Complexity of Finding Nearest Colorful Polytopes

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Abstract

Let $P_1, \dots, P_{d+1} \subset \mathbb{R}^d$ be point sets whose convex hulls each contain the origin. Each set represents a color class. The *Colorful Carathéodory theorem* guarantees the existence of a *colorful choice*, i.e., a set that contains exactly one point from each color class, whose convex hull also contains the origin. The computational complexity of finding such a colorful choice is still unknown. We study a natural generalization of the problem: in the *Nearest Colorful Polytope* problem (NCP), we are given sets $P_1, \dots, P_n \subset \mathbb{R}^d$, and we would like to find a colorful choice whose convex hull minimizes the distance to the origin.

We show that computing local optima of the NCP problem is PLS-complete, while computing a global optimum is NP-hard.

1 Introduction

Let $P \subset \mathbb{R}^d$ be a point set. Carathéodory's theorem [4, Chapter 1] states that if $\vec{0} \in \mathsf{conv}(P)$, we can find a subset $P' \subseteq P$ of at most $d+1$ points with $\vec{0} \in \mathsf{conv}(P')$. Bárány [2] has generalized this result to the colorful setting:

Theorem 1 (Colorful Carathéodory theorem) *Let $P_1, \dots, P_{d+1} \subset \mathbb{R}^d$ be point sets (the color classes). If $\vec{0} \in \mathsf{conv}(P_i)$ for $i = 1, \dots, d+1$, there is a colorful choice C with $\vec{0} \in \mathsf{conv}(C)$. Here, a* colorful choice *is a set with exactly one point from each color class.*

Proof. (*sketch*) Let C be some colorful choice. Assume $\vec{0} \notin \mathsf{conv}(C)$ (otherwise, we are done). Let F be the facet of $\mathsf{conv}(C)$ nearest to the origin and let $p \in C$ be a point in the color class P_i that is no vertex of F. Define h to be the hyperplane through F. Since $\vec{0} \in \mathsf{conv}(P_i)$, there is a point $p' \in P_i$ that is separated from $\mathsf{conv}(C)$ by h. Then, the hull $\mathsf{conv}(C \setminus \{p\} \cup \{p'\})$ is strictly closer to the origin. Since there are only finitely many colorful choices, this concludes the proof. □

Carathéodory's theorem follows by setting $P_1 = \dots = P_{d+1}$. While Carathéodory's theorem can be cast as a linear program and thus be implemented in polynomial time, very little is known about the complexity of the colorful Carathéodory theorem. The problem of finding a colorful choice is contained in *Total Functional NP* (TFNP), the class of total search problems that can be computed in non-deterministic polynomial time. It is well-known that no problem in TFNP can be NP-hard unless NP = coNP [3, Lemma 4]. As a first step towards settling the complexity of Colorful Carathéodory, we consider a further generalization: the *Nearest Colorful Polytope (NCP) problem.* Given a family of color classes $P_1, \dots, P_n \subset \mathbb{R}^d$, find a colorful choice whose convex hull minimizes the distance to the origin. We study this problem in two variants: as a local search problem, in which we want to find colorful choices whose distance of the convex hull to the origin cannot be reduced by exchanging a *single* point with another point of the same color; and as a global search problem, in which we want to compute colorful choices that minimize the distance over all colorful choices. We refer to these problems as L-NCP and G-NCP, respectively. The local search variant is particularly interesting since Bárány's proof of the Colorful Carathéodory theorem gives a local search algorithm and is thus tied closely to L-NCP.

In the first part of Section 2, we show that L-NCP is complete for *Polynomial-Time Local Search* (PLS) [3], a subclass of TFNP. A problem in PLS is defined by

- a set of problem instances $\mathfrak{I}$;
- for each problem instance, a set $\mathfrak{S}$ of polynomial-time verifiable solutions and a polynomial-time algorithm that returns a base solution;
- a polynomial-time computable cost function $\mathfrak{C}$ that, given the instance, weights the solutions; and
- a polynomial-time neighborhood algorithm $\mathfrak{N}$, which, given some solution and the instance, returns a set of neighboring solutions.

The problem is to find a *local optimum*, that is, a solution S^* for which all neighbors $S \in \mathfrak{N}(S^*)$ have

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larger cost (in a minimization problem) or smaller cost (in a maximization problem). The problem definition suggests a simple algorithm: start with the base solution and use $\mathfrak{N}$ to improve until a local optimum is reached. Each iteration takes polynomial-time, but the total number of iterations may be exponential. This is called the *standard algorithm*. There are examples where it is PSPACE-hard to find the solution given by the standard algorithm [1, Chapter 2].

A *PLS-reduction* from a PLS-problem A to a PLS-problem B is defined by two polynomial-time computable functions f and g: f maps problem instances of A to problem instances of B, and g maps local optima of B to local optima of A. Thus, if A is PLS-reducible to B, we can transform any algorithm for B with polynomial-time overhead into an algorithm for A. For a more thorough discussion of PLS and PLS-reductions see [1, 3, 5, 6].

2 Complexity

We start by defining the local search variant of NCP as a PLS problem. Then we prove completeness. We also show NP-hardness of the global search variant.

Definition 1 (L-NCP)

Instances $\mathfrak{I}_{\mathbf{NCP}}$. *Set families $P = \{P_1, \dots, P_n\}$ in $\mathbb{R}^d$, where each $P_i \subset \mathbb{R}^d$ represents a color class.*

Solutions $\mathfrak{S}_{\mathbf{NCP}}$. *All colorful choices, i.e., sets that contain exactly one point from each color class.*

Cost function $\mathfrak{C}_{\mathbf{NCP}}$. *Let S_{NCP} be a colorful choice. Then, $\mathfrak{C}(S_{NCP}) = \|\mathsf{conv}(S_{NCP})\|_1$, where we set $\|\mathsf{conv}(S_{NCP})\|_1 = \min\{\|q\|_1 \mid q \in \mathsf{conv}(S_{NCP})\}$. We want to minimize the cost function.*

Neighborhood $\mathfrak{N}_{\mathbf{NCP}}$. *The neighbors $\mathfrak{N}(S_{NCP})$ of a colorful choice S_{NCP} are all colorful choices that can be obtained by swapping one point with another point of the same color.*

The following PLS problem is used to show completeness of L-NCP. It was shown to be PLS-complete by Schäffer and Yannakakis [6, Corollary 5.12].

Definition 2 (Max-2SAT/Flip)

Instances $\mathfrak{I}_{\mathbf{M2SAT}}$. *All weighted CNF formulas $\bigwedge_{i=1}^{d} C_i$, where each clause C_i is the disjunction of at most two literals and has weight $w_i \in \mathbb{N}_+$.*

Solutions $\mathfrak{S}_{\mathbf{M2SAT}}$. *Let $x_1, x_2, \dots, x_n$ be the variables appearing in the clauses. Then, every complete assignment $\mathcal{A} : \{x_1, \dots, x_n\} \to \{0, 1\}$ of these variables is a solution.*

Cost function $\mathfrak{C}_{\mathbf{M2SAT}}$. *The cost of an assignment is the sum of the weights of all satisfied clauses. We want to maximize the cost function.*

Neighborhood $\mathfrak{N}_{\mathbf{M2SAT}}$. *The neighbors $\mathfrak{N}(\mathcal{A})$ of an assignment $\mathcal{A}$ are all assignments obtained by flipping (i.e., negating) a single variable in $\mathcal{A}$.*

Theorem 2 *L-NCP is PLS-complete.*

Proof. Fix an instance $I_{\text{M2SAT}} = (C_1, \dots, C_d, w_1, \dots, w_d, x_1, \dots, x_n)$ of M2SAT. We construct an instance I_{NCP} of L-NCP in which each colorful choice encodes an assignment to the variables in I_{M2SAT}. Furthermore, the distance to the origin of the convex hull of a colorful choice in I_{NCP} will be the total weight of all unsatisfied clauses of the encoded assignment in I_{M2SAT}.

For each variable x_i, we introduce a color class $P_i = \{p_i, \overline{p_i}\}$ consisting of two points in $\mathbb{R}^d$ that encode whether x_i is set to 1 or 0. We assign the jth dimension to the jth clause and set $(p_i)_j$ to $-nw_j$ if $x_i = 1$ satisfies the jth clause and w_j otherwise. Similarly, $(\overline{p_i})_j = -nw_j$ if $x_i = 0$ satisfies C_j and w_j otherwise. A colorful choice of these sets corresponds to the assignment of variables in I_{M2SAT} in which x_i is set to 1 if $p_i \in P_i$ was chosen and set to 0 if $\overline{p_i} \in P_i$ was picked. More formally, we define a mapping $g : \mathfrak{S}_{\text{NCP}} \to \mathfrak{S}_{M2SAT}$ between the solutions of the L-NCP instance and the M2SAT instance in the following way:

$$g(S_{\text{NCP}})(x_i) = \begin{cases} 1 & \text{if } p_i \in S_{\text{NCP}} \\ 0 & \text{if } \overline{p_i} \in S_{\text{NCP}} \end{cases}$$

The idea of the construction is that in a colorful choice S the convex hull $\mathsf{conv}(S)$ contains the origin in the subspace spanned by the satisfied clauses. In the subspace corresponding to the unsatisfied clauses, the point in $\mathsf{conv}(S)$ closest to the origin has the weight of the jth clause as its jth coordinate. To make this work, we need some additional helper color classes. For each clause C_j, we have a color class $H_j = \{h_j\}$ with a single point, where

$$(h_j)_k = (d+1)\left((n+2) - \frac{d}{d+1}\right) w_j, \quad \text{if } k = j,$$

and w_k otherwise. Finally, the last helper color class $H_{d+1} = \{h_{d+1}\}$ again contains a single point, but this time all coordinates are set to the weights of the clauses, i.e., $(h_{d+1})_j = w_j$. See Fig. 1 for an example.

The remaining proof is divided into two parts:
(i) First, we prove that for every colorful choice S_{NCP} of the L-NCP problem instance $\{P_1, \dots, P_n, H_1, \dots, H_{d+1}\}$, the cost $\mathfrak{C}_{\text{NCP}}(S_{\text{NCP}})$ is lower-bounded by total weight of the unsatisfied clauses in $g(S_{\text{NCP}})$.
(ii) Second, we show that this lower bound is tight, i.e., the distance of the convex hull of any colorful choice S_{NCP} to the origin is at most the total weight of the unsatisfied clauses in $g(S_{\text{NCP}})$.

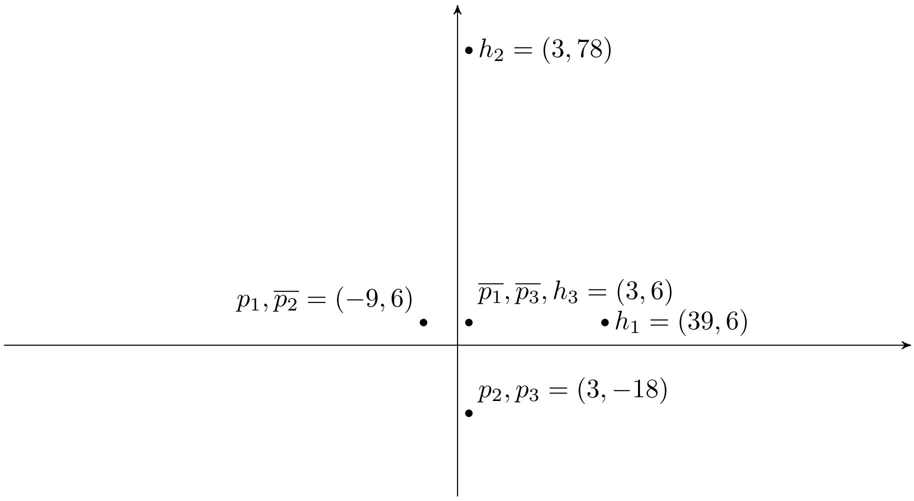

Figure 1: Construction of the point sets corresponding to the M2SAT instance $(x_1 \vee \overline{x_2}) \wedge (x_2 \vee x_3)$ with weights 3 and 6, respectively.

Both claims together imply that $\mathfrak{C}_{\mathrm{NCP}}(S_{\mathrm{NCP}})$ equals the total weight of the unsatisfied clauses for the assignment $g(S_{\mathrm{NCP}})$, which proves the theorem: consider some local optimum S^*_{NCP} of the L-NCP instance. By definition, the costs of all other colorful choices that can be obtained from S^*_{NCP} by exchanging one point with another of the same color are greater or equal to $\mathfrak{C}_{\mathrm{NCP}}(S^*_{\mathrm{NCP}})$. That is, the total weight of the unsatisfied clauses in $g(S^*_{\mathrm{NCP}})$ cannot be decreased by flipping a variable, which is equivalent to $g(S^*_{\mathrm{NCP}})$ being a local optimum of the M2SAT instance.

(i) Let S_{NCP} be a colorful choice and assume some clause C_j is not satisfied by $g(S_{\mathrm{NCP}})$. By construction, the jth coordinate of each point q in S_{NCP} is at least w_j. Thus, the jth coordinate of every convex combination of the points in S_{NCP} is at least w_j. This implies **(i)**.

(ii) Given a colorful choice S_{NCP}, we construct a convex combination of S_{NCP} that gives a point p whose distance to the origin is exactly the total weight of the unsatisfied clauses in $g(S_{\mathrm{NCP}})$.

Let in the following part A_k denote the set of clauses C_j that are satisfied by k literals wrt $g(S_{\mathrm{NCP}})$, for $k = 0, 1, 2$. As a first step towards constructing p, we show the existence of an intermediate point in the convex hull of the helper classes:

Lemma 3 *There is a point $h \in \mathsf{conv}(H_1, \ldots, H_{d+1})$ whose jth coordinate is $(n+2)w_j$ if $j \in A_2$ and w_j otherwise.*

Proof. Take

$$h = \sum_{a \in A_2} \frac{1}{d+1} h_a + \left(1 - \frac{|A_2|}{d+1}\right) h_{d+1}$$

Then, for $j \in A_0 \cup A_1$, we have

$$\begin{aligned}
(h)_j &= \sum_{a \in A_2} \frac{1}{d+1} (h_a)_j + \left(1 - \frac{|A_2|}{d+1}\right) (h_{d+1})_j \\
&\overset{j \notin A_2}{=} \sum_{a \in A_2} \frac{1}{d+1} w_j + \left(1 - \frac{|A_2|}{d+1}\right) w_j \\
&= w_j
\end{aligned}$$

And for $j \in A_2$, we have

$$\begin{aligned}
(h)_j &= \sum_{a \in A_2} \frac{1}{d+1} (h_a)_j + \left(1 - \frac{|A_2|}{d+1}\right) (h_{d+1})_j \\
&= \frac{1}{d+1} h_j + \sum_{a \in A_2 \setminus \{j\}} \frac{1}{d+1} (h_a)_j + \\
&\qquad \left(1 - \frac{|A_2|}{d+1}\right) (h_{d+1})_j \\
&= \left((n+2) - \frac{d}{d+1}\right) w_j + \frac{d}{d+1} w_j \\
&= (n+2) w_j,
\end{aligned}$$

as desired. □

Let $l_i \in P_i$ be the point from P_i in S_{NCP}. Consider

$$p = \sum_{i \in [n]} \frac{1}{n+1} l_i + \frac{1}{n+1} h$$

We show that $(p)_j$ is w_j for $j \in A_0$ and 0 otherwise. Let us start with $j \in A_0$. Since $g(S_{\mathrm{NCP}})$ does not satisfy C_j, the jth coordinate of the points $l_1, \ldots, l_n$ is w_j. Also, $(h)_j = w_j$, by Lemma 3. Thus, $(p)_j = w_j$. Consider now some $j \in A_1$ and let b be s.t. the point

l_b corresponds to the single literal that satisfies C_j.

$$\begin{aligned}(p)_j &= \sum_{i\in[n]} \frac{1}{n+1}(l_i)_j + \frac{1}{n+1}(h)_j \\ &= \frac{1}{n+1}(l_b)_j + \sum_{i\in[n]\setminus\{b\}} \frac{1}{n+1}(l_i)_j + \frac{1}{n+1}(h)_j \\ &= \frac{-n}{n+1}w_j + \frac{n}{n+1}w_j = 0\end{aligned}$$

And finally, consider some $j \in A_2$ and let b_1, b_2 be the indices of the two literals that satisfy C_j:

$$\begin{aligned}(p)_j &= \sum_{i\in[n]} \frac{1}{n+1}(l_i)_j + \frac{1}{n+1}(h)_j \\ &= \frac{1}{n+1}(l_{b_1})_j + \frac{1}{n+1}(l_{b_2})_j + \\ &\qquad \sum_{i\in[n]\setminus\{b_1,b_2\}} \frac{1}{n+1}(l_i)_j + \frac{1}{n+1}(h)_j \\ &= \frac{-2n}{n+1}w_j + \frac{n-2}{n+1}w_j + \frac{n+2}{n+1}w_j = 0\end{aligned}$$

This concludes the proof of (*ii*). □

Remark. We have used the L_1-norm to define $\mathfrak{C}_{\text{NCP}}$. The reduction easily extends to any L_p norm by replacing the occurrence of any clause weight in the coordinates of the created points with its pth root. Note that for $p \neq 1$ the cost functions do not longer coincide.

Theorem 4 *G-NCP is NP-hard.*

Proof. The proof of Theorem 2 can be adapted easily to reduce 3SAT to G-NCP: given a set of clauses $C_1, \ldots, C_d$, we set the weight of each clause to 1 and construct the same point sets as in the PLS reduction. Additionally, we introduce for each clause C_j a new helper color class $H'_j = \{h'_j\}$, where

$$(h'_i)_j = (d+1)\left((2n+2) - \frac{d}{d+1}\right) \qquad \text{if } i = j$$

and 1 otherwise. Let S now be any colorful choice and $A = g(S)$ the corresponding assignment. As in the PLS-reduction, we define the sets A_k, $k = 0, \ldots, 3$, to contain all clauses that are satisfied by exactly k literals in the assignment A. Then, the following point h is contained in the convex hull of the helper points:

$$h = \sum_{a\in A_2} \frac{h_a}{d+1} + \sum_{a'\in A_3} \frac{h'_{a'}}{d+1} + \left(1 - \frac{|A_2|}{d+1}\right) h_{d+1}$$

Again, the convex combination

$$p = \sum_{i\in[n]} \frac{1}{n+1} l_i + \frac{1}{n+1} h$$

results in a point in the convex hull of S whose distance to the origin is the number of unsatisfied clauses, where $l_i \in P_i$ denotes the point from P_i that is contained in S. Together with Claim **(i)** from the proof of Theorem 2, 3SAT can be decided by knowing a global optimum S^* to the NCP problem: if the distance from $\mathsf{conv}(S^*)$ to the origin is 0, $g(S^*)$ is a satisfying assignment. If not, there exists no satisfying assignment at all. □

3 Conclusion

Motivated by the Colorful Carathéodory problem, we have studied the complexity of a generalization, the Nearest Colorful Polytope problem, in two settings: first, we have proved that the corresponding local search problem is PLS-complete by a reduction to Max2SAT. Using an adaptation of the PLS-reduction, we could prove that the problem becomes NP-hard if we restrict the solutions to global optima.

Although the PLS-completeness of the Nearest Colorful Polytope problem together with Bárány's proof indicate that PLS is the right complexity class to show hardness of the Colorful Carathéodory problem, there is a striking difference between the Colorful Carathéodory problem and any known PLS-complete problem: the costs of local optima are known a-priori. While a PLS-complete problem with this property would not lead to a contradiction, this creates a major stumbling block in the construction of a reduction.

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