Some New Operations Over Fermatean Fuzzy Numbers and Application of Fermatean Fuzzy WPM in Multiple Criteria Decision Making

Tapan SENAPATI[1]*, Ronald R. YAGER[2]

[1]*Department of Applied Mathematics with Oceanology and Computer Programming,*
 Vidyasagar University, Midnapore 721102, India
[2]*Machine Intelligence Institute, Iona College, New Rochelle, NY 10801, USA*
e-mail: math.tapan@gmail.com, yager@panix.com

Abstract. Fermatean fuzzy sets (FFSs), proposed by Senapati and Yager (2019a), can handle uncertain information more easily in the process of decision making. They defined basic operations over the Fermatean fuzzy sets. Here we shall introduce three new operations: subtraction, division, and Fermatean arithmetic mean operations over Fermatean fuzzy sets. We discuss their properties in details. Later, we develop a Fermatean fuzzy weighted product model to solve the multi-criteria decision-making problem. Finally, an illustrative example of selecting a suitable bridge construction method is given to verify the approach developed by us and to demonstrate its practicability and effectiveness.
Key words: Fermatean fuzzy set, subtraction operation, division operation, Fermatean arithmetic mean operation, multiple criteria decision making (MCDM), weighted product model (WPM).

1. Introduction

Orthopair fuzzy sets are fuzzy sets in which the membership grades of an element x are pairs of values in the unit interval, $\langle \mu(x), \nu(x) \rangle$, one of which indicates support for membership in the fuzzy set and the other support against membership. Two examples of orthopair fuzzy sets are Atanassov's classic intuitionistic fuzzy sets (IFSs) (Atanassov, 1986, 2012; Atanassov *et al.*, 2013) and a second kind of intuitionistic fuzzy sets (Atanassov, 1983, 2016). This idea has been followed up in Parvathi (2005), Parvathi *et al.* (2012), Vassilev *et al.* (2008), Vassilev (2012, 2013). It is noted that for classic IFSs the sum of the support for and against is bounded by one, while for the second kind, Pythagorean fuzzy sets (PFSs), the sum of the squares of the support for and against is bounded by one. (Yager, 2017) introduced a general class of these sets called q-rung orthopair fuzzy sets in which the sum of the qth power of the support for and the qth power of the support against is bounded by one. He noted that as q increases, the space of acceptable orthopairs

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*Corresponding author.

increases and thus gives the user more freedom in expressing their belief about membership grade. When $q = 3$, Senapati and Yager (2019a) have considered q-rung orthopair fuzzy sets as Fermatean fuzzy sets (FFSs).

PFSs have attracted the attention of many researchers within a short period of time. For example, Yager (2014) built up a helpful decision technique in view of Pythagorean fuzzy aggregation operators to deal with Pythagorean fuzzy MCDM issues. Yager and Abbasov (2013) explained the Pythagorean membership grades (PMGs) and the thoughts identified with PFSs and presented the connection between the PMGs and the complex numbers. Reformat and Yager (2014) applied the PFNs in handling the collaborative-based recommender system. Gou *et al.* (2016) initiated a few Pythagorean fuzzy mappings and studied their basic properties like derivability, continuity, and differentiability in details. Zeng *et al.* (2018) established an aggregation procedure for the PFS and utilized its application in solving MADM problems. Zhang (2016) introduced a MCDM approach in view of the idea of the similarity measure. PFSs have been effectively connected in different fields, for example, the investment decision (Garg, 2016; Peng and Yang, 2015), the candidate selection of Asian Infrastructure Investment Bank (Ren *et al.*, 2016) and the service quality of domestic airline (Zhang and Xu, 2014).

Senapati and Yager (2019a) have given an example to prove the reasonability of the FFS: When a person wants to give his preference for the degree of an alternative x_i to a criterion C_j, he may allow the degree to which the alternative x_i satisfies the criterion C_j as 0.9, and similarly when the alternative x_i dissatisfies the criterion C_j as 0.6. We can definitely get $0.9 + 0.6 > 1$, and, therefore, it does not follow the condition of IFSs. Also, we can get $(0.9)^2 + (0.6)^2 = 0.81 + 0.36 = 1.17 > 1$, which does not obey constraint condition of PFS. However, we can get $(0.9)^3 + (0.6)^3 = 0.729 + 0.216 = 0.945 \leqslant 1$, which is appropriate to engage the FFS to capture it. This is to mention that the FFSs have more uncertainties than IFSs and PFSs, and are capable to handle higher levels of uncertainties.

Senapati and Yager (2019a) defined basic operations over the FFSs and introduced new score functions and accuracy functions of FFSs. They extended the technique for order preference by similarity to ideal solution (TOPSIS) approach to handling the MCDM problem with Fermatean fuzzy information. Senapati and Yager (2019b) introduced several Fermatean fuzzy aggregation operators, for example, the Fermatean fuzzy weighted average (FFWA) operator, Fermatean fuzzy weighted geometric (FFWG) operator, Fermatean fuzzy weighted power average (FFWPA) operator and Fermatean fuzzy weighted power geometric (FFWPG) operator.

The subtraction and division operations of IFS have been introduced by some authors (Atanassov, 2009; Atanassov and Riecan, 2006; Chen, 2007) and finally been established by Atanassov (2012) in his recently published book. Since subtraction and division operations are also necessary for FFS, we are going to introduce those two new operations on FFS. The contributions of this paper are the following: Section 2 displays the preliminary information of the paper. Then, in Section 3, we concentrate on basic knowledge of FFS and their fundamental operations. Section 4 proposes the subtraction operation and the division operation over FFSs. Several operational laws of these two operations are

given. In addition, the relationships between these two operations are also established in this paper. In Section 5, we discuss two theorems on Fermatean arithmetic mean operations over FFNs. Section 6 describes the concept of weighted product models (WPMs). In next Section, we utilize Fermatean fuzzy weighted product model to solve multi-criteria decision-making problems. Section 8 exhibits the reliability and sustainability of the proposed method in fruitful and flawless application by attaching it to an MCDM problem for choosing the most appropriate bridge construction method. Finally, in Section 9, the conclusion and scope of future research are outlined and discussed.

2. IFSs and PFSs

In this section, we have given some definitions, and that, hopefully will come in handy in our work.

IFS, which was invented or formerly explained by Atanassov (1986), is an elaboration of the traditional fuzzy set, and it is a reliable and enormously accepted way to ascertain the uncertainties. It can also be described as follows.

DEFINITION 1. (See Atanassov, 2012.) (IFSs) The intuitionistic fuzzy sets are defined on a non-empty set X as objects having the form $A = \{\langle x, \alpha_A(x), \beta_A(x)\rangle : x \in X\}$, where the functions $\alpha_A(x) : X \to [0, 1]$ and $\beta_A(x) : X \to [0, 1]$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set A, respectively, and $0 \leqslant \alpha_A(x) + \beta_A(x) \leqslant 1$ for all $x \in X$. Obviously, when $\beta_A(x) = 1 - \alpha_A(x)$ for all $x \in X$, the set A becomes a fuzzy set.

Currently, Yager (2013) launched a nonstandard fuzzy set referred to as PFS, whose definition is as follows:

DEFINITION 2. (PFSs) The Pythagorean fuzzy sets defined on a non-empty set X as objects having the form $\mathcal{P} = \{\langle x, \alpha_P(x), \beta_P(x)\rangle : x \in X\}$, where the functions $\alpha_P(x) : X \to [0, 1]$ and $\beta_P(x) : X \to [0, 1]$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set $\mathcal{P}$, respectively, and $0 \leqslant (\alpha_P(x))^2 + (\beta_P(x))^2 \leqslant 1$, for every $x \in X$. For any Pythagorean fuzzy set $\mathcal{P}$ and $x \in X$, $\pi_P(x) = \sqrt{1 - (\alpha_P(x))^2 - (\beta_P(x))^2}$ is called the degree of indeterminacy of x to $\mathcal{P}$.

3. Fermatean Fuzzy Sets

In this section, Fermatean fuzzy sets are defined in detail and their corresponding properties are discussed. Score and accuracy functions of these sets have been defined and compared.

DEFINITION 3. (See Senapati and Yager, 2019a.) Let X be a universe of discourse. A Fermatean fuzzy set $\mathcal{F}$ in X is an object having the form

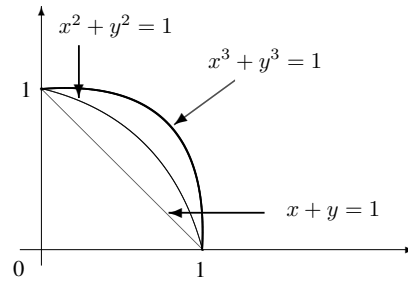$$\mathcal{F} = \big\{\langle x, \alpha_F(x), \beta_F(x)\rangle : x \in X\big\},$$

Fig. 1. Comparison of space of FMGs, PMGs and IMGs.

where $\alpha_F(x) : X \to [0, 1]$ and $\beta_F(x) : X \to [0, 1]$, including the condition

$$0 \leqslant \big(\alpha_F(x)\big)^3 + \big(\beta_F(x)\big)^3 \leqslant 1,$$

for all $x \in X$. The numbers $\alpha_F(x)$ and $\beta_F(x)$ denote, respectively, the degree of membership and the degree of non-membership of the element x in the set $\mathcal{F}$.

For any FFS $\mathcal{F}$ and $x \in X$, $\pi_F(x) = \sqrt[3]{1 - (\alpha_F(x))^3 - (\beta_F(x))^3}$ is identified as the degree of indeterminacy of x to $\mathcal{F}$.

For convenience, Senapati and Yager called $(\alpha_F(x), \beta_F(x))$ a Fermatean fuzzy number (FFN) denoted by $\mathcal{F} = (\alpha_F, \beta_F)$.

We shall point out the membership grades related to Fermatean fuzzy sets as Fermatean membership grades (FMGs).

Theorem 1. (See Senapati and Yager, 2019a.) *The set of FMGs is larger than the set of Pythagorean membership grades (PMGs) and intuitionistic membership grades (IMGs).*

This development can be evidently recognized from Fig. 1. Here we notice that IMGs are all points beneath the line $x + y \leqslant 1$, the PMGs are all points with $x^2 + y^2 \leqslant 1$ and the FMGs are all points with $x^3 + y^3 \leqslant 1$. We see then that the FMGs enable the presentation of a bigger body of nonstandard membership grades than IMGs and PMGs.

DEFINITION 4. (See Senapati and Yager, 2019a.) Let $\mathcal{F} = (\alpha_F, \beta_F)$, $\mathcal{F}_1 = (\alpha_{F_1}, \beta_{F_1})$ and $\mathcal{F}_2 = (\alpha_{F_2}, \beta_{F_2})$ be three FFNs, then their operations are defined as follows:

(i) $\mathcal{F}_1 \cap \mathcal{F}_2 = \big(\min\{\alpha_{F_1}, \alpha_{F_2}\}, \max\{\beta_{F_1}, \beta_{F_2}\}\big)$;
(ii) $\mathcal{F}_1 \cup \mathcal{F}_2 = \big(\max\{\alpha_{F_1}, \alpha_{F_2}\}, \min\{\beta_{F_1}, \beta_{F_2}\}\big)$;
(iii) $\mathcal{F}^c = (\beta_F, \alpha_F)$.

DEFINITION 5. (See Senapati and Yager, 2019a.) Let $\mathcal{F} = (\alpha_F, \beta_F)$, $\mathcal{F}_1 = (\alpha_{F_1}, \beta_{F_1})$ and $\mathcal{F}_2 = (\alpha_{F_2}, \beta_{F_2})$ be three FFNs and $\lambda > 0$, then their operations are interpreted in this way:

(i) $\mathcal{F}_1 \boxplus \mathcal{F}_2 = \Big(\sqrt[3]{\alpha_{F_1}^3 + \alpha_{F_2}^3 - \alpha_{F_1}^3 \alpha_{F_2}^3}, \beta_{F_1}\beta_{F_2}\Big)$;

(ii) $\mathcal{F}_1 \boxtimes \mathcal{F}_2 = \left(\alpha_{F_1} \alpha_{F_2}, \sqrt[3]{\beta_{F_1}^3 + \beta_{F_2}^3 - \beta_{F_1}^3 \beta_{F_2}^3} \right)$;

(iii) $\lambda \mathcal{F} = \left(\sqrt[3]{1 - (1 - \alpha_F^3)^\lambda}, \beta_F^\Lambda \right)$;

(iv) $\mathcal{F}^\lambda = \left(\alpha_F^\lambda, \sqrt[3]{1 - (1 - \beta_F^3)^\lambda} \right)$.

Theorem 2. (See Senapati and Yager, 2019a.) *For three FFNs* $\mathcal{F} = (\alpha_F, \beta_F)$, $\mathcal{F}_1 = (\alpha_{F_1}, \beta_{F_1})$ *and* $\mathcal{F}_2 = (\alpha_{F_2}, \beta_{F_2})$, *the following ones are valid*:

  (i) $\mathcal{F}_1 \boxplus \mathcal{F}_2 = \mathcal{F}_2 \boxplus \mathcal{F}_1$;
 (ii) $\mathcal{F}_1 \boxtimes \mathcal{F}_2 = \mathcal{F}_2 \boxtimes \mathcal{F}_1$;
(iii) $\lambda(\mathcal{F}_1 \boxplus \mathcal{F}_2) = \lambda \mathcal{F}_1 \boxplus \lambda \mathcal{F}_2$, $\lambda > 0$;
 (iv) $(\lambda_1 + \lambda_2)\mathcal{F} = \lambda_1 \mathcal{F} \boxplus \lambda_2 \mathcal{F}$, $\lambda_1, \lambda_2 > 0$;
  (v) $(\mathcal{F}_1 \boxtimes \mathcal{F}_2)^\lambda = \mathcal{F}_1^\lambda \boxtimes \mathcal{F}_2^\lambda$, $\lambda > 0$;
 (vi) $\mathcal{F}^{\lambda_1} \boxtimes \mathcal{F}^{\lambda_2} = \mathcal{F}^{(\lambda_1 + \lambda_2)}$, $\lambda_1, \lambda_2 > 0$.

Theorem 3. (See Senapati and Yager, 2019a.) *For three FFNs* $\mathcal{F}_1 = (\alpha_{F_1}, \beta_{F_1})$, $\mathcal{F}_2 = (\alpha_{F_2}, \beta_{F_2})$ *and* $\mathcal{F}_3 = (\alpha_{F_3}, \beta_{F_3})$, *the following ones are valid*:

  (i) $\mathcal{F}_1 \cap \mathcal{F}_2 = \mathcal{F}_2 \cap \mathcal{F}_1$;
 (ii) $\mathcal{F}_1 \cup \mathcal{F}_2 = \mathcal{F}_2 \cup \mathcal{F}_1$;
(iii) $\mathcal{F}_1 \cap (\mathcal{F}_2 \cap \mathcal{F}_3) = (\mathcal{F}_1 \cap \mathcal{F}_2) \cap \mathcal{F}_3$;
 (iv) $\mathcal{F}_1 \cup (\mathcal{F}_2 \cup \mathcal{F}_3) = (\mathcal{F}_1 \cup \mathcal{F}_2) \cup \mathcal{F}_3$;
  (v) $\lambda(\mathcal{F}_1 \cup \mathcal{F}_2) = \lambda \mathcal{F}_1 \cup \lambda \mathcal{F}_2$;
 (vi) $(\mathcal{F}_1 \cup \mathcal{F}_2)^\lambda = \mathcal{F}_1^\lambda \cup \mathcal{F}_2^\lambda$.

Theorem 4. (See Senapati and Yager, 2019a.) *Let* $\mathcal{F} = (\alpha_F, \beta_F)$, $\mathcal{F}_1 = (\alpha_{F_1}, \beta_{F_1})$ *and* $\mathcal{F}_2 = (\alpha_{F_2}, \beta_{F_2})$ *be three FFNs, then*

  (i) $(\mathcal{F}_1 \cap \mathcal{F}_2)^c = \mathcal{F}_1^c \cup \mathcal{F}_2^c$;
 (ii) $(\mathcal{F}_1 \cup \mathcal{F}_2)^c = \mathcal{F}_1^c \cap \mathcal{F}_2^c$;
(iii) $(\mathcal{F}_1 \boxplus \mathcal{F}_2)^c = \mathcal{F}_1^c \boxtimes \mathcal{F}_2^c$;
 (iv) $(\mathcal{F}_1 \boxtimes \mathcal{F}_2)^c = \mathcal{F}_1^c \boxplus \mathcal{F}_2^c$;
  (v) $(\mathcal{F}^c)^\lambda = (\lambda \mathcal{F})^c$;
 (vi) $\lambda(\mathcal{F}^c) = (\mathcal{F}^\lambda)^c$.

Theorem 5. (See Senapati and Yager, 2019a.) *Let* $\mathcal{F}_1 = (\alpha_{F_1}, \beta_{F_1})$, $\mathcal{F}_2 = (\alpha_{F_2}, \beta_{F_2})$ *and* $\mathcal{F}_3 = (\alpha_{F_3}, \beta_{F_3})$ *be three FFNs, then*

  (i) $(\mathcal{F}_1 \cap \mathcal{F}_2) \boxplus \mathcal{F}_3 = (\mathcal{F}_1 \boxplus \mathcal{F}_3) \cap (\mathcal{F}_2 \boxplus \mathcal{F}_3)$;
 (ii) $(\mathcal{F}_1 \cup \mathcal{F}_2) \boxplus \mathcal{F}_3 = (\mathcal{F}_1 \boxplus \mathcal{F}_3) \cup (\mathcal{F}_2 \boxplus \mathcal{F}_3)$;
(iii) $(\mathcal{F}_1 \cap \mathcal{F}_2) \boxtimes \mathcal{F}_3 = (\mathcal{F}_1 \boxtimes \mathcal{F}_3) \cap (\mathcal{F}_2 \boxtimes \mathcal{F}_3)$;
 (iv) $(\mathcal{F}_1 \cup \mathcal{F}_2) \boxtimes \mathcal{F}_3 = (\mathcal{F}_1 \boxtimes \mathcal{F}_3) \cup (\mathcal{F}_2 \boxtimes \mathcal{F}_3)$.

In order to rank FFNs, we define the score function of the FFN:

DEFINITION 6. For any FFN $\mathcal{F} = (\alpha_F, \beta_F)$, the score function of $\mathcal{F}$ can be defined as follows:

$$score(\mathcal{F}) = \alpha_F^3 - \beta_F^3,$$

where $score(\mathcal{F}) \in [-1, 1]$. For any two FFNs $\mathcal{F}_1 = (\alpha_{F_1}, \beta_{F_1})$ and $\mathcal{F}_2 = (\alpha_{F_2}, \beta_{F_2})$, if $score(\mathcal{F}_1) < score(\mathcal{F}_2)$, then $\mathcal{F}_1 < \mathcal{F}_2$. If $score(\mathcal{F}_1) > score(\mathcal{F}_2)$, then $\mathcal{F}_1 > \mathcal{F}_2$. If $score(\mathcal{F}_1) = score(\mathcal{F}_2)$, then $\mathcal{F}_1 \sim \mathcal{F}_2$.

By Definition 6, we recognize that the score function is well organized to rank the FFNs. But, in some instances, this function is not suitable to suggest which FFN is superior, for example, if $\mathcal{F}_1 = \left(\frac{\sqrt[3]{4}}{2}, \frac{\sqrt[3]{4}}{2}\right)$ and $\mathcal{F}_2 = (0.5, 0.5)$, then $score(\mathcal{F}_1) = score(\mathcal{F}_2) = 0$. More generally, if any FFN satisfies $\alpha_F = \beta_F$, then its score is 0. But we realize that these FFNs are not identical. So impelled by the definition of the score function, an accuracy function for FFN can be described as:

DEFINITION 7. Let $\mathcal{F} = (\alpha_F, \beta_F)$ be an FFN, then the accuracy function of $\mathcal{F}$ may be described as follows:

$$acc(\mathcal{F}) = \alpha_F^3 + \beta_F^3.$$

Clearly, $acc(\mathcal{F}) \in [0, 1]$. In fact, $0 \leqslant acc(\mathcal{F}) = \alpha_F^3 + \beta_F^3 \leqslant 1$. The larger the value of $acc(\mathcal{F})$ is, the higher is the degree of accuracy of the FFS $\mathcal{F}$.

From Definitions 3 and 7, we can get $\pi_F^3 + acc(\mathcal{F}) = 1$. The lower degree of indeterminacy makes the higher accuracy of the FFN $\mathcal{F} = (\alpha_F, \beta_F)$.

Depending upon these score and accuracy functions of FFNs, the ranking technique for any two FFNs can be described as:

DEFINITION 8. Let $\mathcal{F}_1 = (\alpha_{F_1}, \beta_{F_1})$ and $\mathcal{F}_2 = (\alpha_{F_2}, \beta_{F_2})$ be two FFNs. $score(\mathcal{F}_i)$ and $acc(\mathcal{F}_i)$ $(i = 1, 2)$ are the score values and accuracy values of $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, then

1. If $score(\mathcal{F}_1) < score(\mathcal{F}_2)$, then $\mathcal{F}_1 < \mathcal{F}_2$;
2. If $score(\mathcal{F}_1) > score(\mathcal{F}_2)$, then $\mathcal{F}_1 > \mathcal{F}_2$;
3. If $score(\mathcal{F}_1) = score(\mathcal{F}_2)$, then
    (i) If $acc(\mathcal{F}_1) < acc(\mathcal{F}_2)$, then $\mathcal{F}_1 < \mathcal{F}_2$;
    (ii) If $acc(\mathcal{F}_1) > acc(\mathcal{F}_2)$, then $\mathcal{F}_1 > \mathcal{F}_2$;
    (iii) If $acc(\mathcal{F}_1) = acc(\mathcal{F}_2)$, then $\mathcal{F}_1 = \mathcal{F}_2$.

4. Subtraction and Division Operations over FFNs

The subtraction and division operations in IFSs had been firstly suggested by Atanassov and Riecan (2006). Later, Chen (2007) additionally brought comparable operations for IFSs. Liao and Xu (2014) utilized these operations over hesitant fuzzy sets. Here we extend subtraction and division operations over FFNs.

DEFINITION 9. Let $\mathcal{F}_1 = (\alpha_{F_1}, \beta_{F_1})$ and $\mathcal{F}_2 = (\alpha_{F_2}, \beta_{F_2})$ be two FFNs, then

(i) $\mathcal{F}_1 \boxminus \mathcal{F}_2 = \left(\sqrt[3]{\dfrac{\alpha_{F_1}^3 - \alpha_{F_2}^3}{1 - \alpha_{F_2}^3}}, \dfrac{\beta_{F_1}}{\beta_{F_2}} \right)$ if $\alpha_{F_1} \geqslant \alpha_{F_2},\ \beta_{F_1} \leqslant \min\left\{ \beta_{F_2}, \dfrac{\beta_{F_2}\pi_1}{\pi_2} \right\}$;

(ii) $\mathcal{F}_1 \boxslash \mathcal{F}_2 = \left(\dfrac{\alpha_{F_1}}{\alpha_{F_2}}, \sqrt[3]{\dfrac{\beta_{F_1}^3 - \beta_{F_2}^3}{1 - \beta_{F_2}^3}} \right)$ if $\alpha_{F_1} \leqslant \min\left\{ \alpha_{F_2}, \dfrac{\alpha_{F_2}\pi_1}{\pi_2} \right\},\ \beta_{F_1} \geqslant \beta_{F_2}$.

Theorem 6. *Let $\mathcal{F} = (\alpha_F, \beta_F)$, $\mathcal{F}_1 = (\alpha_{F_1}, \beta_{F_1})$ and $\mathcal{F}_2 = (\alpha_{F_2}, \beta_{F_2})$ be three FFNs, and $\lambda, \lambda_1, \lambda_2 > 0$, then*

(i) $\lambda(\mathcal{F}_1 \boxminus \mathcal{F}_2) = \lambda\mathcal{F}_1 \boxminus \lambda\mathcal{F}_2$ *if* $\alpha_{F_1} \geqslant \alpha_{F_2},\ \beta_{F_1} \leqslant \min\left\{ \beta_{F_2}, \dfrac{\beta_{F_2}\pi_1}{\pi_2} \right\}$;

(ii) $(\mathcal{F}_1 \boxslash \mathcal{F}_2)^\lambda = \mathcal{F}_1^\lambda \boxslash \mathcal{F}_2^\lambda$ *if* $\alpha_{F_1} \leqslant \min\left\{ \alpha_{F_2}, \dfrac{\alpha_{F_2}\pi_1}{\pi_2} \right\},\ \beta_{F_1} \geqslant \beta_{F_2}$;

(iii) $\lambda_1\mathcal{F} \boxminus \lambda_2\mathcal{F} = (\lambda_1 \boxminus \lambda_2)\mathcal{F}$ *if* $\lambda_1 \geqslant \lambda_2$;

(iv) $\mathcal{F}^{\lambda_1} \boxslash \mathcal{F}^{\lambda_2} = \mathcal{F}^{(\lambda_1 - \lambda_2)}$ *if* $\lambda_1 \geqslant \lambda_2$.

Proof. We will present the proofs of (i) and (iii). Let $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ be three FFNs, and $\lambda, \lambda_1, \lambda_2 > 0$.

(i) Since $\alpha_{F_1} \geqslant \alpha_{F_2},\ \beta_{F_1} \leqslant \min\left\{ \beta_{F_2}, \dfrac{\beta_{F_2}\pi_1}{\pi_2} \right\}$ we have

$\beta_{F_1}\pi_2 \leqslant \beta_{F_2}\pi_1$

$\Rightarrow\quad \beta_{F_1}^3 \pi_2^3 \leqslant \beta_{F_2}^3 \pi_1^3$

$\Rightarrow\quad \beta_{F_1}^3 \beta_{F_2}^3 + \beta_{F_1}^3 \pi_2^3 \leqslant \beta_{F_1}^3 \beta_{F_2}^3 + \beta_{F_2}^3 \pi_1^3$

$\Rightarrow\quad \beta_{F_1}^3 (\beta_{F_2}^3 + \pi_2^3) \leqslant \beta_{F_2}^3 (\beta_{F_1}^3 + \pi_1^3)$

$\Rightarrow\quad \beta_{F_1}^3 (1 - \alpha_{F_2}^3) \leqslant \beta_{F_2}^3 (1 - \alpha_{F_1}^3)$

$\Rightarrow\quad \left(\dfrac{\beta_{F_1}^3}{\beta_{F_2}^3} \right)^\lambda \leqslant \left(\dfrac{1 - \alpha_{F_1}^3}{1 - \alpha_{F_2}^3} \right)^\lambda$

$\Rightarrow\quad 1 - \left(\dfrac{1 - \alpha_{F_1}^3}{1 - \alpha_{F_2}^3} \right)^\lambda + \left(\dfrac{\beta_{F_1}^3}{\beta_{F_2}^3} \right)^\lambda \leqslant 1$

$\Rightarrow\quad \left(\sqrt[3]{1 - \left(\dfrac{1 - \alpha_{F_1}^3}{1 - \alpha_{F_2}^3} \right)^\lambda} \right)^3 + \left(\dfrac{\beta_{F_1}^\lambda}{\beta_{F_2}^\lambda} \right)^3 \leqslant 1.$

Then from Definitions 9 and 5, we get

$$\Lambda(\mathcal{F}_1 \boxminus \mathcal{F}_2) = \Lambda\left(\sqrt[3]{\dfrac{\alpha_{F_1}^3 - \alpha_{F_2}^3}{1 - \alpha_{F_2}^3}}, \dfrac{\beta_{F_1}}{\beta_{F_2}} \right) = \left(\sqrt[3]{1 - \left(1 - \dfrac{\alpha_{F_1}^3 - \alpha_{F_2}^3}{1 - \alpha_{F_2}^3} \right)^\lambda}, \left(\dfrac{\beta_{F_1}}{\beta_{F_2}} \right)^\lambda \right)$$

$$= \left(\sqrt[3]{1 - \left(\dfrac{1 - \alpha_{F_1}^3}{1 - \alpha_{F_2}^3} \right)^\lambda}, \dfrac{\beta_{F_1}^\lambda}{\beta_{F_2}^\lambda} \right)$$

and

$$\lambda\mathcal{F}_1 \boxminus \lambda\mathcal{F}_2 = \left(\sqrt[3]{1-\left(1-\alpha_{F_1}^3\right)^\lambda}, \beta_{F_1}^\Lambda\right) \boxminus \left(\sqrt[3]{1-\left(1-\alpha_{F_2}^3\right)^\lambda}, \beta_{F_2}^\lambda\right)$$

$$= \left(\sqrt[3]{\frac{1-(1-\alpha_{F_1}^3)^\lambda - 1 + (1-\alpha_{F_2}^3)^\lambda}{1-1+(1-\alpha_{F_2}^3)^\lambda}}, \frac{\beta_{F_1}^\lambda}{\beta_{F_2}^\lambda}\right)$$

$$= \left(\sqrt[3]{\frac{(1-\alpha_{F_2}^3)^\lambda - (1-\alpha_{F_1}^3)^\lambda}{(1-\alpha_{F_2}^3)^\lambda}}, \frac{\beta_{F_1}^\lambda}{\beta_{F_2}^\lambda}\right)$$

$$= \left(\sqrt[3]{1-\left(\frac{1-\alpha_{F_1}^3}{1-\alpha_{F_2}^3}\right)^\lambda}, \frac{\beta_{F_1}^\lambda}{\beta_{F_2}^\lambda}\right).$$

Hence, $\lambda(\mathcal{F}_1 \boxminus \mathcal{F}_2) = \lambda\mathcal{F}_1 \boxminus \lambda\mathcal{F}_2$.

(iii) $\lambda_1\mathcal{F} \boxminus \lambda_2\mathcal{F} = \left(\sqrt[3]{1-(1-\alpha_F^3)^{\lambda_1}}, \beta_F^{\lambda_1}\right) \boxminus \left(\sqrt[3]{1-(1-\alpha_F^3)^{\lambda_2}}, \beta_F^{\lambda_2}\right)$

$$= \left(\sqrt[3]{\frac{1-(1-\alpha_F^3)^{\lambda_1} - 1 + (1-\alpha_F^3)^{\lambda_2}}{1-1+(1-\alpha_F^3)^{\lambda_2}}}, \frac{\beta_F^{\lambda_1}}{\beta_F^{\lambda_2}}\right)$$

$$= \left(\sqrt[3]{1-(1-\alpha_F^3)^{\lambda_1-\lambda_2}}, \beta_F^{\lambda_1-\lambda_2}\right)$$

$$= (\lambda_1 \boxminus \lambda_2)\mathcal{F}.$$

The other assertions are proved analogously.   □

Theorem 7. *Let $\mathcal{F}_1 = (\alpha_{F_1}, \beta_{F_1})$ and $\mathcal{F}_2 = (\alpha_{F_2}, \beta_{F_2})$ be two FFNs, then*

   (i) $(\mathcal{F}_1 \boxminus \mathcal{F}_2)^c = \mathcal{F}_1^c \boxslash \mathcal{F}_2^c$ *if* $\alpha_{F_1} \geqslant \alpha_{F_2}, \beta_{F_1} \leqslant \min\left\{\beta_{F_2}, \frac{\beta_{F_2}\pi_1}{\pi_2}\right\}$;

   (ii) $(\mathcal{F}_1 \boxslash \mathcal{F}_2)^c = \mathcal{F}_1^c \boxslash \mathcal{F}_2^c$ *if* $\alpha_{F_1} \leqslant \min\left\{\alpha_{F_2}, \frac{\alpha_{F_2}\pi_1}{\pi_2}\right\}, \beta_{F_1} \geqslant \beta_{F_2}$.

Proof. We will present the proof of (i). Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two FFNs, then

$$(\mathcal{F}_1 \boxminus \mathcal{F}_2)^c = \left(\sqrt[3]{\frac{\alpha_{F_1}^3 - \alpha_{F_2}^3}{1-\alpha_{F_2}^3}}, \frac{\beta_{F_1}}{\beta_{F_2}}\right)^c = \left(\frac{\beta_{F_1}}{\beta_{F_2}}, \sqrt[3]{\frac{\alpha_{F_1}^3 - \alpha_{F_2}^3}{1-\alpha_{F_2}^3}}\right)$$

$$= (\beta_{F_1}, \alpha_{F_1}) \boxslash (\beta_{F_2}, \alpha_{F_2}) = \mathcal{F}_1^c \boxslash \mathcal{F}_2^c.$$

Assertion (ii) is proved analogously.   □

Theorem 8. *Let $\mathcal{F}_1 = (\alpha_{F_1}, \beta_{F_1})$ and $\mathcal{F}_2 = (\alpha_{F_2}, \beta_{F_2})$ be two FFNs, then*

   (i) $(\mathcal{F}_1 \cup \mathcal{F}_2) \boxminus (\mathcal{F}_1 \cap \mathcal{F}_2) = \mathcal{F}_1 \boxminus \mathcal{F}_2$ *if* $\alpha_{F_1} \geqslant \alpha_{F_2}, \beta_{F_1} \leqslant \min\left\{\beta_{F_2}, \frac{\beta_{F_2}\pi_1}{\pi_2}\right\}$;

(ii) $(\mathcal{F}_1 \cup \mathcal{F}_2) \boxslash (\mathcal{F}_1 \cap \mathcal{F}_2) = \mathcal{F}_1 \boxslash \mathcal{F}_2 \ \text{if} \ \alpha_{F_1} \leqslant \min\left\{\alpha_{F_2}, \frac{\alpha_{F_2}\pi_1}{\pi_2}\right\}, \ \beta_{F_1} \geqslant \beta_{F_2}.$

Proof. We will present the proof of (i). Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two FFNs. Since $\alpha_{F_1} \geqslant \alpha_{F_2}$, $\beta_{F_1} \leqslant \min\left\{\beta_{F_2}, \frac{\beta_{F_2}\pi_1}{\pi_2}\right\}$, then

$$(\mathcal{F}_1 \cup \mathcal{F}_2) \boxminus (\mathcal{F}_1 \cap \mathcal{F}_2)$$

$$= \left(\max\{\alpha_{F_1}, \alpha_{F_2}\}, \min\{\beta_{F_1}, \beta_{F_2}\}\right) \boxminus \left(\min\{\alpha_{F_1}, \alpha_{F_2}\}, \max\{\beta_{F_1}, \beta_{F_2}\}\right)$$

$$= \left(\sqrt[3]{\frac{(\max\{\alpha_{F_1}, \alpha_{F_2}\})^3 - (\min\{\alpha_{F_1}, \alpha_{F_2}\})^3}{1 - (\min\{\alpha_{F_1}, \alpha_{F_2}\})^3}}, \frac{\min\{\beta_{F_1}, \beta_{F_2}\}}{\max\{\beta_{F_1}, \beta_{F_2}\}}\right)$$

$$= \left(\sqrt[3]{\frac{\max\left\{\alpha_{F_1}^3, \alpha_{F_2}^3\right\} - \min\left\{\alpha_{F_1}^3, \alpha_{F_2}^3\right\}}{1 - \min\left\{\alpha_{F_1}^3, \alpha_{F_2}^3\right\}}}, \frac{\beta_{F_1}}{\beta_{F_2}}\right)$$

$$= \left(\sqrt[3]{\frac{\alpha_{F_1}^3 - \alpha_{F_2}^3}{1 - \alpha_{F_2}^3}}, \frac{\beta_{F_1}}{\beta_{F_2}}\right) = \mathcal{F}_1 \boxminus \mathcal{F}_2.$$

Assertion (ii) is proved analogously. □

Theorem 9. *Let $\mathcal{F}_1 = (\alpha_{F_1}, \beta_{F_1})$ and $\mathcal{F}_2 = (\alpha_{F_2}, \beta_{F_2})$ be two FFNs, then*

(i) $(\mathcal{F}_1 \cap \mathcal{F}_2) \boxminus (\mathcal{F}_1 \cup \mathcal{F}_2) = \mathcal{F}_2 \boxminus \mathcal{F}_1 \ \text{if} \ \alpha_{F_1} \geqslant \alpha_{F_2}, \ \beta_{F_1} \leqslant \min\left\{\beta_{F_2}, \frac{\beta_{F_2}\pi_1}{\pi_2}\right\};$

(ii) $(\mathcal{F}_1 \cap \mathcal{F}_2) \boxslash (\mathcal{F}_1 \cup \mathcal{F}_2) = \mathcal{F}_1 \boxslash \mathcal{F}_2 \ \text{if} \ \alpha_{F_1} \leqslant \min\left\{\alpha_{F_2}, \frac{\alpha_{F_2}\pi_1}{\pi_2}\right\}, \ \beta_{F_1} \geqslant \beta_{F_2}.$

Proof. We will present the proof of (*i*). Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two FFNs. Since $\alpha_{F_1} \geqslant \alpha_{F_2}$, $\beta_{F_1} \leqslant \min\left\{\beta_{F_2}, \frac{\beta_{F_2}\pi_1}{\pi_2}\right\}$, then

$$(\mathcal{F}_1 \cap \mathcal{F}_2) \boxminus (\mathcal{F}_1 \cup \mathcal{F}_2)$$

$$= \left(\min\{\alpha_{F_1}, \alpha_{F_2}\}, \max\{\beta_{F_1}, \beta_{F_2}\}\right) \boxminus \left(\max\{\alpha_{F_1}, \alpha_{F_2}\}, \min\{\beta_{F_1}, \beta_{F_2}\}\right)$$

$$= \left(\sqrt[3]{\frac{(\min\{\alpha_{F_1}, \alpha_{F_2}\})^3 - (\max\{\alpha_{F_1}, \alpha_{F_2}\})^3}{1 - (\max\{\alpha_{F_1}, \alpha_{F_2}\})^3}}, \frac{\max\{\beta_{F_1}, \beta_{F_2}\}}{\min\{\beta_{F_1}, \beta_{F_2}\}}\right)$$

$$= \left(\sqrt[3]{\frac{\min\left\{\alpha_{F_1}^3, \alpha_{F_2}^3\right\} - \max\left\{\alpha_{F_1}^3, \alpha_{F_2}^3\right\}}{1 - \max\left\{\alpha_{F_1}^3, \alpha_{F_2}^3\right\}}}, \frac{\beta_{F_1}}{\beta_{F_2}}\right)$$

$$= \left(\sqrt[3]{\frac{\alpha_{F_2}^3 - \alpha_{F_1}^3}{1 - \alpha_{F_1}^3}}, \frac{\beta_{F_2}}{\beta_{F_1}}\right) = \mathcal{F}_2 \boxminus \mathcal{F}_1.$$

Assertion (ii) is proved analogously. □

Theorem 10. *Let* $\mathcal{F}_1 = (\alpha_{F_1}, \beta_{F_1})$ *and* $\mathcal{F}_2 = (\alpha_{F_2}, \beta_{F_2})$ *be two FFNs, then*

(i) $(\mathcal{F}_1 \cap \mathcal{F}_2) \cup \mathcal{F}_2 = \mathcal{F}_2$;

(ii) $(\mathcal{F}_1 \cup \mathcal{F}_2) \cap \mathcal{F}_2 = \mathcal{F}_2$;

(iii) $(\mathcal{F}_1 \boxminus \mathcal{F}_2) \boxplus \mathcal{F}_2 = \mathcal{F}_1$ *if* $\alpha_{F_1} \geqslant \alpha_{F_2}$, $\beta_{F_1} \leqslant \min\left\{\beta_{F_2}, \frac{\beta_{F_2} \pi_1}{\pi_2}\right\}$;

(iv) $(\mathcal{F}_1 \boxslash \mathcal{F}_2) \boxtimes \mathcal{F}_2 = \mathcal{F}_1$ *if* $\alpha_{F_1} \leqslant \min\left\{\alpha_{F_2}, \frac{\alpha_{F_2} \pi_1}{\pi_2}\right\}$, $\beta_{F_1} \geqslant \beta_{F_2}$.

Proof. We will present the proofs of (i) and (iii). Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two FFNs, then

(i) $(\mathcal{F}_1 \cap \mathcal{F}_2) \cup \mathcal{F}_2 = \big(\min\{\alpha_{F_1}, \alpha_{F_2}\}, \max\{\beta_{F_1}, \beta_{F_2}\}\big) \cup (\alpha_{F_2}, \beta_{F_2})$

$= \big(\max\big\{\min\{\alpha_{F_1}, \alpha_{F_2}\}, \alpha_{F_2}\big\}, \min\big\{\max\{\beta_{F_1}, \beta_{F_2}\}, \beta_{F_2}\big\}\big)$

$= (\alpha_{F_2}, \beta_{F_2}) = \mathcal{F}_2$;

(iii) Since $\alpha_{F_1} \geqslant \alpha_{F_2}$, $\beta_{F_1} \leqslant \min\left\{\beta_{F_2}, \frac{\beta_{F_2}\pi_1}{\pi_2}\right\}$ we have

$$(\mathcal{F}_1 \boxminus \mathcal{F}_2) \boxplus \mathcal{F}_2 = \left(\sqrt[3]{\frac{\alpha_{F_1}^3 - \alpha_{F_2}^3}{1 - \alpha_{F_2}^3}}, \frac{\beta_{F_1}}{\beta_{F_2}}\right) \boxplus (\alpha_{F_2}, \beta_{F_2})$$

$$= \left(\sqrt[3]{\frac{\alpha_{F_1}^3 - \alpha_{F_2}^3}{1 - \alpha_{F_2}^3} + \alpha_{F_2}^3 - \frac{\alpha_{F_1}^3 - \alpha_{F_2}^3}{1 - \alpha_{F_2}^3}\alpha_{F_2}^3}, \frac{\beta_{F_1}}{\beta_{F_2}}\beta_{F_2}\right)$$

$$= \left(\sqrt[3]{\frac{\alpha_{F_1}^3 - \alpha_{F_2}^3 + \alpha_{F_2}^3 - \alpha_{F_2}^6 - \alpha_{F_1}^3\alpha_{F_2}^3 + \alpha_{F_2}^6}{1 - \alpha_{F_2}^3}}, \beta_{F_1}\right)$$

$$= \left(\sqrt[3]{\frac{\alpha_{F_1}^3(1 - \alpha_{F_2}^3)}{(1 - \alpha_{F_2}^3)}}, \beta_{F_1}\right) = (\alpha_{F_1}, \beta_{F_1}) = \mathcal{F}_1.$$

The other assertions are proved analogously. $\qquad\square$

Theorem 11. *Let* $\mathcal{F}_1 = (\alpha_{F_1}, \beta_{F_1})$, $\mathcal{F}_2 = (\alpha_{F_2}, \beta_{F_2})$ *and* $\mathcal{F}_3 = (\alpha_{F_3}, \beta_{F_3})$ *be three FFNs, then*

(i) $(\mathcal{F}_1 \cap \mathcal{F}_2) \boxminus \mathcal{F}_3 = (\mathcal{F}_1 \boxminus \mathcal{F}_3) \cap (\mathcal{F}_2 \boxminus \mathcal{F}_3)$, *if* $\alpha_{F_3} \leqslant \min\{\alpha_{F_1}, \alpha_{F_2}\}$, $\max\{\beta_{F_1}, \beta_{F_2}\}$
$\leqslant \{\beta_{F_3}, \beta_{F_3}\frac{\pi_1}{\pi_3}, \beta_{F_3}\frac{\pi_2}{\pi_3}\}$, $\frac{\min\{\alpha_{F_1}^3, \alpha_{F_2}^3\} - \alpha_{F_3}^3}{1 - \alpha_{F_3}^3} + \frac{\max\{\beta_{F_1}^3, \beta_{F_2}^3\}}{\beta_{F_3}^3} \leqslant 1$;

(ii) $(\mathcal{F}_1 \cup \mathcal{F}_2) \boxminus \mathcal{F}_3 = (\mathcal{F}_1 \boxminus \mathcal{F}_3) \cup (\mathcal{F}_2 \boxminus \mathcal{F}_3)$, *if* $\alpha_{F_3} \leqslant \min\{\alpha_{F_1}, \alpha_{F_2}\}$, $\max\{\beta_{F_1}, \beta_{F_2}\}$
$\leqslant \{\beta_{F_3}, \beta_{F_3}\frac{\pi_1}{\pi_3}, \beta_{F_3}\frac{\pi_2}{\pi_3}\}$, $\frac{\max\{\alpha_{F_1}^3, \alpha_{F_2}^3\} - \alpha_{F_3}^3}{1 - \alpha_{F_3}^3} + \frac{\min\{\beta_{F_1}^3, \beta_{F_2}^3\}}{\beta_{F_3}^3} \leqslant 1$;

(iii) $(\mathcal{F}_1 \cap \mathcal{F}_2) \boxslash \mathcal{F}_3 = (\mathcal{F}_1 \boxslash \mathcal{F}_3) \cap (\mathcal{F}_2 \boxslash \mathcal{F}_3)$, *if* $\beta_{F_3} \leqslant \min\{\beta_{F_1}, \beta_{F_2}\}$, $\max\{\alpha_{F_1}, \alpha_{F_2}\}$
$\leqslant \{\alpha_{F_3}, \alpha_{F_3}\frac{\pi_1}{\pi_3}, \alpha_{F_3}\frac{\pi_2}{\pi_3}\}$, $\frac{\max\{\beta_{F_1}^3, \beta_{F_2}^3\} - \beta_{F_3}^3}{1 - \beta_{F_3}^3} + \frac{\min\{\alpha_{F_1}^3, \alpha_{F_2}^3\}}{\alpha_{F_3}^3} \leqslant 1$;

(iv) $(\mathcal{F}_1 \cup \mathcal{F}_2) \boxslash \mathcal{F}_3 = (\mathcal{F}_1 \boxslash \mathcal{F}_3) \cup (\mathcal{F}_2 \boxslash \mathcal{F}_3)$, *if* $\beta_{F_3} \leqslant \min\{\beta_{F_1}, \beta_{F_2}\}$, $\max\{\alpha_{F_1}, \alpha_{F_2}\}$
$\leqslant \{\alpha_{F_3}, \alpha_{F_3}\frac{\pi_1}{\pi_3}, \alpha_{F_3}\frac{\pi_2}{\pi_3}\}$, $\frac{\min\{\beta_{F_1}^3, \beta_{F_2}^3\} - \beta_{F_3}^3}{1 - \beta_{F_3}^3} + \frac{\max\{\alpha_{F_1}^3, \alpha_{F_2}^3\}}{\alpha_{F_3}^3} \leqslant 1$.

Proof. We will present the proofs of (i) and (iii). Let $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ be three FFNs.

(i) Since $\alpha_{F_3} \leqslant \min\{\alpha_{F_1}, \alpha_{F_2}\}$, $\max\{\beta_{F_1}, \beta_{F_2}\} \leqslant \{\beta_{F_3}, \beta_{F_3}\frac{\pi_1}{\pi_3}, \beta_{F_3}\frac{\pi_2}{\pi_3}\}$, $\frac{\min\{\alpha_{F_1}^3, \alpha_{F_2}^3\} - \alpha_{F_3}^3}{1 - \alpha_{F_3}^3} + \frac{\max\{\beta_{F_1}^3, \beta_{F_2}^3\}}{\beta_{F_3}^3} \leqslant 1$, then

$$(\mathcal{F}_1 \cap \mathcal{F}_2) \boxminus \mathcal{F}_3 = \big(\min\{\alpha_{F_1}, \alpha_{F_2}\}, \max\{\beta_{F_1}, \beta_{F_2}\}\big) \boxminus (\alpha_{F_3}, \beta_{F_3})$$

$$= \left(\sqrt[3]{\frac{\min\{\alpha_{F_1}^3, \alpha_{F_2}^3\} - \alpha_{F_3}^3}{1 - \alpha_{F_3}^3}}, \frac{\max\{\beta_{F_1}, \beta_{F_2}\}}{\beta_{F_3}}\right)$$

$$= \left(\min\left\{\sqrt[3]{\frac{\alpha_{F_1}^3 - \alpha_{F_3}^3}{1 - \alpha_{F_3}^3}}, \sqrt[3]{\frac{\alpha_{F_2}^3 - \alpha_{F_3}^3}{1 - \alpha_{F_3}^3}}\right\}, \max\left\{\frac{\beta_{F_1}}{\beta_{F_3}}, \frac{\beta_{F_2}}{\beta_{F_3}}\right\}\right);$$

$$(\mathcal{F}_1 \boxminus \mathcal{F}_3) \cap (\mathcal{F}_2 \boxminus \mathcal{F}_3)$$

$$= \left(\sqrt[3]{\frac{\alpha_{F_1}^3 - \alpha_{F_3}^3}{1 - \alpha_{F_3}^3}}, \frac{\beta_{F_1}}{\beta_{F_3}}\right) \cap \left(\sqrt[3]{\frac{\alpha_{F_2}^3 - \alpha_{F_3}^3}{1 - \alpha_{F_3}^3}}, \frac{\beta_{F_2}}{\beta_{F_3}}\right)$$

$$= \left(\min\left\{\sqrt[3]{\frac{\alpha_{F_1}^3 - \alpha_{F_3}^3}{1 - \alpha_{F_3}^3}}, \sqrt[3]{\frac{\alpha_{F_2}^3 - \alpha_{F_3}^3}{1 - \alpha_{F_3}^3}}\right\}, \max\left\{\frac{\beta_{F_1}}{\beta_{F_3}}, \frac{\beta_{F_2}}{\beta_{F_3}}\right\}\right).$$

Hence, $(\mathcal{F}_1 \cap \mathcal{F}_2) \boxminus \mathcal{F}_3 = (\mathcal{F}_1 \boxminus \mathcal{F}_3) \cap (\mathcal{F}_2 \boxminus \mathcal{F}_3)$.

(iii) Since $\beta_{F_3} \leqslant \min\{\beta_{F_1}, \beta_{F_2}\}$, $\max\{\alpha_{F_1}, \alpha_{F_2}\} \leqslant \{\alpha_{F_3}, \alpha_{F_3}\frac{\pi_1}{\pi_3}, \alpha_{F_3}\frac{\pi_2}{\pi_3}\}$, $\frac{\max\{\beta_{F_1}^3, \beta_{F_2}^3\} - \beta_{F_3}^3}{1 - \beta_{F_3}^3} + \frac{\min\{\alpha_{F_1}^3, \alpha_{F_2}^3\}}{\alpha_{F_3}^3} \leqslant 1$, then

$$(\mathcal{F}_1 \cap \mathcal{F}_2) \boxslash \mathcal{F}_3 = \big(\min\{\alpha_{F_1}, \alpha_{F_2}\}, \max\{\beta_{F_1}, \beta_{F_2}\}\big) \boxslash (\alpha_{F_3}, \beta_{F_3})$$

$$= \left(\frac{\min\{\alpha_{F_1}, \alpha_{F_2}\}}{\alpha_{F_3}}, \sqrt[3]{\frac{\max\{\beta_{F_1}^3, \beta_{F_2}^3\} - \beta_{F_3}^3}{1 - \beta_{F_3}^3}}\right)$$

$$= \left(\min\left\{\frac{\alpha_{F_1}}{\alpha_{F_3}}, \frac{\alpha_{F_2}}{\alpha_{F_3}}\right\}, \max\left\{\sqrt[3]{\frac{\beta_{F_1}^3 - \beta_{F_3}^3}{1 - \beta_{F_3}^3}}, \sqrt[3]{\frac{\beta_{F_2}^3 - \beta_{F_3}^3}{1 - \beta_{F_3}^3}}\right\}\right);$$

$$(\mathcal{F}_1 \boxslash \mathcal{F}_3) \cap (\mathcal{F}_2 \boxslash \mathcal{F}_3)$$

$$= \left(\frac{\alpha_{F_1}}{\alpha_{F_3}}, \sqrt[3]{\frac{\beta_{F_1}^3 - \beta_{F_3}^3}{1 - \beta_{F_3}^3}}\right) \cap \left(\frac{\alpha_{F_2}}{\alpha_{F_3}}, \sqrt[3]{\frac{\beta_{F_2}^3 - \beta_{F_3}^3}{1 - \beta_{F_3}^3}}\right)$$

$$= \left(\min\left\{ \frac{\alpha_{F_1}}{\alpha_{F_3}}, \frac{\alpha_{F_2}}{\alpha_{F_3}} \right\}, \max\left\{ \sqrt[3]{\frac{\beta_{F_1}^3 - \beta_{F_3}^3}{1 - \beta_{F_3}^3}}, \sqrt[3]{\frac{\beta_{F_2}^3 - \beta_{F_3}^3}{1 - \beta_{F_3}^3}} \right\} \right).$$

Hence, $(\mathcal{F}_1 \cap \mathcal{F}_2) \boxslash \mathcal{F}_3 = (\mathcal{F}_1 \boxslash \mathcal{F}_3) \cap (\mathcal{F}_2 \boxslash \mathcal{F}_3)$.

The other assertions are proved analogously.   □

Theorem 12. *Let* $\mathcal{F}_1 = (\alpha_{F_1}, \beta_{F_1})$, $\mathcal{F}_2 = (\alpha_{F_2}, \beta_{F_2})$ *and* $\mathcal{F}_3 = (\alpha_{F_3}, \beta_{F_3})$ *be three FFNs, then*

  (i) $\mathcal{F}_1 \boxplus \mathcal{F}_2 \boxplus \mathcal{F}_3 = \mathcal{F}_1 \boxplus \mathcal{F}_3 \boxplus \mathcal{F}_2$;

  (ii) $\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{F}_3 = \mathcal{F}_1 \boxtimes \mathcal{F}_3 \boxtimes \mathcal{F}_2$;

  (iii) $\mathcal{F}_1 \boxminus \mathcal{F}_2 \boxminus \mathcal{F}_3 = \mathcal{F}_1 \boxminus \mathcal{F}_3 \boxminus \mathcal{F}_2$, *if* $\alpha_{F_1} \geqslant \max\{\alpha_{F_2}, \alpha_{F_3}\}$, $\beta_{F_1} \leqslant \min\{\beta_{F_2}\beta_{F_3}$,

       $\beta_{F_2}\frac{\pi_1}{\pi_2}, \beta_{F_3}\frac{\pi_1}{\pi_3}\}$, $\frac{\alpha_{F_1}^3 - \alpha_{F_2}^3 - \alpha_{F_3}^3 + \alpha_{F_2}^3 \alpha_{F_3}^3}{(1 - \alpha_{F_2}^3)(1 - \alpha_{F_3}^3)} + \frac{\beta_{F_1}^3}{\beta_{F_2}^3 \beta_{F_3}^3} \leqslant 1$;

  (iv) $\mathcal{F}_1 \boxslash \mathcal{F}_2 \boxslash \mathcal{F}_3 = \mathcal{F}_1 \boxslash \mathcal{F}_3 \boxslash \mathcal{F}_2$, *if* $\beta_{F_1} \geqslant \max\{\beta_{F_2}, \beta_{F_3}\}$, $\alpha_{F_1} \leqslant \min\{\alpha_{F_2}\alpha_{F_3}$,

       $\alpha_{F_2}\frac{\pi_1}{\pi_2}, \alpha_{F_3}\frac{\pi_1}{\pi_3}\}$, $\frac{\beta_{F_1}^3 - \beta_{F_2}^3 - \beta_{F_3}^3 + \beta_{F_2}^3 \beta_{F_3}^3}{(1 - \beta_{F_2}^3)(1 - \beta_{F_3}^3)} + \frac{\alpha_{F_1}^3}{\alpha_{F_2}^3 \alpha_{F_3}^3} \leqslant 1$.

Proof. We will present the proofs of (i) and (iii). Let $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ be three FFNs.

(i) $\mathcal{F}_1 \boxplus \mathcal{F}_2 \boxplus \mathcal{F}_3$

$$= (\alpha_{F_1}, \beta_{F_1}) \boxplus (\alpha_{F_2}, \beta_{F_2}) \boxplus (\alpha_{F_3}, \beta_{F_3})$$

$$= \left(\sqrt[3]{\alpha_{F_1}^3 + \alpha_{F_2}^3 - \alpha_{F_1}^3 \alpha_{F_2}^3}, \beta_{F_1}\beta_{F_2} \right) \boxplus (\alpha_{F_3}, \beta_{F_3})$$

$$= \left(\sqrt[3]{\alpha_{F_1}^3 + \alpha_{F_2}^3 - \alpha_{F_1}^3 \alpha_{F_2}^3 + \alpha_{F_3}^3 - \alpha_{F_3}^3(\alpha_{F_1}^3 + \alpha_{F_2}^3 - \alpha_{F_1}^3 \alpha_{F_2}^3)}, \beta_{F_1}\beta_{F_2}\beta_{F_3} \right)$$

$$= \left(\sqrt[3]{\alpha_{F_1}^3 + \alpha_{F_2}^3 + \alpha_{F_3}^3 - \alpha_{F_1}^3 \alpha_{F_2}^3 - \alpha_{F_1}^3 \alpha_{F_3}^3 - \alpha_{F_2}^3 \alpha_{F_3}^3 + \alpha_{F_1}^3 \alpha_{F_2}^3 \alpha_{F_3}^3}, \beta_{F_1}\beta_{F_2}\beta_{F_3} \right)$$

$$= \left(\sqrt[3]{\alpha_{F_1}^3 + \alpha_{F_3}^3 - \alpha_{F_1}^3 \alpha_{F_3}^3 + \alpha_{F_2}^3 - \alpha_{F_2}^3(\alpha_{F_1}^3 + \alpha_{F_3}^3 - \alpha_{F_1}^3 \alpha_{F_3}^3)}, \beta_{F_1}\beta_{F_3}\beta_{F_2} \right)$$

$$= \left(\sqrt[3]{\alpha_{F_1}^3 + \alpha_{F_3}^3 - \alpha_{F_1}^3 \alpha_{F_3}^3}, \beta_{F_1}\beta_{F_3} \right) \boxplus (\alpha_{F_2}, \beta_{F_2})$$

$$= \mathcal{F}_1 \boxplus \mathcal{F}_3 \boxplus \mathcal{F}_2.$$

(iii) Since $\alpha_{F_1} \geqslant \max\{\alpha_{F_2}, \alpha_{F_3}\}$, $\beta_{F_1} \leqslant \min\{\beta_{F_2}\beta_{F_3}, \beta_{F_2}\frac{\pi_1}{\pi_2}, \beta_{F_3}\frac{\pi_1}{\pi_3}\}$,
$\frac{\alpha_{F_1}^3 - \alpha_{F_2}^3 - \alpha_{F_3}^3 + \alpha_{F_2}^3 \alpha_{F_3}^3}{(1 - \alpha_{F_2}^3)(1 - \alpha_{F_3}^3)} + \frac{\beta_{F_1}^3}{\beta_{F_2}^3 \beta_{F_3}^3} \leqslant 1$, then

$$\mathcal{F}_1 \boxminus \mathcal{F}_2 \boxminus \mathcal{F}_3 = \left(\sqrt[3]{\frac{\alpha_{F_1}^3 - \alpha_{F_2}^3}{1 - \alpha_{F_2}^3}}, \frac{\beta_{F_1}}{\beta_{F_2}} \right) \boxminus (\alpha_{F_3}, \beta_{F_3})$$

$$= \left(\sqrt[3]{\frac{\frac{\alpha_{F_1}^3 - \alpha_{F_2}^3}{1 - \alpha_{F_2}^3} - \alpha_{F_3}^3}{1 - \alpha_{F_3}^3}}, \frac{\beta_{F_1}}{\beta_{F_2}\beta_{F_3}} \right)$$

$$= \left(\sqrt[3]{\frac{\alpha_{F_1}^3 - \alpha_{F_2}^3 - \alpha_{F_3}^3 + \alpha_{F_2}^3 \alpha_{F_3}^3}{(1 - \alpha_{F_2}^3)(1 - \alpha_{F_3}^3)}}, \frac{\beta_{F_1}}{\beta_{F_2}\beta_{F_3}} \right);$$

and

$$\mathcal{F}_1 \boxminus \mathcal{F}_3 \boxminus \mathcal{F}_2 = \left(\sqrt[3]{\frac{\alpha_{F_1}^3 - \alpha_{F_3}^3}{1 - \alpha_{F_3}^3}}, \frac{\beta_{F_1}}{\beta_{F_3}} \right) \boxminus (\alpha_{F_2}, \beta_{F_2})$$

$$= \left(\sqrt[3]{\frac{\frac{\alpha_{F_1}^3 - \alpha_{F_3}^3}{1 - \alpha_{F_3}^3} - \alpha_{F_2}^3}{1 - \alpha_{F_2}^3}}, \frac{\beta_{F_1}}{\beta_{F_2}\beta_{F_3}} \right)$$

$$= \left(\sqrt[3]{\frac{\alpha_{F_1}^3 - \alpha_{F_3}^3 - \alpha_{F_2}^3 + \alpha_{F_2}^3 \alpha_{F_3}^3}{(1 - \alpha_{F_2}^3)(1 - \alpha_{F_3}^3)}}, \frac{\beta_{F_1}}{\beta_{F_2}\beta_{F_3}} \right).$$

Therefore, $\mathcal{F}_1 \boxminus \mathcal{F}_2 \boxminus \mathcal{F}_3 = \mathcal{F}_1 \boxminus \mathcal{F}_3 \boxminus \mathcal{F}_2$.

The other assertions are proved analogously. $\square$

Theorem 13. *Let $\mathcal{F}_1 = (\alpha_{F_1}, \beta_{F_1})$, $\mathcal{F}_2 = (\alpha_{F_2}, \beta_{F_2})$ and $\mathcal{F}_3 = (\alpha_{F_3}, \beta_{F_3})$ be three FFNs, then*

(i) $\mathcal{F}_1 \boxminus \mathcal{F}_2 \boxminus \mathcal{F}_3 = \mathcal{F}_1 \boxminus (\mathcal{F}_2 \boxplus \mathcal{F}_3)$, *if* $\alpha_{F_1} \geqslant \max\{\alpha_{F_2}, \alpha_{F_3}\}$, $\beta_{F_1} \leqslant \min\{\beta_{F_2}\beta_{F_3}$, $\beta_{F_2}\frac{\pi_1}{\pi_2}, \beta_{F_3}\frac{\pi_1}{\pi_3}\}$, $\frac{\alpha_{F_1}^3 - \alpha_{F_2}^3 - \alpha_{F_3}^3 + \alpha_{F_2}^3 \alpha_{F_3}^3}{(1 - \alpha_{F_2}^3)(1 - \alpha_{F_3}^3)} + \frac{\beta_{F_1}^3}{\beta_{F_2}^3 \beta_{F_3}^3} \leqslant 1$;

(ii) $\mathcal{F}_1 \boxdiv \mathcal{F}_2 \boxdiv \mathcal{F}_3 = \mathcal{F}_1 \boxdiv (\mathcal{F}_3 \boxtimes \mathcal{F}_2)$, *if* $\beta_{F_1} \geqslant \max\{\beta_{F_2}, \beta_{F_3}\}$, $\alpha_{F_1} \leqslant \min\{\alpha_{F_2}\alpha_{F_3}$, $\alpha_{F_2}\frac{\pi_1}{\pi_2}, \alpha_{F_3}\frac{\pi_1}{\pi_3}\}$, $\frac{\beta_{F_1}^3 - \beta_{F_2}^3 - \beta_{F_3}^3 + \beta_{F_2}^3 \beta_{F_3}^3}{(1 - \beta_{F_2}^3)(1 - \beta_{F_3}^3)} + \frac{\alpha_{F_1}^3}{\alpha_{F_2}^3 \alpha_{F_3}^3} \leqslant 1$.

Proof. We will present the proof of (i), and (ii) can be proved analogously. Let $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ be three FFNs.

Since $\alpha_{F_1} \geqslant \max\{\alpha_{F_2}, \alpha_{F_3}\}$, $\beta_{F_1} \leqslant \min\{\beta_{F_2}\beta_{F_3}, \beta_{F_2}\frac{\pi_1}{\pi_2}, \beta_{F_3}\frac{\pi_1}{\pi_3}\}$, $\frac{\alpha_{F_1}^3 - \alpha_{F_2}^3 - \alpha_{F_3}^3 + \alpha_{F_2}^3 \alpha_{F_3}^3}{(1 - \alpha_{F_2}^3)(1 - \alpha_{F_3}^3)} + \frac{\beta_{F_1}^3}{\beta_{F_2}^3 \beta_{F_3}^3} \leqslant 1$, then

$$\mathcal{F}_1 \boxminus \mathcal{F}_2 \boxminus \mathcal{F}_3 = \left(\sqrt[3]{\frac{\alpha_{F_1}^3 - \alpha_{F_2}^3 - \alpha_{F_3}^3 + \alpha_{F_2}^3 \alpha_{F_3}^3}{(1 - \alpha_{F_2}^3)(1 - \alpha_{F_3}^3)}}, \frac{\beta_{F_1}}{\beta_{F_2}\beta_{F_3}} \right)$$

and

$$\mathcal{F}_1 \boxminus (\mathcal{F}_2 \boxplus \mathcal{F}_3) = (\alpha_{F_1}, \beta_{F_1}) \boxminus \left(\sqrt[3]{\alpha_{F_2}^3 + \alpha_{F_3}^3 - \alpha_{F_2}^3 \alpha_{F_3}^3}, \beta_{F_2} \beta_{F_3} \right)$$

$$= \left(\sqrt[3]{\frac{\alpha_{F_1}^3 - \alpha_{F_2}^3 - \alpha_{F_3}^3 + \alpha_{F_2}^3 \alpha_{F_3}^3}{1 - \alpha_{F_2}^3 - \alpha_{F_3}^3 + \alpha_{F_2}^3 \alpha_{F_3}^3}}, \frac{\beta_{F_1}}{\beta_{F_2} \beta_{F_3}} \right)$$

$$= \left(\sqrt[3]{\frac{\alpha_{F_1}^3 - \alpha_{F_2}^3 - \alpha_{F_3}^3 + \alpha_{F_2}^3 \alpha_{F_3}^3}{(1 - \alpha_{F_2}^3)(1 - \alpha_{F_3}^3)}}, \frac{\beta_{F_1}}{\beta_{F_2} \beta_{F_3}} \right).$$

Therefore, $\mathcal{F}_1 \boxminus \mathcal{F}_2 \boxminus \mathcal{F}_3 = \mathcal{F}_1 \boxminus (\mathcal{F}_2 \boxplus \mathcal{F}_3)$. $\square$

5. Fermatean Arithmetic Mean Operations over FFNs

In this section, Fermatean arithmetic mean operations over FFNs are defined in detail and two corresponding theorems are proved.

DEFINITION 10. Let $\mathcal{F}_1 = (\alpha_{F_1}, \beta_{F_1})$ and $\mathcal{F}_2 = (\alpha_{F_2}, \beta_{F_2})$ be two FFNs, then average operator of $\mathcal{F}_1$ and $\mathcal{F}_2$ is defined as

$$\mathcal{F}_1 @ \mathcal{F}_2 = \left(\frac{\alpha_{F_1}^3 + \alpha_{F_2}^3}{2}, \frac{\beta_{F_1}^3 + \beta_{F_2}^3}{2} \right).$$

Theorem 14. *Let* $\mathcal{F}_1 = (\alpha_{F_1}, \beta_{F_1})$, $\mathcal{F}_2 = (\alpha_{F_2}, \beta_{F_2})$ *and* $\mathcal{F}_3 = (\alpha_{F_3}, \beta_{F_3})$ *be three FFNs, then*

   (i) $(\mathcal{F}_1 \cap \mathcal{F}_2) @ \mathcal{F}_3 = (\mathcal{F}_1 @ \mathcal{F}_3) \cap (\mathcal{F}_2 @ \mathcal{F}_3)$;
   (ii) $(\mathcal{F}_1 \cup \mathcal{F}_2) @ \mathcal{F}_3 = (\mathcal{F}_1 @ \mathcal{F}_3) \cup (\mathcal{F}_2 @ \mathcal{F}_3)$;
  (iii) $(\mathcal{F}_1 @ \mathcal{F}_2)^c = \mathcal{F}_1^c @ \mathcal{F}_2^c$.

Proof. Let $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ be three FFNs.

(i) $(\mathcal{F}_1 \cap \mathcal{F}_2) @ \mathcal{F}_3 = (\min\{\alpha_{F_1}, \alpha_{F_2}\}, \max\{\beta_{F_1}, \beta_{F_2}\}) @ (\alpha_{F_3}, \beta_{F_3})$;

$$= \left(\frac{\min\{\alpha_{F_1}^3, \alpha_{F_2}^3\} + \alpha_{F_3}^3}{2}, \frac{\max\{\beta_{F_1}^3, \beta_{F_2}^3\} + \beta_{F_3}^3}{2} \right)$$

$$= \left(\min\left\{ \frac{\alpha_{F_1}^3 + \alpha_{F_3}^3}{2}, \frac{\alpha_{F_2}^3 + \alpha_{F_3}^3}{2} \right\}, \max\left\{ \frac{\beta_{F_1}^3 + \beta_{F_3}^3}{2}, \frac{\beta_{F_2}^3 + \beta_{F_3}^3}{2} \right\} \right)$$

$$= \left(\frac{\alpha_{F_1}^3 + \alpha_{F_3}^3}{2}, \frac{\beta_{F_1}^3 + \beta_{F_3}^3}{2} \right) \cap \left(\frac{\alpha_{F_2}^3 + \alpha_{F_3}^3}{2}, \frac{\beta_{F_2}^3 + \beta_{F_3}^3}{2} \right)$$

$$= (\mathcal{F}_1 @ \mathcal{F}_3) \cap (\mathcal{F}_2 @ \mathcal{F}_3)$$

(iii) $(\mathcal{F}_1 @ \mathcal{F}_2)^c = \left(\dfrac{\alpha_{F_1}^3 + \alpha_{F_2}^3}{2}, \dfrac{\beta_{F_1}^3 + \beta_{F_2}^3}{2} \right)^c = \left(\dfrac{\beta_{F_1}^3 + \beta_{F_2}^3}{2}, \dfrac{\alpha_{F_1}^3 + \alpha_{F_2}^3}{2} \right)$

$\qquad = (\beta_{F_1}, \alpha_{F_1}) @ (\beta_{F_2}, \alpha_{F_2}) = \mathcal{F}_1^c @ \mathcal{F}_2^c.$

Assertion (ii) is proved analogously. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Theorem 15. *Let* $\mathcal{F}_1 = (\alpha_{F_1}, \beta_{F_1})$ *and* $\mathcal{F}_2 = (\alpha_{F_2}, \beta_{F_2})$ *be two FFNs, then*

$$\big((\mathcal{F}_1 \cap \mathcal{F}_2) \boxplus (\mathcal{F}_1 \cup \mathcal{F}_2)\big) @ \big((\mathcal{F}_1 \cap \mathcal{F}_2) \boxtimes (\mathcal{F}_1 \cup \mathcal{F}_2)\big) = \mathcal{F}_1 @ \mathcal{F}_2.$$

Proof. We know that for any two real numbers a and b,

$$\min(a, b) + \max(a, b) = a + b,$$
$$\min(a, b) \cdot \max(a, b) = a \cdot b.$$

Now,

$(\mathcal{F}_1 \cap \mathcal{F}_2) \boxplus (\mathcal{F}_1 \cup \mathcal{F}_2)$

$\quad = \big(\min\{\alpha_{F_1}, \alpha_{F_2}\}, \max\{\beta_{F_1}, \beta_{F_2}\} \big) \boxplus \big(\max\{\alpha_{F_1}, \alpha_{F_2}\}, \min\{\beta_{F_1}, \beta_{F_2}\} \big)$

$\quad = \Big(\sqrt[3]{\min\{\alpha_{F_1}^3, \alpha_{F_2}^3\} + \max\{\alpha_{F_1}^3, \alpha_{F_2}^3\} - \min\{\alpha_{F_1}^3, \alpha_{F_2}^3\} \max\{\alpha_{F_1}^3, \alpha_{F_2}^3\}},$

$\qquad \max\{\beta_{F_1}, \beta_{F_2}\} \min\{\beta_{F_1}, \beta_{F_2}\} \Big) = \Big(\sqrt[3]{\alpha_{F_1}^3 + \alpha_{F_2}^3 - \alpha_{F_1}^3 \alpha_{F_2}^3}, \beta_{F_1} \beta_{F_2} \Big);$

$(\mathcal{F}_1 \cap \mathcal{F}_2) \boxtimes (\mathcal{F}_1 \cup \mathcal{F}_2)$

$\quad = \big(\min\{\alpha_{F_1}, \alpha_{F_2}\}, \max\{\beta_{F_1}, \beta_{F_2}\} \big) \boxtimes \big(\max\{\alpha_{F_1}, \alpha_{F_2}\}, \min\{\beta_{F_1}, \beta_{F_2}\} \big)$

$\quad = \Big(\min\{\alpha_{F_1}, \alpha_{F_2}\} \max\{\alpha_{F_1}, \alpha_{F_2}\},$

$\qquad \sqrt[3]{\max\{\beta_{F_1}^3, \beta_{F_2}^3\} + \min\{\beta_{F_1}^3, \beta_{F_2}^3\} - \max\{\beta_{F_1}^3, \beta_{F_2}^3\} \min\{\beta_{F_1}^3, \beta_{F_2}^3\}} \Big)$

$\quad = \Big(\alpha_{F_1} \alpha_{F_2}, \sqrt[3]{\beta_{F_1}^3 + \beta_{F_2}^3 - \beta_{F_1}^3 \beta_{F_2}^3} \Big).$

Therefore,

$\big((\mathcal{F}_1 \cap \mathcal{F}_2) \boxplus (\mathcal{F}_1 \cup \mathcal{F}_2)\big) @ \big((\mathcal{F}_1 \cap \mathcal{F}_2) \boxtimes (\mathcal{F}_1 \cup \mathcal{F}_2)\big)$

$\quad = \Big(\sqrt[3]{\alpha_{F_1}^3 + \alpha_{F_2}^3 - \alpha_{F_1}^3 \alpha_{F_2}^3}, \beta_{F_1} \beta_{F_2} \Big) @ \Big(\alpha_{F_1} \alpha_{F_2}, \sqrt[3]{\beta_{F_1}^3 + \beta_{F_2}^3 - \beta_{F_1}^3 \beta_{F_2}^3} \Big)$

$\quad = \left(\dfrac{\alpha_{F_1}^3 + \alpha_{F_2}^3}{2}, \dfrac{\beta_{F_1}^3 + \beta_{F_2}^3}{2} \right) = \mathcal{F}_1 @ \mathcal{F}_2. \qquad\qquad\qquad\qquad$ □

6. Weighted Product Model

The weighted product model (WPM) is one of the best known and often applied MCDM methods for evaluating a number of alternatives in terms of a number of decision criteria. Each decision alternative is compared with the others by multiplying a number of ratios, one for each decision criterion. Each ratio is raised to the power equivalent to the relative weight of the corresponding criterion. Some of the first references to this method are due to Bridgman (1922) and Miller and Starr (1969). More details on this method are given in the MCDM book by Triantaphyllou (2000).

Its application first requires development of a decision/evaluation matrix, $X = [x_{ij}]_{m*n}$ where x_{ij} is the performance of i-th alternative with respect to j-th criterion, m is the number of alternatives and n is the number of criteria. To make the performance measures comparable and dimensionless, all the elements in the decision matrix are normalized using the following two equations:

$$\overline{x}_{ij} = \frac{x_{ij}}{\max_i x_{ij}}, \quad \text{for beneficial criteria,} \tag{1}$$

$$\overline{x}_{ij} = \frac{\min_i x_{ij}}{x_{ij}}, \quad \text{for non-beneficial criteria,} \tag{2}$$

where $\overline{x}_{ij}$ is the normalized value of x_{ij}.

According to WPM the total relative importance of alternative i, denoted as Q_i, is defined as follows:

$$Q_i = \prod_{j=1}^{n} (\overline{x}_{ij})^{w_j}, \tag{3}$$

where w_j denotes the weight (relative importance) of j-th criterion.

7. WPM Approach to MCDM Problem with Fermatean Fuzzy Data

In this section, I am going to introduce the MCDM problem under Fermatean fuzzy environment. Then, an effective decision-making method is hereby indicated to handle such MCDM problems.

7.1. *Description of the MCDM Problem with FFNs*

The main work done in most of the MCDM problem is to rank one or more alternatives from a collection of possible alternatives regarding multiple criteria. For a stated MCDM problem under Fermatean fuzzy domain, presume that there are m alternatives S_i $(i = 1, 2, \ldots, m)$ and n criterion C_j $(j = 1, 2, \ldots, n)$ with the criteria weight vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $0 \leqslant w_j \leqslant 1$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} w_j = 1$. We express the assessment values of the alternative S_i with respect to the criterion C_j

by $x_{ij} = (u_{ij}, v_{ij})$, and $R = (x_{ij})_{m \times n}$ is a Fermatean fuzzy decision matrix. Thus, the MCDM problem with FFNs can be considered as the subsequent matrix form:

$$R = (x_{ij})_{m \times n} = \begin{array}{c} \\ S_1 \\ S_2 \\ \vdots \\ S_m \end{array} \begin{array}{c} C_1 \qquad C_2 \qquad \cdots \qquad C_n \end{array} \\ \begin{pmatrix} (u_{11}, v_{11}) & (u_{12}, v_{12}) & \cdots & (u_{1n}, v_{1n}) \\ (u_{21}, v_{21}) & (u_{22}, v_{22}) & \cdots & (u_{2n}, v_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (u_{m1}, v_{m1}) & (u_{m2}, v_{m2}) & \cdots & (u_{mn}, v_{mn}) \end{pmatrix}, \tag{4}$$

where each of elements $x_{ij} = (u_{ij}, v_{ij})$ is an FFN, which implies that those degrees should the alternative S_i fulfills the attributes C_j will be the value u_{ij} and the degree should the alternative S_i disappoints the attributes C_j may be the worth v_{ij}.

7.2. *The Proposed Decision Method*

For an effective solution of the MCDM problem, mentioned above, we suggest a Fermatean fuzzy WPM (FF-WPM) method.

The first step in FF-WPM is to normalize the $R = (x_{ij})_{m \times n}$ matrix with a linear method. Let the criteria be classified to a subset of benefit criteria, B, and a subset of cost criteria, C. The linear normalization for any $j \in B$ is defined as follows:

$$\overline{x}_{ij} = x_{ij} \boxslash \max_i x_{ij}, \tag{5}$$

where $\max_i x_{ij}$ is determined as $\max_i x_{ij} = (\max_i u_{ij}, \min_i v_{ij})$.

And the linear normalization for any $j \in C$ is defined as follows:

$$\overline{x}_{ij} = \min_i x_{ij} \boxslash x_{ij}, \tag{6}$$

where $\min_i x_{ij}$ is determined as $\min_i x_{ij} = (\min_i u_{ij}, \max_i v_{ij})$.

By applying score functions, we can easily get the result that $x_{ij} \leqslant \max_i x_{ij}$ and $\min_i x_{ij} \leqslant x_{ij}, \forall i, j : i = 1, 2, \ldots, m, j \in C$. The division operations in Eq. (5) and Eq. (6) are done based on the Definition 9(ii). The decision matrix R is transformed into the normalized matrix $\overline{R}$:

$$\overline{R} = \left(\overline{x}_{ij}\right)_{m \times n} = \begin{pmatrix} \overline{x}_{11} & \overline{x}_{12} & \cdots & \overline{x}_{1n} \\ \overline{x}_{21} & \overline{x}_{22} & \cdots & \overline{x}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{x}_{m1} & \overline{x}_{m2} & \cdots & \overline{x}_{mn} \end{pmatrix}. \tag{7}$$

According to WPM the total relative importance of alternative i, denoted as $\overline{Q}_i$, is defined as

$$\overline{Q}_i = \prod_{j=1}^{n} (\overline{x}_{ij})^{w_j}. \tag{8}$$

Table 1
Fermatean fuzzy decision matrix.

Criteria	Optimization direction	Weight	S_1	S_2	S_3	S_4	max/min x_{ij}
C_1	max	0.142875	$(0.38, 0.52)$	$(0.73, 0.28)$	$(0.78, 0.23)$	$(0.57, 0.44)$	$(0.78, 0.23)$
C_2	min	0.059524	$(0.62, 0.22)$	$(0.56, 0.87)$	$(0.08, 0.81)$	$(0.35, 0.65)$	$(0.08, 0.87)$
C_3	min	0.214251	$(0.24, 0.80)$	$(0.25, 0.47)$	$(0.21, 0.65)$	$(0.42, 0.57)$	$(0.21, 0.80)$
C_4	min	0.095238	$(0.51, 0.47)$	$(0.25, 0.71)$	$(0.46, 0.48)$	$(0.40, 0.51)$	$(0.25, 0.71)$
C_5	max	0.142875	$(0.68, 0.35)$	$(0.39, 0.33)$	$(0.65, 0.34)$	$(0.29, 0.63)$	$(0.68, 0.33)$
C_6	min	0.119048	$(0.16, 0.75)$	$(0.23, 0.79)$	$(0.38, 0.45)$	$(0.86, 0.61)$	$(0.16, 0.79)$
C_7	min	0.059524	$(0.25, 0.42)$	$(0.68, 0.29)$	$(0.21, 0.75)$	$(0.17, 0.87)$	$(0.17, 0.87)$
C_8	min	0.166665	$(0.11, 0.90)$	$(0.40, 0.31)$	$(0.38, 0.63)$	$(0.23, 0.76)$	$(0.11, 0.90)$

In Eq. (8), the power operation $(\overline{x}_{ij})^{w_j}$, $j = 1, 2, \ldots, n$ is done based on Definition 5(iv), while the product operation of obtained values is performed based on Definition 5(i). It is notable that $\overline{Q}_i$ is a FFN.

In the last step, the score and accuracy functions are calculated for $\overline{Q}_i$, $i = 1, 2, \ldots, m$ and the final rankings of alternatives are determined based on the descending order of $\overline{Q}_i$, $i = 1, 2, \ldots, m$.

8. Applications to Bridge Construction Selection

In this section, we expand the implementation of the proposed WPM approach with a numerical example, which is discussed in Chen (2012). There are four available bridge construction methods, including the advanced shoring method (S_1), the incremental launching method (S_2), the balanced cantilever method (S_3), and the precast segmental method (S_4). The set of all candidate methods is denoted by $S = \{S_1, S_2, S_3, S_4\}$. The criteria for evaluating bridge construction methods contain durability (C_1), damage cost (C_2), construction cost (C_3), traffic conflict (C_4), site condition (C_5), weather condition (C_6), landscape (C_7), and environmental effect (C_8). Among these criteria, C_1 and C_5 are the benefit criteria, whereas the others are cost criteria. The set of evaluative criteria is denoted by $C = \{C_1, C_2, \ldots, C_8\}$ with $C_b = \{C_1, C_5\}$ and $C_c = \{C_2, C_3, C_4, C_6, C_7, C_8\}$. In order to avoid influencing each other, the decision makers are required to evaluate the four available bridge construction methods S_i ($i = 1, 2, 3, 4$) under the above eight criterions and the decision matrix $R = (x_{ij})$ is presented in the middle of Table 1, where x_{ij} ($i = 1, 2, 3, 4$, $j = 1, 2, \ldots, 8$) are in the form of FFNs. The weight vector of C_j $j = 1, 2, \ldots, 8$ is $w = (0.142875, 0.059524, 0.214251, 0.095238, 0.142875, 0.119048, 0.059524, 0.166665)^T$.

The normalized decision matrix will be obtained by applying Eq. (5) and Eq. (6) based on Definition 9(ii), as shown in Table 2.

Finally, the FF-WPM is calculated for all of the alternatives. The results are shown in Table 3. The rankings of alternatives in Table 3 are obtained by calculating the score and accuracy functions of FF-WPM. It can be seen that the most preferred alternative is S_1 and then the alternatives S_2, S_3 and S_4 are in the 3rd, 2nd and 4th places, respectively. Therefore, the advanced shoring method will be preferred to other alternatives.

Table 2
Normalized decision matrix.

Criteria	S_1	S_2	S_3	S_4
C_1	$(0.38, 0.52)$	$(0.73, 0.28)$	$(0.78, 0.23)$	$(0.57, 0.44)$
C_2	$(0.08, 0.87)$	$(0.08, 0.87)$	$(0.08, 0.87)$	$(0.08, 0.87)$
C_3	$(0.21, 0.80)$	$(0.21, 0.80)$	$(0.21, 0.80)$	$(0.21, 0.80)$
C_4	$(0.25, 0.71)$	$(0.25, 0.71)$	$(0.25, 0.71)$	$(0.25, 0.71)$
C_5	$(0.68, 0.35)$	$(0.39, 0.33)$	$(0.65, 0.34)$	$(0.29, 0.63)$
C_6	$(0.16, 0.79)$	$(0.16, 0.79)$	$(0.16, 0.79)$	$(0.16, 0.79)$
C_7	$(0.17, 0.87)$	$(0.17, 0.87)$	$(0.17, 0.87)$	$(0.17, 0.87)$
C_8	$(0.11, 0.90)$	$(0.11, 0.90)$	$(0.11, 0.90)$	$(0.11, 0.90)$

Table 3
Numerical results obtained by FF-WPM.

Alternatives	FF-WPM	Score	Rank
S_1	$(0.7088603, 0.558145642)$	0.182312972	1
S_2	$(0.5582420, 0.693929450)$	-0.16018621	3
S_3	$(0.6793039, 0.693499322)$	-0.02006513	2
S_4	$(0.4626257, 0.691596519)$	-0.23178229	4

9. Conclusions

In this article, we have introduced the subtraction, division and Fermatean arithmetic mean operations over FFNs. A few operative rules of these three operations have been provided. In addition, the relationships between these operations have also been established. These operations can be instantly prolonged into interval-valued Fermatean fuzzy sets. Finally, we propose a FF-WPM method and apply it to bridge construction selection. In the future, we will combine others methods like VIKOR Method (Liu and Qin, 2017), MUL-TIMOORA Method (Stanujkic *et al.*, 2017), SWARA Method (Karabasevic *et al.*, 2016), SECA method (Keshavarz-Ghorabaee *et al.*, 2018), GDM models (Capuano *et al.*, 2018; Moral *et al.*, 2018; Zhang *et al.*, 2018), with FFNs.

References

Atanassov, K.T. (1983). A second type of intuitionistic fuzzy sets. *BUSEFAL*, 56, 66–70.

Atanassov, K.T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20, 87–96.

Atanassov, K.T. (2009). Remark on operations "subtraction" over intuitionistic fuzzy sets. *Notes on Intuitionistic Fuzzy Sets*, 15(3), 20–24.

Atanassov, K.T. (2012). *On Intuitionistic Fuzzy Sets Theory*. Springer-Verlag.

Atanassov, K.T. (2016). Geometrical interpretation of the elements of the intuitionistic fuzzy objects. *International Journal Bioautomation*, 20(S1), S27–S42.

Atanassov K.T., Riecan, B. (2006). On two operations over intuitionistic fuzzy sets. *Journal of Applied Mathematics, Statistics and Informatics*, 2(2), 145–148.

Atanassov, K.T., Vassilev, P.M., Tsvetkov, R.T. (2013). *Intuitionistic Fuzzy Sets, Measures and Integrals*. Professor Marin Drinov Academic Publishing House, Sofia, Bulgaria.

Bridgman, P.W. (1922). *Dimensional Analysis*. Yale University Press, New Haven.

Capuano, N., Chiclana, F., Fujita, H., Herrera-Viedma, E., Loia, V. (2018). Fuzzy group decision making with incomplete information guided by social influence. *IEEE Transactions on Fuzzy Systems*, 26(3), 1704–1718.

Chen, T.Y. (2007). Remarks on the subtraction and division operations over intuitionistic fuzzy sets and interval-valued fuzzy sets. *International Journal of Fuzzy Systems*, 9(3), 169–172.

Chen, T.Y. (2012). Nonlinear assignment-based methods for interval-valued intuitionistic fuzzy multi-criteria decision analysis with incomplete preference information. *International Journal of Information Technology & Decision Making*, 11(4), 821–855.

Garg, H. (2016). A new generalized Pythagorean fuzzy information aggregation using Einstein operations and its application to decision making. *International Journal of Intelligent Systems*, 31, 886–920.

Gou, X., Xu, Z., Ren, P. (2016). The properties of continuous pythagorean fuzzy information. *International Journal of Intelligent Systems*, 31, 401–424.

Karabasevic, D., Zavadskas, E.K., Turskis, Z., Stanujkic, D. (2016). The framework for the selection of personnel based on the SWARA and ARAS methods under uncertainties. *Informatica*, 27(1), 49–65.

Keshavarz-Ghorabaee, M., Amiri, M., Zavadskas, E.K., Turskis, Z., Antucheviciene, J. (2018). Simultaneous evaluation of criteria and alternatives (SECA) for multi-criteria decision-making. *Informatica*, 29(2), 265–280.

Liao, H., Xu, Z. (2014). Subtraction and division operations over hesitant fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 27, 65–72.

Liu, P., Qin, X. (2017). An extended VIKOR method for decision making problem with interval-valued linguistic intuitionistic fuzzy numbers based on entropy. *Informatica*, 28(4), 665–685.

Miller, D.W., Starr, M.K. (1969). *Executive Decisions and Operations Research*. Prentice-Hall, Inc., Englewood Cliffs, NJ.

Moral, M.J., Chiclana, F., Tapia, J.M., Herrera-Viedma, E. (2018). A comparative study on consensus measures in group decision making. *International Journal of Intelligent Systems*, 33(8), 1624–1638.

Parvathi, R., (2005). *Theory of Operators on Intuitionistic Fuzzy Sets of Second Type and Their Applications to Image Processing*. PhD dissertation, Dept. Math., Alagappa Univ., Karaikudi, India.

Parvathi, R., Vassilev, P., Atanassov, K.T. (2012). A note on the bijective correspondence between intuitionistic fuzzy sets and intuitionistic fuzzy sets of p-th type. In: *New Developments in Fuzzy Sets*, *Intuitionistic Fuzzy Sets*, *Generalized Nets and Related Topics*, Vol. I. Foundations, SRI PAS IBS PAN, Warsaw, pp. 143–147.

Peng, X., Yang, Y. (2015). Some results for pythagorean fuzzy sets. *International Journal of Intelligent Systems*, 30, 1133–1160.

Reformat, M.Z., Yager, R.R. (2014). Suggesting recommendations using Pythagorean fuzzy sets illustrated using netflix movie data. In: *Information Processing and Management of Uncertainty in Knowledge-Based Systems*. Springer, pp. 546–556.

Ren, P., Xu, Z., Gou, X. (2016). Pythagorean fuzzy TODIM approach to multi-criteria decision making. *Applied Soft Computing*, 42, 246–259.

Senapati, T., Yager, R.R. (2019a). *Fermatean Fuzzy Sets*. Communicated.

Senapati, T., Yager, R.R. (2019b). *Fermatean Fuzzy Weighted Averaging/Geometric Operators and Its Application in Multi-Criteria Decision-Making Methods*. Communicated.

Stanujkic, D., Zavadskas, E.K., Smarandache, F., Brauers, W.K.M., Karabasevic, D. (2017). A neutrosophic extension of the MULTIMOORA method. *Informatica*, 28(1), 181–192.

Triantaphyllou, E. (2000). *Multi-Criteria Decision Making: A Comparative Study*. Kluwer Academic Publishers (now Springer), Dordrecht, The Netherlands, 320 pp., ISBN 0-7923-6607-7.

Vassilev, P., Parvathi, R., Atanassov, K.T. (2008). Note on intuitionistic fuzzy sets of p-th type. *Issues Intuitionistic Fuzzy Sets Generalized Nets*, 6, 43–50.

Vassilev, P. (2012). On the intuitionistic fuzzy sets with metric type relation between the membership and non-membership functions. *Notes on Intuitionistic Fuzzy Sets*, 18(3), 30–38.

Vassilev, P. (2013). *Intuitionistic Fuzzy Sets with Membership and Nonmembership Functions in Metric Relation*. PhD dissertation (in Bulgarian).

Yager, R.R. (2013). Pythagorean fuzzy subsets. In: *Proceedings Joint IFSA World Congress and NAFIPS Annual Meeting*, Edmonton, Canada, pp. 57–61.

Yager, R.R. (2014). Pythagorean membership grades in multicriteria decision making. *IEEE Transactions on Fuzzy Systems*, 22, 958–965.

Yager, R.R. (2017). Generalized orthopair fuzzy sets. *IEEE Transactions on Fuzzy Systems*, 25(5), 1222–1230.

Yager, R.R., Abbasov, A.M. (2013). Pythagorean membership grades, complex numbers, and decision making. *International Journal of Intelligent Systems*, 28, 436–452.

Zeng, S., Mu, Z., Balezentis, T. (2018). A novel aggregation method for Pythagorean fuzzy multiple attribute group decision making. *International Journal of Intelligent Systems*, 33(3), 573–585.

Zhang, X. (2016). A novel approach based on similarity measure for pythagorean fuzzy multiple criteria group decision making. *International Journal of Intelligent Systems*, 31, 593–611.

Zhang, X., Xu, Z. (2014). Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets. *International Journal of Intelligent Systems*, 29, 1061–1078.

Zhang, H., Dong, Y., Herrera-Viedma, E. (2018). Consensus building for the heterogeneous large-scale GDM with the individual concerns and satisfactions. *IEEE Transactions on Fuzzy Systems*, 26(2), 884–898.

 T. Senapati received the BSc, MSc and PhD degrees in mathematics all from the Vidyasagar University, India, in 2006, 2008 and 2013, respectively. He has published more than 50 articles in international journals. His research interests include aggregation operators, fuzzy logic, fuzzy algebraic structures, fuzzy decision making, and their applications.

 R.R. Yager received the BE degree from the City College of New York, USA, in 1963, and the PhD degree in systems science from the Polytechnic University of New York, USA, in 1968. He is currently the director of the Machine Intelligence Institute and a professor of Information Systems at Iona College. He is the chief editor of the *International Journal of Intelligent Systems*. He has published over 900 papers and edited over 30 books in areas related to fuzzy sets, human behavioural modelling, decision-making under uncertainty and the fusion of information. He was the recipient of the IEEE Computational Intelligence Society Pioneer award in Fuzzy Systems. He received the special honorary medal of the 50-th Anniversary of the Polish Academy of Sciences. He received the lifetime outstanding achievement award from International Fuzzy Systems Association. Dr. Yager is a fellow of the IEEE, the New York Academy of Sciences and the Fuzzy Systems Association. He has served at the National Science Foundation as program director in the Information Sciences program. He was a NASA/Stanford visiting fellow and a research associate at the University of California, Berkeley. He has been a lecturer at NATO Advanced Study Institutes. He is a visiting distinguished scientist at King Saud University, Riyadh, Saudi Arabia. He is a distinguished honorary professor at the Aalborg University, Denmark.