Minimal Geometric Graph Representations of Order Types*

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Abstract

We consider the problem of characterizing small geometric graphs whose structure uniquely determines the order type of its vertex set. We describe a set of edges that prevent the order type from changing by continuous movement and identify properties of the resulting graphs.

1 Introduction

Let $S, T \subset \mathbb{R}^2$ be two sets of n labeled points in the plane, not all on a common line. We say that S and T *have the same order type* if there is a bijection $\varphi : S \to T$ such that any triple $(p, q, r) \in S^3$ has the same orientation (clockwise, counterclockwise, or collinear) as the image $(\varphi(p), \varphi(q), \varphi(r)) \in T^3$ [7]. This induces an equivalence relation on planar sets of n points, with a finite number of equivalence classes, the *order types*. For example, the order type of a point set S determines which geometric graphs can be drawn on S without crossings. This makes order types relevant for extremal problems on geometric graphs.

Suppose we have discovered an interesting order type, and we want to illustrate it in a publication. One solution might be to give explicit coordinates of a representative point set S. This is unlikely to satisfy most readers. Thus, we would rather present S as a set of dots in a figure. For some point sets (particularly those with extremal properties), the reader may find it difficult to discern the orientation of an almost collinear point triple. To mend this, we could draw all lines spanned by two points in S. In fact, it suffices to show only the *segments* between the point pairs (the complete geometric graph on S). The orientation of a triple can then be obtained by inspecting the corresponding triangle; see Figure 1(a). In

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general, our drawing will be rather dense, and we may have trouble following an edge from one point to the next. Some edges, however, are redundant. Without them, we can still "see" the order type of the underlying point set.

We would like to understand which edges are essential for order type representation. To this end we provide a formal definition of this concept, identify a superset of these *non-redundant* edges, and provide a classification and some properties. While these edges prevent changing the order type by moving the points continuously (intuitively justified by the motivation above), we fall short of proving that their structure fully determines the order type.

Definitions. Let S be a set of n labeled points in the plane. A *geometric graph* on S is a graph with vertex set S whose edges are represented as line segments between their endpoints. A geometric graph is thus a drawing of an abstract graph. Two geometric graphs G and H are *topologically equivalent* if there is a homeomorphism of the plane transforming G into H. Each class of this equivalence relation may be described combinatorially by the cyclic orders of the edge segments around vertices and crossings, and by the incidences of vertices, crossings, edge segments, and faces. In the following we will consider topology-preserving deformations. An *ambient isotopy* of the real plane is a continuous map $f : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ such that $f(\cdot,t)$ is a homeomorphism for every $t \in [0,1]$ and $f(\cdot,0) = \mathrm{Id}$.

▶ **Definition 1.1.** Let G be a geometric graph on a point set S. We say that G *supports* S (or that G is *supporting*) if every ambient isotopy of $\mathbb{R}^2$ that keeps the images of the edges of G straight and preserves topological equivalence to the drawing G also preserves the order type of the vertex set.

Every complete geometric graph is supporting. A supporting graph need not be connected, and two distinct minimal supporting graphs can be drawings of the same abstract graph; see Figure 2 (b,c), and Figure 4. Thus, the structure of the drawing is crucial.

▶ **Definition 1.2.** Let G be a geometric graph on a set S of n points. We say that G *forces* S (or that G is *forcing*) if every n-point set S' that is the vertex set of a geometric graph topologically equivalent to G has the same order type as S.

Clearly, every forcing geometric graph is also supporting.

Related work and outline. The connection between order types and straight-line drawings has been studied intensively, both for planar drawings and for drawings minimizing the number of crossings. For example, it is NP-complete to decide whether a planar graph can be embedded on a given point set [4]. Continuous movements of the vertices of plane geometric graphs have also been considered [1]. The continuous movement of points maintaining the order type was considered by Mnëv [12], who showed that there are point sets with the same order type such that there is no ambient isotopy between them preserving the order type (settling a conjecture by Ringel [13]). The orientations of triples that have to be fixed to determine the order type are strongly related to the concept of *minimal reduced systems* [3].

We describe a notion of *exit edges* for a given point set. Although the resulting *exit graphs* are always supporting, they are not necessarily minimal with this property. One reason is that the topological structure of a geometric graph is not completely determined by the order type of its vertex set, whereas the exit edges are derived solely from the order type. Furthermore, some exit edges are rendered unnecessary by nonstretchability of certain pseudoline arrangements. This concept and the subsequent difficulties are discussed in

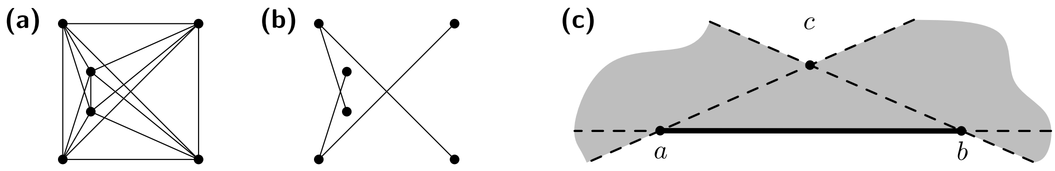

Figure 1 A set of six points with (a) all segments and (b) only exit edges drawn. (c) If the gray region is empty of points, then the edge ab is an exit edge.

Section 2. Despite being non-minimal in general, we argue that exit edges are good candidates for supporting graphs by discussing their dual representation in pseudoline arrangements (Section 3). We provide some further properties in Section 4. We conjecture that graphs based on exit edges are not only supporting but also forcing.

2 Exit edges

To obtain a supporting graph, we select edges so that no vertex of the resulting geometric graph can be moved to change the order type while preserving topological equivalence. In this section we will assume point sets to be in *general position*, that is, with no three collinear points, unless stated otherwise.

▶ **Definition 2.1.** Let S be a finite point set in general position, and let a, b, c be three distinct points from S. We say that ab is an *exit edge* with *witness* c if there is no point $p \in S$ such that the line $\overline{ap}$ separates b from c or the line $\overline{bp}$ separates a from c. The geometric graph on S with edge set formed by the exit edges is called the *exit graph* of S.

Equivalently, ab is an *exit edge* with *witness* c if and only if the double-wedge through a between b and c and the double-wedge through b between a and c contain no point of S in their interior; see Figure 1(c). An exit edge has at most two witnesses. Also, if $|S| \geq 4$ and ab is an exit edge in S with witness c, neither ac nor bc can be an exit edge with witness b or a, respectively. We illustrate the set of exit edges for a set of 6 points in Figure 1(b).

Exit edges can be characterized via 4-holes. For an integer $k \geq 3$, a *k-hole* in a point set S is a simple polygon spanned by k points of S whose interior contains no point of S. A pair ab from S is *extremal* in S if it lies on the boundary of the convex hull of S. A pair of points from S that is not extremal in S is *internal* in S.

▶ **Theorem 2.2.** *The edge ab is not an exit edge of S if and only if the following holds.*

1. *If ab is extremal in S, then it is incident to at least one convex 4-hole in S.*
2. *If ab is internal in S, then it is incident to at least one general 4-hole on each side such that the reflex angle (if any) is incident to ab.*

We remark that an internal exit edge either has a witness on both sides or is incident to at least one general 4-hole on one side. Due to space constraints, the proof of Theorem 2.2 is deferred to the full version of the paper.

▶ **Proposition 2.3.** *Let $S \subset \mathbb{R}^2$ be a finite point set in general position and for every $t \in [0, 1]$, let $S(t)$ be a continuous deformation of S at time t; more formally, let $S(t)$ be the point set $\{f(s, t); s \in S\}$ given by some ambient isotopy $f : \mathbb{R}^2 \times [0, 1] \to \mathbb{R}^2$. Let (a, b, c) be the first triple to become collinear, at time $t_0 > 0$. If c lies on the segment ab in $S(t_0)$, then ab is an exit edge of $S(0)$ with witness c.*

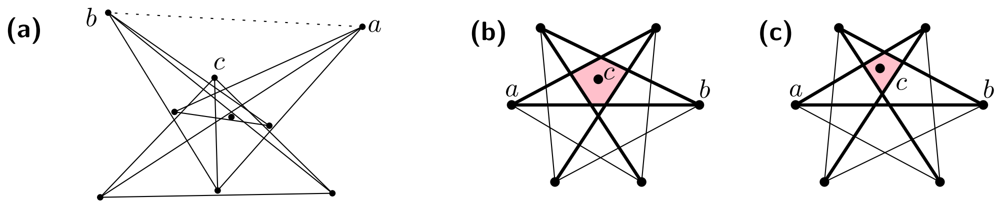

Figure 2 (a) Moving c over ab to make (a, b, c) oriented clockwise, without changing the orientation of other triples, would contradict Pappus's theorem [13]. (The corresponding abstract order type is not realizable.) (b, c) The segment ab is an exit edge with witness c. In (c), we cannot move c continuously to ab without first changing the order type, unless we also move other points.

Proof. For $t \in [0, t_0)$, the triple orientations in $S(t)$ remain unchanged. In $S(t_0)$, the point c lies on ab. Thus, for $t \in [0, t_0)$, there is no line through two points of $S(t)$ that strictly separates the relative interior of ab from c. In particular, there is no such separating line through a or b in $S(0)$. Hence, ab is an exit edge with witness c. ◀

▶ **Corollary 2.4.** *The exit graph of every point set is supporting.*

The proof of Proposition 2.3 also shows that if a line separates c from the relative interior of ab, then there is such a line through a or b. This may suggest that the exit edges are necessary for a supporting graph. However, this is not true in general. For example, in Figure 2(a), we see a construction by Ringel [13]: ab is an exit edge with witness c, but c cannot move over ab without violating Pappus' theorem. There are also point sets where two or more other line segments prevent a witness c from crossing an exit edge ab, for example, see Figure 2(c). In general, this cannot be inferred from the order type of the underlying point set. While c cannot move to ab without changing the order type in Figure 2(c), we could first change the point set to the one in Figure 2(b) and then move c over ab. So ab indeed has to be part of a graph supporting the set. Note that by Definition 1.1, it also prevents the point set to be transformed to the other one.

3 Exit edges and empty triangles

For a point set S in the Euclidean plane, add a line ℓ_∞ to obtain the real projective plane. By taking the projective dual of S and ℓ_∞, we get a projective line arrangement S^* where one cell, the *marked* cell, contains the dual point ℓ_∞^* at vertical infinity. The combinatorial structure of S^*, together with the marked cell, determines the order type of S. Dual to the proof of Proposition 2.3, we continuously move the lines without crossing ℓ_∞^*. The combinatorial structure changes when a line crosses a vertex of S^*. Before that, there is a triangular cell T bounded by three lines, dual to the endpoints of an exit edge and its witness. In S, the witness is the point that is between the other two points when the set becomes collinear. If we project S^* to the Euclidean plane by choosing a *line at infinity* through ℓ_∞^* that does not intersect T, the witness corresponds to the bounding line of T with *median slope*. Alternatively, the witness corresponds to the line containing the leftmost and the rightmost vertex of T.

Hence, the number of *triangles* in a simple projective line arrangement gives an upper bound on the number of exit edges of a point set. One triangle could contain ℓ_∞^*, and there could be pairs of triangles that share a crossing in such a way that leads to only one exit edge for the primal point set. Any projective arrangement of $n \geq 4$ lines has at least n triangles, as each line is incident to at least three triangles [10], which is tight. Therefore, any set of

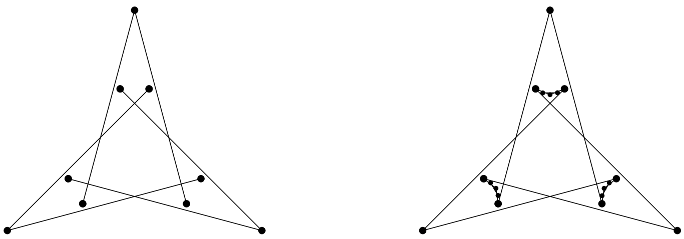

Figure 3 Constructions of point sets with $n-3$ exit edges, currently the smallest known number.

$n \geq 4$ points has at least $\lceil \frac{n-1}{2} \rceil$ exit-edges. A more careful counting of exit edges with one and two witnesses gives a lower bound of $\frac{3n}{5} - O(1)$ for the number of exit edges. The proof can be found in the full version of the paper. This bound is not proven tight, since so far we only know of point sets with $n-3$ exit edges for $n \geq 9$; see Figure 3.

The number of triangles in a simple arrangement is at most $\frac{n(n-1)}{3}$ [8]. Roudneff [14] and Harborth [9] showed that this is also tight. Thus, this is an upper bound on the number of exit edges. Possibly, this upper bound can be improved, as constructions showing tightness of the bound have many pairs of triangles sharing a vertex and corresponding to the same exit edge. However, there are also line arrangements with no such pair of triangles [11]. In the full version of the paper, we adapt a construction from [2] to show that the tight upper bound on the number of supporting edges for n points is in $\Theta(n^2)$.

4 (Counter-)Examples and properties

We present some results on general supporting graphs (and thus on exit graphs).

▶ **Theorem 4.1.** *Any geometric graph supporting a point set S, $|S| \geq 9$, contains a crossing.*

Proof. Let G be a geometric graph with vertex set S without crossings. There is a point set S' with a different order type that also admits G: Dujmović showed that every plane graph admits a plane straight-line embedding with at least $\sqrt{n/2}$ points on a line [5]; as we have a point set with a collinear triple that admits G, there are at least two point sets in general position with a different order type that admit G. Moreover, one can continuously morph S to S' while keeping the corresponding geometric graph planar and topologically equivalent to G (see, for example, [1]). Therefore, G does not support S. ◀

▶ **Proposition 4.2.** *Let S be a point set in general position in $\mathbb{R}^2$ and let G be its exit graph. Every vertex in the unbounded face of G is extremal, that is, it lies on the boundary of the convex hull of S.*

Note that, as shown in Figure 2(a), an analogous statement does not hold for general supporting graphs. The proof of Proposition 4.2 is deferred to the full version of the paper.

So far, we have few results for characterizing graphs that force a point set S, but we conjecture that the graph of exit edges not only supports S, but also forces it. However, even if we are given all the exit edges and their witnesses (in the dual, this means having all triangles of a line arrangement and their orientations), we cannot always infer the order type of S. A counterexample is sketched in Figure 4 as a dual (stretchable) pseudoline arrangement of 14 lines in the projective plane, based on an example by Felsner and Weil [6]. It consists of two arrangements of six lines in the Euclidean plane that are combinatorially different, but share the set of triangles and their orientations. While the exit edges are the same for the two order types, the corresponding exit graphs are not topologically equivalent.

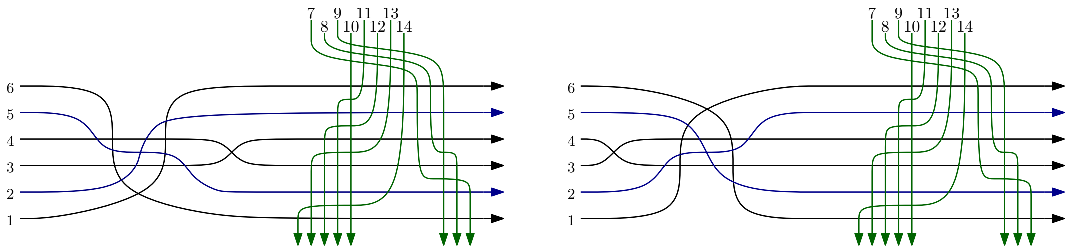

Figure 4 Two arrangements of 14 pseudolines with the same set of triangles (extending [6, Figure 3]). The green arrangements are the same. There is no triangle crossed by the line at infinity.

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