Constructing Large Matchings via Query Access to a Maximal Matching Oracle

FSTTCS 2020

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University of BRISTOL

Matchings

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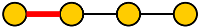

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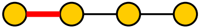

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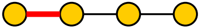

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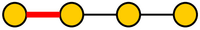

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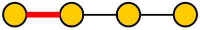

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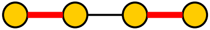

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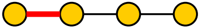

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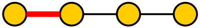

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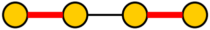

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Property: $|\text{maximal matching}| \geq \frac{1}{2} \, |\text{maximum matching}|$

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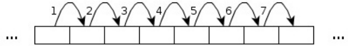

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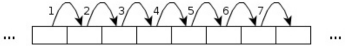
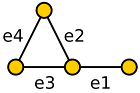


- Input stream: Sequence of edges of input graph $G = (V, E)$ with $n = |V|$ in arbitrary order

$$S = e_2 e_1 e_4 e_3$$



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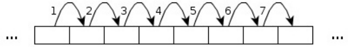

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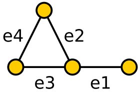

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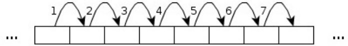

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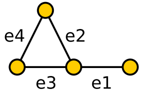

- Goal: Few passes algorithms with small space
- Streaming Maximal Matching Algorithm: Insert current edge into initially empty matching if possible ($\textsc{Greedy}$), using space $\tilde{O}(n)$

State of the Art Streaming Matching Algorithms

# passes	Approximation	det/rand	Reference
Bipartite Graphs			
1	$\frac{1}{2}$	det	Greedy, folklore
2	$2-\sqrt{2} \approx 0.5857$	rand	Konrad '18
3	0.6067	rand	Konrad '18
$O(\frac{1}{\epsilon^2})$	$1-\epsilon$	det	Assadi, Liu, Tarjan '21
General Graphs			
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How large a matching can we compute if we solely invoke Greedy in each pass?

Maximal Matching Oracle

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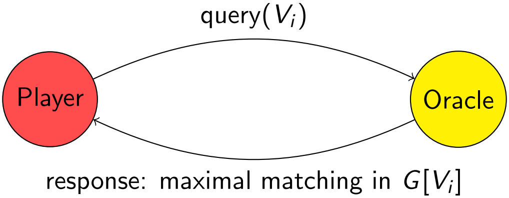

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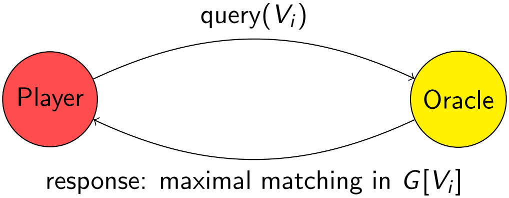

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Research Question: What is the trade-off between the number of rounds and the approximation ratio?

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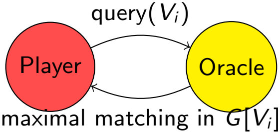

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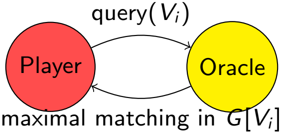

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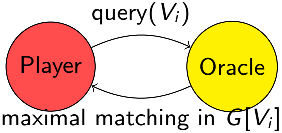

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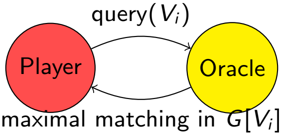

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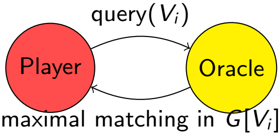

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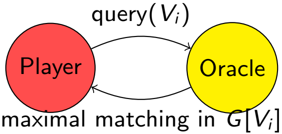

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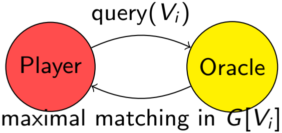

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Upper Bounds for General Graphs: ✗ (except 1 round $\frac{1}{2}$-approx.)

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Outline:

1. $\Omega(\frac{1}{\epsilon})$ rounds are needed for a $(1-\epsilon)$-approximation
2. $\Omega(n)$ rounds needed for $(\frac{1}{2}+\epsilon)$-approx. in general graphs, $\epsilon > 0$
3. 0.6-approximation lower bound for 3 rounds (main technical result)

Lower Bound for Computing a $(1 - \epsilon)$-approximation

Theorem. Any query algorithm with approximation factor $1 - \epsilon$ requires at least $\frac{1}{\epsilon} - 1$ queries,even in bipartite graphs.

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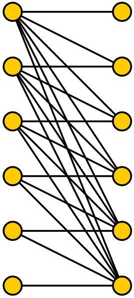
Proof. Consider semi-complete graph on $2c$ vertices



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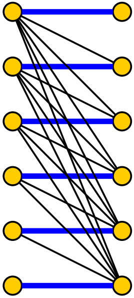

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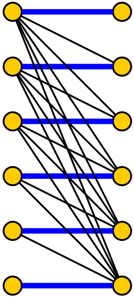

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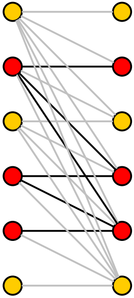

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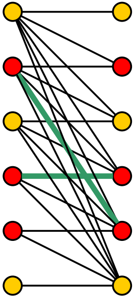

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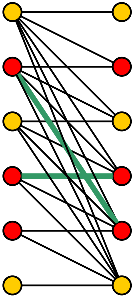

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- Hence, to achieve a $(1 - \epsilon)$-approximation, for $\epsilon = \frac{1}{c+1}$, $c = \frac{1}{\epsilon} - 1$ queries are needed

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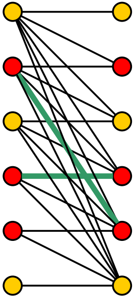

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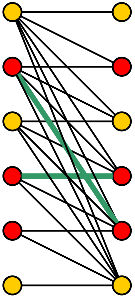
- Hence, to achieve a $(1-\epsilon)$-approximation, for $\epsilon = \frac{1}{c+1}$, $c = \frac{1}{\epsilon}-1$ queries are needed
- Use multiple disjoint gadgets for arbitrary n

Lower Bound for Computing a $(1-\epsilon)$-approximation

Theorem. Any query algorithm with approximation factor $1-\epsilon$ requires at least $\frac{1}{\epsilon}-1$ queries,even in bipartite graphs.

Proof. Consider semi-complete graph on $2c$ vertices



- Unique perfect matching M^* of size c
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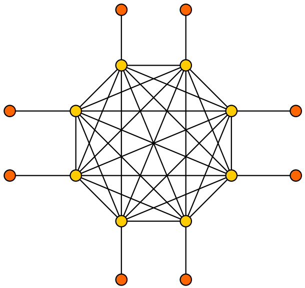

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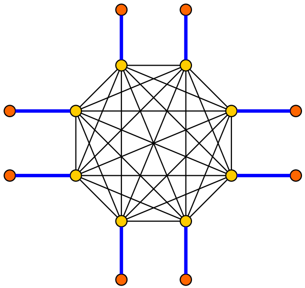

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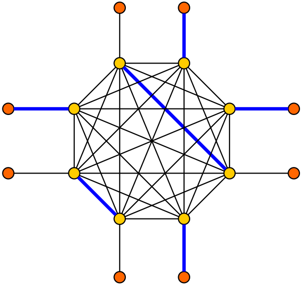

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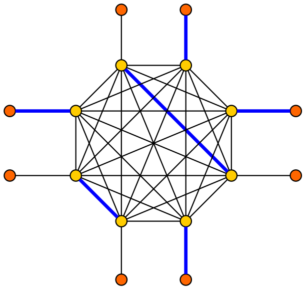

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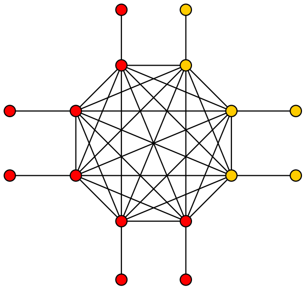

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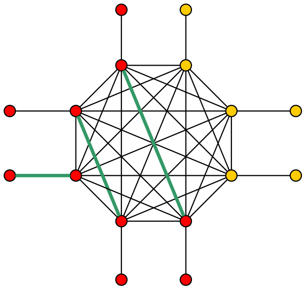

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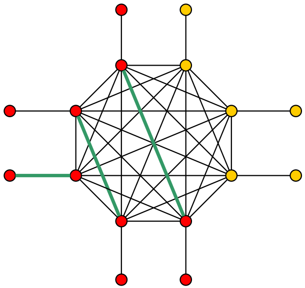

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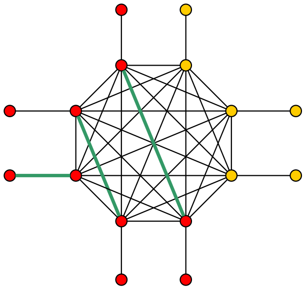

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Remarks

Deterministic / Randomized Query Algorithms:

- Lower bounds on previous slides hold even if the input graph is known by the player
- They also hold for randomized query algorithms

Lower Bound for 3 Rounds on Bipartite Graphs:

- More subtle argument
- Oracle builds graph that depends on the queries
- Lower bound therefore only holds for deterministic algorithms

Three Round Query Algorithm for Bipartite Graphs

Algorithm (input graph $G = (A, B, E)$)

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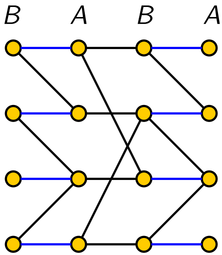

Input graph $G = (A, B, E)$ with perfect matching M^*

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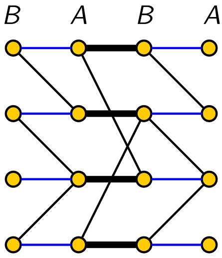

1st query: Matching M

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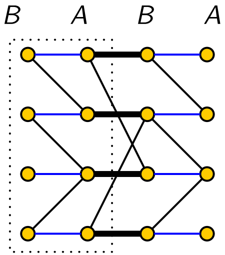

Subgraph $G[A(M) \cup \overline{M(B)}]$

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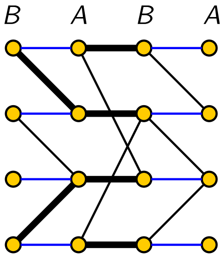

2nd query: Matching M_L

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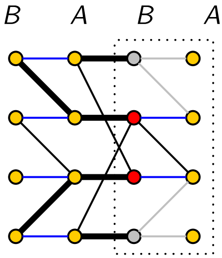

B' and Subgraph $G[\overline{A(M)} \cup B']$

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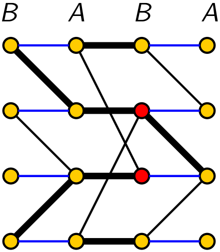

Matching M_R

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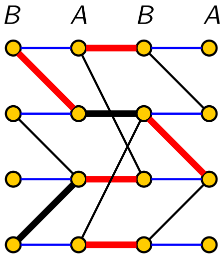

Largest matching in $M \cup M_L \cup M_R$ (M augmented with $M_L \cup M_R$)

Three Round Query Algorithm for Bipartite Graphs (2)

Analysis:

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Analysis: $\frac{3}{5}$-approximation algorithm [Kale and Tirodkar, '17]

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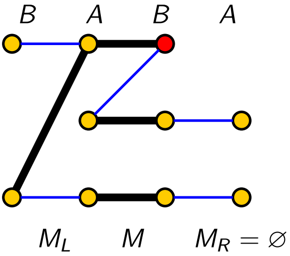

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Strategy: Bound "knowledge" about input graph after each query ("structure graph"); ensure perfect matching can be added

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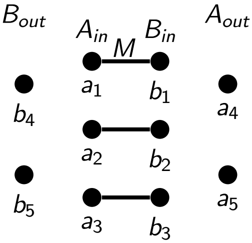
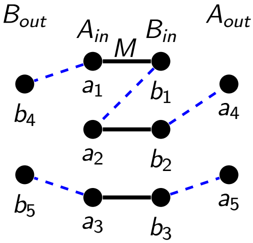
First Query:

- Oracle commits to structure below and returns subset of edges M (no edges between A_{out} and B_{out})
- A perfect matching (blue edges) can be added, which implies that approximation factor is 3/5 at best after first query

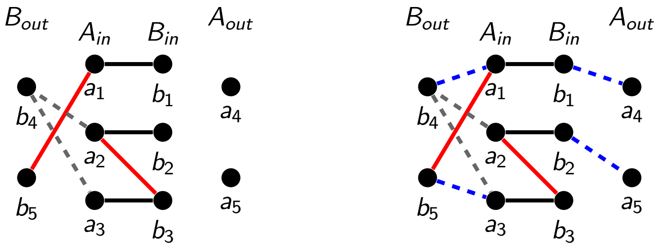




Lower Bound Construction - Second Query

Second Query:

- Information can be bounded by structure below - grey edges indicate that edges are not present in output graph
- Again, perfect matching can be added, which implies that approximation factor is 3/5 at best after second query



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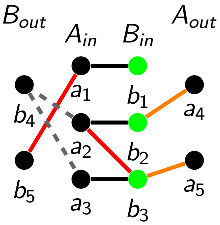

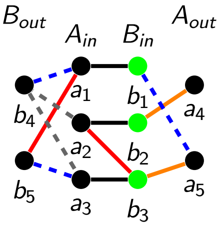

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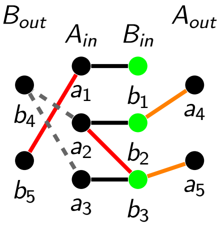

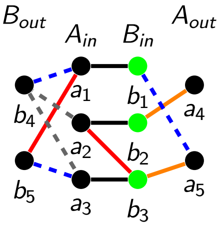

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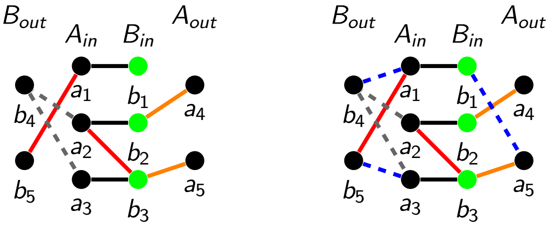

$\square$

Key Technique: Structural properties that allow eliminating cases

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Outlook:

- Extensions: Edge queries instead of vertex queries
- Randomization?

Thank you for your attention.