Recognizing Generalized Transmission Graphs of Line Segments and Circular Sectors[*]

Katharina Klost[1], Wolfgang Mulzer[1][0000−0002−1948−5840]

Institut für Informatik, Freie Universität Berlin, Germany
{kathklost, mulzer}@inf.fu-berlin.de

Abstract. Suppose we have an arrangement $\mathcal{A}$ of n geometric objects $x_1, \ldots, x_n \subseteq \mathbb{R}^2$ in the plane, with a distinguished point p_i in each object x_i. The *generalized transmission graph* of $\mathcal{A}$ has vertex set $\{x_1, \ldots, x_n\}$ and a *directed* edge $x_i x_j$ if and only if $p_j \in x_i$. Generalized transmission graphs provide a generalized model of the connectivity in networks of directional antennas.

The complexity class $\exists \mathbb{R}$ contains all problems that can be reduced in polynomial time to an existential sentence of the form $\exists x_1, \ldots, x_n : \phi(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ range over $\mathbb{R}$ and ϕ is a propositional formula with signature $(+, -, \cdot, 0, 1)$. The class $\exists \mathbb{R}$ aims to capture the complexity of the *existential theory of the reals*. It lies between **NP** and **PSPACE**.

Many geometric decision problems, such as recognition of disk graphs and of intersection graphs of lines, are complete for $\exists \mathbb{R}$. Continuing this line of research, we show that the recognition problem of generalized transmission graphs of line segments and of circular sectors is hard for $\exists \mathbb{R}$. As far as we know, this constitutes the first such result for a class of *directed* graphs.

1 Introduction

Let $\mathcal{A}$ be an arrangement of n geometric objects $x_1, \ldots, x_n$ in the plane. The *intersection graph* of $\mathcal{A}$ has one vertex for each object and an undirected edge between two objects x_i and x_j if and only if x_i and x_j intersect. In particular, if the objects are (unit) disks, we speak of *(unit) disk graphs*. These are often used as a symmetric model for antenna reachability. In some cases, however, this symmetry is not desired, since it does not accurately model the properties of the network. For omnidirectional antennas, there is an asymmetric model called *transmission graphs* [2]. Transmission graphs are also defined on disks: as in disk graphs, there is one vertex per disk, and the edges indicate directed reachability. There is a *directed* edge between two disks if the first disk contains the center of the second disk.

Here, we present a new class of *generalized transmission graphs*. Now, the objects may be arbitrary sets in $\mathbb{R}^2$, and the points that decide about the existence of an edge can be arbitrary points in the objects.

For a given graph class, the *recognition problem* is as follows: given a combinatorial graph $G = (V, E)$, decide whether G belongs to this class. For the recognition of geometrically defined graphs, it turned out that the complexity class $\exists\mathbb{R}$ plays a major role. The class $\exists\mathbb{R}$ was formally introduced by Schaefer [7]. It consists of all problems that are polynomial-time reducible to the set of all true sentences of the form $\exists x_1, \ldots, x_n : \Phi(x_1, \ldots, x_n)$. Here, Φ is a quantifier-free formula with signature $(+, -, \cdot, 0, 1)$ additional to the standard boolean signature. The variables range over the reals. Hardness for this class is defined via polynomial reduction.

There are multiple classes of intersection graphs for which the recognition problem is $\exists\mathbb{R}$-complete. Kang and Müller showed this for intersection graphs of k-spheres [1], and Schaefer proved a similar result for intersection graphs of line segments and convex sets [7].

One prototypical $\exists\mathbb{R}$-complete problem that serves as the starting point of many reductions is STRETCHABILITY, which was among the first known $\exists\mathbb{R}$-hard problems. The original hardness-proof is due to Mnëv [6], and it was restated in terms of $\exists\mathbb{R}$ by Matoušek [5].

Here, we show that the recognition of generalized transmission graphs of line segments and of a certain type of arrangements of circular sectors is hard for $\exists\mathbb{R}$. For this, we need to extend the known proofs significantly, and we need to develop new tools to reason about geometric realizations of directed graphs. With some further work the inclusion of these problems in $\exists\mathbb{R}$ could be shown. For details see the master thesis of the first author [3].

2 Preliminaries

2.1 Graph classes

Let $x_1, \ldots, x_n \subseteq \mathbb{R}^2$ be a set of n objects, and suppose that there is a distinguished point $p(x_i) \in x_i$, in every object x_i. The *generalized transmission graph* of these objects is a directed graph $G = (V, E)$ with

$$V = \{x_1, \ldots, x_n\} \text{ and } E = \{(x_i, x_j) \mid p(x_j) \in x_i, 1 \le i, j \le n\}.$$

We will consider generalized transmission graphs for line segments and circular sectors. In these cases, the distinguished points $p(x_i)$ are defined as follows: for line segments, we choose one fixed endpoint; for circular sectors, we choose the apex.

When constructing arrangements of line segments and of circular sectors below, in Sections 3 and 4, we need some notation. A line segment ℓ is described by an *endpoint* $p(\ell)$, a *length* $r(\ell)$, and a *direction* $u(\ell)$. A *circular sector* c is presented by an *apex* $p(c)$, a *radius* $r(c)$, an *opening angle* $\alpha(c)$, and a direction $u(c)$. The direction is a vector in $\mathbb{R}^2$, and it indicates the direction of the bisector. We will call the bounding line segments the *outer* line segments of c. Let $B(c)$ be the smallest rectangle with two sides parallel to $u(c)$ that contains c, the *bounding box* of c.

2.2 Stretchability and combinatorial descriptions

Let $\mathcal{L}$ be an arrangement of n non-vertical lines, such that no two lines in $\mathcal{L}$ are parallel. We define the *combinatorial description* $D(\mathcal{L})$ of $\mathcal{L}$ as follows:

Let g be a vertical line that lies to the left of all intersection points of $\mathcal{L}$. We number the lines $\ell_1, \ldots, \ell_n$ in the order in which they intersect g, from top to bottom. This ordering corresponds to the ascending order of the slopes. For each line ℓ_i, $i = 1, \ldots, n$, we have a list O^i of the following form:

$$O^i = (o_1^i, \ldots, o_k^i) \qquad\qquad o_j^i \subseteq \{1, \ldots, n\}$$

$$\bigcup_{j=1}^{k} o_j^i = \{1, ..., n\} \qquad\qquad o_j^i \cap o_{j'}^i = \emptyset, \text{ for } j \neq j'.$$

For $i = 1, \ldots, n$, the order of the indices in O^i indicates the order in which the lines ℓ_j cross ℓ_i, as we travel along ℓ_i from left to right. The lists O^i, for $i = 1, \ldots, n$, form the *combinatorial description* of the arrangement $\mathcal{L}$. If $\mathcal{L}$ is simple, each o_j^i is a singleton.

Given a combinatorial description $\mathcal{D}$ as above, it is relatively easy to detect whether it comes from an arrangement of *pseudo-lines*. This can be done by checking a few simple axioms [4]. However, the decision problem STRETCHABILITY of deciding if $\mathcal{D}$ originates from an actual arrangement of *lines* turns out to be significantly harder. If all sets o_j^i are singletons, the same problem is called SIMPLE-STRETCHABILITY. Both variants of the problem are complete for $\exists\mathbb{R}$ [5,6].

3 Line segments

We now present our first result on the recognition of intersection graphs of line segments.

Theorem 3.1. *Recognizing a generalized transmission graph of line segments is* $\exists\mathbb{R}$*-hard.*

Proof. The proof proceeds by a reduction from SIMPLE-STRETCHABILITY. Given an alleged description $\mathcal{D}$ of a simple arrangement of lines, we construct a graph $G_L = (V_L, E_L)$ such that $\mathcal{D}$ is realizable as a line arrangement if and only if G_L is the generalized transmission graph of an arrangement of line segments. We set $V_L = A \cup B \cup C$ with

$$\begin{aligned}
A &= \{a_{\{i,k\}} \mid 1 \leq i \neq k \leq n\}, \\
B &= \{b_k^i \quad \mid 1 \leq i \leq n, 1 \leq k \leq n-1\}, \\
C &= \{c_i \quad \mid 1 \leq i \leq n\},
\end{aligned}$$

where the c_i are numbered in order given by $\mathcal{D}$. The { } in the indices of the $a_{\{i,k\}}$ indicates that $a_{\{i,k\}} = a_{\{k,i\}}$.

Before defining the edges, we describe the intuitive meaning of the different vertices. The line segments associated with C correspond to the lines ℓ_i of the

arrangement. The endpoints of the line segment associated with $a_{\{i,k\}}$ will enforce that there is an intersection of the line segments for c_i and c_k, for $1 \leq i \neq k \leq n$. The endpoints of the line segments for the b_k^i, $k = 1, \ldots, n-1$, will be placed between the $a_{\{i,k\}}$ on c_i and thus enforce the order of the intersection. When it is clear from the context, we will not explicitly distinguish between a vertex of the graph and the associated line segment. Now we define the edges:

$$
\begin{aligned}
E_L = \quad & \{(c_i, a_{\{i,k\}}), (c_i, b_k^i), (b_k^i, c_i) \mid 1 \leq i \neq k \leq n\} \\
& \cup \{(b_{o_k^i}^i, b_{o_l^i}^i), (b_{o_k^i}^i, a_{\{i,o_l^i\}}) \qquad \mid 1 \leq i \leq n, 1 \leq l < k \leq n-1\}
\end{aligned}
$$

Given $\mathcal{D}$, the sets V_L and E_L can be constructed in polynomial time. It remains to show correctness. Suppose first that $\mathcal{D}$ is realizable, and let $\mathcal{L} = (\ell_1, \ldots, \ell_n)$ be a simple line arrangement with $\mathcal{D} = \mathcal{D}(\mathcal{L})$. We show that there exists an arrangement $\mathcal{C}$ of line segments that realizes G_L. Let D be a disk that contains all vertices of $\mathcal{L}$, with ∂D having a positive distance from each vertex.

The circular order of the intersections between $\ell_1, \ldots, \ell_n$ and ∂D is $\ell_1, \ldots, \ell_n$, $\ell_1, \ldots, \ell_n$. There is no vertical line in $\mathcal{L}$, so we can add a virtual vertical line ℓ' that divides the intersection points along ∂D into a "left" set $D_l = \{q_1^l, q_2^l, \ldots, q_n^l\}$ and a "right" set $D_r = \{q_1^r, q_2^r, \ldots, q_n^r\}$ such that each set contains exactly one intersection with each line ℓ_i, $i = 1, \ldots, n$.

For $i = 1, \ldots, n$, we set c_i to $\ell_i \cap D$, with $p(c_i) = q_i^l$. The $a_{\{i,k\}}$ are constructed such that $p(a_{\{i,k\}})$ is the intersection point of ℓ_i and ℓ_k. The direction and length are chosen in such a way that $a_{\{i,k\}}$ intersects no other lines. Now we place the line segments $b_{o_k^i}^i$. They are positioned such that $p(b_{o_k^i}^i)$ lies between $p(a_{\{i,o_k^i\}})$ and $p(a_{\{i,o_{k+1}^i\}})$, for $k = 1, \ldots, n-2$. Furthermore, we place $p(b_{o_{n-1}})$ to the right of $a_{\{i,o_{n-1}^i\}}$. The line segments lie on the lines ℓ_i such that $p(c_i)$ lies in the relative interior of b_k^i. For an example of this construction, see Figure 1. It follows from

(a) Complete line segment construction for three lines

(b) Closeup of c_2. The line segments b_1^2 and b_3^2 are shifted upwards to show their positioning.
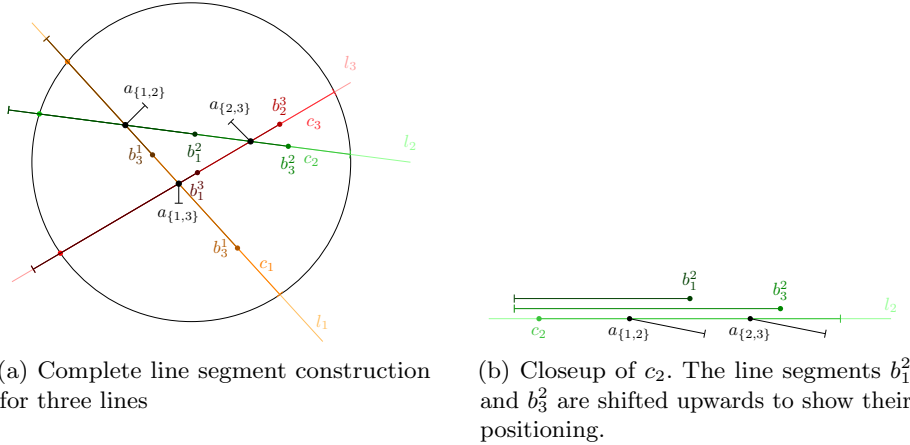
Fig. 1. Construction of the line segments.

the construction that the generalized transmission graph of $\mathcal{C}$ is indeed G_L.

Now consider an arrangement $\mathcal{C}$ of line segments realizing G_L. Let $\mathcal{L}' = (\ell_1', \ldots, \ell_n')$ be the arrangement of lines where ℓ_i' is the supporting line of c_i, for $i = 1, \ldots, n$. We claim that $\mathcal{D} = \mathcal{D}(\mathcal{L}')$.

We first consider the role of the line segments $a_{\{i,k\}}$. Since $p(a_{\{i,k\}})$ lies on c_i and c_k, we have $p(a_{\{i,k\}}) = c_i \cap c_k$, and therefore ℓ_i' and ℓ_k' intersect in $p(a_{\{i,k\}})$. This ensures that all pairs of lines have an intersection point that is also the endpoint of an $a_{\{i,k\}}$. Next, we have to show that the order of the intersections along each line ℓ_i', for $i = 1, \ldots, n$, is in the order as given by $\mathcal{D}$. This is guaranteed by the line segments b_k^i as follows: By the definition of E_L, namely by the edges (c_i, b_k^i) and (b_k^i, c_i), it is ensured that all $p(b_k^i)$ lie on the same line as c_i. The definition also enforces the order of the $p(a_{\{i,k\}})$ and $p(b_k^i)$ along the line. Since $p(a_{\{i,o_k\}})$ lies on $b_{o_{k+1}}^i$ but not on $b_{o_k}^i$ and since all lie on the same line c_i, it has to lie between the corresponding endpoints. This enforces the correct order of the intersections.
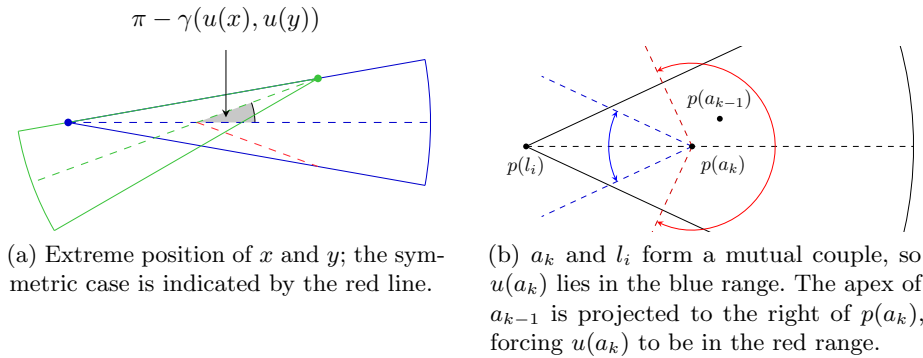
4 Circular sectors

We now consider the problem of recognizing generalized transmission graphs of circular sectors. The reduction extends the proof for Theorem 3.1, but we need to be more careful in order to enforce the correct order of intersection.

We will only consider circular sectors with opening angle $\alpha \leq \pi/4$. If x and y are circular sectors with $p(x) \in y$ and $p(y) \in x$, we call x and y a *mutual couple* of circular sectors. We write $\gamma(u(x), u(y))$ for the counter-clockwise angle between the vectors $u(x)$ and $u(y)$.

Observation 4.1. *Let x and y be a mutual couple of circular sectors, then*

$$|\pi - \gamma(u(x), u(y))| \leq (\alpha(x) + \alpha(y))/2.$$

The argument is visualized in Figure 2a.

(a) Extreme position of x and y; the symmetric case is indicated by the red line.

(b) a_k and l_i form a mutual couple, so $u(a_k)$ lies in the blue range. The apex of a_{k-1} is projected to the right of $p(a_k)$, forcing $u(a_k)$ to be in the red range.

Observation 4.2. *Let x and y be circular sectors whose bisectors intersect at an acute angle of $\beta > \max\{\alpha(x), \alpha(y)\}/2$. Then, the acute angle between the outer line segments of x and the bisector of y is at least $\beta - \max\{\alpha(x), \alpha(y)\}/2$.*

Lemma 4.3. *Let l be a circular sector and let $a_1, \ldots, a_n$ be circular sectors with*

$$p(a_i) \in l, \quad 1 \le i \le n,$$
$$p(a_i) \in a_j, \; 1 \le i < j \le n, \; \text{ and}$$
$$p(l) \in a_j, \; 1 \le j \le n.$$

Then, the projection of the $p(a_i)$ onto the directed line ℓ defined by $u(l)$ has the order

$$O = o_1, \ldots, o_n = a_1, \ldots, a_n.$$

Proof. Each a_i forms a mutual couple with l. Thus, with Observation 4.1, we get

$$|\pi - \gamma(u(a_i), u(l))| \le \pi/4. \tag{1}$$

Assume that the order of the projection differs from O. Let $O' = o'_1, \ldots, o'_n$ be the actual order of the projection of the $p(a_i)$ onto ℓ. Let j be the first index with $o'_j \ne o_j$ and $o'_j = a_k$. Then, there is an o'_i, $i > j$, with $o'_i = a_{k-1}$. By definition, $p(a_{k-1})$ has to be included in a_k, while still being projected on ℓ to the right of p_k. This is only possible if

$$|\pi - \gamma(u(a_j), \ell)| > \frac{\pi}{2} - \frac{\alpha(a_k)}{2} \ge \frac{\pi}{2} - \frac{\pi}{8} = \frac{3\pi}{8} > \frac{\pi}{4}$$

This is a contradiction to (1), and consequently the order of the projection is as claimed. The possible ranges of the angles are illustrated in Figure 2b. $\square$

An arrangement $\mathcal{C}$ of circular sectors is called *equiangular* if $\alpha(c) = \alpha(c')$ for all circular sectors $c, c' \in \mathcal{C}$.

Let c, c' be two circular sectors of $\mathcal{C}$, and assume that $d \in \mathcal{C}$ is a circular sector with $p(d) \in c$ and $p(d) \in c'$, such that c and c' do not form both a mutual couple with the same circular sector. Moreover let $\beta_{\min}$ be the smallest acute angle between the bisector of any pair c, c' with this property. We will call the arrangement *wide spread* if

$$\beta_{\min} \ge 2 \cdot \max_{c \in \mathcal{C}}(\alpha(c))$$

The possible situations are depicted in Figure 2.

Definition 4.4. *The recognition problem of the generalized transmission graphs of equiangular, wide spread circular sectors is called SECTOR.*

Now we want to show that SECTOR is hard for $\exists \mathbb{R}$. This is done in three steps. First, we give a polynomial-time construction that creates an arrangement of circular sectors from an alleged combinatorial description of a line arrangement. Then we show that this construction is indeed a reduction and therefore show the $\exists \mathbb{R}$-hardness of SECTOR.
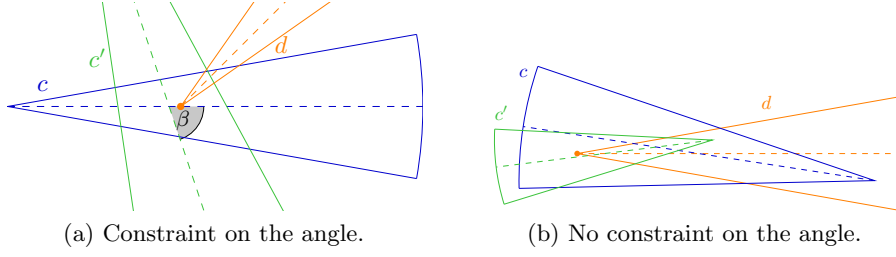
(a) Constraint on the angle. (b) No constraint on the angle.

Fig. 2. The wide spread condition.

Construction 4.5. Given a description $\mathcal{D}$ where all o_i are singletons, we construct a graph $G_L = (V_L, E_L)$. For this construction, let $1 \leq i, k, l \leq n$, $1 \leq m, m', m'' \leq 3$. The set of vertices is defined as follows:

$$V_L = \{c_{im}\} \cup \{a_{km'}^{im} \mid i \neq k\} \cup \{b_{km'}^{im} \mid i \neq k\}$$

As for the line segments, we do not distinguish between the vertices and the circular sectors. For the vertices $a_{km'}^{im}$ and $b_{km'}^{im}$, the upper index indicates the c_{im} with whom $a_{km'}^{im}$ and $b_{km'}^{im}$ form a mutual couple. The lower index hints at a relation to $c_{km'}$. In most cases, the upper index is im and the lower index differs. For better readability, the indices are marked bold ($a_{\mathbf{im}}^{\mathbf{km'}}$), if im is the lower index.

The bisectors of the circular sectors c_{i2} will later define the lines of the arrangement. The circular sectors $a_{km'}^{im}$ and $a_{\mathbf{im}}^{\mathbf{km'}}$ have a similar role as the $a_{\{i,k\}}$ in the construction for the line segments. They enforce the intersection of c_{im} and $c_{km'}$. Similar to the b_k^i, the $b_{km'}^{im}$ help enforcing the intersection order.

We describe E_L on a high level. For a detailed technical description, refer to Appendix A.1. We divide the edges of the graph into *categories*. The first category, E_I, contains the edges that enforce an intersection between two circular sectors c_{im} and $c_{km'}$, for $k < l$. The edges of the next category E_C enforce that each $a_{km'}^{im}$ and each $b_{km'}^{im}$ forms a mutual couple with c_{im}.

$$E_I = \{(c_{im}, a_{km'}^{im}) \mid i \neq k\} \qquad E_C = \{(a_{km'}^{im}, c_{im}) \mid i \neq k\}$$
$$\cup \{(c_{im}, a_{im}^{km'}) \mid i \neq k\} \qquad \cup \{(c_{im}, b_{km'}^{im}) \mid i \neq k\}$$
$$\cup \{(b_{km'}^{im}, c_{im}) \mid i \neq k\}$$

The edges in the next categories enforce the local order. The first category, called E_{GO}, enforces a global order in the sense that the apexes of all $a_{o_j m'}^{im}$ and $b_{o_j m'}^{im}$ will be projected to the left of any $a_{o_k m'}^{im}$ and $b_{o_k m'}^{im}$ with $k > j$. Additionally, all $a_{\mathbf{im}}^{\mathbf{o_j m'}}$ will be included in $a_{o_k m'}^{im}$ and $b_{o_k m'}^{im}$. The projection order is enforced by the construction described in Lemma 4.3, the inclusion is enforced by adding the appropriate edges.

It remains to consider the local order of the six circular sectors $(a_{j1}^{im}, \dots, a_{j3}^{im}, b_{j1}^{im}, \dots, b_{j3}^{im})$ that are associated with c_{im} for each intersecting circular sector

c_{j2}. The projection order of these is either "1, 2, 3" or "3, 2, 1", depending on the order of l_i and l_j on the vertical line. If l_j is below l_i, the order on c_{im} is "1, 2, 3"; in the other case, it is "3, 2, 1". This is again enforced by adding the edges as defined in Lemma 4.3. For a possible realization of this graph, see Figures 3 and 4. This construction can be carried out in polynomial time.

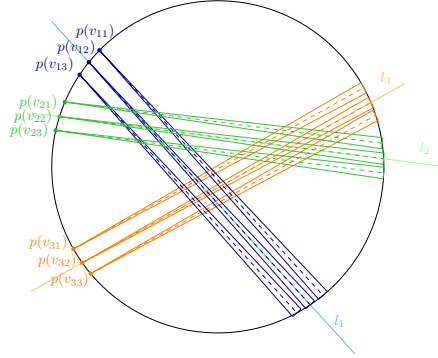Now we show that Construction 4.5 gives us indeed a reduction:

Fig. 3. Construction of the circular sectors c_{im} based on a given line arrangement

Lemma 4.6. *Suppose there is a line arrangement* $\mathcal{L} = \{\ell_1, \ldots, \ell_n\}$ *realizing* $\mathcal{D}$, *then there is an equiangular, wide spread arrangement* $\mathcal{C}$ *of circular sectors realizing* G_L *as defined in Construction 4.5.*

Proof. We construct the containing disk D, and the sets of intersection points D_l and D_r as in the proof of Theorem 3.1. By ℓ_{im}, we denote the directed line through the bisector of the circular sector c_{im}. Let $\alpha_{\min}$ be the smallest acute angle between any two lines of $\mathcal{L}$. The angle α for $\mathcal{C}$ will be set depending on $\alpha_{\min}$ and the placement of the constructed circular sectors c_{im}.

In the first step, we place the circular sectors c_{i2}. They are constructed such that their apexes are on q_i^l and their bisectors are exactly the line segments $\ell_i \cap D$. We place $p(c_{i1})$ in clockwise direction next to $p(c_{i2})$ onto the boundary of D. The distance between $p(c_{i1})$ and $p(c_{i2})$ on ∂D is some small $\tau > 0$. The point $p(c_{i3})$ is placed in the same way, but in counter-clockwise direction from $p(c_{i2})$. The bisectors of all c_{im} are parallel. The radii for c_{i1} and c_{i3} are chosen to be the length of the line segments $\ell_{i1} \cap D$ and $\ell_{i3} \cap D$.

The distance τ must be small enough so that no intersection of any two original lines lies between ℓ_{i1} and ℓ_{i3}. Let β be the largest angle such that if the angle of all c_{im} is set to β, there is always at least one point in c_{im} between the bounding boxes B of two circular sectors with consecutively intersecting bisectors.

Since $\mathcal{L}$ is a simple line arrangement, this is always possible. The angle α for the construction is now set to $\min\{\alpha_{\min}/2, \beta\}$. This first part of the construction is illustrated in Figure 3.

Now we place the remaining circular sectors. Their placement can be seen in Figure 4. The points $p(a^{im}_{km'})$ all lie on ℓ_{im} with a distance of δ to the left of the intersection of ℓ_{im} and $\ell_{km'}$. By "to the left", we mean that the point lies closer to $p(c_{im})$ on the line ℓ_{im} than the intersection point. The distance δ is chosen small enough such that $p(a^{im}_{km'})$ lies inside of all $a^{\mathbf{km}'}_{\mathbf{im}}$ that have a larger distance to $p(c_{\mathbf{km}'})$ than $p(a^{im}_{km'})$. The direction of the circular sector $a^{im}_{km'}$ is set to $-u(c_{im})$, and its radius is set to $r(a^{im}_{o_k m'}) = \operatorname{dist}(p(a^{im}_{km'}), p(c_{im})) + \varepsilon$, for $\varepsilon > 0$. This lets $p(c_{im})$ lie on the bisecting line segment of every circular sector $a^{im}_{km'}$. The directions and radii for the $b^{im}_{km'}$ are chosen in the same way as for the $a^{im}_{km'}$. The apexes of $b^{im}_{km'}$ are placed such that they lie between the corresponding bounding boxes $B(c_{km'})$. For α small enough, this is always possible.
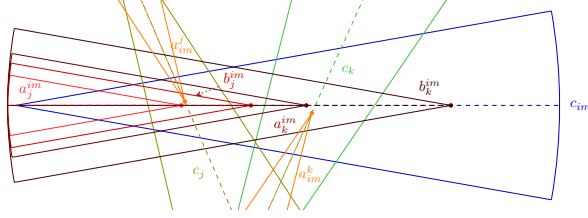

Fig. 4. Detailed construction inside of one circular sector c_{im}.

It follows directly from the construction that the generalized transmission graph of this arrangement is G_L. A detailed argument can be found in Appendix A.2.

Lemma 4.7. *Suppose there is an equiangular, wide spread arrangement $\mathcal{C}$ of circular sectors realizing G_L as defined in Construction 4.5, then there is an arrangement of lines realizing $\mathcal{D}$.*

Proof. From $\mathcal{C}$, we construct an arrangement $\mathcal{L} = (\ell_1, \ldots, \ell_n)$ of lines such that $\mathcal{D}(\mathcal{L}) = \mathcal{D}$ by setting ℓ_i to the line spanned by $u(c_{i2})$. Now, we show that this line arrangement indeed satisfies the description, e.g., that the intersection order of the lines is as indicated by the description.

All $a^{im}_{km'}$ and $b^{im}_{km'}$ form mutual couples with c_{im}. Thus, Lemma 4.3 can be applied to them. It follows that the order of the projections of the apexes of the circular sectors is known. In particular, the order of projections of the $p(a^{i2}_{j2})$ onto ℓ_i is the order given by $\mathcal{D}$ and $p(b^{i2}_{o_j 2})$ is projected between $p(a^{i2}_{o_j 2})$ and $p(a^{i2}_{o_{j+1} 2})$.

Now, we have to show that the order of intersections of the lines corresponds to the order of the projections of the $p(a^{i2}_{j2})$. This will be done through a contradiction. We consider two circular sectors c_{j2} and c_{k2}. Assume that the order of the projection of the apexes of a^{i2}_{j2} and a^{i2}_{k2} onto ℓ_i is $p(a^{i2}_{j2})$, $p(a^{i2}_{k2})$, while the order of intersection of the lines is ℓ_k, ℓ_j.

Note that by the definition of the edges of G_L, c_{j2} and c_{k2} share the apexes of a_{j2}^{k2} and a_{k2}^{j2}, but there is no circular sector they both form a mutual couple with and thus the angle between their bisecting line segments is large.

There are two main cases to consider, based on the position of the intersection point p of ℓ_j and ℓ_k relative to c_{i2}:

Case one $p \notin c_{i2}$: If p does not lie in c_{i2}, then ℓ_j and ℓ_k divide c_{i2} into three parts. Let s_j, s_k be the outer line segments of c_{j2} and c_{k2} that lie in the middle part of this decomposition. A schematic of this situation can be seen in Figure 5a.

(a) Case one, a_{k2}^{i2} cannot reach a_{j2}^{i2}.   (b) Case two, b_{j2}^{i2} cannot lie in F_1 or F_3.
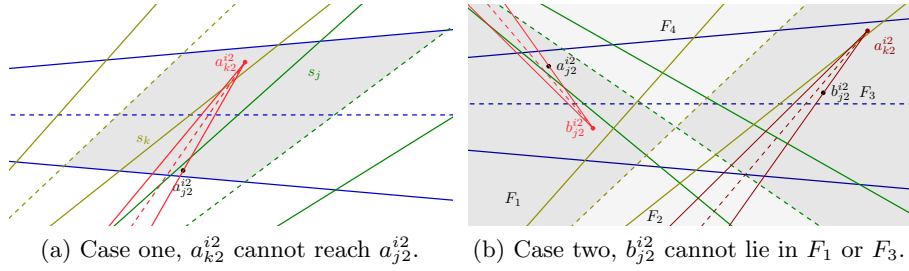
From Observation 4.2 and since $\mathcal{C}$ is an equiangular, wide spread arrangement it follows that $|\pi - \gamma(s_j, u(c_i))| > 3\alpha/2$ and $|\pi - \gamma(s_k, u(c_i))| > 3\alpha/2$.

In order to have an intersection order that differs from the projection order, the circular sector a_{k2}^{i2} has to reach $p(a_{j2}^{i2})$. The latter point is projected to the left of a_{k2}^{i2} but lies right of s_k. The directed line segment d from $p(a_{k2}^{i2})$ to $p(a_{k2}^{j2})$ has to intersect s_j and s_k, and thus it has to hold that $|\pi - \gamma(d, u(c_{i2}))| \geq 3\alpha/2$. The line segment d has to lie inside of a_{k2}^{i2}, which is only possible if $|\pi - \gamma(u(a_{k2}^{i2}), u(c_i))| > \alpha$. However, this is a contradiction to $|\pi - \gamma(u(a_k^i), u(c_i))| \leq \alpha$, which follows from Observation 4.1.

Case two $p \in c_{i2}$: W.l.o.g., let $u(c_{i2}) = \lambda \cdot (1,0)$, $\lambda > 0$, and let $\mathcal{F} = \{F_1, F_2, F_3, F_4\}$ be the decomposition of the plane into faces induced by ℓ_j and ℓ_k. Here, F_1 is the face with $p(c_{i2})$, and the faces are numbered in counter-clockwise order.

We consider the possible placements of $p(b_{j2}^{i2})$ in one of the face. First, we show that $p(b_{j2}^{i2})$ cannot lie in F_1 or in F_3. From the form of E_{GO}, we know that $p(a_{j2}^{i2})$ has to be projected left of $p(b_{j2}^{i2})$ and $p(a_{j2}^{i2})$ has to lie inside of b_{j2}^{i2}; see Figure 5b for a schematic of the situation. If $p(b_{j2}^{i2})$ lies in F_1, the line segment in b_{j2}^{i2} that connects $p(b_{j2}^{i2})$ and $p(a_{j2}^{i2})$ has to cross an outer line segment of c_{j2}. This yields the same contradiction as in the first case. If $p(b_{j2}^{i2})$ were in F_3, an analogous argument holds for $p(b_{j2}^{i2})$, which has to lie inside of a_{k2}^{i2}.

This leaves F_2 and F_4 as possible positions for b_{j2}^{i2}. W.l.o.g., let b_{j2}^{i2} be located in F_4. We divide c_{j2} and c_{k2} by ℓ_k or ℓ_j, respectively, into two parts, and denote

the parts containing the line segments that are incident to F_4 by J and K. Then, again by using that the arrangement is wide spread, it can be seen that $p(a_{j2}^{i2})$ and $p(a_{k2}^{i2})$ are located in J and K. The possible placement is visualized in Figure 5a.

(a) The localization of a_{j2}^{i2} and a_{k2}^{i2}.
(b) p can not lie in c_{i1}.
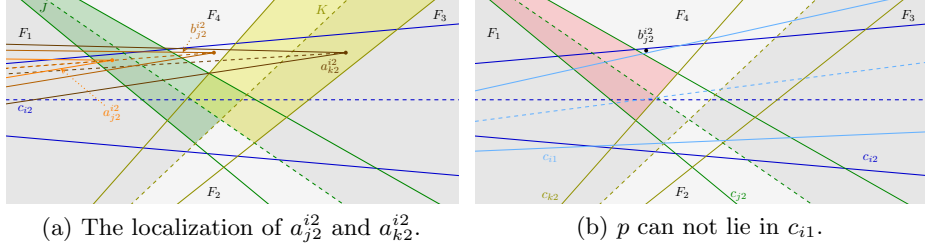
The argument so far yields that if $p \in c_{i2}$, then the intersection order of ℓ_j and ℓ_k with ℓ_i is the same as the order of projection if ℓ_i lies above p, and is the inverse order if ℓ_i lies below p. The uncertainty of this situation is not desirable. By considering the circular sectors c_{i1} and c_{i3}, we will now show that such a situation cannot occur.

First, we show that c_{i1} and c_{i3} cannot contain the intersection point of ℓ_j and ℓ_k. W.l.o.g., assume that the intersection point lies in c_{i1}. Then, b_{j2}^{i1} is included in either F_2 or F_4. Consider the case that b_{j2}^{i1} lies in F_4. Since $u(c_{i2}) = \lambda \cdot (1, 0)$ and since one of the outer line segments of c_{i2} has to lie beneath p, there is only one outer line segment of c_{i2} that intersects $F_4 \setminus (J \cup K)$, J and K. There are at most two intersection points of this outer line segment with ∂c_{i1}. This implies that there is no intersection point of ∂c_{i2} and ∂c_{i1} in at least one of J, K, and $F_4 \setminus (J \cup K)$. If there is no intersection point, then c_{i1} and c_{i2} overlap in this interval. W.l.o.g., let this area be J, and let $c_{i1} \cap J$ be fully contained in $c_{i2} \cap J$. Then, $p(a_{j1}^{im})$ cannot be placed. Consequently, this situation is not possible. The argument is depicted in Figure 5b.

If $p(b_{j2}^{i1})$ was included in F_2, then the order of projection of $p(a_{k2}^{i2})$ and $p(a_{j2}^{i2})$ would be the same order as the order of intersections of ℓ_j and ℓ_k with a parallel line to ℓ_i that lies below ℓ_i. This order is the inverse order of the order of projection in c_{i2}. Since the order of the projection as defined by E_{GO} depends only on k and i, the order of projection of $p(a_{j2}^{im})$ and $p(a_{k2}^{im})$ has to be the same in all c_{im}. This implies that $p(b_{k2}^{i1})$ is not included in F_2.

Now, we know that c_{i1} and c_{i3} do not contain the intersection point. This implies that the argument from the case $p \notin c_{i2}$ can be applied to them and the order of intersection in c_{i1} and c_{i3} is the same as the order of the projections of $p(a_{j2}^{i1})$ and $p(a_{k2}^{i1})$. This order is the same in all three c_{im}, and thus the bisectors of c_{i1} and c_{i3} have to lie on the same side of the intersection point. Furthermore, the points $p(a_{j2}^{i1})$ and $p(a_{j2}^{i3})$ have to lie in J but outside of c_{i2}. This implies that ℓ_{i1} and ℓ_{i3} both intersect ℓ_j and ℓ_k either before ℓ_i or after ℓ_i, while $p(b_{j2}^{i2})$ lies in F_4.

The edges for the local order define that the order of projection onto ℓ_j is $p(a_{j2}^{i1})$, $p(a_{j2}^{i2})$, $p(a_{j2}^{i3})$ (or the reverse), and the analogous statement holds for ℓ_k. This order is not possible with c_{i1} and c_{i3}, both lying above or below c_{i2}, which implies that the intersection point cannot lie in c_{i2}. Since the order of intersection is the same as the order of the projection, if $p \notin c_{i2}$ and a situation with $p \in c_{i2}$ is not possible, we have shown that $\mathcal{D}(\mathcal{L}) = \mathcal{D}$.

With the tools from above, we can now give the proof of the main result of this section:

Theorem 4.8. SECTOR *is hard for* $\exists\mathbb{R}$.

Proof. The theorem follows from Construction 4.5 and lemmas 4.6 and 4.7.

5 Conclusion

We have defined the new graph class of generalized transmission graphs as a model for directed antennas with arbitrary shapes. We showed that the recognition of generalized transmission graphs of line segments and a special form of circular sectors is $\exists\mathbb{R}$-hard.

For the case of circular sectors, we needed to impose certain conditions on the underlying arrangements. The wide spread condition in particular seems to be rather restrictive. We assume that this condition can be weakened, if not dropped, while the problem remains $\exists\mathbb{R}$-hard.

Ours are the first $\exists\mathbb{R}$-hardness results on directed graphs that we are aware of. We believe that this work could serve as a starting point for a broader investigation into the recognition problem for geometrically defined directed graph models, and to understand further what makes these problems hard.

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A Missing proofs and constructions

A.1 Full construction for SECTOR

Let the vertices of the construction be defined as in Construction 4.5. We divide the edges of the graph into *categories*. The first category E_I contains the edges that enforce an intersection of two circular sectors c_{im} and $c_{km'}$ for $k < l$.

$$E_I = \left\{ \left(c_{im}, a_{km'}^{im}\right), \left(c_{im}, a_{im}^{km'}\right) \,\middle|\, i \neq k \right\}.$$

The edges E_C enforce that each $a_{km'}^{im}$ and each $b_{km'}^{im}$ forms a mutual couple with c_{im}.

$$E_C = \left\{ \left(a_{km'}^{im}, c_{im}\right), \left(c_{im}, b_{km'}^{im}\right), \left(b_{km'}^{im}, c_{im}\right) \,\middle|\, i \neq k \right\}.$$

The edges of E_{GO} will enforce the order of the projection of the apexes of $a_{o_k m'}^{im}$, $a_{o_l m''}^{im}$, $b_{o_k m'}^{im}$, and $b_{o_l m''}^{im}$ for $k > l$ onto the bisector of c_{im}. They are chosen such that $p(a_{o_k m'}^{im})$ will be projected closer to $p(c_{im})$ than $p(a_{o_l m''}^{im})$, for $k < l$. Also included in E_{GO} are edges that enforce that all $p(a_{\mathbf{im}}^{\mathbf{o_k m'}})$ are included in the circular sectors $a_{o_l m''}^{im}$ and $b_{o_l m''}^{im}$.

$$
\begin{aligned}
E_{\mathrm{GO}} = \quad & \left\{ (a_{o_k m'}^{im}, a_{o_l m''}^{im}), (a_{o_k m'}^{im}, a_{\mathbf{im}}^{\mathbf{o_l m''}}), (a_{o_k m'}^{im}, b_{o_l m''}^{im}), \right. \\
& \left. (b_{o_k m'}^{im}, a_{o_l m''}^{im}), (b_{o_k m'}^{im}, a_{\mathbf{im}}^{\mathbf{o_l m''}}) \qquad\qquad \middle|\, i \neq k, k > l \right\}.
\end{aligned}
$$

The last two categories of edges will enforce the projection order of the apexes of $a_{o_k 1}^{im}$, $a_{o_k 2}^{im}$, $a_{o_k 3}^{im}$, and $b_{o_k 1}^{im}$, $b_{o_k 2}^{im}$, $b_{o_k 3}^{im}$ onto the bisector of c_{im}. This order is $a_{o_k 1}^{im}$, $b_{o_k 1}^{im}$, $a_{o_k 2}^{im}$, $b_{o_k 2}^{im}$, $a_{o_k 3}^{im}$ $b_{o_k 3}^{im}$, if $o_k > i$, and the inverse order, otherwise. The edges for the first case are E_{LOI}, and the edges for the second case are E_{LOD}. We set

$$
\begin{aligned}
E_{\mathrm{LOI}} = \quad & \left\{ (a_{o_k m'}^{im}, a_{o_k m''}^{im}), (a_{o_k m'}^{im}, a_{\mathbf{im}}^{\mathbf{o_k m''}}), \right. \\
& \left. (a_{o_k m'}^{im}, b_{o_k m''}^{im}), (b_{o_k m'}^{im}, b_{o_k m''}^{im}) \,\middle|\, i \neq k, m'' < m', o_k > i \right\} \\
& \cup \left\{ (b_{o_k m'}^{im}, a_{o_k m''}^{im}), (b_{o_k m'}^{im}, a_{\mathbf{im}}^{\mathbf{o_k m''}}) \,\middle|\, i \neq k, m'' \leq m', o_k > i \right\}
\end{aligned}
$$

and

$$
\begin{aligned}
E_{\mathrm{LOD}} = \quad & \left\{ (a_{o_k m'}^{im}, a_{o_k m''}^{im}), (a_{o_k m'}^{im}, a_{\mathbf{im}}^{\mathbf{o_k m''}}), \right. \\
& \left. (a_{o_k m'}^{im}, b_{o_k m''}^{im}), (b_{o_k m'}^{im}, b_{o_k m''}^{im}) \,\middle|\, i \neq k, m'' > m', o_k < i \right\} \\
& \cup \left\{ (b_{o_k m'}^{im}, a_{o_k m''}^{im}), (b_{o_k m'}^{im}, a_{\mathbf{im}}^{\mathbf{o_k m''}}) \,\middle|\, i \neq k, m'' \geq m', o_k < i \right\}.
\end{aligned}
$$

The set of all edges is defined as

$$E_L = E_I \cup E_C \cup E_{\mathrm{GO}} \cup E_{\mathrm{LOI}} \cup E_{\mathrm{LOD}}.$$

A.2 Remaining proof for Lemma 4.6

Lemma A.1. *The generalized transmission graph of the arrangement $\mathcal{C}$ of circular sectors constructed in Lemma 4.6 is G_L*

Proof. As δ is chosen small enough that $a_{km'}^{im}$ and $a_{\mathbf{im}}^{\mathbf{km}'}$ lie in c_{im}, the edges of E_I are created. Since $b_{km'}^{im}$ and $a_{km'}^{im}$ have the inverse direction of c_{im} and the radii are large enough, $p(c_{im})$ is included in $a_{km'}^{im}$ and in $b_{km'}^{im}$. Hence all edges in E_C are created.

By the choice of the radii and the direction, $a_{o_k m'}^{im}$ includes all apexes of circular sectors that lie on ℓ_{im} and closer to $p(c_m)$ than $p(a_{o_k m'}^{im})$. Furthermore, δ is small enough such that all $a_{\mathbf{im}}^{\mathbf{o_l m}''}$, $l < k$, are included in $a_{o_k m'}^{im}$. This implies that edges from E_{GO} are present in the generalized transmission graph of $\mathcal{C}$.

The only edges that have not been considered yet are the edges in E_{LOI} and E_{LOD}. For a circular sector $a_{o_k m'}^{im}$ with $o_k > i$, the slope of ℓ_{o_k} is larger than the slope of ℓ_i. By the counter-clockwise construction, $\ell_{o_k 1}$ lies above $\ell_{o_k 2}$. This implies that the intersection point of $\ell_{o_k 1}$ and ℓ_{im} lies closer to $p(c_{im})$ than the intersection points with $\ell_{o_k 2}$ or $\ell_{o_k 3}$. The presence of the edges can now be seen by the same argument as for the edges of E_{GO}. Symmetrical considerations can be made for the edges of E_{LOD}.

It remains to show that no additional edges are created. Note that all apexes of the circular sectors lie inside of D and that all $a_{km'}^{im} \cap D$ and $b_{km'}^{im} \cap D$ are included in the boxes $B(c_{im})$.

Since only the apexes of $a_{km'}^{im}$, $a_{\mathbf{im}}^{\mathbf{km}'}$, and $b_{km'}^{im}$ lie in c_{im}, there are no additional edges starting at c_{im}. The rectangles $B(c_{im})$ are disjoint on the boundary of D and all $a_{km'}^{im} \cap D$ and $b_{km'}^{im} \cap D$ lie inside of $B(c_{im})$. This implies that there are no additional edges ending at c_{im}. Now, we have to consider additional edges starting at $a_{km'}^{im}$ and $b_{km'}^{im}$. Note that $\alpha \leq \pi/4$ enforces that no circular sector $a_{km'}^{im}$ or $b_{km'}^{im}$ can reach an apex having a larger distance to $p(c_{im})$. Also, note that there are edges for all circular sectors with smaller distances in E_{GO}, E_{LOD} or E_{LOI}. This covers all possible additional edges.