The Geometry of Perfect Parking *

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Abstract

How much extra length (above the length of your car) do you need to parallel park? This report shows how to write down a formula for this.

1 Introduction

I want to parallel park, and I've found a space. The road is wide, but the space looks narrow. I'm not interested in shuffling back and forth to get into the space: I want to reverse into the space at full lock, and then drive straight forward into the middle of the space to park. How narrow can the space be so that I can do this? This report uses some straightforward geometry to compute the smallest length that the space can be. I use some basic facts about circles, and I use Pythagoras's Theorem (see Figure 1 for a reminder). I hope you enjoy seeing how mathematics can be used to answer questions that might occur to you in real life; if you do, I've added some more comments at the end of this report.

*This research was commissioned by Vauxhall Motors, who own the copyright. A disclaimer: The work described in this paper is one theoretical model for parking behaviour. Please take sensible precautions before trying to experimentally verify the theory!

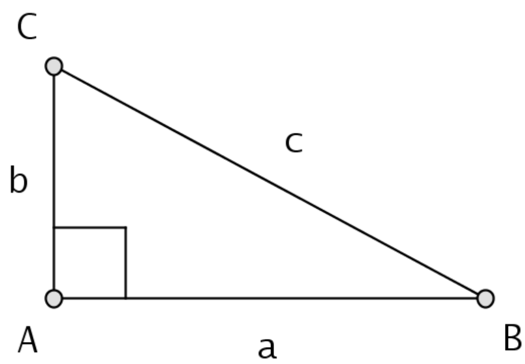

Figure 1: Pythagoras's Theorem: If ABC is a right angled triangle (so the angle at A is 90 degrees) then the distances a, b and c satisfy $a^2 + b^2 = c^2$.

Before I start with details, here is the formula I'm trying to prove. The length of the parking space must be at least the length of my car plus

$$\sqrt{(r^2 - \ell^2) + (\ell + k)^2 - \left(\sqrt{r^2 - \ell^2} - w\right)^2} - \ell - k, \tag{1}$$

where r is the radius of my car's kerb-to-kerb turning circle, ℓ is my car's wheel-base (the distance between the centres of the front wheel and the corresponding back wheel), k is the distance from the centre of the front wheel to the front of the car, and w is the width of one of the parked cars: the one near the front of my car once I've parked.

All of r, ℓ, k and w should be measured using the same units (metres, say). To put some typical numbers into the formula, a car might have kerb-to-kerb turning circle of diameter 10.8m (the radius is half the diameter, so $r = 5.4$m) and wheelbase $\ell = 2.6$m. The distance from the front wheels to the front of the car might be $k = 1.3$m and the parked car might have width $w = 1.7$m. In this case the length of the space needs to be at least the length of the car plus 1.43m.

The formula is justified by the diagram in Figure 2. The diagram shows the situation looking directly down from above, just as I've stopped reversing and I'm preparing to move forward into the middle of the space. In this diagram, the kerb is the line through AEFD, my car is the rectangle ABCD,

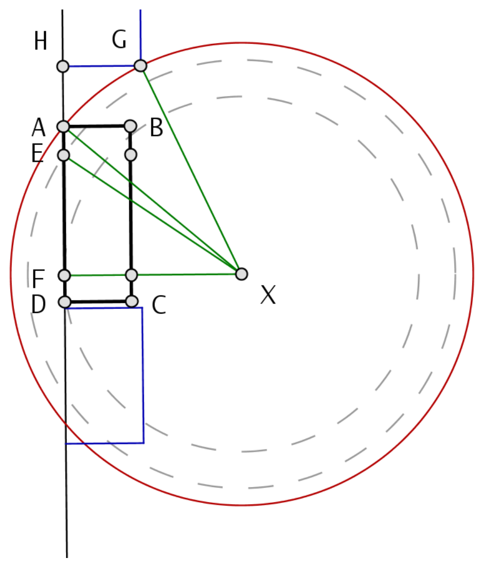

Figure 2: A perfect parallel parking manoeuvre

and the blue lines are the edges of the cars at either side of my parking space. As I reverse into the space, the corner A of my car travels along part of the red circle, just missing a car at G. My wheels at E and F have travelled along parts of the dashed circles. The red and two dashed circles all have centre X: the centre of the turning circle of my car. We know the following distances: $|\mathrm{EX}| = r$, $|\mathrm{EF}| = \ell$, $|\mathrm{AE}| = k$, $|\mathrm{GH}| = w$. The distance we are trying to find is $|\mathrm{AH}|$.

In the next section of this report, I'll make some comments on why I've reduced the perfect parallel parking problem to the geometry of Figure 2. In Section 3, I'll justify the formula (1). Finally, in Section 4, I'll make some more comments, and talk about other questions you might ask about the geometry of parking manoeuvres.

2 What does it mean to park perfectly?

Why have I modelled the problem in the way I have? Firstly, what does perfect parking mean? I've never enjoyed toing and froing in a narrow space, getting inches nearer the kerb each time, so I decided that there would be none of this in my definition of a perfect manoeuvre (but see Section 4). And

I'm not allowing handbrake turns or any other forms of skidding! Once this was decided, I wanted to define a problem that was simple enough to analyse, but still captured the essence of parallel parking. I was especially interested in thinking about exactly how a car moved when it was reversing.

I've made some simplifying assumptions: certainly the road is flat, the kerb is straight, and the cars parked each side of the space are parked neatly parallel to the kerb. I'm assuming that the road is wide (so there's no problem with swinging in at any angle). I'm assuming that the kerb is low (so the body of the car can swing over it). The cars (or at least the extreme points of them) will be rectangles. I'm assuming the wheels on my car (at E and F in Figure 2) line up closely with the extreme points A and D on the side of the car: from looking at cars in my local car park, this seems reasonable.

My car has its rear wheels fixed in orientation parallel to the side of the car. (Some cars have four wheel steering, where this is not true.) I'm also assuming that when the car turns a corner, the front wheels turn in an 'Ackermann' fashion: if we draw two imaginary lines through the hub caps of both front wheels, and an imaginary line through the centres of both rear wheels, all lines meet in a single point, which is the point about which the car turns. [When a car turns in a circle, the inside front wheel travels on a tighter circle than the outer wheel. So, if you want to avoid excessive friction, you need to turn this inside wheel more than the outer wheel. This was already known in the age of horses and carriages: the Ackermann linkage is an arrangment of links[1] that was patented in 1818 to accomplish this mechanically [1]. Modern cars do not quite do this: they make a compromise between reducing friction in low speed manoeuvres and steering methods that improve cornering at high speeds.] In particular, I'll make use of the fact that the line through the two rear wheels goes through the centre X of the turning circle. I'm assuming the width of the car is less than the radius of its turning circle. This is certainly true of all normal cars (the turning circle of a car will have a radius of about 5.5 metres, much more than a car's width). I do this so that all parts of the car are turning in the same direction (clockwise in my reversing manoeuvre in Figure 2). I'm ignoring the fact that a car's wheel on full lock might run into the kerb before

[1]There is a wonderful mathematical theory of linkages that is very accessible: I'd recommend reading the recent popular textbook *How round is your circle?* [4] if you are interested in hearing more. This book is a great read. Linkages (configurations of rigid rods and pivots) appear in two obvious places in a modern car: windscreen wipers, and the door hinges.

the car is parallel with it. Maybe a perfect parking manoeuvre should not quite touch the kerb anyway (I never like to hear the scrape of alloy against concrete when I park), so we could redefine the line AEFD in Figure 2 as being a line a few centimetres to the right of the true kerb. If we do this, it would still make sense to assume that the left hand sides of the parked cars on either side of my space go along this line, as they have also been perfectly parked!

Finally, I chose to express the problem in terms of the distances r, ℓ, k and w because these values are all easy to obtain from the web, or at least easy to measure yourself.

3 Justifying the formula

Here are the detailed steps prove the formula:

- Use Pythagoras's Theorem on the triangle EFX, to show that $|\mathrm{FX}| = \sqrt{r^2 - \ell^2}$.
- Use Pythagoras's Theorem on the triangle AFX, to show that $|\mathrm{AX}| = \sqrt{(\ell + k)^2 + (r^2 - \ell^2)}$.
- Observe that $|\mathrm{GX}| = |\mathrm{AX}|$, as they are radii of the same circle.
- Add a new line down from G, parallel to the kerb, until it meets the line FX (see the light blue line in Figure 3). Let K be the point where the new line intersects FX.
- Observe, from the diagram in Figure 3, that $|\mathrm{KX}| = |\mathrm{FX}| - w$.
- Use Pythagoras's Theorem on the triangle GKX, to show that

$$|\mathrm{GK}| = \sqrt{(r^2 - \ell^2) + (\ell + k)^2 - \left(\sqrt{r^2 - \ell^2} - w\right)^2}.$$

- Show that $|\mathrm{AH}| = |\mathrm{GK}| - \ell - k$ (see Figure 3).

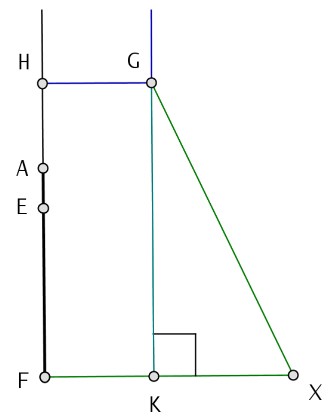

Figure 3: Drawing a new line

4 Some further comments

Some questions you might ask:

I parallel park by pointing my car at the kerb at some angle, reversing straight back, then putting on full lock at the last minute. What does your parking formula say about this method? You can use the geometry in Figure 2 to say what the best angle of approach to the kerb should be. You put your car on full lock at the point where the corner A of the car has just passed the point G, so you need to calculate the angle the car makes with the kerb at this time. It's the difference between two angles: the angle a tangent to the red circle at G makes when it intersects the kerb, minus the angle a tangent to the red circle at A makes with the kerb. You can calculate this using some straightforward trigonometry, though the resulting formula does not look particularly simple.

Why doesn't your formula involve the width of your car? I've always found wider cars more difficult to park, so maybe it is surprising that the formula doesn't depend on my car's width. There are several factors where width does make a difference when parallel parking: in narrow roads, the opposite side of the road makes it more difficult to swing into the space at the optimal angle; the width of my car *does* make a difference when you start toing and froing (see below); in a bigger car it might be more difficult to estimate the position of the edges of the car, so I might have to leave more room for error.

I don't mind shuttling to and fro when I park: I get satisfaction from parking in a very narrow space, no matter how many movements I use. Can this be analysed? It's not difficult to show that it's possible for you (in my model) to park in any space that is longer than your car (no matter how small the extra length you have to play with), provided you are patient enough to use lots of parking movements. Maybe you can be satisfied that your parking is perfect if you've used the minimum number of movements to get into your space. You could analyse your initial sweep into your space just as I have done in my formula, but now the kerb is further to the left, you are *given* the extra length c that you have to play with (so $|\mathrm{AH}| = c$), and w is redefined to be the distance w that you move inwards towards the kerb from the edge of the parked cars as you swing in: you want to calculate the maximum value w that is possible. Once you're looked at this, and your car is partially in your space, you need to find out how far you can move towards the kerb using each extra shuttling movement. It's best to think in terms of the movement of the mid-point between your two rear wheels, which can be modelled as moving along two opposite arcs of circles, each of radius $\sqrt{r^2 - \ell^2} - \frac{1}{2}n$, where n is the distance between the rear wheels of your car. Figure 4 (where I've used different notation from Figure 2) shows the geometrical situation on a forward movement: the mid-point of your two rear wheels moves along arcs AB and Ba of circles with centres X and x respectively. The distance between the lines AX and ax is the distance c that you have to play with; the distance you can travel inwards towards the kerb in each forward or backward movement is $2\,|\mathrm{PA}|$, where P is the foot of a perpendicular dropped from B to the line AX. You have to take into account an annoying factor: in a forward movement, the front of your car is sometimes further to the left of its position at the end of the forward motion. If you don't want to run over the pavement, this restricts you towards the end of your manoeuvre (indeed, you can't end up exactly on the kerb after a curved forward movement: you have to get to the kerb by reversing).

What about parking nose first in a multi-storey carpark, for example? The geometry of this kind of parking is very different to parallel parking. Suppose I'm turning my car ABCD into my space EFGH, as in Figure 5. I might start off by turning my car at full lock towards the space, but during the most delicate part of the manoeuvre, I steer so that the extreme point B of the car travels in a line (the dashed line on the diagram) as close as possible to EF. (Once I'm in the space, I will steer the car hard to the left

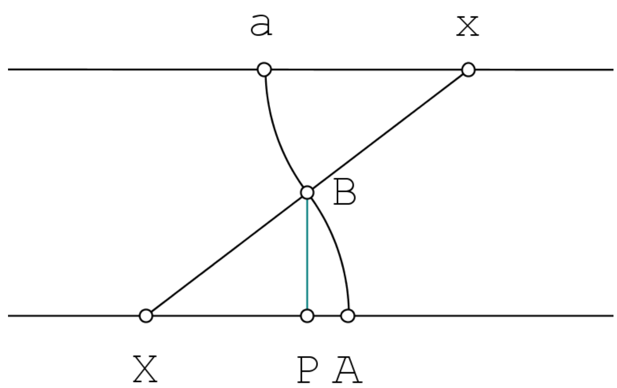

Figure 4: Shuffling towards the kerb

to make it parallel to the parked cars.) When I'm steering B along this line, the rear point C of the car travels along part of an interesting curve called a *tractrix*[2]; see Figure 6. You can define the tractrix as follows. Draw lots of circles of the same radius, whose centres all lie on one horizontal line. Then the tractrix is the curve that will cross all these circles at right angles. The black curves in Figure 6 are parts of these circles; in our situation their common radius is equal to the length of my car. I'd like to make sure that the left side AD of my car travels below G in Figure 5. It is not difficult to write down an equation for the path of a point on this side of my car in terms of the tractrix and the width of my car; a typical path for the point D, for example, is also shown in Figure 6. The mathematics of this situation is more complicated than the parallel parking situation I've looked at above, but it's very interesting (especially to the designers of multi-storey carparks and traffic junctions).

I hope you've enjoyed reading about geometry applied to parking problems. If you have, see [4] for more geometry coming from engineering and see [6] for more about curves. At a higher level, see [2, 3, 5] for a discussion of another traffic problem, or see [7] for much more on ground vehicle mobility.

[2]If you are interested in finding out about this curve, and many others, I'd strongly recommend reading the classic *A Book of Curves*[6]

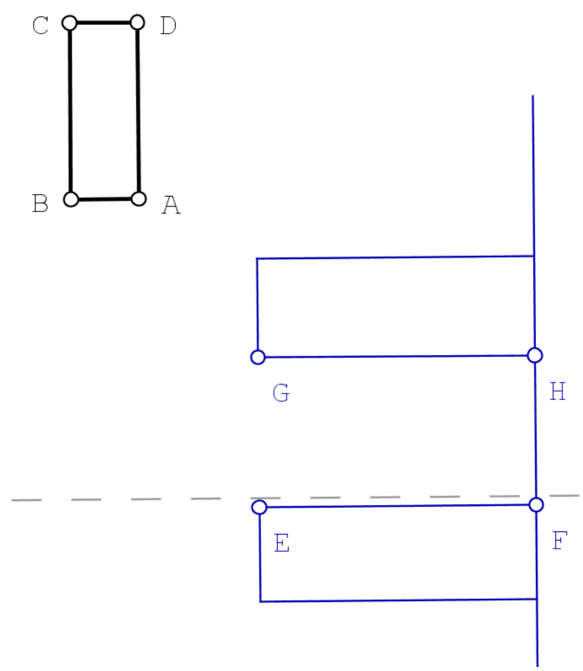

Figure 5: Parking nose first

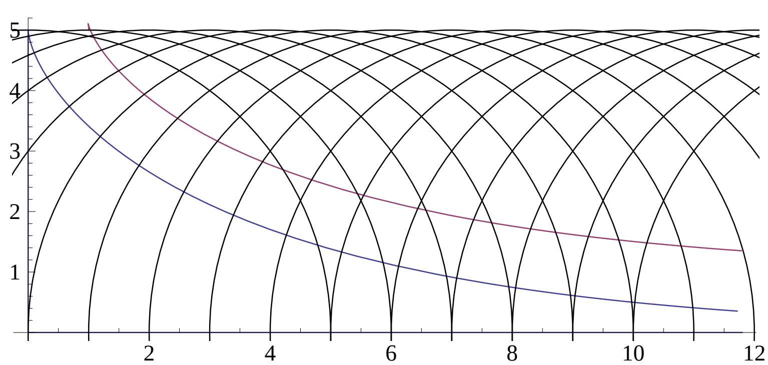

Figure 6: The tractrix (the lower curve, in blue) and the path (in purple) of the rear corner of the car

References

[1] *Ackermann Steering Geometry*, `http://en.wikipedia.org/wiki/Ackermann_steering_geometry`

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[6] E.H. Lockwood, *A Book of Curves*, Cambridge University Press, 1961.

[7] J.Y. Wong, *The Theory of Ground Vehicles (4th Edition)*, John Wiley & Sons, 2008.