) introduce

$$Y(x) = \frac{e^{-x^2}}{x} \int_0^x e^{t^2}\, dt \tag{12}$$

so that for real argument

$$Z(x) = i\sqrt{\pi} e^{-x^2} - 2xY(x) \tag{13}$$

6 Asymptotic formulas

6.1 For $|x| \ll 1$ (series expansion)

See *Abramowitz and Stegun* (1965, 7.1.8):

$$w(x) = \sum_{n=0}^{\infty} \frac{(ix)^n}{\Gamma\left(\frac{n}{2} + 1\right)} \tag{14}$$

The even terms give e^{-x^2}. To collect odd terms, note that for $n \geq 0$:

$$\Gamma\left(n + \frac{3}{2}\right) = \frac{\sqrt{\pi}}{2} \frac{(2n+1)!!}{2^n} \tag{15}$$

$$\Gamma\left(n + \frac{1}{2}\right) = \sqrt{\pi} \frac{(2n-1)!!}{2^n} \tag{16}$$

where we define $(2n+1)!! = 1 \cdot 3 \cdot 5 \cdots (2n+1)$ and $(-1)!! = 1$. We have

$$w(x) = e^{-x^2} + \frac{2ix}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-2x^2)^n}{(2n+1)!!} \tag{17}$$

$$Z(x) = i\sqrt{\pi} e^{-x^2} - 2x \sum_{n=0}^{\infty} \frac{(-2x^2)^n}{(2n+1)!!} \tag{18}$$

The first few terms are

$$w(x) \approx e^{-x^2} + \frac{2ix}{\sqrt{\pi}}\left(1 - \frac{2x^2}{3} + \frac{4x^4}{15} - \dots\right) \tag{19}$$

$$Z(x) \approx i\sqrt{\pi}e^{-x^2} - 2x\left(1 - \frac{2x^2}{3} + \frac{4x^4}{15} - \dots\right) \tag{20}$$

The Jackson function (8) also has a nice expansion

$$G(x) = \sqrt{\pi}\sum_{n=0}^{\infty}\frac{(ix)^n}{\Gamma\left(\frac{n+1}{2}\right)} \tag{21}$$

$$= i\sqrt{\pi}xe^{-x^2} + \sum_{n=0}^{\infty}\frac{(-2x^2)^n}{(2n-1)!!} \tag{22}$$

$$\approx 1 + i\sqrt{\pi}x - 2x^2 + \dots \tag{23}$$

6.2 For $|x| \gg 1$

These formulas are valid for $-\pi/4 < \arg x < 5\pi/4$ (*Abramowitz and Stegun*, 1965, 7.1.23), i.e., around positive imaginary axis:

$$w(x) = \frac{i}{\sqrt{\pi}x}\sum_{m=0}^{\infty}\frac{(2m-1)!!}{(2x^2)^m}$$

$$\approx \frac{i}{\sqrt{\pi}x}\left(1 + \frac{1}{2x^2} + \frac{3}{4x^4} + \dots\right) + e^{-x^2} \tag{24}$$

$$Z(x) = -\frac{1}{x}\sum_{m=0}^{\infty}\frac{(2m-1)!!}{(2x^2)^m}$$

$$\approx -\frac{1}{x}\left(1 + \frac{1}{2x^2} + \frac{3}{4x^4} + \dots\right) + i\sqrt{\pi}e^{-x^2} \tag{25}$$

$$G(x) = -\sum_{m=1}^{\infty}\frac{(2m-1)!!}{(2x^2)^m}$$

$$\approx -\frac{1}{2x^2} - \frac{3}{4x^4} - \dots + i\sqrt{\pi}xe^{-x^2} \tag{26}$$

7 Gordeyev's integral

Gordeyev's integral $G_\nu(\omega, \lambda)$ (*Gordeyev*, 1952) is defined as

$$G_\nu(\omega, \lambda) = \omega\int_0^{\infty}\exp\left[i\omega t - \lambda(1 - \cos t) - \frac{\nu t^2}{2}\right]dt, \qquad \Re\nu > 0 \tag{27}$$

and is calculated in terms of the plasma dispersion function Z (*Paris*, 1998):

$$G_\nu(\omega,\lambda) = \frac{-i\omega}{\sqrt{2\nu}} e^{-\lambda} \sum_{n=-\infty}^{\infty} I_n(\lambda) Z\left(\frac{\omega - n}{\sqrt{2\nu}}\right) \tag{28}$$

8 Plasma permittivity

The dielectric permittivity of hot (Maxwellian) plasma is (*Jackson*, 1960)

$$\epsilon(\omega, \mathbf{k}) = 1 + \sum_s \Delta\epsilon_s = 1 + \sum_s \frac{1}{k^2\lambda_s^2} G(x_s) \tag{29}$$

The summation is over charged species. For each species, we have introduced the Debye length

$$\lambda = \sqrt{\frac{\epsilon_0 T}{N q^2}} = \frac{v}{\Pi}$$

where $v = \sqrt{T/m}$ is the thermal velocity and $\Pi = \sqrt{Nq^2/(m\epsilon_0)}$ is the plasma frequency; and

$$x = \frac{\omega - (\mathbf{k}\cdot\mathbf{u})}{\sqrt{2}kv}$$

where $\mathbf{u}$ is the species drift velocity.

For warm components, $x \gg 1$, we have

$$\Delta\epsilon = -\frac{\Pi^2}{[\omega - (\mathbf{k}\cdot\mathbf{u})]^2}\left(1 + \frac{3k^2v^2}{[\omega - (\mathbf{k}\cdot\mathbf{u})]^2}\right)$$

The dispersion relation for plasma oscillations is obtained by equating $\epsilon = 0$. For example, for warm electron plasma at rest,

$$\epsilon = 1 - \frac{\Pi^2}{\omega^2}\left(1 + \frac{3k^2v^2}{\omega^2}\right)$$

and we have the dispersion relation (for $\omega \approx \Pi$):

$$\omega^2 = \Pi^2 + 3k^2v^2 = \Pi^2 + k^2\langle v^2\rangle$$

9 Ion acoustic waves

Assume that for electrons, $\omega \ll kv$ but for ions still $\omega \gg kV$ (we'll see later that in means $T_i \ll T_e$). For $x \ll 1$, we use $G(x) \approx 1 + i\sqrt{\pi}x$:

$$\epsilon = 1 + \frac{\Pi_e^2}{k^2v^2}\left(1 + i\sqrt{\pi}\frac{\omega}{\sqrt{2}kv}\right) - \frac{\Pi_i^2}{\omega^2}\left(1 + \frac{3k^2V^2}{\omega^2}\right) \tag{30}$$

where v and V are thermal velocities of electrons and ions, respectively. From $\epsilon = 0$ we have

$$1 + \frac{3k^2V^2}{\omega^2} = \frac{\omega^2}{k^2v_s^2}\left(1 + k^2\lambda_e^2 + i\sqrt{\frac{\pi}{2}}\frac{\omega}{\Pi_e k\lambda_e}\right)$$

where $v_s = \lambda_e\Pi_i = \sqrt{ZT_e/M}$. If we neglect V and imaginary part, then we get

$$\omega = \frac{kv_s}{\sqrt{1 + k^2\lambda_e^2}} \tag{31}$$

For long wavelengths, it reduces to the usual relation $\omega = kv_s$. If we substitute this into the imaginary part, we get

$$\omega \approx \frac{kv_s}{\sqrt{1 + i\sqrt{\frac{\pi Zm}{2M}}}} \tag{32}$$

The attenuation coefficient for the ion-acoustic waves is small:

$$\gamma = -\Im\omega = kv_s\sqrt{\frac{\pi Zm}{8M}} \ll \Re\omega$$

Neglecting V is equivalent to $kV \ll \omega$, i.e., $V \ll v_s$ or $T_i \ll T_e$. Otherwise, the ion-acoustic waves do not exist.

10 Fourier transform of $Z(x)$

In the plasma dielectric permittivity expression, the argument of $G(x) = -Z'(x)$ is $x = [\omega - (\mathbf{k}\cdot\mathbf{u})]/[\sqrt{2}kv]$. Thus, the plasma dispersion function is usually applied in the frequency domain. The argument is dimensionless, but is proportional to ω. Let us consider an inverse Fourier transform of $\tilde{F}(\omega) =$

$Z(\omega/[\sqrt{2}\Omega])$, where Ω is a parameter which has the same dimensionality as ω. (In the plasma dielectric permittivity expression, we have $\Omega \equiv kv$.)

Quick reminder of the time-frequency and space-wavevector Fourier transforms:

$$\begin{aligned}\tilde{X}(\omega, \mathbf{k}) &= \iint X(t, \mathbf{r}) e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}}\, dt\, d^3\mathbf{r} \\ X(t, \mathbf{r}) &= \iint \tilde{X}(\omega, \mathbf{k}) e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}} \frac{d\omega}{2\pi} \frac{d^3\mathbf{k}}{(2\pi)^3}\end{aligned}$$

where integrals are from $-\infty$ to $+\infty$.

Using expression (6), we have

$$\tilde{F}(\omega) = Z\left(\frac{\omega}{\sqrt{2}\Omega}\right) = \frac{2\pi}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{e^{-\frac{\omega'^2}{2\Omega^2}}}{\omega' - \omega - i\Delta} \frac{d\omega'}{2\pi}$$

where the condition that $\Im\omega > 0$ is implemented by adding a small imaginary part to ω, i.e., $\omega \to \omega + i\Delta$. We notice that the above expression is a convolution which simply gives a product in the t-domain:

$$\tilde{F}(\omega) = \int_{-\infty}^{+\infty} \tilde{F}_1(\omega')\tilde{F}_2(\omega - \omega') \frac{d\omega'}{2\pi} \quad \Longrightarrow \quad F(t) = F_1(t)F_2(t)$$

where

$$\tilde{F}_1(\omega) = 2\pi \frac{1}{\sqrt{2\pi}\Omega} e^{-\frac{\omega^2}{2\Omega^2}} \quad \Longrightarrow \quad F_1(t) = e^{-\frac{\Omega^2 t^2}{2}}$$

and

$$\tilde{F}_2(\omega) = \sqrt{2} i\Omega \frac{i}{\omega + i\Delta} \quad \Longrightarrow \quad F_2(t) = \sqrt{2} i\Omega H(t)$$

where $H(t)$ is the Heaviside (step) function, defined to be $H(t) = 0$ for $t < 0$ and $H(t) = 1$ for $t > 0$. (The value at $t = 0$ is not important, but most often is assumed to be $1/2$.) The last inverse Fourier trasform is accomplished by using the usual technique of integrating over a closed contour in the plane of complex ω around the pole at $-i\Delta$ and taking a residue. Note that the Fourier transform between $F_2(t)$ and $\tilde{F}_2(\omega)$ illuminates the physical sense of the trick used in equation (5).

Thus,

$$\tilde{F}(\omega) = Z\left(\frac{\omega}{\sqrt{2}\Omega}\right) \quad \Longrightarrow \quad F(t) = \sqrt{2} i\Omega H(t) e^{-\frac{\Omega^2 t^2}{2}}$$

The derivative $G(x) = -Z'(x)/2$ is found by using $d/d\omega \to it$:

$$\tilde{F}_G(\omega) = G\left(\frac{\omega}{\sqrt{2}\Omega}\right) = -\frac{\Omega}{\sqrt{2}}\frac{d}{d\omega}\tilde{F}(\omega) \quad \Longrightarrow \quad F_G(t) = \Omega^2 t H(t) e^{-\frac{\Omega^2 t^2}{2}}$$

The shift by $\Delta\omega = (\mathbf{k} \cdot \mathbf{u})$ is accomplished using the property

$$\tilde{F}(\omega - \Delta\omega) \quad \Longrightarrow \quad e^{-i\Delta\omega t} F(t)$$

Finally, we have the dielectric permittivity in time-wavevector domain:

$$\epsilon(t, \mathbf{k}) = \delta(t) + \sum_s \Delta\epsilon_s(t, \mathbf{k}) = \delta(t) + tH(t) \sum_s \Pi_s^2 e^{-\frac{k^2 v_s^2 t^2}{2} - i(\mathbf{k}\cdot\mathbf{u}_s)t} \qquad (33)$$

where the delta function is obtained from transforming "1". The same answer may be obtained from the first principles by calculating the polarization created by a delta-function electric field in the time-space domain and converting $\mathbf{r} \to \mathbf{k}$. But this is a completely different topic.

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