$\mathcal{LOCAL}$ Approximation of Independent Set and Coloring

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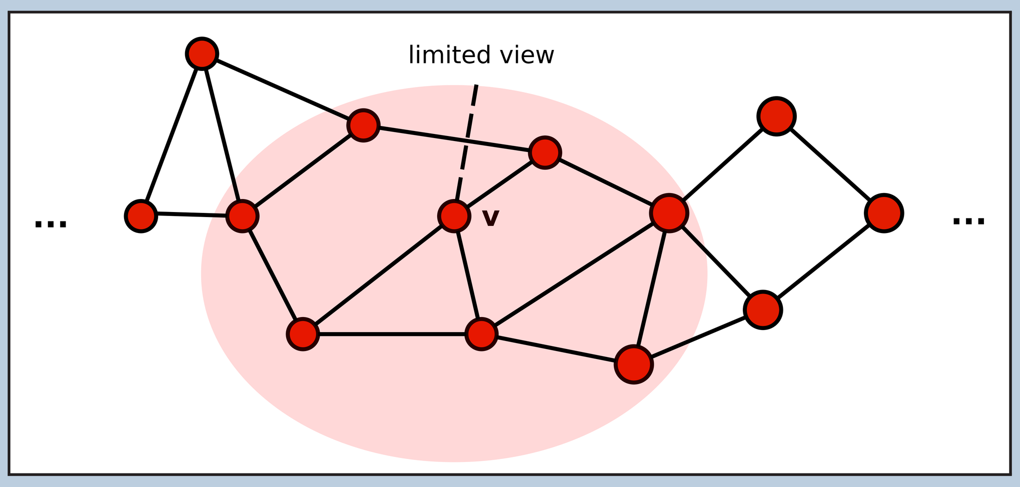
The $\mathcal{LOCAL}$ Model

Model:

- O(1) communication rounds
- Unbounded message sizes
- Unbounded computational power



Focus:

Locality of computational problems

Independent Sets/Colorings

Hardness. The maximum independent set and the minimum vertex coloring problems are NP-hard, and they are even hard to approximate within a factor of $n^{1-\epsilon}$.

Exponential Time. Under the assumption that $P \neq NP$, every local algorithm with non-trivial approximation ratio for either problem has to use exponential time computations.

Related Work

Most works on distributed independent sets and colorings consider the maximal independent set problem and the $(\Delta+1)$-coloring problem. These problems can easily be solved sequentially.

The work of Barenboim [ICALP, 2012] is closest to our work and presents a $O(n^{1/2+\epsilon})$-approximation local algorithm for the minimum vertex coloring problem (using exponential time computations).

Results

Upper Bounds:

We present local randomized $O(n^\epsilon)$-approximation algorithms for the maximum independent set and the minimum vertex coloring problems, for any $\epsilon > 0$, which run in $O(3^{\frac{1}{\epsilon}})$ rounds.

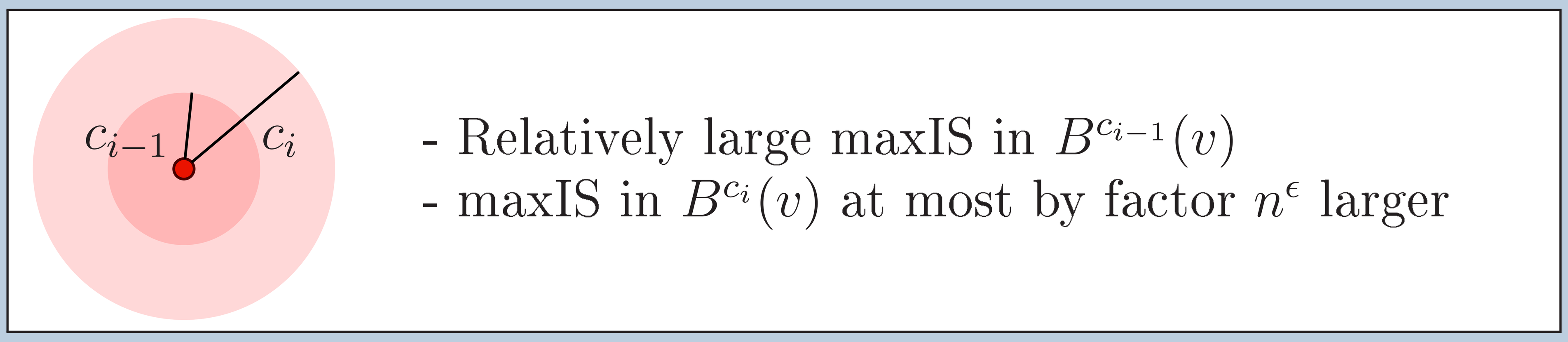
Lower Bounds:

We prove that both algorithms are optimal in that no local algorithm can achieve $n^{o(1)}$-approximations for either problem.

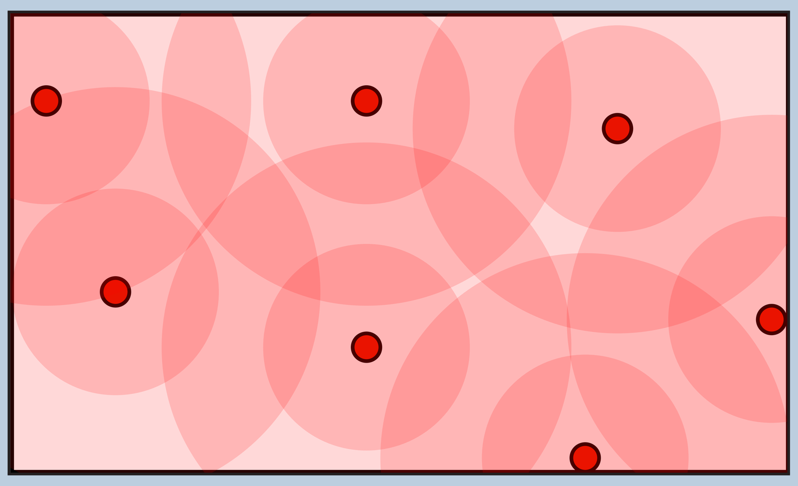
Distributed Maximum Independent Set Approximation

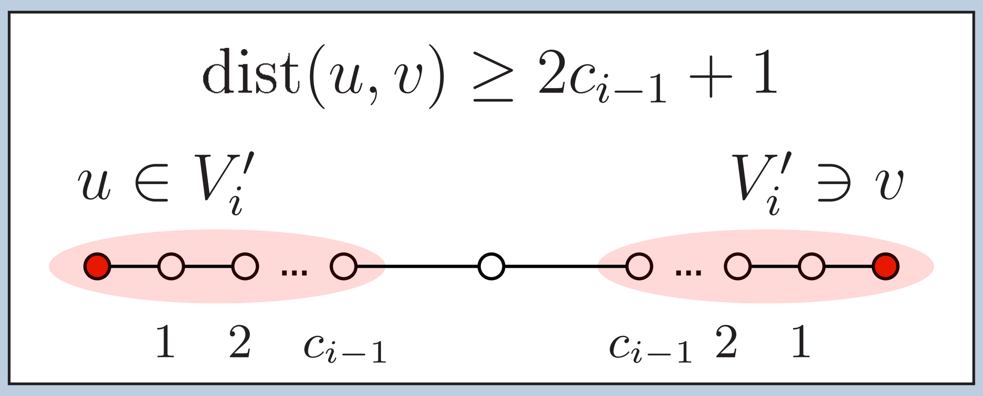
Suppose that every node computes a maximum independent set in its k-neighborhood. How can we combine these locally optimal solutions to a coherent global solution?

1. **New Vertex Decomposition**
   For a constant $k = \Theta(\frac{1}{\epsilon})$, partition V into disjoint sets $V_1, V_2, \dots, V_k$ so that $v \in V_j$ if j is the smallest i such that
   $$\text{maxIS}(N^{c_{i-1}}(v)) \geq n^{(i-1)/k}, \text{ and } \text{maxIS}(N^{c_i}(v)) \leq n^{i/k},$$
   where $(c_i)_i$ is an exponentially increasing sequence, and $N^d(v)$ denotes the d-neighborhood of v.



2. **Ruling Set Algorithm**
   For $i = 1, \dots, k$, we treat the sets V_i separately. Using an algorithm by Gfeller and Vicari [PODC 2007], in O(1) rounds, we compute a $(2c_{i-1}+1)$-independent subset $V_i' \subseteq V_i$ which *essentially* c_i-dominates V_i.





   Then, a large independent set $I_i = \bigcup_{v \in V_i'} \text{maxIS}(B^{c_{i-1}}(v))$ is established. Let I^* be a a maximum independent set in the input graph. We show that:
   $$n^\epsilon |I_i| \geq |I^* \cap V_i|, \tag{1}$$

3. **Merging the Independent Sets**
   From the sets $I_1, \dots, I_k$, we compute an independent set I so that $|I| \geq |I_i|$, for every i. Since there exists an i such that $|I^* \cap V_i| \geq |I^*|/k$, and using Inequality 1, I is a $(k \cdot n^\epsilon)$-approximation.

Distributed Minimum Vertex Coloring Approximation

We make use of the following connection between between minimum vertex coloring and *network decompositions*.

Definition A (d,c)-network decomposition is a partitioning of the vertices of the input graph into clusters of maximal diameter d so that the graph obtained when contracting the clusters into vertices can be colored with at most c colors.

Theorem (Barenboim [ICALP, 2012]) Suppose that nodes of a graph $G = (V,E)$ know their color in a (d,c)-network decomposition. Then, there is an $O(d)$-rounds distributed algorithm that computes a c-approximate minimum vertex coloring.

Barenboim [ICALP, 2012] showed that there is a sampling-based, local algorithm that computes a $(O(1), n^{\frac{1}{2}+\epsilon})$-network decomposition, implying a local $O(n^{\frac{1}{2}+\epsilon})$-approximation algorithm for minimum vertex coloring.

Our Result We show that via a recursive sampling-based approach similar to Barenboim's method, a $(O(1), n^\epsilon)$-network decomposition can be computed, leading to a local $O(n^\epsilon)$-approximation algorithm for minimum vertex coloring.