Distributed Minimum Vertex Coloring and Maximum Independent Set in Chordal Graphs

MFCS 2019

Christian Konrad and Victor Zamaraev

University of Bristol

Durham University

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Outline:

1. Introduction: LOCAL Model and Vertex Coloring
2. Results: Minimum Vertex Coloring and Maximum Independent Set in Chordal Graphs
3. Discussion: Tree Decomposition and Distributed Computing

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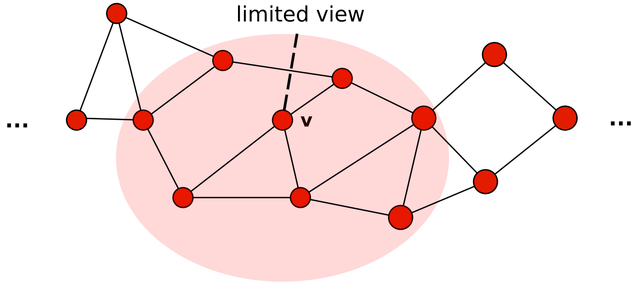
The LOCAL Model

Input: Network $G = (V, E)$, $n = |V|$, max degree Δ



- Nodes host processors and have unique IDs
- Synchronous communication along edges, individual messages of unbounded sizes
- **Running time:** Number of communication rounds
- r rounds $\Leftrightarrow$ compute output from distance-r neighborhood

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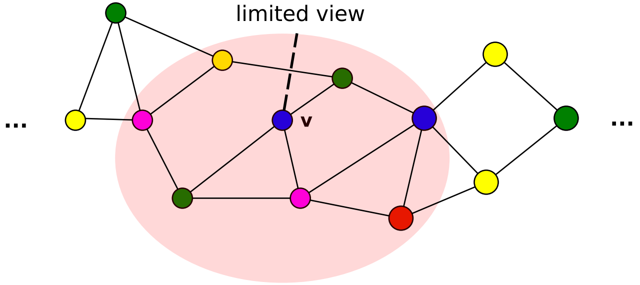

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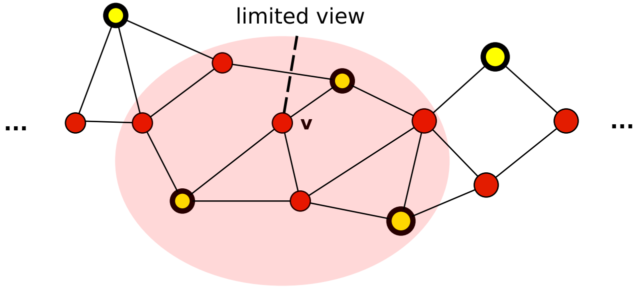

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Vertex Coloring Problems

$(\Delta + 1)$-**coloring:**

$2^{O(\sqrt{\log \log n})}$ rounds **[Chang, Li, Pettie, 2018]**

Δ-**coloring:** (assuming no $\Delta + 1$ clique, $\Delta \geq 3$)

$O(\log \Delta) + 2^{O(\sqrt{\log \log n})}$ rounds **[Ghaffari et al., 2018]**

Fewer colors:

- Arboricity a: $O(a)$-coloring in $O(a \log n)$ rounds **[Barenb., Elkin, 2010]**
- 3-coloring trees, 6-coloring planar graphs, ...

Minimum Vertex Coloring

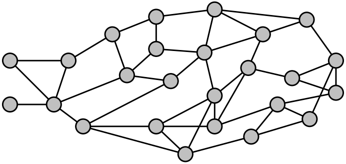
Chromatic number $\chi(G)$: smallest c such that there is a c-coloring

Minimum Vertex Coloring (MVC): find $\chi(G)$-coloring

- NP-hard **[Karp, 1972]**
- Hard to approximate within factor $n^{1-\epsilon}$ **[Håstad, 1999]**

Distributed MVC: Network-decomposition **[Linial, Saks, 1993]**

- Partition vertices $V = V_1 \dot\cup \cdots \dot\cup V_k$ into clusters, $\mathrm{O}(\log^2 n)$ rounds
- Each cluster $G[V_i]$ has diameter $\mathrm{O}(\log n)$
- Cluster graph colored with $\mathrm{O}(\log n)$ colors:



- $\mathrm{O}(\log n)$-approximation in $\mathrm{O}(\log^2 n)$ rounds to MVC
- Poly-time if graph class admits poly-time approximations

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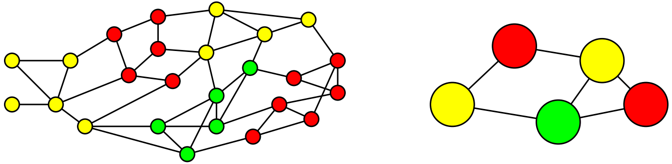

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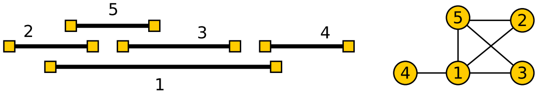
MVC on Interval Graphs

Interval Graphs: Intersection graph of intervals on the line



[Halldórsson, Konrad, 2014,2017] :

- $(1+\epsilon)$-approximation in $\mathrm{O}(\frac{1}{\epsilon}\log^* n)$ rounds (for $\epsilon > \frac{2}{\chi(G)}$)
- Lower Bound: $\Omega(\frac{1}{\epsilon} + \log^* n)$ rounds

Research Questions: Can we...

- Improve approximation factor $\mathrm{O}(\log n)$ on general graphs?
- Get $\mathrm{O}(1)$ or $(1+\epsilon)$-approximations on other graph classes?

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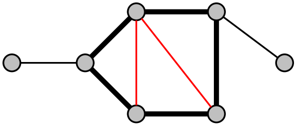
MVC on Chordal Graphs

Chordal Graphs: Every cycle of at least 4 vertices contains a chord:



[Konrad, Zamaraev, MFCS 2019] : **MVC**

- $(1+\epsilon)$-approximation in $O(\frac{1}{\epsilon} \log n)$ rounds (for $\epsilon > \frac{2}{\chi(G)}$)
- *Lower Bound:* $\Omega(\frac{1}{\epsilon} + \log n)$ *rounds (known results)*

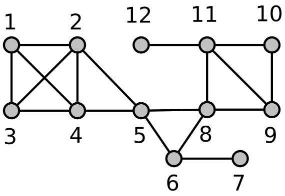
Chordal Graphs vs. Interval Graphs:

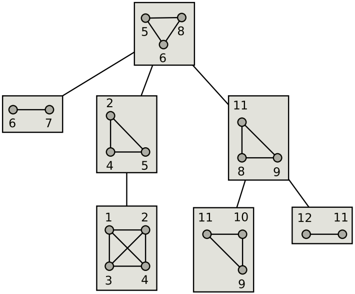
- Chordal graphs contain trees, interval graphs don't
- Linial's tree coloring LB applies: coloring trees with $O(1)$ colors requires $\Omega(\log n)$ rounds **[Linial, 1992]**

Technique: Tree Decomposition

Tree Decomposition of Chordal Graphs

Chordal Graph: Clique Tree





1. Set of bags = set of maximal cliques
2. Bags containing any vertex v induces a subtree

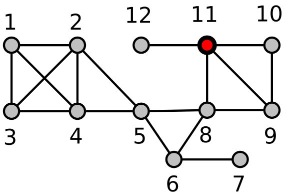
Distributed Processing:

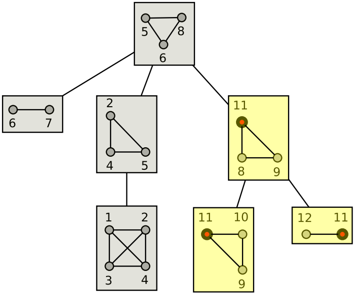
- Nodes compute local view of (global) clique tree
- Locality property: Diameter of each bag is 1

Interval Graph: Clique tree is a path

Tree Decomposition of Chordal Graphs

Chordal Graph: Clique Tree





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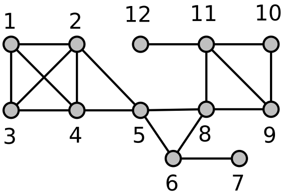
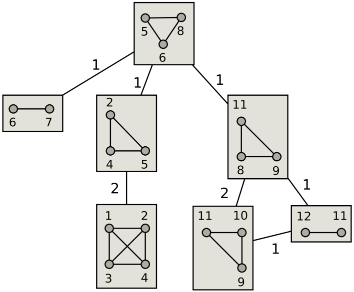
Local View of Clique Tree

Weighted Clique Intersection Graph:

- Let $\mathcal{C}$ be the maximal cliques in chordal graph G
- Let $W_G = (\mathcal{C}, \mathcal{E})$ be the *weighted clique intersection graph* of G, i.e., there is an edge of weight k ($k \geq 1$) between cliques C_i, C_j if $|C_i \cap C_j| = k$

$\mathcal{T}$ is a clique tree $\Leftrightarrow$ $\mathcal{T}$ is a maximum weight spanning tree in W_G





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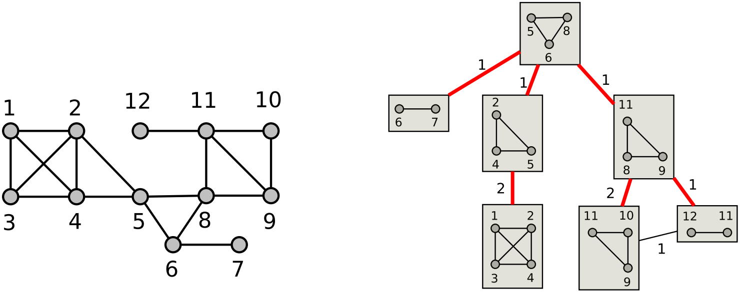

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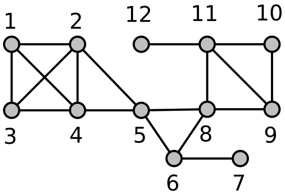
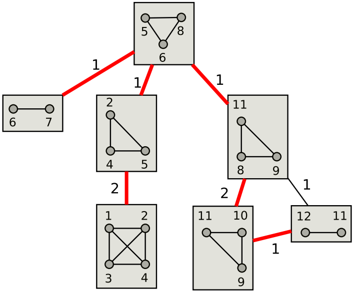




Local View of Clique Tree (2)

Important Property:

Nodes agree on the same maximum weight spanning tree

- Use node identifiers to distinguish between maximum weight spanning trees
- Local View of v: For each $u \in \Gamma^r(v)$: r : desired distance

Algorithm:

1. Compute maximal cliques that u is contained in;
2. Compute maximum weight spanning tree $\mathcal{T}_u$ in clique intersection graph of these cliques;
3. Add $\mathcal{T}_u$ to local view of global spanning tree .

Distributed Algorithm

$(1+\epsilon)$-**approximation Algorithm for MVC:**

1. **Peeling Phase:** Partition vertex set V into layers $V_1, V_2, \ldots, V_{\log n}$ such that $G[V_i]$ is an interval graph in $\mathrm{O}(\frac{1}{\epsilon} \log n)$ rounds
2. **Coloring Phase:** Color each interval graph $G[V_i]$ independently and separately (compute a $(1+\epsilon)$-approximation to MVC) in $\mathrm{O}(\frac{1}{\epsilon} \log^* n)$ rounds using **[Halldórsson, Konrad, 2017]**
3. **Color Correction Phase:** Resolve coloring conflicts between the layers in $\mathrm{O}(\frac{1}{\epsilon} \log n)$ rounds

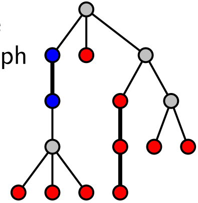
Overall Runtime: $\mathrm{O}(\frac{1}{\epsilon} \log n)$ rounds

Peeling Phase

Definition: Let $\mathcal{T}$ be the clique tree of G

- **Pendant Path:** incident to a leaf, degrees at most 2
- **Internal Path:** not incident to a leaf, degrees at most 2

Lemma: Graph induced by vertices whose subtrees are contained in pendant or internal path is an interval graph



Peeling Process: Let $\mathcal{T}_1 = \mathcal{T}$. For $i = 1 \ldots \log n$ do:

- Remove all pendant paths, and all "long enough" internal paths from $\mathcal{T}_i$.
  (nodes can decide this in $\mathrm{O}(\frac{1}{\epsilon})$ rounds)
- Let V_i be all vertices whose corresponding subtree in $\mathcal{T}_i$ is included in a pendant/long enough internal path
- Let $\mathcal{T}_{i+1}$ be clique tree of residual graph (can be obtained by removing pendant/internal paths)

Lemma: Peeling process terminates after $\log n$ rounds.
(each step number of nodes of degree ≥ 3 halves)

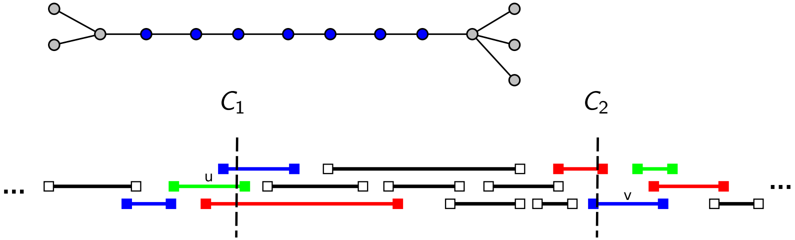
Color Correction Phase

Algorithm:

1. Leave colors of layer $V_{\log n}$ unchanged
2. Correct colors layer by layer from $V_{\log n - 1}$ downwards to layer V_1

Correcting layer i**:**

Layer i corresponds to pendant and internal paths in $\mathcal{T}_i$



Lemma:[Halldórsson, Konrad, 2017] : Only intervals at distance $O(\frac{1}{\epsilon})$ from boundary cliques need to change colors to resolve all coloring conflicts

Algorithm: Summary

Three phases:

1. Peeling Phase: $\log n$ iterations, each requiring $\mathrm{O}(\frac{1}{\epsilon})$ rounds
2. Coloring Phase: $\mathrm{O}(\frac{1}{\epsilon} \log^* n)$
3. Color Correction Phase: $\log n$ iterations, each requiring $\mathrm{O}(\frac{1}{\epsilon})$ rounds

[Konrad, Zamaraev, MFCS 2019] : **MVC**
$(1+\epsilon)$-approximation in chordal graphs in $O(\frac{1}{\epsilon} \log n)$ rounds

Maximum Independent Set

Adapt Technique to Maximum Independent Set: (MaxIS)

[Konrad, Zamaraev, MFCS 2019] : **MaxIS**

- $(1+\epsilon)$-approximation in $\mathrm{O}(\frac{1}{\epsilon}\log(\frac{1}{\epsilon})\log^* n)$ rounds
- On the way: $(1+\epsilon)$-approx. on interval graphs in $\mathrm{O}(\frac{1}{\epsilon}\log^* n)$ rounds
- Lower Bound: $\Omega(\frac{1}{\epsilon})$ rounds

Intuition:

- **Lemma:** Layers $1 \ldots O(\log(\frac{1}{\epsilon}))$ contain $1(+\epsilon)$-approximate independent set
- Develop maximum independent set algorithm for interval graph
- Apply algorithm on these layers and make sure that not much is lost at intersections

Lower Bound for Maximum Independent Set

Indistinguishability Argument: Consider a path P_n

- Assume every vertex is assigned a unique label from $\{1, 2, \ldots, n\}$
- Vertices far enough from boundary have same local views (in expectation over labellings), same probability to be chosen

- Local neighborhoods of u and v are disjoint, therefore whether u and v are chosen is independent

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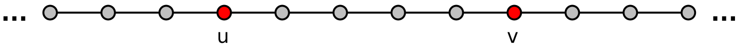
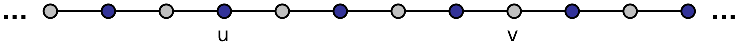


- Local neighborhoods of u and v are disjoint, therefore whether u and v are chosen is independent
- If both u and v are chosen then solution is suboptimal



[Konrad, Zamaraev, MFCS 2019] : **MaxIS** Computing a $(1+\epsilon)$-approximation to MaxIS on a path requires $\Omega(\frac{1}{\epsilon})$ rounds

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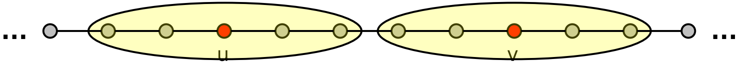

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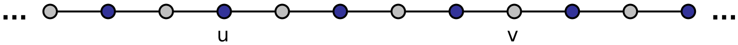

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Tree Decomposition in Distributed Computing

How useful are Tree Decompositions for Distributed Algorithms?

- Only few papers make use of tree decompositions
- Perfect tool for chordal graphs
- Can we handle other graph classes as well using tree decompositions?

Obstacle:

- Tree decomposition of cycle of length k contains bags that are at distance $\Omega(k)$ in the original graph
- Impossible for nodes to obtain coherent local views of global tree decomposition in $o(k)$ rounds

Outlook: Tree Length

- Graph of tree length k has tree decomposition where diameter of every bag is at most k
- Contains k-chordal graphs

Our Results

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