Clarkson's Theorem

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Let $P \subseteq \mathbb{R}^2$ be a planar point set of size n. The set $S_{\leq k}$ of $(\leq k)$-*sets* of P is defined as

$$S_{\leq k} := \{Q \subseteq P \mid |Q| \leq k \text{ and } Q = P \cap h, h \text{ open halfplane}\}.$$

Clarkson's theorem gives an upper bound on the number of possible $(\leq k)$-sets.

Theorem 1. *We have* $|S_{\leq k}| = O(nk)$.

Proof. We may assume that $2 \leq k \leq n-2$, since otherwise the theorem clearly holds.

We begin with a definition: Let $0 \leq \ell \leq k$. A pair $(p,q) \in P^2$ of distinct points in P is called ℓ-*edge* if and only if $|P \cap h^+_{\overrightarrow{pq}}| = \ell$. Here, $h^+_{\overrightarrow{pq}}$ denotes the open halfplane to the left of the oriented line $\overrightarrow{pq}$. Let $L_{\leq k}$ be the set of all $(\leq k)$-edges.

We have $|S_{\leq k}| \leq 2|L_{\leq k}|$. We can assign to each ℓ-edge (p,q) one ℓ- and one $(\ell+1)$-set, namely the ℓ-set $P \cap h_{\overrightarrow{pq}}$, and the $(\ell+1)$-set that is cut off from P after slightly rotating $\overrightarrow{pq}$ clockwise around p. Every $(\leq k)$-set Q can be obtained this way. To see this, take a line g that bounds the halfplane defining Q. Translate g away from Q until it hits a point from P, then rotate g counterclockwise until it hits another point from P.

Let $R \subseteq P$ a random subset of P containing each point $p \in P$ independently with probability $1/k$. We consider the set $E(\mathrm{CH}(R))$ of the edges on the convex hull of R, and we bound the size of $E(\mathrm{CH}(R))$ in two different ways.

On the one hand, we have

$$\mathbf{E}[|E(\mathrm{CH}(R))|] \leq \mathbf{E}[|R|] = n/k,$$

since the convex hull of R has at most $|R|$ edges, and each point from P was chosen with probability $1/k$.

Now let $(p,q) \in P^2$ be a pair of distinct points in P, and let $I_{(p,q)}$ be the indicator random variable for the event that (p,q) defines a (clockwise) edge on $\mathrm{CH}(R)$. Then,

$$\mathbf{E}[|E(\mathrm{CH}(R))|] = \sum_{(p,q)\in P^2} \mathbf{E}[I_{(p,q)}] \geq \sum_{(p,q)\in L_{\leq k}} \mathbf{E}[I_{(p,q)}],$$

by linearity of expectation. For a $(\leq k)$-edge (p,q) we have that $\mathbf{E}[I_{(p,q)}]$ is precisely the probability of the event $(p,q) \in E(\mathrm{CH}(R))$. For this event to happen, we must have (i) $p,q \in R$; and (ii) $R \cap h^+_{\overrightarrow{pq}} = \emptyset$. The probability for this is at least $k^{-2}(1-1/k)^k$, since $|P \cap h^+_{\overrightarrow{pq}}| \leq k$ and since the points in R were chosen independently.

It follows that

$$\mathbf{E}[|E(\mathrm{CH}(R))|] \geq \sum_{(p,q)\in L_{\leq k}} \mathbf{E}[I_{(p,q)}] \geq \sum_{(p,q)\in L_{\leq k}} k^{-2}(1-1/k)^k \geq |L_{\leq k}|/4k^2,$$

as $k \geq 2$. Hence, $|L_{\leq k}| \leq 4nk$ and $|S_{\leq k}| \leq 8nk$. □