Maximum matching in disk graphs of bounded depth

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Let $\mathcal{U}$ be a family of n disks in the plane. The ***disk graph*** $G_{\mathcal{U}}$ for $\mathcal{U}$ is the undirected graph with vertex set $\mathcal{U}$ and edge set

$$E(G_{\mathcal{U}}) \;=\; \{UV \mid U, V \in \mathcal{U},\, U \cap V \neq \emptyset\}.$$

If the disks in $\mathcal{U}$ are partitioned into two sets, one can also define the *bipartite* disk graph, a subgraph of $G_{\mathcal{U}}$, in the obvious way. The ***depth*** (ply) of $\mathcal{U}$, $\rho(\mathcal{U})$, is the largest number of disks that cover a single point,

$$\rho(\mathcal{U}) \;=\; \max_{p \in \mathbb{R}^2} |\{U \in \mathcal{U} \mid p \in U\}|.$$

Our goal is to compute a maximum matching in *subgraphs* of disk graphs, assuming a geometric representation. In the bipartite case, the algorithm of Efrat, Itai, and Katz [1], using augmenting paths and blocking flows as well as data structures for weighted nearest neighbors [4], can find a maximum matching in $O(n^{3/2} \operatorname{polylog} n)$ time. Moreover, if $\mathcal{U}$ has bounded depth, i.e., $\rho(\mathcal{U}) = O(1)$, a standard packing argument shows that $G_{\mathcal{U}}$ has $O(n)$ edges and thus can be constructed explicitly in $O(n \log n)$ time, using a plane sweep to check for disk containment and for intersecting disk boundaries. Then, the algorithm of Micali-Vazirani needs $O(\sqrt{n}|E(G_{\mathcal{U}})|) = O(n^{3/2})$ time to find a maximum matching [6].

We investigate whether faster algorithms are possible if we consider the depth $\rho(\mathcal{U})$ as an additional parameter. Naturally, the case $\rho(\mathcal{U}) = O(1)$ of bounded depth is of particular interest.

Mucha and Sankowski [7] gave an $O(n^{\omega/2})$-time randomized algorithm for maximum matchings in n-vertex planar graphs, where ω is the matrix multiplication exponent. Their method needs linear algebra over finite fields and Gaussian elimination; the planarity is used to obtain nested dissections [2]. This was extended by Yuster and Zwick [8] to H-minor-free graphs. In fact, it works in $O(n^{\beta\omega})$ time for *hereditary* graph families (closed under taking subgraphs) that have bounded average degree and separators of size $O(n^{\beta})$ that can be found in linear time, for $1/2 < \beta < 1$.

Using that disk graphs of bounded depth have separators of size $O(\sqrt{n})$ and constant average degree, one can adapt the algorithms of [7, 8] to find a maximum matching in a subgraph of a disk graph with n disks and depth $O(1)$ in $O(n^{\omega/2}) = O(n^{1.19})$ time. The dependency on ρ is polynomial, but it is not easy to trace it precisely.

There are several interesting features to this approach. For one, we solve a geometric problem with algebraic tools, namely linear algebra over finite fields. The use of geometry is limited to finding separators and bounding the degree. Moreover, the required properties of the disks are very weak, and the approach extends to low-density families of planar objects using the separators by Har-Peled and Quanrud [3].

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