Flipping Plane Spanning Paths*

Oswin Aichholzer[†1], Kristin Knorr[‡2], Maarten Löffler[3], Zuzana Masárová[4], Wolfgang Mulzer[§2], Johannes Obenaus[¶2], Rosna Paul[‖1], and Birgit Vogtenhuber[1]**

1 **Institute of Software Technology, Graz University of Technology, Austria**
`{oaich,ropaul,bvogt}@ist.tugraz.at`

2 **Department of Computer Science, Freie Universität Berlin, Germany**
`{first name}.{family name}@fu-berlin.de`

3 **Utrecht University, the Netherlands**
`m.loffler@uu.nl`

4 **Institute of Science and Technology Austria**
`zuzana.masarova@ist.ac.at`

Abstract

Let S be a planar point set in general position, and let $\mathcal{P}(S)$ be the set of all plane (straight-line) spanning paths for S. A flip in a path $P \in \mathcal{P}(S)$ is the operation of removing an edge $e \in P$ and replacing it with a new edge f on S such that the resulting graph is again a path in $\mathcal{P}(S)$. Towards the question whether any two plane spanning paths of $\mathcal{P}(S)$ can be transformed into each other by a sequence of flips, we give positive answers if S is a wheel set, an ice cream cone, or a double chain. On the other hand, we show that in the general setting, it is sufficient to prove the statement for plane spanning paths with fixed first edge.

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1 Introduction

Let S be a set of n points in the plane such that no three points in S are collinear (this property is called *general position* of S). Let $\mathcal{P}(S)$ be the set of all plane (i.e., crossing-free), straight-line spanning paths for S. A *flip* on a path $P \in \mathcal{P}(S)$ is the operation of removing one edge $e \in P$ and replacing it with a new edge f on S such that the resulting graph is again a plane spanning path form $\mathcal{P}(S)$ (note that e and f might cross). Unless stated otherwise, all paths in this paper are plane, spanning, and straight-line.

The question we consider is the following. Given two paths $P_s, P_t \in \mathcal{P}(S)$, can we always transform the starting path P_s into the target path P_t by a sequence of flips? Or, to phrase it in a more graph-theoretic manner: the *flip-graph* (on $\mathcal{P}(S)$) is defined to have vertex set $\mathcal{P}(S)$ and two vertices form an edge if and only if the corresponding paths differ by a single flip. Then, we are concerned with the question, whether the flip-graph is connected for any point set S.

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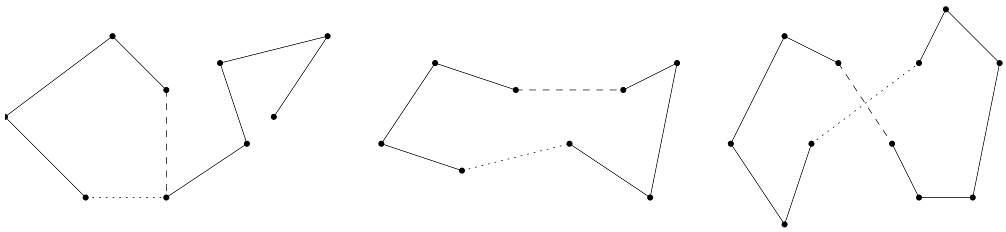


Figure 1 For Type 1 flips, the two involved edges share a common endpoint. For Type 2, the union of both paths is a plane spanning cycle (note that Type 2 flips can be simulated by a sequence of Type 1 flips). For Type 3, the two involved flipping edges cross, while the rest of the cycle is plane.

Flips in geometric graphs are a local but very powerful operation, see the survey of Bose and Hurtado [3]. On page 71, Bose and Hurtado write: "We are unaware of any progress for the same problem (obtaining a connected flip-graph for Hamiltonian crossing-free paths) on generic point sets."

Akl, Islam, and Meijer [2] showed that the flip graph is connected with diameter at most $2n-5$ for any $n \geq 3$ points in convex position, and for any $n \leq 8$ points in general position. Slightly later, tight bounds were derived by Chang and Wu [4]: if S is in convex position, then the diameter of the flip graph of $\mathcal{P}(S)$ is exactly $2n-5$, for $n = 3, 4$, and exactly $2n-6$, for $n \geq 5$.

There are different types of possible flips in a plane spanning path $P \in \mathcal{P}(S)$, but here we will mostly focus on *Type 1 flips* (see Figure 1 for other types of flips): enumerate the vertices of P as $p_1, \ldots, p_n$. Then, a *Type 1* flip consists of replacing an edge $p_{i-1}p_i$ of P, $i > 2$, by the edge p_1p_i. It results in the path $p_{i-1}, \ldots, p_1, p_i, \ldots, p_n$ (of course, the flip is only valid if the resulting path is still plane). In other words, a Type 1 flip inverts a contiguous chunk from one of the two ends of P.[1]

Our Results. First, we verify by a computer assisted proof with the help of the order type database [1] that the flip graph is connected for any set of $n \leq 10$ points.

For the general setting, we pursue two directions. On the one hand, we extend the proof of Akl et al. [2] to point sets in wheel, ice cream cone, and double chain configuration, the result for the double chains being the main contribution (see Theorem 6 in Section 2).

On the other hand, we show that it is sufficient to consider the flip-graph for paths where the first edge is fixed. More precisely: for distinct $p, q \in S$, let $\mathcal{P}(S, p)$ be the set of all plane spanning paths for S that start at p, and let $\mathcal{P}(S, p, q)$ be the set of all plane spanning paths for S that start at p and have pq as their first edge. We conjecture:

▶ **Conjecture 1** ([2]). *For any finite set $S \subset \mathbb{R}^2$ in general position and any two paths $P_s, P_t \in \mathcal{P}(S)$, there is a sequence of flips transforming P_s into P_t.*

▶ **Conjecture 2.** *For any finite set $S \subset \mathbb{R}^2$ in general position, any $p \in S$, and any two paths $P_s, P_t \in \mathcal{P}(S, p)$, there is a sequence of flips transforming P_s into P_t such that all intermediate paths are in $\mathcal{P}(S, p)$.*

[1] The corresponding flip at the other end of the path replaces an edge of the form p_jp_{j+1}, $j < n-1$ by the edge p_jp_n, resulting in the path $p_1, \ldots, p_j, p_n, \ldots, p_{j+1}$.

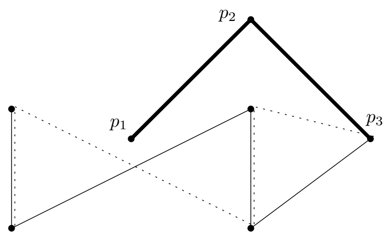


Figure 2 Example where the flip graph is disconnected if the first three points of the paths are fixed. The solid path cannot be flipped, but there is at least one other path (dotted) with the same three starting points.

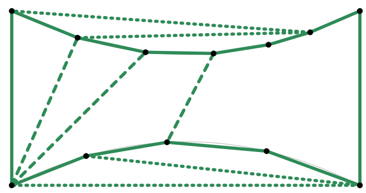


Figure 3 A double chain. Boundary edges are solid, bridge edges dashed, and chordal edges dotted (not all edges are drawn).

▶ **Conjecture 3.** *For any finite set $S \subset \mathbb{R}^2$ in general position, any distinct $p, q \in S$, and any two paths $P_s, P_t \in \mathcal{P}(S, p, q)$, there is a sequence of flips transforming P_s into P_t such that all intermediate paths are in $\mathcal{P}(S, p, q)$.*

We show (Lemmas 8 and 9 in Section 3) that for any fixed n, a positive answer to Conjecture 3 implies a positive answer to Conjecture 2, and similarly a positive answer to Conjecture 2 implies a positive answer to Conjecture 1.

Given Conjectures 1–3, one might think that an analogous statement for paths with a common starting sequence $p_1, p_2, \ldots, p_k$ of $k \geq 3$ points might also hold. Figure 2, however, shows a counter-example with 7 points for $k = 3$.

2 Special classes of point sets

We prove the connectedness of the flip-graph for wheel sets, ice cream cones, and double chains. Due to space constraints, however, we focus only on our main result, namely double chains. Our strategy is always to transform some path to a *canonical* path (consisting only of certain edges).

A *double chain* consists of two convex chains (each containing at least two points) with opposed concavity such that (i) the convex hull forms a quadrilateral (where the left and right endpoints of upper and lower chain form the *extreme vertices*) and (ii) no line through two points of the same chain separates the other chain (see e.g. [5, 3]). We classify the edges as follows: *boundary* edges are the edges between consecutive points on the upper and the lower chain, as well as the two *special* boundary edges between the two left and between the two right extreme points; *bridge* edges are the edges that connect the upper and the lower chain (except for the leftmost and rightmost such edge); all the other edges are *chordal* edges. A crucial property of double chains is the fact that boundary edges are uncrossed. We denote the class of double chains by DC (see Figure 3 for an illustration).

We define a (combinatorial) *distance* on the boundary of S, the plane cycle formed by the

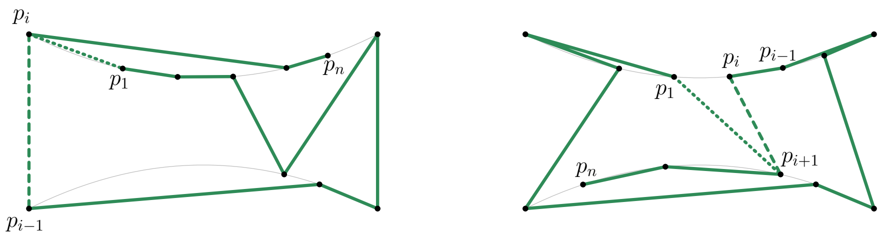


Figure 4 Illustration of Observation 4 (left) and Observation 5 (right). Replacing the dashed edge by the dotted forms a valid flip.

boundary edges[2]: let $S \in \mathsf{DC}$, and let $p, q \in S$ be two points on the boundary of S.[3] Further, let $o \in \{\text{cw}, \text{ccw}\}$ be an *orientation*. We define the *distance* between p and q in direction o, denoted by $d^o(p, q)$, as the number of boundary edges along the boundary that lie between p and q in direction o. Also, let the *distance* between p and q be

$$d(p, q) = \min\{d^{\text{cw}}(p, q), d^{\text{ccw}}(p, q)\}.$$

Note that neighboring vertices (along the boundary) have distance 1. Associating the pairs of vertices with an edge, we may also speak of the distance of an edge, i.e., the distance of an edge is just the distance between its endpoints. The *total* or *overall* distance (of a plane spanning path) is just the sum of all distances of its edges.

Let $S \in \mathsf{DC}$, and let $P = p_1, \ldots, p_n \in \mathcal{P}(S)$. For $i = 1, \ldots, n-1$, we call the two vertices p_i, p_{i+1} *consecutive* along P, and we say that p_i is the *predecessor* of p_{i+1} and that p_{i+1} is the *successor* of p_i. We emphasize that the terms *consecutive*, *predecessor*, and *successor* are reserved for the order along paths, whereas the terms *neighboring* and *neighbors* always refer to vertices that are incident to a common boundary edge of S.

The following observations will be useful to verify the validity of a flip (the first holds because no boundary edge is crossed by another edge on S), see also Figure 4:

▶ **Observation 4.** *Let $S \in \mathsf{DC}$, and let $P = p_1, \ldots, p_n \in \mathcal{P}(S)$ be a plane spanning path on S. Let p_i, $i \neq 2$, be a neighbor of p_1. Then, the edge $p_{i-1}p_i$ can be flipped to the edge p_1p_i, i.e., replacing $p_{i-1}p_i$ by p_1p_i results in a valid plane spanning path for S.*

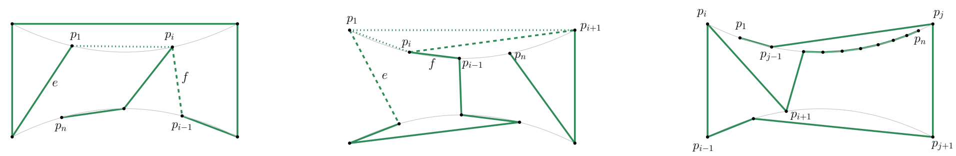
▶ **Observation 5.** *Let $S \in \mathsf{DC}$, and let $P = p_1, \ldots, p_n \in \mathcal{P}(S)$ be a plane spanning path on S. Let p_1, p_i, $i \neq n$, be neighbors on the same chain. Then, the only edge of P that p_1p_{i+1} may cross is $p_{i-1}p_i$. In particular, if $p_{i-1}p_i$ is a boundary edge, replacing p_ip_{i+1} by p_1p_{i+1} forms a valid flip.*

The following theorem constitutes the main result of this section. We illustrate the main ideas and structure of the proof, but postpone the most involved cases to the full version.

▶ **Theorem 6.** *Let $S \in \mathsf{DC}$, and let $P, Q \in \mathcal{P}(S)$ be two plane spanning paths on S. Then, P can be transformed to Q in $O(n^2)$ flips.*

Proof. Let $P = p_1, \ldots, p_n \in \mathcal{P}(S)$, and consider the edge $e = p_1p_2$. Let $p_i, i \neq 2$ be a neighbor of p_1 and whenever we have the choice, we pick p_i to be a neighbor such that p_1p_i does not form a special boundary edge (if both neighbors fulfill this property, pick one arbitrary). We denote $f = p_{i-1}p_i$. We describe a process where in each iteration we either:

[2] We emphasize that the boundary of S is distinct from the convex hull of S.
[3] Note, in the setting of double chains, any vertex is on the boundary.



(a) Case 1 **(b)** Case 2.1 **(c)** Case 2.2

Figure 5 Illustration of the three cases of Theorem 6. The solid paths together with the dashed edges form the initial path. Then, the dashed edges are replaced by the dotted (in (b), pay attention to the order of the flips). (c) illustrates some of the intricacies, if the starting edge is a boundary edge. None of the flips in the previous cases are valid here (no matter from which endpoint the path is viewed).

(i) increase the number of boundary edges (while not increasing the overall distance of P), or
(ii) decrease the overall distance of P (while not decreasing the number of boundary edges).

We can assume, w.l.o.g., that the endpoints of P are not neighbors, since otherwise we add the edge p_1p_n and remove an arbitrary (non-boundary) edge. We distinguish the following cases:

Case 1 f is not a boundary edge.
Then, we can simply replace f by p_1p_i (forming a proper flip by Observation 4). This increases the number of boundary edges and decreases the overall distance (recall that boundary edges have distance one and all other edges distance at least two).

Case 2 f is a boundary edge.
Then, the edge p_ip_{i+1} is not a boundary edge, since p_i already has the two neighbors p_1 and p_{i-1}.

Case 2.1 e is not a boundary edge.
Note that, since e is not a boundary edge, p_1p_i is not a special boundary edge.
We apply the following flips:
- replace p_ip_{i+1} by p_1p_{i+1} and
- replace e by p_1p_i.

The first flip is valid by Observation 5 and the second flip by Observation 4. The first flip may increase the overall distance by at most one, but the second flip decreases the overall distance by at least one. Hence, the overall distance does not increase. On the other hand, we increase the number of boundary edges.

Case 2.2 e is a boundary edge.
The case where e and f are both boundary edges is surprisingly intricate (especially when p_1 and p_i lie on different chains). It is easy to see that either $d(p_1, p_{i+1}) < d(p_i, p_{i+1})$ or $d(p_{i-1}, p_{i+1}) < d(p_i, p_{i+1})$ holds and our goal is to perform the corresponding flip that decreases the distance. However, if p_1 and p_i lie on different chains, we need to be very careful in order to preserve planarity (the details can be found in the full version of this paper).

Recursively applying above process, we will eventually transform P to a canonical path that consists only of boundary edges (the only paths with minimum overall distance). Doing the same for Q and noting that any pair of canonical paths can be transformed into each other by a single flip, the connectedness of the flip-graph follows.

Concerning the required number of flips, note that any edge has distance at most $\frac{n}{2} - 1$ and the path has $n-1$ edges. Hence, the total number of iterations to transform P into a canonical path is at most

$$\left((n-1) \cdot \left(\frac{n}{2} - 2\right) + (n-1)\right) \in O(n^2)$$

Furthermore, any iteration requires at most two flips and hence, the total number of flips to transform P into Q is still in $O(n^2)$. ◀

3 A sufficient condition

In this section, we prove the sufficient condition of considering only paths with a fixed starting edge (recall that we consider point sets in general position now). We need one preliminary lemma, whose proof can be found in the full version of this paper:

▶ **Lemma 7.** *For any two points p_1 and p_2 of S there exists a path $P \in \mathcal{P}(S)$ which has p_1 as starting and p_2 as target point.*

▶ **Lemma 8.** *A positive answer to Conjecture 2 implies a positive answer to Conjecture 1.*

Proof. Let P_s and P_t be the two paths of Conjecture 1. If they have a common endpoint, we can directly use Conjecture 2 and the statement follows. So assume that P_s has the endpoints p_a and p_b, and P_t has the endpoints p_c and p_d, which are all distinct. By Lemma 7 there exists a path P_m having the two endpoints p_a and p_c. By Conjecture 2 there is a flip sequence from P_s to P_m with the common endpoint p_a, and again by Conjecture 2 there is a further flip sequence from P_m to P_t with the common endpoint p_c. This implies the statement. ◀

▶ **Lemma 9.** *A positive answer to Conjecture 3 implies a positive answer to Conjecture 2.*

The general strategy to prove Lemma 9 is similar to the one of Lemma 8, but of course more involved as we also need to handle the position of the common starting point (again, we defer the details to the full version).

4 Conclusion

In this paper, we made progress towards a positive answer of Conjecture 1, though it still remains open. A natural way to prove Conjecture 1 would be to prove Conjecture 3 by induction. We can assume all three conjectures to hold for all sets of size at most $n-1$ and only need to show that Conjecture 3 holds for n.

Concerning the approach of special classes of point sets, of course one can try to further adapt the ideas to other classes.

Lastly, there are several other directions for further research conceivable, e.g. considering simple drawings (or other types of drawings) instead of straight-line drawings.

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