# The Rainbow at the End of the Line — A PPAD Formulation of the Colorful Carathéodory Theorem with Applications

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**Abstract**

Let $C_1, ..., C_{d+1}$ be $d+1$ point sets in $\mathbb{R}^d$, each containing the origin in its convex hull. A subset $C$ of $\bigcup_{i=1}^{d+1} C_i$ is called a colorful choice (or rainbow) for $C_1, \ldots, C_{d+1}$, if it contains exactly one point from each set $C_i$. The colorful Carathéodory theorem states that there always exists a colorful choice for $C_1, \ldots, C_{d+1}$ that has the origin in its convex hull. This theorem is very general and can be used to prove several other existence theorems in high-dimensional discrete geometry, such as the centerpoint theorem or Tverberg's theorem. The colorful Carathéodory problem (ColorfulCarathéodory) is the computational problem of finding such a colorful choice. Despite several efforts in the past, the computational complexity of ColorfulCarathéodory in arbitrary dimension is still open.

We show that ColorfulCarathéodory lies in the intersection of the complexity classes PPAD and PLS. This makes it one of the few geometric problems in PPAD and PLS that are not known to be solvable in polynomial time. Moreover, it implies that the problem of computing centerpoints, computing Tverberg partitions, and computing points with large simplicial depth is contained in PPAD $\cap$ PLS. This is the first nontrivial upper bound on the complexity of these problems.

Finally, we show that our PPAD formulation leads to a polynomial-time algorithm for a special case of ColorfulCarathéodory in which we have only two color classes $C_1$ and $C_2$ in $d$ dimensions, each with the origin in its convex hull, and we would like to find a set with half the points from each color class that contains the origin in its convex hull.

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## 1 Introduction

Let $P \subset \mathbb{R}^d$ be a $d$-dimensional point set. We say $P$ *embraces* a point $\boldsymbol{p} \in \mathbb{R}^d$ or $P$ is $\boldsymbol{p}$*-embracing* if $\boldsymbol{p} \in \text{conv}(P)$, and we say $P$ *ray-embraces* $\boldsymbol{p}$ if $\boldsymbol{p} \in \text{pos}(P)$, where $\text{pos}(P) = \left\{ \sum_{\boldsymbol{p} \in P} \alpha_{\boldsymbol{p}} \boldsymbol{p} \mid \alpha_{\boldsymbol{p}} \geq 0 \text{ for all } \boldsymbol{p} \in P \right\}$. Carathéodory's theorem [10, Theorem 1.2.3] states that if $P$ embraces the origin, then there exists a subset $P' \subseteq P$ of size $d+1$ that also embraces the origin. This was generalized by Bárány [1] to the *colorful* setting: let $C_1, \ldots, C_{d+1} \subset \mathbb{R}^d$ be point sets that each embrace the origin. We call a set $C = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{d+1}\}$ a *colorful choice* (or *rainbow*) for $C_1, \ldots, C_{d+1}$, if $\boldsymbol{c}_i \in C_i$, for $i = 1, \ldots, d+1$. The *colorful* Carathéodory theorem states that there always exists a $\mathbf{0}$-embracing colorful choice that contains the origin in its convex hull. Bárány also gave the following generalization.

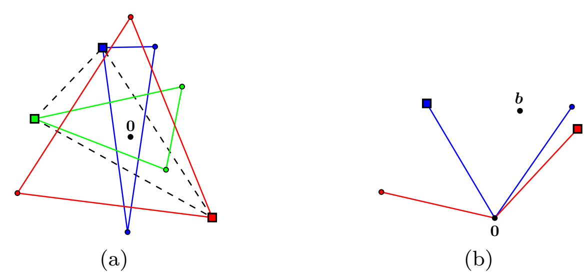


Figure 1: (a) Example of the convex version of Theorem 1.1 in two dimensions. (b) Example of the cone version of Theorem 1.1 in two dimensions.

Theorem 1.1. (Colorful Carathéodory Theorem, Cone Version [1]) *Let $C_1, \ldots, C_d \subset \mathbb{R}^d$ be point sets and $\boldsymbol{b} \in \mathbb{R}^d$ a point with $\boldsymbol{b} \in \text{pos}(C_i)$, for $i = 1, \ldots, d$. Then, there is a colorful choice $C$ for $C_1, \ldots, C_d$ that ray-embraces $\boldsymbol{b}$.*

The classic (convex) version of the colorful Carathéodory theorem follows easily from Theorem 1.1: lift the sets $C_1, \ldots, C_{d+1} \subset \mathbb{R}^d$ to $\mathbb{R}^{d+1}$ by appending a 1 to each element, and set $\boldsymbol{b} = (0, \ldots, 0, 1)^T$. See Figure 1 for an example of both versions in two dimensions.

Even though the cone version of the colorful Carathéodory theorem guarantees the existence of a colorful choice that ray-embraces the point $\boldsymbol{b}$, it is far from clear how to find it efficiently. We call this computational problem the *colorful Carathéodory problem* (ColorfulCarathéodory). To this day, settling the complexity of ColorfulCarathéodory remains an intriguing open problem, with a potentially wide range of consequences. We can use linear programming to check in polynomial time whether a given colorful choice ray-embraces a point, so ColorfulCarathéodory lies in *total function NP* (TFNP) [16], the complexity class of total search problems that can be solved in non-deterministic polynomial time. This implies that ColorfulCarathéodory cannot be NP-hard unless NP = coNP [8]. However, the complexity landscape inside TFNP is far from understood, and there exists a rich body of work that studies subclasses of TFNP meant to capture different aspects of mathematical existence proofs, such as the pigeonhole principle (PPP), potential function arguments (PLS, CLS), or various parity arguments (PPAD, PPA, PPADS) [6, 8, 16].

While the complexity of ColorfulCarathéodory remains elusive, related problems are known to be complete for PPAD or for PLS. For example, given $d+1$ point sets $C_1, \dots, C_{d+1} \subset \mathbb{Q}^d$ consisting of two points each and a colorful choice $C$ for $C_1, \dots, C_{d+1}$ that embraces the origin, it is PPAD-complete to find another colorful choice that embraces the origin [13]. Furthermore, given $d+1$ point sets $C_1, \dots, C_{d+1} \subset \mathbb{Q}^d$, we call a colorful choice $C$ for $C_1, \dots, C_{d+1}$ *locally optimal* if the $L_1$-distance of $\operatorname{conv}(C)$ to the origin cannot be decreased by swapping a point of color $i$ in $C$ with another point from the same color. Then, computing a locally optimal colorful choice is PLS-complete [15].

Understanding the complexity of ColorfulCarathéodory becomes even more interesting in the light of the fact that the colorful Carathéodory theorem plays a crucial role in proving several other prominent theorems in convex geometry, such as Tverberg's theorem [18] (and hence the centerpoint theorem [17]) and the first selection lemma [10, 1]. In fact, these proofs can be interpreted as polynomial time reductions from the respective computational problems, Tverberg, Centerpoint, and SimplicialCenter, to ColorfulCarathéodory. See the full version for more details.

Several approximation algorithms have been proposed for ColorfulCarathéodory. Bárány and Onn [2] describe an exact algorithm that can be stopped early to find a colorful choice whose convex hull is "close" to the origin. More precisely, let $\varepsilon, \rho > 0$ be parameters. We call a set *$\varepsilon$-close* if its convex hull has $L_2$-distance at most $\varepsilon$ to the origin. Given sets $C_1, \dots, C_{d+1} \subset \mathbb{R}^d$ such that (i) each $C_i$ contains a ball of radius $\rho$ centered at the origin in its convex hull; and (ii) all points $\boldsymbol{p} \in \bigcup_{i=1}^{d+1} C_i$ fulfill $1 \leq \|\boldsymbol{p}\| \leq 2$ and can be encoded using $L$ bits, one can find an $\varepsilon$-close colorful choice in time $O(\operatorname{poly}(L, \log(1/\varepsilon), 1/\rho))$ on the Word-Ram with logarithmic costs. For $\varepsilon = 0$, the algorithm actually finds a solution to ColorfulCarathéodory in finite time, and, more interestingly, if $1/\rho = O(\operatorname{poly}(L))$, the algorithm finds a solution to ColorfulCarathéodory in polynomial time. In the same spirit, Barman [3] showed that if the points have constant norm, an $\varepsilon$-close colorful choice can be found by solving $d^{O(1/\varepsilon^2)}$ convex programs. Mulzer and Stein [15] considered a different notion of approximation: a set is called *$m$-colorful* if it contains at most $m$ points from each $C_i$. They showed that for all fixed $\varepsilon > 0$, an $\lceil \varepsilon d \rceil$-colorful choice that contains the origin in its convex hull can be found in polynomial time.

**Our Results.** We provide a new upper bound on the complexity of ColorfulCarathéodory by showing that the problem is contained in PPAD ∩ PLS, implying the first nontrivial upper bound on the computational complexity of computing centerpoints or finding Tverberg partitions.

The traditional proofs of the colorful Carathéodory theorem all proceed through a potential function argument. Thus, it may not be surprising that ColorfulCarathéodory lies in PLS, even though a detailed proof that can deal with degenerate instances requires some care. On the other hand, showing that ColorfulCarathéodory lies in PPAD calls for a completely new approach. Even though there are proofs of the colorful Carathéodory theorem that use topological methods usually associated with PPAD (such as certain variants of Sperner's lemma) [7, 9], these proofs involve existential arguments that have no clear algorithmic interpretation. Thus, we present a new proof of the colorful Carathéodory theorem that proceeds similarly as the usual proof for Sperner's lemma [5]. This new proof has an algorithmic interpretation that leads to a formulation of ColorfulCarathéodory as a PPAD-problem.

Finally, we consider the special case of ColorfulCarathéodory that we are given two color classes $C_1, C_2 \subset \mathbb{R}^d$ of $d$ points each and a vector $\boldsymbol{b} \in \mathbb{R}^d$ such that both $C_1$ and $C_2$ ray-embrace $\boldsymbol{b}$. We describe an algorithm that solves the following problem in polynomial time: given $k \in [d]$, find a set $C \subseteq C_1 \cup C_2$ with $|C \cap C_1| = k$ and $|C \cap C_2| = d - k$ such that $C$ ray-embraces $\boldsymbol{b}$. Note that this is a special case of ColorfulCarathéodory since we can just take $k$ copies of $C_1$ and $d-k$ copies of $C_2$ in a problem instance for ColorfulCarathéodory.

## 2 Preliminaries

**The Complexity Class PPAD.** The complexity class *polynomial parity argument in a directed graph*

(PPAD) [16] is a subclass of TFNP that contains search problems that can be modeled as follows: let $G = (V, E)$ be a directed graph in which each node has indegree and outdegree at most one. That is, $G$ consists of paths and cycles. We call a node $v \in V$ a *source* if $v$ has indegree 0 and we call $v$ a *sink* if it has outdegree 0. Given a source in $G$, we want to find another source or sink. By a parity argument, there is an even number of sources and sinks in $G$ and hence another source or sink must exist. However, finding this sink or source is nontrivial since $G$ is defined implicitly and the total number of nodes may be exponential.

More formally, a problem in PPAD is a relation $\mathcal{R}$ between a set $\mathcal{I} \subseteq \{0,1\}^\star$ of *problem instances* and a set $\mathcal{S} \subset \{0,1\}^\star$ of *candidate solutions*. Assume further the following.

- The set $\mathcal{I}$ is polynomial-time verifiable. Furthermore, there is an algorithm that on input $I \in \mathcal{I}$ and $s \in \mathcal{S}$ decides in time $\mathrm{poly}(|I|)$ whether $s$ is a *valid* candidate solution for $I$. We denote with $\mathcal{S}_I \subseteq \mathcal{S}$ the set of all valid candidate solutions for a fixed instance $I$.

- There exist two polynomial-time computable functions pred and succ that define the edge set of $G$ as follows: on input $I \in \mathcal{I}$ and $s \in \mathcal{S}_I$, pred and succ return a valid candidate solution from $\mathcal{S}_I$ or $\perp$. Here, $\perp$ means that $v$ has no predecessor/successor.

- There is a polynomial-time algorithm that returns for each instance $I$ a valid candidate solution $s \in \mathcal{S}_I$ with $\mathrm{pred}(s) = \perp$. We call $s$ the *standard source*.

Now, each instance $I \in \mathcal{I}$ defines a graph $G_I = (V, E)$ as follows. The set of nodes $V$ is the set of all valid candidate solutions $\mathcal{S}_I$ and there is a directed edge from $u$ to $v$ if and only if $v = \mathrm{succ}(u)$ and $u = \mathrm{pred}(v)$. Clearly, each node in $G_I$ has indegree and outdegree at most one. The relation $\mathcal{R}$ consists of all tuples $(I, s)$ such that $s$ is a sink or source other than the standard source in $G_I$.

The definition of a PPAD-problem suggests a simple algorithm, called the *standard algorithm*: start at the standard source and follow the path until a sink is reached. This algorithm always finds a solution but the length of the traversed path may be exponential in the size of the input instance.

**Polyhedral Complexes and Subdivisions.** We call a finite set of polyhedra $\mathcal{P}$ in $\mathbb{R}^d$ a *polyhedral complex* if and only if (i) for all polyhedra $f \in \mathcal{P}$, all faces of $f$ are contained in $\mathcal{P}$; and (ii) for all $f, f' \in \mathcal{P}$, the intersection $f \cap f'$ is a face of both. Note that the first requirement implies that $\emptyset \in \mathcal{P}$. Furthermore, we say $\mathcal{P}$ has *dimension* $k$ if there exists some polyhedron $f \in \mathcal{P}$ with $\dim f = k$ and all other polyhedra in $\mathcal{P}$ have dimension at most $k$. We call $\mathcal{P}$ a *polytopal complex* if it is a polyhedral complex and all elements are polytopes. Similarly, we say $\mathcal{P}$ is a *simplicial complex* if it is a polytopal complex whose elements are simplices. Finally, we say $\mathcal{P}$ *subdivides* a set $Q \subseteq \mathbb{R}^d$ if $\bigcup_{f \in \mathcal{P}} f = Q$. For more details, see [19, Section 5.1].

**Linear Programming.** Let $A \in \mathbb{R}^{d \times n}$ be a matrix and $F$ a set of column vectors from $A$. Then, we denote with $\mathrm{ind}\,(F) \subseteq [n]$ the set of column indices in $F$ and for an index set $I \subseteq [n]$, we denote with $A_I$ the submatrix of $A$ that consists of the columns indexed by $I$. Similarly, for a vector $\boldsymbol{c} \in \mathbb{R}^n$ and an index set $I \subset [n]$, we denote with $\boldsymbol{c}_I$ the subvector of $\boldsymbol{c}$ with the coordinates indexed by $I$. Now, let $L'$ denote a system of linear equations

$$L' : A\boldsymbol{x} = \boldsymbol{b},$$

where $A \in \mathbb{Q}^{d \times n}$, $\boldsymbol{b} \in \mathbb{Q}^d$ and $\mathrm{rank}(A) = k$. By multiplying with the least common denominator, we may assume in the following that $A \in \mathbb{Z}^{d \times n}$ and $\boldsymbol{b} \in \mathbb{Z}^d$. We call a set of $k$ linearly independent column vectors $B$ of $A$ a *basis* and we say that $A$ is *non-degenerate* if $k = d$ and for all bases $B$ of $A$, no coordinate of the corresponding solution $\boldsymbol{x}_{\mathrm{ind}(B)}$ is 0. In particular, if $L'$ is non-degenerate, then $\boldsymbol{b}$ is not contained in the linear span of any set of $d' < d$ column vectors from $A$ and hence if $d > n$, the linear system $L'$ has no solution. In the following, we assume that $L'$ is non-degenerate and that $d \leq n$.

We denote with $L$ the linear program obtained by extending the linear system $L'$ with the constraints $\boldsymbol{x} \geq \boldsymbol{0}$ and with a cost vector $\boldsymbol{c} \in \mathbb{Q}^n$:

$$L : \min \boldsymbol{c}^T \boldsymbol{x} \text{ subject to } A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}.$$

We say a set of column vectors $B$ is a *basis* for $L$ if $B$ is a basis for $L'$. Let $\boldsymbol{x} \in \mathbb{R}^n$ be the corresponding solution, i.e., let $\boldsymbol{x}$ be such that $A\boldsymbol{x} = \boldsymbol{b}$ and $x_i = 0$ for $i \in [n] \backslash \mathrm{ind}\,(B)$. We call $\boldsymbol{x}$ a *basic feasible solution*, and $B$ a *feasible basis*, if $\boldsymbol{x} \geq \boldsymbol{0}$. Furthermore, we say $L$ is *non-degenerate* if for all feasible bases $B$, the corresponding basic feasible solutions have strictly positive values in the coordinates of $B$. Now, let $R = [n] \setminus \mathrm{ind}\,(B)$ be the column indices not in $B$. The *reduced cost vector* $\boldsymbol{r}_{B,\boldsymbol{c}} \in \mathbb{Q}^{n-d}$ with respect to $B$ and $\boldsymbol{c}$ is then defined as

$$\boldsymbol{r}_{B,\boldsymbol{c}} = \boldsymbol{c}_R - \left(A^{-1}_{\mathrm{ind}(B)} A_R\right)^T \boldsymbol{c}_{\mathrm{ind}(B)}. \tag{2.1}$$

It is well-known that $B$ is optimal for $\boldsymbol{c}$ if and only if $\boldsymbol{r}_{B,\boldsymbol{c}}$ is non-negative in all coordinates [12]. For technical reasons, we consider in the following the *extended reduced cost vector* $\hat{\boldsymbol{r}}_{B,\boldsymbol{c}} \in \mathbb{Q}^n$ that has a 0 in dimensions $\mathrm{ind}\,(B)$ and otherwise equals $\boldsymbol{r}_{B,\boldsymbol{c}}$ to align the coordinates of the reduced cost vector with the column indices in $A$. More formally, we set

$$(\hat{\boldsymbol{r}}_{B,\boldsymbol{c}})_j = \begin{cases} 0 & \text{if } j \in \operatorname{ind}(B) \text{, and} \\ (\boldsymbol{r}_{B,\boldsymbol{c}})_{j'} & \text{otherwise,} \end{cases}$$

where $j'$ is the rank of $j$ in $R$, that is, $(\boldsymbol{r}_{B,\boldsymbol{c}})_{j'}$ is the coordinate of $\boldsymbol{r}_{B,\boldsymbol{c}}$ that corresponds to the $j'$th non-basis column with column index $j$ in $A$.

Geometrically, the feasible solutions for the linear program $L$ define an $(n-d)$-dimensional polyhedron $\mathcal{P}$ in $\mathbb{R}^n$. Since $L$ is non-degenerate, $\mathcal{P}$ is simple. Let $f \subseteq \mathcal{P}$ be a $k$-face of $\mathcal{P}$. Then, $f$ has an associated set $\operatorname{supp}(f) \subseteq [n]$ of $k$ column indices such that $f$ consists precisely of the feasible solutions for the linear program $A_{\operatorname{supp}(f)}\boldsymbol{x'} = \boldsymbol{b}$, $\boldsymbol{x'} \geq \boldsymbol{0}$, lifted to $\mathbb{R}^n$ by setting the coordinates with indices not in $\operatorname{supp}(f)$ to 0. We call $\operatorname{supp}(f)$ the *support* of $f$ and we say the columns in $A_{\operatorname{supp}(f)}$ *define* $f$. Furthermore, for all subfaces $\check{f} \subseteq f$, we have $\operatorname{supp}\left(\check{f}\right) \subseteq \operatorname{supp}(f)$ and in particular, all bases that define vertices of $f$ are $d$-subsets of columns from $A_{\operatorname{supp}(f)}$.

Moreover, we say a nonempty face $f \subseteq \mathcal{P}$ is *optimal* for a cost vector $\boldsymbol{c}$ if all points in $f$ are optimal for $\boldsymbol{c}$. We can express this condition using the reduced cost vector. Let $B$ be a basis for a vertex in $f$. Then $f$ is optimal for $\boldsymbol{c}$ if and only if

$$(\hat{\boldsymbol{r}}_{B,\boldsymbol{c}})_j = 0 \text{ for } j \in \operatorname{supp}(f)\text{, and } (\hat{\boldsymbol{r}}_{B,\boldsymbol{c}})_j \leq 0 \text{ otherwise.}$$

## 3 Overview of the PPAD-Formulation

We give a new constructive proof of the cone version of the colorful Carathéodory theorem based on Sperner's lemma. Using this, we can obtain a PPAD-formulation of ColorfulCarathéodory, by adapting Papadimitriou's formulation of Sperner's lemma as a PPAD problem.

Recall the statement of Sperner's lemma: let $\mathcal{S}$ be a simplicial subdivision of the $d$-dimensional standard simplex $\Delta^d = \operatorname{conv}(\boldsymbol{e}_1, \dots, \boldsymbol{e}_{d+1}) \subset \mathbb{R}^{d+1}$, where $\boldsymbol{e}_i$ is the $i$th canonical basis vector. We call a function $\lambda$ that assigns to each vertex in $\mathcal{S}$ a label from $[d+1]$ a *Sperner labeling* if for each vertex $\boldsymbol{v}$ of $\mathcal{S}$ contained in $\operatorname{conv}(\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_k})$, we have $\lambda(\boldsymbol{v}) \in \{i_1, \dots, i_k\}$, for all $\{i_1, \dots, i_k\} \subseteq [d+1], k \in [d+1]$. For a simplex $\sigma \in \mathcal{S}$, we set $\lambda(\sigma)$ to be the set of labels of the vertices of $\sigma$. We call $\sigma$ *fully-labeled* if $\lambda(\sigma) = [d+1]$.

Theorem 3.1. (Strong Sperner's Lemma [5]) *The number of fully-labeled simplices is odd.*

Now suppose we are given an instance $I = (C_1, \dots, C_d, \boldsymbol{b})$ of (the cone version of) ColorfulCarathéodory, where $\boldsymbol{b} \in \mathbb{R}^d$, $\boldsymbol{b} \neq \boldsymbol{0}$, and each

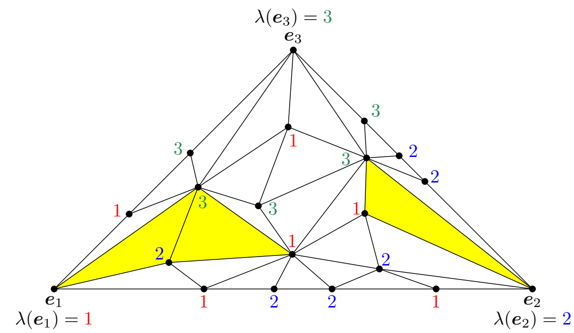


Figure 2: An example of Sperner's lemma in two dimensions. The fully-labeled simplices are marked yellow.

$C_i \subset \mathbb{Q}^d$, $i \in [d]$, ray-embraces $\boldsymbol{b}$. In the full version [14], we show that w.l.o.g. each set $C_i$ has size $d$. We now describe how to define a simplicial complex $\mathcal{S}$ and a Sperner labeling $\lambda$ for $I$ such that a fully labeled simplex will encode a colorful choice that contains the vector $\boldsymbol{b}$ in its positive span.

In the following, we call $\mathbb{R}^d$ the *parameter space* and a vector $\boldsymbol{\mu} \in \mathbb{R}^d$ a *parameter vector*. We define a family of linear programs $\{L^{\mathrm{CC}}_{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathbb{R}^d\}$, where each linear program $L^{\mathrm{CC}}_{\boldsymbol{\mu}}$ has the same linear system

$$L^{\mathrm{CC}} : A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}, \tag{3.2}$$

as constraints and differs only in its cost vector $\boldsymbol{c}_{\boldsymbol{\mu}}$. The cost vector $\boldsymbol{c}_{\boldsymbol{\mu}}$ is defined by a linear function in $\boldsymbol{\mu} \in \mathbb{R}^d$. Let $A = (C_1\ C_2\ \dots\ C_d) \in \mathbb{Q}^{d \times d^2}$ be the matrix that has the vectors from $C_1$ in the first $d$ columns, the vectors from $C_2$ in the second $d$ columns, and so on. Then, we denote with $L^{\mathrm{CC}}_{\boldsymbol{\mu}}$ the linear program

$$L^{\mathrm{CC}}_{\boldsymbol{\mu}} : \min \boldsymbol{c}^T_{\boldsymbol{\mu}}\boldsymbol{x}, \text{ subject to } L^{\mathrm{CC}}, \tag{3.3}$$

and we denote with $\mathcal{P}^{\mathrm{CC}} \subset \mathbb{R}^{d^2}$ the polyhedron that is defined by the linear system $L^{\mathrm{CC}}$. We can think of the $i$th coordinate of the parameter vector $\boldsymbol{\mu} \in \mathbb{R}^d$ as the weight of color $i$, i.e., the costs of columns from $A$ with color $i$ decrease if $(\boldsymbol{\mu})_i$ increases. To each face $f$ of $P$, we assign the set of parameter vectors $\Phi(f) \subset \mathbb{R}^d$ such that for all $\boldsymbol{\mu} \in \Phi(f)$, the face $f$ is optimal for the linear program $L^{\mathrm{CC}}_{\boldsymbol{\mu}}$ that has $L^{\mathrm{CC}}$ as constraints and $\boldsymbol{c}_{\boldsymbol{\mu}}$ as cost vector. We call $\Phi(f)$ the *parameter region* of $f$. The cost vector is designed to control the colors that appear in the support of optimal faces for a specific subset of parameter vectors. Let $\mathcal{M} = \left\{\boldsymbol{\mu} \in \mathbb{R}^d \,\middle|\, \boldsymbol{\mu} \geq \boldsymbol{0}, \|\boldsymbol{\mu}\|_\infty = 1\right\}$ denote the faces of the unit cube in which at least one coordinate is set to 1. Then, no face $f$ that is assigned to a parameter vector $\boldsymbol{\mu} \in \mathcal{M}$ with $(\boldsymbol{\mu})_{i^\times} = 0$ has a column from $A$ with color $i^\times$ in its defining set $A_{\text{supp}(f)}$. This property will become crucial when we define a Sperner labeling later on. Now, we define a polyhedral subcomplex $\mathcal{F}$ of $\mathcal{P}^{\text{CC}}$ that consists of all faces $f$ of $\mathcal{P}^{\text{CC}}$ such that $\Phi(f)\cap\mathcal{M} \neq \emptyset$. Furthermore, the intersections of the parameter regions with $\mathcal{M}$ induce a polytopal complex $\mathcal{Q}$ that is in a dual relationship to $\mathcal{F}$. By performing a central projection with the origin as center of $\mathcal{Q}$ onto the standard simplex $\Delta^{d-1}$, we obtain a polytopal subdivision $\mathcal{Q}_\Delta$ of $\Delta^{d-1}$. To get the desired *simplicial* subdivision of $\Delta^{d-1}$, we take the barycentric subdivision $\operatorname{sd}\mathcal{Q}_\Delta$ of $\mathcal{Q}_\Delta$.

We construct a Sperner labeling $\lambda$ for $\operatorname{sd}\mathcal{Q}_\Delta$ as follows: let $\boldsymbol{v}$ be a vertex in $\operatorname{sd}\mathcal{Q}_\Delta$, and let $f$ be the face of $\mathcal{F}$ that corresponds to $\boldsymbol{v}$. Then, we set $\lambda(\boldsymbol{v}) = i$ if the $i$th color appears most often in the support of $f$. The color controlling property of the cost function $c_{\boldsymbol{\mu}}$ then implies that $\lambda$ is a Sperner labeling. Furthermore, using the properties of the barycentric subdivision and the correspondence between $\mathcal{Q}_\Delta$ and $\mathcal{F}$, we can show that one vertex of a fully-labeled $(d-1)$-simplex in $\operatorname{sd}\mathcal{Q}_\Delta$ encodes a colorful feasible basis of the ColorfulCarathéodory instance $I$. This concludes a new constructive proof of the colorful Carathéodory theorem using Sperner's lemma.

To show that ColorfulCarathéodory is in PPAD however, we need to be able to traverse $\operatorname{sd}\mathcal{Q}_\Delta$ efficiently. For this, we introduce a combinatorial encoding of the simplices in $\mathcal{Q}_\Delta$ that represents neighboring simplices in a similar manner. Furthermore, we describe how to generalize the orientation used in the PPAD formulation of 2D-Sperner [16] to our setting. This finally shows that ColorfulCarathéodory is in PPAD.

To ensure that the complexes that appear in our algorithms are sufficiently generic, we prove several perturbation lemmas that give a deterministic way of achieving this. Our PPAD-formulation also shows that the special case of ColorfulCarathéodory involving two colors can be solved in polynomial time. Indeed, we will see that in this case the polytopal complex $\mathcal{Q}_\Delta$ can be made 1-dimensional. Then, binary search can be used to find a fully-labeled simplex in $\mathcal{Q}_\Delta$. In order to prove that the binary search terminates after a polynomial number of steps, we use methods similar to our perturbation techniques to obtain a bound on the length of the 1-dimensional fully-labeled simplex.

## 4 The Colorful Carathéodory Problem is in PPAD

As before, let $I = (C_1, \dots, C_d, \boldsymbol{b})$ denote an instance for the cone version of ColorfulCarathéodory. Our formulation of ColorfulCarathéodory as a PPAD-problem requires $I$ to be in general position. In particular, we assume that (P1) all color classes $C_i \subset \mathbb{Z}^d$ consist of $d$ points and all points have integer coordinates. Furthermore, we assume that (P2) there exist no subset $P \subset \bigcup_{i=1}^d C_i$ of size $d-1$ that ray-embraces $\boldsymbol{b}$. We show in the full version [14] how to ensure the properties by an explicit deterministic perturbation of polynomial bit-complexity.

### 4.1 The Polytopal Complex

Let $N = d!m^d$, where $m$ is the largest absolute value that appears in $A$ and $\boldsymbol{b}$. Then, we define $\boldsymbol{c}_{\boldsymbol{\mu}} \in \mathbb{R}^{d^2}$ as

$$(\boldsymbol{c}_{\boldsymbol{\mu}})_j = 1 + (1 - (\boldsymbol{\mu})_i)\, dN^2 + \varepsilon^j, \tag{4.4}$$

where $j \in [d^2]$, $i$ is the color of the $j$th column in $A$, and $0 < \varepsilon \leq N^{-3}$ is a suitable perturbation that ensures non-degeneracy of the reduced costs (see [4]). As stated in the overview, the cost function controls the colors in the support of the optimal faces for parameter vectors in $\mathcal{M}$. The proof of the following lemma can be found in the full version [14].

Lemma 4.1. *Let $i^\times \in [d]$ be a color and let $\boldsymbol{\mu} \in \mathcal{M}$ be a parameter vector with $\boldsymbol{\mu}_{i^\times} = 0$. Furthermore, let $B^\star$ be an optimal feasible basis for $L^{\text{CC}}_{\boldsymbol{\mu}}$. Then, $B^\star \cap C_{i^\times} = \emptyset$.*

We denote for a face $f \subseteq \mathcal{P}^{\text{CC}}$, $f \neq \emptyset$, with $\Phi(f) = \{\boldsymbol{\mu} \in \mathbb{R}^d \mid f \text{ is optimal for } L_{\boldsymbol{\mu}}\}$ the set of all parameter vectors for which $f$ is optimal. We call this the *parameter region* for $f$. Using the reduced cost vector, we can express $\Phi(f)$ as solution space to the following linear system, where $B$ is a feasible basis of some vertex of $f$ and the $d$ coordinates of the parameter vector $\boldsymbol{\mu}$ are the variables:

$$\begin{aligned} L^{\Phi}_{B,f} : (\hat{\boldsymbol{r}}_{B,\boldsymbol{c}_{\boldsymbol{\mu}}})_j &= 0 \text{ for } j \in \operatorname{supp}(f) \setminus \operatorname{ind}(B), \\ (\hat{\boldsymbol{r}}_{B,\boldsymbol{c}_{\boldsymbol{\mu}}})_j &\leq 0 \text{ for } \left[d^2\right] \setminus \operatorname{supp}(f). \end{aligned} \tag{4.5}$$

Then, we define $\mathcal{F}$ as the set of all faces that are optimal for some parameter vector in $\mathcal{M}$:

$$\mathcal{F} = \left\{ f \mid f \text{ is a face of } \mathcal{P}^{\text{CC}},\ \Phi(f) \cap \mathcal{M} \neq \emptyset \right\}.$$

By definition, $\mathcal{F} \cup \{\emptyset\}$ is a polyhedral subcomplex of $\mathcal{P}^{\text{CC}}$. The intersections of the parameter regions with faces of $\mathcal{M}$ induce a subdivision $\mathcal{Q}$ of $\mathcal{M}$:

$$\mathcal{Q} = \{\Phi(f) \cap g \mid f \in \mathcal{F},\ g \text{ is a face of } \mathcal{M}\}.$$

In the full version [14], we show that $\mathcal{Q}$ is a $(d-1)$-dimensional polytopal complex. Next, we construct $\mathcal{Q}_\Delta$ through a central projection with the origin as center of $\mathcal{Q}$ onto the $(d-1)$-dimensional standard simplex $\Delta \subset \mathbb{R}^d$. It is easy to see that this projection is a bijection. For a parameter vector $\boldsymbol{\mu} \in \mathbb{R}^d$, we denote with $\Delta(\boldsymbol{\mu}) = \boldsymbol{\mu}/\|\boldsymbol{\mu}\|_1$ its projection onto $\Delta$. Similarly, we denote with $\mathcal{M}(\boldsymbol{\mu}) = \boldsymbol{\mu}/\|\boldsymbol{\mu}\|_\infty$ the projection of $\boldsymbol{\mu}$ onto $\mathcal{M}$ and we use the same notation to denote the element-wise projection of sets. Then, we can write the projection $\mathcal{Q}_\Delta$ of $\mathcal{Q}$ onto $\Delta$ as $\mathcal{Q}_\Delta = \{\Delta(q) \mid q \in \mathcal{Q}\}$. Furthermore, let $\mathcal{S} = \{\Delta(g) \mid g \text{ is a face of } \mathcal{M}\}$ denote the projections of the faces of $\mathcal{M}$ onto $\Delta$. For $f \in \mathcal{F}$, let $\Phi_\Delta(f) = \Delta(\Phi(f) \cap \mathcal{M})$ denote the projection of all parameter vectors in $\mathcal{M}$ for which $f$ is optimal onto $\Delta$. Please refer to Table 1 on Page 10 for an overview of the current and future notation. The following results are proved in the full version [14].

LEMMA 4.2. *Let $q \neq \emptyset$ be an element from $\mathcal{Q}_\Delta$. Then, there exists unique pair $(f, g)$ where $f$ is a face of $\mathcal{F}$ and $g$ is a face of $\mathcal{S}$ such that $q = \Phi_\Delta(f) \cap g$. Moreover, $q$ is a simple polytope of dimension $\dim g - \dim f$ and, if $\dim q > 0$, the set of facets of $q$ can be written as*

$$\left\{\Phi_\Delta(f) \cap \check{g} \neq \emptyset \,\middle|\, \check{g} \text{ is a facet of } g\right\} \cup \left\{\Phi_\Delta\left(\hat{f}\right) \cap g \neq \emptyset \,\middle|\, f \text{ is a facet of } \hat{f} \in \mathcal{F}\right\}.$$

LEMMA 4.3. *The set $\mathcal{Q}_\Delta$ is a $(d-1)$-dimensional polytopal complex that decomposes $\Delta$.*

**4.2 The Barycentric Subdivision** The *barycentric subdivision* [11, Definition 1.7.2] is a well-known method to subdivide a polytopal complex into simplices. We define $\operatorname{sd}\mathcal{Q}_\Delta$ as the set of all simplices $\operatorname{conv}(\boldsymbol{v}_0, \dots, \boldsymbol{v}_k)$, $k \in [d]$, such that there exists a chain $q_0 \subset \cdots \subset q_k$ of polytopes in $\mathcal{Q}_\Delta$ with $\dim q_{i-1} < \dim q_i$ and such that $\boldsymbol{v}_i$ is the barycenter of $q_i$ for $i \in [k]$. We define the label of a vertex $\boldsymbol{v} \in \operatorname{sd}\mathcal{Q}_\Delta$ as follows. By Lemma 4.2, there exists a unique pair $f \in \mathcal{F}$ and $g \in \mathcal{S}$ with $\boldsymbol{v} = \Phi_\Delta(f) \cap g$. Then, the label $\lambda(\boldsymbol{v})$ of $\boldsymbol{v}$ is defined as

$$\lambda(\boldsymbol{v}) = \underset{i \in [d]}{\arg\max} \left|\operatorname{ind}(C)_i \cap \operatorname{supp}(f)\right|. \tag{4.6}$$

In case of a tie, we take the smallest $i \in [d]$ that achieves the maximum. Lemma 4.1 implies that $\lambda(\cdot)$ is a Sperner labeling of $\operatorname{sd}\mathcal{Q}_\Delta$. In fact, $\lambda$ is a Sperner labeling for any fixed simplicial subdivision of $\Delta$. Now, Theorem 3.1 guarantees the existence of a $(d-1)$-simplex $\sigma \in \operatorname{sd}\mathcal{Q}_\Delta$ whose vertices have all $d$ possible labels. The next lemma shows that then one of the vertices of $\sigma$ defines a solution to the COLORFULCARATHÉODORY instance. Here, we use specific properties of the barycentric subdivision.

LEMMA 4.4. *Let $\sigma \in \operatorname{sd}\mathcal{Q}_\Delta$ be a fully-labeled $(d-1)$-simplex and let $\boldsymbol{v}_{d-1}$ denote the vertex of $\sigma$ that is the barycenter of a $(d-1)$-face $q_{d-1} = \Phi_\Delta(f_{d-1}) \cap g_{d-1} \in \mathcal{Q}_\Delta$, where $f_{d-1} \in \mathcal{F}$ and $g_{d-1} \in \mathcal{S}$. Then, the columns from $A_{\operatorname{supp}(f_{d-1})}$ are a colorful choice that ray-embraces $\boldsymbol{b}$.*

Our discussion up to now already yields a new Sperner-based proof of the colorful Carathéodory theorem. However, in order to show that COLORFULCARATHÉODORY $\in$ PPAD, we need to replace the invocation of Theorem 3.1 by a PPAD-problem. Note that it is not possible to use the formulation of Sperner from [16, Theorem 2] directly, since it is defined for a fixed simplicial subdivision of the standard simplex. In our case, the simplicial subdivision of $\Delta$ depends on the input instance. In the following, we generalize the PPAD formulation of Sperner in [16] to $\mathcal{Q}_\Delta$ by mimicking the proof of Theorem 3.1. For this, we need to be able to find simplices in $\operatorname{sd}\mathcal{Q}_\Delta$ that share a given facet. We begin with a simple encoding of simplices in $\operatorname{sd}\mathcal{Q}_\Delta$ that allows us to solve this problem completely combinatorially.

We first show how to encode a polytope $q \in \mathcal{Q}_\Delta$. By Lemma 4.2, there exists a unique pair of faces $f \in \mathcal{F}$ and $g \in S$ such that $q = \Phi_\Delta(f) \cap g$. Since $\mathcal{M}(g)$ is a face of the unit cube, the value of $d - \dim g$ coordinates in $\mathcal{M}(g)$ is fixed to either 0 or 1. Let $I_j \subseteq [d]$, $j = 0, 1$, denote the indices of the coordinates that are fixed to $j$. Then, the encoding of $q$ is defined as $\operatorname{enc}(q) = (\operatorname{supp}(f), I_0, I_1)$. We use this to define an encoding of the simplices in $\mathcal{Q}_\Delta$ as follows. Let $\sigma \in \mathcal{Q}_\Delta$ be a $k$-simplex and let $q_0 \subset \cdots \subset q_k$ be the corresponding face chain in $\mathcal{Q}_\Delta$ such that the $i$th vertex of $\sigma$ is the barycenter of $q_i$. Then, the encoding $\operatorname{enc}(\sigma)$ is defined as

$$\operatorname{enc}(\sigma) = (\operatorname{enc}(q_0), \dots, \operatorname{enc}(q_k)). \tag{4.7}$$

In the proof of Theorem 3.1, we traverse only a subset of simplices in the simplicial subdivision, namely $(k-1)$-simplices that are contained in the face $\Delta_{[k]} = \operatorname{conv}\{\boldsymbol{e}_i \mid i \in [k]\}$ of $\Delta$ for $k \in [d]$. Let $\Sigma_k = \left\{\sigma \in \operatorname{sd}\mathcal{Q}_\Delta \,\middle|\, \dim(\sigma) = k-1,\ \sigma \subseteq \Delta_{[k]}\right\}$ denote the set of $(k-1)$-simplices in $\operatorname{sd}\mathcal{Q}_\Delta$ that are contained in the $(k-1)$-face, where $k \in [d]$, and let $\Sigma = \bigcup_{k=1}^{d} \Sigma_k$ be the collection of all those simplices. In the following, we give a precise characterization of the encodings of the simplices in $\Sigma_k$. For two disjoint index sets $I_0, I_1 \subseteq [d]$, we denote with $g(I_0, I_1) = \{\boldsymbol{\mu} \in \mathcal{M} \mid j = 0, 1,\ (\boldsymbol{\mu})_i = j \text{ for } i \in I_j\}$ the face of $\mathcal{M}$ that we obtain by fixing the coordinates in dimensions $I_0 \cup I_1$. Let now $T = (Q_0, \dots, Q_{k-1})$, $k \in [d-1]$, be a tuple, where $Q_i = \left(S^{(i)}, I_0^{(i)}, I_1^{(i)}\right)$, $S^{(i)} \subset \left[d^2\right]$, and $I_0^{(i)}, I_1^{(i)}$ are disjoint subsets of $[d]$ with $I_1^{(i)} \neq \emptyset$ for $i \in [k-1]_0$. We say $T$ is *valid* if and only if $T$ has the following properties.

(i) We have $I_0^{(k-1)} = [d] \setminus [k]$, $\left|I_1^{(k-1)}\right| = 1$, and the columns in $A_{S^{(k-1)}}$ are a feasible basis for a vertex $f$. Moreover, the intersection $\Phi(f) \cap$

$g\left(I_0^{(k-1)} \cup I_1^{(k-1)}\right)$ is nonempty.

(ii) For all $i \in [k-1]$, we either have

(ii.a) $I_0^{(i-1)} = I_0^{(i)}$, $I_1^{(i-1)} = I_1^{(i)}$, and $S^{(i-1)} = S^{(i)} \cup \{a_{i-1}\}$ for some index $a_{i-1} \in \left[d^2\right] \setminus S^{(i)}$,

(ii.b) or $S^{(i-1)} = S^{(i)}$ and there is an index $j_{i-1} \in [d] \setminus \left(I_0^{(i)} \cup I_1^{(i)}\right)$ such that either $I_0^{(i-1)} = I_0^{(i)}$ and $I_1^{(i-1)} = I_1^{(i)} \cup \{j_{i-1}\}$, or $I_1^{(i-1)} = I_1^{(i)}$ and $I_0^{(i-1)} = I_0^{(i)} \cup \{j_{i-1}\}$.

LEMMA 4.5. *For $k \in [d]$, the function* $\mathrm{enc}(\cdot)$ *restricted to the simplices in $\Sigma_k$ is a bijection from $\Sigma_k$ to the set of valid $k$-tuples.*

Using our characterization of encodings as valid tuples, it becomes an easy task to check whether a given candidate encoding corresponds to a simplex in $\Sigma$.

LEMMA 4.6. *Let $T = (Q_0, \ldots, Q_{k-1})$, $k \in [d-1]$, be a tuple, where $Q_i = \left(S^{(i)}, I_0^{(i)}, I_1^{(i)}\right)$, $S^{(i)} \subset \left[d^2\right]$, and $I_0^{(i)}, I_1^{(i)}$ are disjoint subsets of $[d]$ with $I_1^{(i)} \neq \emptyset$ for $i \in [k-1]_0$. Then, we can check in polynomial time whether $T$ is a valid $k$-tuple.*

In the full version [14], we show that simplices in $\Sigma$ that share a facet have similar encodings that differ only in one element of the encoding tuples. Using this fact, we can traverse $\Sigma$ efficiently by manipulating the respective encodings.

LEMMA 4.7. *Let $\sigma \in \Sigma_k$ be a simplex and let $q_0 \subset \cdots \subset q_{k-1}$ be the corresponding face chain in $\mathcal{Q}_\Delta$ such that the $i$th vertex $\boldsymbol{v}_i$ of $\sigma$ is the barycenter of $q_i$, where $k \in [d]$ and $i \in [k-1]_0$. Then, we can solve the following problems in polynomial time: (i) Given* $\mathrm{enc}(\sigma)$ *and $i$, compute the encoding of the simplex $\sigma' \in \Sigma_k$ that shares the facet* $\mathrm{conv}\{\boldsymbol{v}_j \mid j \in [k-1]_0,\, j \neq i\}$ *with $\sigma$ or state that there is none; (ii) Assuming that $k < d$ and given* $\mathrm{enc}(\sigma)$*, compute the encoding of the simplex $\hat{\sigma} \in \Sigma_{k+1}$ that has $\sigma$ as facet; and (iii) Assuming that $k > 1$ and given* $\mathrm{enc}(\sigma)$*, compute the encoding of the simplex $\check{\sigma} \in \Sigma_{k-1}$ that is a facet of $\sigma$ or state that there is none.*

**4.3 The PPAD graph** Using our tools from the previous sections, we now describe the PPAD graph $G = (V, E)$ for the COLORFULCARATHÉODORY instance. The definition of $G$ follows mainly the ideas from the formulation of Sperner as a PPAD-problem [16, Theorem 2] and the proof of Theorem 3.1.

The graph has one node per simplex in $\Sigma$ that has all labels or all but the largest possible label. That is, we have one node for each $(k-1)$-simplex $\sigma$ in $\Sigma_k$ with $[k-1] \subseteq \lambda(\sigma)$. Two simplices are connected by an edge if one simplex is the facet of the other or if both simplices share a facet that has all but the largest possible label. More formally, for $k \in [d]$, we set $V_k = \{\mathrm{enc}(\sigma) \mid \sigma \in \Sigma_k,\, [k-1] \subseteq \lambda(\sigma)\}$, the set of all encodings for $(k-1)$-simplices in $\Sigma_k$ whose vertices have all or all but the largest possible label. Then, $V$ is the union of all $V_k$ for $k \in [d]$. There are two types of edges: edges within a set $V_k$, $k \in [d]$, and edges connecting nodes from $V_k$ to nodes in $V_{k-1}$ and $V_{k+1}$. Let $\mathrm{enc}(\sigma), \mathrm{enc}(\sigma')$ be two vertices in $V_k$ for some $k \in [d]$. Then, there is an edge between $\mathrm{enc}(\sigma)$ and $\mathrm{enc}(\sigma')$ if the encoded simplices $\sigma, \sigma' \in \Sigma_k$ share a facet $\check{\sigma}$ with $\lambda(\check{\sigma}) = [k-1]$, i.e., both simplices are connected by a facet that has all but the largest possible label. Now, let $\mathrm{enc}(\sigma) \in V_k$ and $\mathrm{enc}(\sigma') \in V_{k+1}$ for some $k \in [d-1]$. Then, there is an edge between $\mathrm{enc}(\sigma)$ and $\mathrm{enc}(\sigma')$ if $\lambda(\sigma) = [k]$ and $\sigma$ is a facet of $\sigma'$. In the next lemma, we show that $G$ consists only of paths and cycles. Please consult the full version for the proof [14].

LEMMA 4.8. *Let* $\mathrm{enc}(\sigma) \in V$ *be a node. If* $\mathrm{enc}(\sigma) \in V_1$ *or* $\mathrm{enc}(\sigma) \in V_d$ *with $\lambda(\sigma) = [d]$, then* $\deg \mathrm{enc}(\sigma) = 1$. *Otherwise,* $\deg \mathrm{enc}(\sigma) = 2$.

This already shows that COLORFULCARATHÉODORY $\in$ PPA. By generalizing the orientation from [16] to our setting, we obtain a function dir that orients the edges of $G$ such that only vertices with degree one in $G$ are sinks or sources in the oriented graph. In the full version [14], we show how to compute this function in polynomial time. This finally yields our main result.

THEOREM 4.1. *COLORFULCARATHÉODORY, CENTERPOINT, TVERBERG, and SIMPLICIALCENTER are in PPAD $\cap$ PLS.*

*Proof.* We give a formulation of COLORFULCARATHÉODORY as PPAD-problem. See the full version for a formulation of COLORFULCARATHÉODORY as PLS-problem [14]. Using the classic proofs discussed in the full version, this then also implies the statement for the other problems.

The set of problem instances $\mathcal{I}$ consists of all tuples $I = (C_1, \ldots, C_d, \boldsymbol{b})$, where $d \in \mathbb{N}$, the set $C_i \subset \mathbb{Q}^d$ ray-embraces $\boldsymbol{b} \in \mathbb{Q}^d$ and $\boldsymbol{b} \neq \boldsymbol{0}$. Let $I^{\approx} = (C_1^{\approx}, \ldots, C_d^{\approx}, \boldsymbol{b}^{\approx})$ denote then the COLORFULCARATHÉODORY instance that we obtain by applying our perturbation techniques to $I$ (see the full version). Then, $I^{\approx}$ has the general position properties (P1) and (P2). The set of candidate solutions $\mathcal{S}$ consists of all tuples $(Q_0, \ldots, Q_{k-1})$, where $k \in \mathbb{N}$ and $Q_i$ is a tuple $\left(S^{(i)}, I_0^{(i)}, I_1^{(i)}\right)$ with

$S^{(i)}, I_0^{(i)}, I_1^{(i)} \subset \mathbb{N}$. Furthermore, $\mathcal{S}$ contains all $d$-subsets $C \subset \mathbb{Q}^d$ for $d \in \mathbb{N}$. We define the set of valid candidate solutions $\mathcal{S}_I$ for the instance $I$ to be the set of all valid $k$-tuples with respect to the instance $I^{\approx}$ and the set of all colorful choices with respect to $I$ that ray-embrace $\boldsymbol{b}$, where $k \in [d]$. Let $s \in \mathcal{S}$ be a candidate solution. If it is a tuple, we first use the algorithm from Lemma 4.6 to check in polynomial time in the length of $I^{\approx}$ and hence in the length of $I$ whether $s \in \mathcal{S}_I$. If affirmative, we check whether the simplex has all or all but the largest possible label. Using the encoding, this can be carried out in polynomial time. If $s$ is a set of points, we can determine in polynomial time with linear programming whether the points in $s$ ray-embrace $\boldsymbol{b}$.

We set as standard source the 0-simplex $\{\boldsymbol{e}_1\}$. We can assume without loss of generality that $\{\boldsymbol{e}_1\}$ is a source (otherwise we invert the orientation).

Given a valid candidate solution $s \in \mathcal{S}_I$, we compute its predecessor and successor with the algorithms from Lemma 4.7 and the orientation function discussed above, with one modification: if a node $s \in V$ is a source different from the standard source in the graph $G$, it encodes by the above discussion a colorful choice $C^{\approx}$ that ray-embraces $\boldsymbol{b}^{\approx}$. Let $C$ be the corresponding colorful choice for $I$ that ray-embraces $\boldsymbol{b}$. Then, we set the predecessor of $s$ to $C$. The properties of our perturbation ensure that we can compute $C$ in polynomial time. Similarly, if $s$ is a sink in $G$, we set its successor to the corresponding solution for the instance $I$.

## 5 A Polynomial-Time Case

We show that for a special case of ColorfulCarathéodory, our formulation of ColorfulCarathéodory as a PPAD problem has algorithmic implications. Let $C_1, C_2 \in \mathbb{R}^d$ be two color classes and let $C \subseteq C_1 \cup C_2$ be a set. We call $C$ an $(k, d-k)$-*colorful choice* for $C_1$ and $C_2$ if there are two subsets $C_1' \subseteq C_1$, $C_2' \subseteq C_2$ with $|C_1'| \leq k$ and $|C_2'| \leq d-k$. Now, given two color classes $C_1, C_2$ that each ray-embrace a point $\boldsymbol{b} \in \mathbb{R}^d$ and a number $k \in [d]_0$, we want to find an $(k, d-k)$-colorful choice that ray-embraces $\boldsymbol{b}$. It is a straightforward consequence of the colorful Carathéodory theorem that such a colorful choice always exists.

Using our techniques from Section 4, we present a weakly polynomial-time algorithm for this case. As described in Section 4.1, we construct implicitly a 1-dimensional polytopal complex, where at least one edge corresponds to a solution. Then, we apply binary search to find this edge. Since the length of the edges can be exponentially small in the length of the input, this results in a weakly polynomial-time algorithm.

Theorem 5.1. *Let $\boldsymbol{b} \in \mathbb{Q}^d$ be a point and let $C_1, C_2 \subset \mathbb{Q}^d$ be two sets of size $d$ that ray-embrace $\boldsymbol{b}$. Furthermore, let $k \in [d-1]$ be a parameter. Then, there is an algorithm that computes a $(k, d-k)$-colorful choice $C$ that ray-embraces $\boldsymbol{b}$ in weakly-polynomial time.*

For Sperner's lemma, it is well-known that a fully-labeled simplex can be found if there are only two labels by binary search. Essentially, this is also what the presented algorithm does: reducing the problem to Sperner's lemma and then applying binary search to find the right simplex. Since the computational problem Sperner is PPAD-complete even for $d = 2$, a polynomial-time generalization of this approach to three colors must use specific properties of the colorful Carathéodory instance under the assumption that no PPAD-complete problem can be solved in polynomial time.

## 6 Conclusion

We have shown that ColorfulCarathéodory lies in the intersection of PPAD and PLS. This also immediately implies that several illustrious problems associated with ColorfulCarathéodory, such as finding centerpoints or Tverberg partitions, belong to PPAD $\cap$ PLS.

Previously, the intersection PPAD $\cap$ PLS has been studied in the context of *continuous local search*: Daskalakis and Papadimitriou [6] define a subclass CLS $\subseteq$ PPAD $\cap$ PLS that "captures a particularly benign kind of local optimization". Daskalakis and Papadimitriou describe several interesting problems that lie in CLS but are not known to be solvable in polynomial time. Unfortunately, our results do not show that ColorfulCarathéodory lies in CLS, since we reduce ColorfulCarathéodory in $d$ dimensions to Sperner in $d-1$ dimensions, and since Sperner is not known to be in CLS. Indeed, if Sperner's lemma could be shown to be in CLS, this would imply that PPAD $=$ CLS $\subseteq$ PLS, solving a major open problem. Thus, showing that ColorfulCarathéodory lies in CLS would require fundamentally new ideas, maybe exploiting the special structure of the resulting Sperner instance. On the other hand, it appears that Sperner is a more difficult problem than ColorfulCarathéodory, since Sperner is PPAD-complete for every fixed dimension larger than 1, whereas ColorfulCarathéodory becomes hard only in unbounded dimension. On the positive side, our perturbation results show that a polynomial-time algorithm for ColorfulCarathéodory, even under strong general position assumptions, would lead to polynomial-time algorithms for several well-studied problems in high-dimensional computational geometry.

Finally, it would also be interesting to find further special cases of ColorfulCarathéodory that are amenable to polynomial-time solutions. For example, can we extend our algorithm for two color classes to *three* color classes? We expect this to be difficult, due to an analogy between 1D-Sperner, which is in P, and 2D-Sperner, which is PPAD-complete. However, there seems to be no formal justification for this intuition.

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| Symbol | Definition |
|---|---|
| $C_i$ | The $i$th color class. The $d$-set $C_i \subset \mathbb{R}^d$ ray-embraces $\boldsymbol{b}$. |
| $A$ | The $(d \times d^2)$-matrix with $C_1$ as first $d$ columns, $C_2$ as second $d$ columns, and so on. |
| $c_{\boldsymbol{\mu}}$ | The cost vector parameterized by a parameter vector $\boldsymbol{\mu} \in \mathbb{R}^d$. See (4.4). |
| $L^{\mathrm{CC}}$; $L^{\mathrm{CC}}_{\boldsymbol{\mu}}$ | $L^{\mathrm{CC}}$ refers to the linear system $A\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$ (see (3.2) and (3.3)). $L^{\mathrm{CC}}_{\boldsymbol{\mu}}$ denotes the linear program $\max c_{\boldsymbol{\mu}}^T \boldsymbol{x}$ s.t. $L^{\mathrm{CC}}$. |
| $\mathcal{P}^{\mathrm{CC}}$ | The polytope defined by $L^{\mathrm{CC}}$. |
| $f$; $\mathrm{supp}\,(f)$; $\mathrm{ind}\,(B)$ | For a face $f \subseteq \mathcal{P}^{\mathrm{CC}}$, we denote with $\mathrm{supp}\,(f)$ the indices of the columns in $A$ that define it. For a set of columns $B$ of $A$, we denote with $\mathrm{ind}\,(B)$ the indices of these columns. |
| $\Phi(f)$; $L^{\Phi}_{B,f}$ | For a face $f$ of $\mathcal{P}^{\mathrm{CC}}$, $\Phi(f)$ denotes the set of parameter vectors $\boldsymbol{\mu} \in \mathbb{R}^d$ such that $f$ is optimal for $L^{\mathrm{CC}}_{\boldsymbol{\mu}}$. The set $\Phi(f)$ can be described as the solution space to the linear system $L^{\Phi}_{B,f}$, where $B$ is a feasible basis of a vertex of $f$. |
| $\mathcal{M}$ | The set $\mathcal{M}$ contains all faces from the unit cube in $\mathbb{R}^d$ that set at least one coordinate to 1. Parameters from $\mathcal{M}$ control the colors of the defining columns of optimal faces (see Lemma 4.1). |
| $\mathcal{F}$ | The set of faces $f$ of $\mathcal{P}^{\mathrm{CC}}$ of that are optimal for some parameter vector in $\mathcal{M}$, i.e., the set of faces $f$ with $\Phi(f) \cap \mathcal{M} \neq \emptyset$. $\mathcal{F}$ is a $(d-1)$-dimensional polyhedral complex. |
| $\mathcal{Q}$ | The $(d-1)$-dimensional polytopal complex that consists of all elements $q = \Phi(f) \cap g$, where $f \in \mathcal{F}$ and $g$ is a face of $\mathcal{M}$. |
| $\Delta$; $\Delta_{[k]}$ | $\Delta$ denotes the $(d-1)$-dimensional standard simplex and $\Delta_{[k]}$ denotes the face $\mathrm{conv}\{\boldsymbol{e}_i \mid i \in [k]\}$ of $\Delta$. |
| $\mathcal{S}$ | The set $\mathcal{S}$ contains the central projections of the faces of $\mathcal{M}$ onto $\Delta$ with the origin as center. |
| $\Phi_\Delta$; $\mathcal{Q}_\Delta$ | $\Phi_\Delta(f)$ denotes the central projection of $\Phi(f) \cap \mathcal{M}$ onto $\Delta$ with center $\mathbf{0}$. The $(d-1)$-dimensional polytopal complex $\mathcal{Q}_\Delta$ consists of the projections of the elements in $\mathcal{Q}$ onto $\Delta$. Each element $q$ of $\mathcal{Q}_\Delta$ can be uniquely written as $q = \Phi_\Delta(f) \cap g$, where $f \in \mathcal{F}$ and $g \in \mathcal{S}$. |
| $\lambda$ | The labeling function, see (4.6). |
| $\Sigma$; $\Sigma_k$; $\mathrm{enc}\,(\sigma)$ | The set $\Sigma_k$, $k \in [d]$, consists of all $(k-1)$-simplices in $\mathrm{sd}\,\mathcal{Q}_\Delta$ that are contained in the face $\Delta_{[k]}$ of $\Delta$. The set $\Sigma$ is the union of all $\Sigma_k$. For a simplex $\sigma \in \Sigma$, we denote with $\mathrm{enc}\,(\sigma)$ its combinatorial encoding (see (4.7)). |

Table 1: Notation reference.