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$\mathcal{X}$-Armed Bandits: Optimizing Quantiles, CVaR and Other Risks
Supplementary Material

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1. Details about the regularity hypothesis

In the classical setting the Optimized Certainty Equivalent is defined as

$$S_u(Y) = \sup_z \Big\{ z + \mathbb{E}\big(u(Y - z)\big) \Big\},$$

with u a concave function. Here we assume u is concave and k-lipschitzian ($k - \mathrm{Lip}$). Let us consider two random variables Y_{x_1} and Y_{x_2}, then

$$\begin{aligned}|S_u(Y_{x_1}) - S_u(Y_{x_2})| &= \Big| \sup_z \Big\{ z + \mathbb{E}\big(u(Y_{x_1} - z)\big) \Big\} - \sup_z \Big\{ z + \mathbb{E}\big(u(Y_{x_2} - z)\big) \Big\} \Big| \\ &\leq \sup_z \Big\{ \big| \mathbb{E}\big(u(Y_{x_1} - z)\big) - \mathbb{E}\big(u(Y_{x_2} - z)\big) \big| \Big\}.\end{aligned}$$

Using the *Kantorovich-Rubinstein* representation one obtains

$$\begin{aligned}\sup_z \Big\{ \big| \mathbb{E}\big(u(Y_{x_1} - z)\big) - \mathbb{E}\big(u(Y_{x_2} - z)\big) \big| \Big\} &\leq k \times \mathcal{W}_1(Y_{x_1} - z, Y_{x_2} - z) \\ &= k \times \mathcal{W}_1(Y_{x_1}, Y_{x_2})\end{aligned}$$

with $\mathcal{W}_1$ the Wasserstein distance associated with $p = 1$. Thus if $g = S_u$, then a sufficient condition to satisfied (1) is $\mathcal{W}_1(Y_{x*}, Y_x) \leq \frac{\beta}{k} \|x^* - x\|^\gamma$, for all $x \in \mathcal{X}$.

To treat the case of the CVaR_τ, we use the fact that if $u(z) = \dfrac{\min(z, 0)}{1 - \tau}$ then we have the equality $S_u = -\,\mathrm{CVaR}_\tau$.

In the case of the conditional expectation the same kind of condition can be sufficient. Indeed we have

$$|\mathbb{E}\big(Y_{x_1}\big) - \mathbb{E}\big(Y_{x_2}\big)| \leq \sup_{\|f\| \in 1-\mathrm{Lip}} \Big\{ \big| \mathbb{E}\big(f(Y_{x_1})\big) - \mathbb{E}\big(f(Y_{x_2})\big) \big| \Big\} = \mathcal{W}_1(Y_{x_1}, Y_{x_2}).$$

2. Proofs related to the generic analysis of StoROO

Proof of Proposition 2

Let us define $\mathcal{P}_{h^*,j^*}$ the partition containing x^*. Assume that the partition $\mathcal{P}_{h,j}$ has been selected, thus

$$\bar{U}_\eta^{h,j}(t) \geq \bar{U}_\eta^{h^*,j^*}(t).$$

By definition $\bar{U}_\eta^{h^*,j^*}(t) \geq g^*$, thus $\bar{U}_\eta^{h,j}(t) \geq g^*$. Conditionally on $\mathcal{A}_\eta$, $L_\eta^{h,j}(t)) \leq g(x_{h,j}(t))$ that implies

$$g^* - g(x_{h,j}) \leq \bar{U}_\eta^{h,j}(t) - L_\eta^{h,j}(t) \leq U_\eta^{h,j}(t) + \hat{\beta}\,\delta(h)^{\hat{\gamma}} - L_\eta^{h,j}(t) \leq 2\,\hat{\beta}\,\delta(h)^{\hat{\gamma}}.$$

Note that the last inequality is obtained because the partition is expanded, which implies that

$$U(x_{h,j})(t) - L(x_{h,j})(t) \leq \hat{\beta}\,\delta(h)^{\hat{\gamma}}.$$

Finally:

$$g^* \leq g(x_{h,j}) + 2\,\hat{\beta}\,\delta(h)^{\hat{\gamma}},$$

thus $x_{h,j}$ belongs to J_h. ■

Proof of Proposition 3

$$\begin{aligned} T = \sum_{h,j \in \mathcal{T}_T} N_{h,j}(t) &\leq \sum_{h,j \in \mathcal{T}_T} n_{\eta,h} \quad \text{because } N_{h,j}(t) \leq n_{\eta,h} \\ &\leq \sum_{h'=0}^{\text{depth}(\mathcal{T}_T)-1} K|\mathcal{T}_T \cap J_h| n_{\eta,h'+1} \quad \text{StoROO has not expanded all the sampled nodes} \\ &\leq \sum_{h'=0}^{\text{depth}(\mathcal{T}_T)-1} K|J_h| n_{\eta,h'+1} = S_{\text{depth}(\mathcal{T}_T)-1}. \end{aligned}$$

Thus $S_{H_\eta} \leq S_{\text{depth}(\mathcal{T}_T)-1} \leq S_{\text{depth}(\mathcal{T}_T)}$ so $H_\eta \leq \text{depth}(\mathcal{T}_T)$. There is at least an expanded node of depth $H_\eta^* \geq H_\eta$ after a budget T was used. ■

Proof of Proposition 4

Proposition 2 implies that the center of an expanded partition is in J_h. Proposition 3 implies that a partition of depth at least H_η^* has been expanded. Thus StoROO has expanded a node in $J_{H_\eta^*}$. At the end of the budget StoROO returns the node having the highest LCB among the nodes that have been expanded and not the deepest node among those that have been expanded. But

$$g^* - g(x_{h,j}) \leq \bar{U}_{H_\eta^*(T),j'} - L_{h,j} \leq \bar{U}_{H_\eta^*(T),j'} - L_{H_\eta^*(T),j'} \leq 2\,\hat{\beta}\,\delta(H_\eta^*(T))^{\hat{\gamma}}.$$

That ensure the node having the highest LCB has the same theoretical regret as the node of maximal depth among those that have been expanded. ■

Proof of Proposition 6

According to the assumption 2, each cell $\mathcal{P}_{h,j}$ contains a ball of radius $\nu\delta(h)$ centered in $x_{h,j}$ that is a $\ell_{\hat{\beta},\hat{\gamma}}$-ball of radius $\hat{\beta}(\nu\delta(h))^{\hat{\gamma}}$ centered in $x_{h,j}$. If d is the $\nu^{\hat{\gamma}}/2$ near optimality dimension then there is at most $C[2\,\hat{\beta}\,\delta(h)^{\hat{\gamma}}]^{-d}$ disjoint $\ell_{\hat{\beta},\hat{\gamma}}$- balls of radius $\hat{\beta}(\nu\delta(h))^{\hat{\gamma}}$ inside $\mathcal{X}_{2\,\hat{\beta}\,\delta(h)^{\hat{\gamma}}}$. Thus if $|J_h| = |x_{h,j} \in \mathcal{X}_{2\,\hat{\beta}\,\delta(h)^{\hat{\gamma}}}| > C[\hat{\beta}\,\delta(h)^{\hat{\gamma}}]^{-d}$ this implies there is more than $C[2\,\hat{\beta}\,\delta(h)^{\hat{\gamma}}]^{-d}$ disjoint $\ell_{\hat{\beta},\hat{\gamma}}$ balls of radius $\hat{\beta}(\nu\delta(h))^{\hat{\gamma}}$ with center in $\mathcal{X}_{2\,\hat{\beta}\,\delta(h)^{\hat{\gamma}}}$, that is a contradiction.

■

Proof of Therorem 7

$$
\begin{aligned}
T &\le \sum_{h=0}^{H^*} K|J_h| n_{\eta,h+1} && \text{by definition of } H^* \\
&\le \sum_{h=0}^{H^*} KC[2\,\hat{\beta}\,\delta(h)^{\hat{\gamma}}]^{-d} n_{\eta,h+1} && \text{using Proposition 6} \\
&= \sum_{h=0}^{H^*} KC[2\,\hat{\beta}(c\rho^h)^{\hat{\gamma}}]^{-d} n_{\eta,h+1} && \text{using the exponential decay of the diameter of the cells} \\
&\le \sum_{h=0}^{H^*} KC[2\,\hat{\beta}(c\rho^h)^{\hat{\gamma}}]^{-d} \times \kappa^{\alpha} \frac{\log(T^2/\eta)}{(\hat{\beta}(c\rho^h)^{\hat{\gamma}})^{\alpha}} && \text{using Definition 1} \\
&= \log(T^2/\eta) \frac{KC\kappa^{\alpha}[2\,\hat{\beta}\,c^{\hat{\gamma}}]^{-d}}{\hat{\beta}\,c^{\hat{\gamma}\,\alpha}} \sum_{h=0}^{H^*} \rho^{h(-d\,\hat{\gamma} - \hat{\gamma}\,\alpha)} \\
&= \log(T^2/\eta) \frac{KC\kappa^{\alpha}[2\,\hat{\beta}\,c^{\hat{\gamma}}]^{-d}}{\hat{\beta}\,c^{\hat{\gamma}\,\alpha}} \times \frac{\rho^{(H^*+1)(-d\,\hat{\gamma} - \hat{\gamma}\,\alpha)} - 1}{\rho^{-d\,\hat{\gamma} - \hat{\gamma}\,\alpha} - 1} && \text{rewriting the sum} \\
&\le \frac{\log(T^2/\eta)}{(1 - \rho^{d\,\hat{\gamma} + \hat{\gamma}\,\alpha})} \frac{KC\kappa^{\alpha}[2\,\hat{\beta}\,c^{\hat{\gamma}}]^{-d}}{\hat{\beta}\,c^{\hat{\gamma}\,\alpha}} \times \rho^{H^*(-d\,\hat{\gamma} - \hat{\gamma}\,\alpha)} \\
&= \frac{\log(T^2/\eta)}{(1 - \rho^{d\,\hat{\gamma} + \hat{\gamma}\,\alpha})} \frac{KC\kappa^{\alpha}[2\,\hat{\beta}]^{-d}}{\hat{\beta}} \times \delta(H^*)^{-d\,\hat{\gamma} - \hat{\gamma}\,\alpha}.
\end{aligned}
$$

Finally

$$
\left[\frac{KC\kappa^{\alpha}[2\,\hat{\beta}]^{-d}}{\hat{\beta}(1 - \rho^{d\,\hat{\gamma} + \hat{\gamma}\,\alpha})}\right]^{\frac{1}{d\,\hat{\gamma} + \hat{\gamma}\,\alpha}} \left[\frac{\log(T^2/\eta)}{T}\right]^{\frac{1}{d\,\hat{\gamma} + \hat{\gamma}\,\alpha}} \ge \delta(H^*).
$$

Using Proposition 4 we obtain

$$
r_T \le c_1 \Big[\frac{\log(T^2/\eta)}{T}\Big]^{\frac{1}{\alpha+d}}.
$$

■

3. Optimizing quantiles

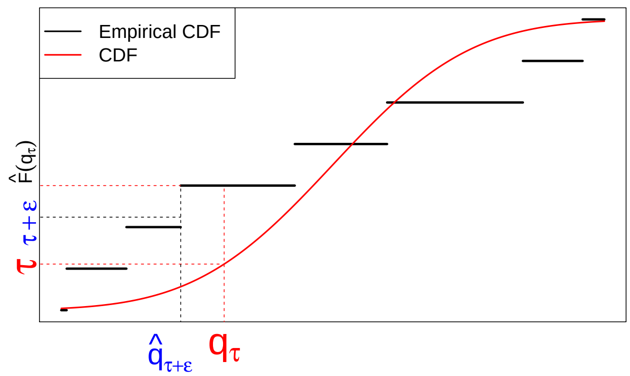

Figure 1: Illustration of the equivalence (4).

Proof of Proposition 8

Let us consider the event

$$\begin{aligned} \xi_\eta \;\; &= \;\; \{\forall\ h \geq 0, \forall\ 0 \leq j \leq K^h, \forall\ 1 \leq t \leq T, \\ &\hat{F}^t_{h,j}\Big(q_{h,j}(\tau)\Big) \geq \tau + \epsilon^\eta_{N_{h,j}(t)} \text{ or } \hat{F}^t_{h,j}\Big(q_{h,j}(\tau)\Big) < \tau - \epsilon^\eta_{N_{h,j}(t)}\}. \end{aligned}$$

$$\begin{aligned} \mathbb{P}\big(\xi_\eta\big) = \mathbb{P}\Bigg(&\forall h \geq 0, \forall 0 \leq j \leq K^h, \forall 1 \leq t \leq T,\ \hat{F}^t_{h,j}\Big(q_{h,j}(\tau)\Big) \geq \tau + \epsilon^\eta_{N_{h,j}(t)} \text{ or },\\ &\hat{F}^t_{h,j}\Big(q_{h,j}(\tau)\Big) < \tau - \epsilon^\eta_{N_{h,j}(t)}\Bigg)\\ \leq \;\; &\mathbb{P}\Bigg(\forall h \leq 0, \forall 0 \leq j \leq K^h, \forall 1 \leq t \leq T,\ \hat{F}^t_{h,j}\Big(q_{h,j}(\tau)\Big) \geq \tau + \epsilon^\eta_{N_{h,j}(t)})\Bigg)\\ + \;&\mathbb{P}\Bigg(\forall h \geq 0, \forall 0 \leq j \leq K^h, \forall 1 \leq t \leq T,\ \hat{F}^t_{h,j}\Big(q_{h,j}(\tau)\Big) < \tau - \epsilon^\eta_{N_{h,j}(t)}\Bigg) \end{aligned}$$

Define $m \leq T$ the number of nodes expanded throughout the algorithm, define for $1 \leq w \leq m$, ζ^s_w as the time when the cell w has been selected for the s-th time and define $Y_w(\zeta^s_w)$ the reward obtained at that time at the point x_w. Then one can write

$$\mathbb{P}\Bigg(\hat{F}^t_{h,j}\Big(q_{h,j}(\tau)\Big) \geq \tau + \epsilon^{\eta,T}_{N_{h,j}(t)}\Bigg) = \mathbb{P}\Bigg(\frac{1}{N_{h,j}(t)} \sum_{s=1}^{N_{h,j}(t)} \mathbb{1}_{Y_{h,j}(\zeta^s_{h,j}) \leq q_{h,j}(\tau)} \geq \tau + \epsilon^\eta_{N_{h,j}(t)}\Bigg).$$

Using this notation, we have:

$$\mathbb{P}\Bigg(\forall h \geq 0, \forall 0 \leq j \leq K^h, \forall 1 \leq t \leq T,\ \hat{F}^t_{h,j}\Big(q_{h,j}(\tau)\Big) \geq \tau + \epsilon^\eta_{N_{h,j}(t)}\Bigg)$$

$$\leq \ \mathbb{P}\Big(\exists\, 1 \leq w \leq T,\ \exists\, 1 \leq u \leq T,\ \frac{1}{u}\sum_{s=1}^{u} \mathbb{1}_{Y_w(\zeta_w^s) \leq q_w(\tau)} \geq \tau + \epsilon_u^\eta\Big)$$

$$\leq \ \sum_{w=1}^{T}\sum_{u=1}^{T}\mathbb{P}\Big(\frac{1}{u}\sum_{s=1}^{u} \mathbb{1}_{Y_w(\zeta_w^s) \leq q_w(\tau)} \geq \tau + \epsilon_u^\eta\Big)$$

By Hoeffding's inequality, if

$$\epsilon_u^\eta = \sqrt{\frac{\log(2T^2/\eta)}{2u}},$$

we obtain

$$\mathbb{P}\bigg(\forall h \leq 0, \forall 0 \leq j \leq K^h, \forall 1 \leq t \leq T,\ \hat{F}_{h,j}^t\Big(q_{h,j}(\tau)\Big) \geq \tau + \epsilon_{N_{h,j}(t)}^\eta\bigg) \leq \frac{\eta}{2}.$$

Now using Equation (4) we can express this inequality directly in terms of quantiles:

$$\mathbb{P}\bigg(\forall h \leq 0, \forall 0 \leq j \leq K^h, \forall 1 \leq t \leq T,\ q_{h,j}(\tau) \geq U_{h,j}^\eta(t)\bigg) \leq \frac{\eta}{2}.$$

Using the same scheme of proof with Inequality (5), we obtain:

$$\mathbb{P}\bigg(\forall h \geq 0, \forall 0 \leq j \leq K^h, \forall 1 \leq t \leq T,\ q_{h,j}(\tau) \leq L_{h,j}^\eta(t)\bigg) \leq \frac{\eta}{2},$$

and hence $\mathbb{P}\big(\mathcal{A}_\eta\big) = 1 - \mathbb{P}\big(\xi_\eta\big) \geq 1 - \eta$. ■

Proof of Proposition 9

Without loss of generality let us assume $\tau > 0.5$. Assume the node $x_{h,j}$ has been sampled $N_{h,j} \geq M_\tau = \max(n_\tau, n_{1-\tau})$ times, with

$$n_\tau > \frac{2\log(2T^2/\eta)}{\tau^2} \quad \text{and} \quad n_{1-\tau} > \frac{2\log(2T^2/\eta)}{(1-\tau)^2}$$

thus

$$\tau + 2\sqrt{\frac{\log(2T^2/\eta)}{2N_{h,j}}} < 1 \text{ and } \tau - 2\sqrt{\frac{\log(2T^2/\eta)}{2N_{h,j}}} > 0.$$

That implies

$$q_{h,j}\left(\tau + 2\sqrt{\frac{\log(2T^2/\eta)}{2N_{h,j}}}\right) < +\infty \text{ and } q_{h,j}\left(\tau - 2\sqrt{\frac{\log(2T^2/\eta)}{2N_{h,j}}}\right) > -\infty,$$

and in particular

$$U_{h,j}^\eta < +\infty \text{ and } L_{h,j}^\eta > -\infty.$$

Then define the event

$$\mathcal{C}_\eta = \bigcap_{T \geq t \geq 1}\ \bigcap_{\mathcal{P}_{h,j} \in \mathcal{T}_t} \Big\{q_{h,j}\big(\tau + 2\epsilon_{N_{h,j}(t)}^{\eta,T})\big) \geq U_{h,j}^\eta(t) \geq q_{h,j}(\tau) \geq L_{h,j}^\eta(t) \geq q_{h,j}\big(\tau - 2\epsilon_{N_{h,j}(t)}^{\eta,T}\big)\Big\},$$

with

$$\epsilon^{\eta,T}_{N_{h,j}(t)} = \sqrt{\frac{\log(2T^2/\eta)}{2N_{h,j}(t)}}.$$

Using equivalences (4) and (5), one can write:

$$q_{h,j}\big(\tau + 2\epsilon^{\eta,T}_{N_{h,j}(t)}\big) \geq U^{\eta}_{h,j}(t) \geq q_{h,j}(\tau) \geq L^{\eta}_{h,j}(t) \geq q_{h,j}\big(\tau - 2\epsilon^{\eta,T}_{N_{h,j}(t)}\big)$$
$$\Leftrightarrow \hat{F}(q_{h,j}(\tau + 2\epsilon^{\eta,T}_{N_{h,j}(t)})) \geq \tau + \epsilon^{\eta,T}_{N_{h,j}(t)} > \hat{F}(q_{h,j}(\tau) \geq \tau - \epsilon^{\eta,T}_{N_{h,j}(t)} > \hat{F}(q_{h,j}(\tau + 2\epsilon^{\eta,T}_{N_{h,j}(t)})).$$

Thus

$$\begin{aligned}\mathbb{P}(\mathcal{C}_\eta) \geq &\quad 1 - \mathbb{P}(\forall h \geq 0, \forall 0 \leq j \leq K^h, \forall 1 \leq t \leq T, \sup_{y = q_\tau, q_{\tau + \epsilon^{\eta,T}_{N_{h,j}(t)}}} |F_{h,j}(y) - \hat{F}^t_{h,j}(y)| \geq \epsilon^{\eta,T}_{N_{h,j}(t)} \;) \\ \geq &\quad 1 - \mathbb{P}(\forall h \geq 0, \forall 0 \leq j \leq K^h, \forall 1 \leq t \leq T, \sup_{y\in[0,1]} |F_{h,j}(y) - \hat{F}^t_{h,j}(y)| \geq \epsilon^{\eta,T}_{N_{h,j}(t)} \;).\end{aligned}$$

Using the same notation as in the proof of Proposition 8, one can write

$$\geq \quad 1 - \sum_{w=1}^{T}\sum_{u=1}^{T} \mathbb{P}(\sup_{y\in[0,1]} |F_w(y) - \frac{1}{u}\sum_{s=1}^{u} \mathbb{1}_{Y_w(\zeta^s_w) \leq q_w(\tau)}| \geq \epsilon^{\eta,T}_u).$$

Now by applying the Massart's inequality to bound

$$\mathbb{P}(\sup_{y\in[0,1]} |F_w(y) - \sum_{s=1}^{u} \mathbb{1}_{Y_w(\zeta^s_w) \leq q_w(\tau)}| \geq \epsilon^{\eta,T}_u),$$

one obtain $\mathbb{P}(\mathcal{C}_\eta) \geq \;\; 1 - \eta$. Thus with probability $1 - \eta$, we have:

$$U^{\eta}_{h,j}(t) - L^{\eta}_{h,j}(t) \leq q_{h,j}\left(\tau + 2\epsilon^{\eta,T}_{N_{h,j}(t)}\right) - q_{h,j}\left(\tau - 2\epsilon^{\eta,T}_{N_{h,j}(t)}\right). \tag{1}$$

Assuming that $q_{h,j}$ is differentiable in τ, by the mean value theorem, we deduce

$$q_{h,j}(\tau + 2\sqrt{\frac{\log(2T^2/\eta)}{2N_{h,j}}}) - q_{h,j}(\tau - 2\sqrt{\frac{\log(2T^2/\eta)}{2N_{h,j}}}) \leq 4\sqrt{\frac{\log(2T^2/\eta)}{2N_{h,j}}} \max_{\tau' \in [\tau - 2\epsilon^{\eta,T}_{n_\tau}, \tau + 2\epsilon^{\eta,T}_{n_{1-\tau}}]} \frac{1}{f_{x_{h,j}} \circ F^{-1}_{x_{h,j}}(\tau')}.$$

Next, using (1) it is possible to write that with probability $1 - \eta$:

$$U^{\eta}_{h,j} - L^{\eta}_{h,j} \leq 4\sqrt{\frac{\log(2T^2/\eta)}{2N_{h,j}}} \frac{1}{\bar{f}_{x_{h,j}}} \leq 4\sqrt{\frac{\log(2T^2/\eta)}{2N_{h,j}}} \frac{1}{\min_{x\in\mathcal{X}} \bar{f}(x)}.$$

We define $n'_{\eta,h}$ as the smallest n such that

$$4\sqrt{\frac{\log(2T^2/\eta)}{2n}} \frac{1}{\inf_{x\in\mathcal{X}} \bar{f}(x)} \leq \hat{\beta}\, \delta(h)^{\hat{\gamma}},$$

that is

$$n'_{\eta,h} = \log(2T^2/\eta)\left(\frac{2\sqrt{2}}{\hat{\beta}\,\delta(h)^{\hat{\gamma}}\min_{x\in\mathcal{X}}\bar{f}(x)}\right)^2.$$

A proper $n_{\eta,h}$ has to verify

$$n_{\eta,h} \geq M_\tau \text{ and } n_{\eta,h} \geq \log(2T^2/\eta)\left(\frac{2\sqrt{2}}{\hat{\beta}\,\delta(h)^{\hat{\gamma}}\min_{x\in\mathcal{X}}\bar{f}(x)}\right)^2.$$

To satisfy this constraint we define

$$\begin{aligned} n_{\eta,h} =&\ \log(2T^2/\eta)\left(\frac{\sqrt{8\min(1-\tau,\tau)^2 + 4\big(\hat{\beta}\,\mathrm{diam}(\mathcal{X})^{\hat{\gamma}}\min_{x\in\mathcal{X}}\bar{f}(x)\big)^2}}{\hat{\beta}\,\delta(h)^{\hat{\gamma}}\min_{x\in\mathcal{X}}\bar{f}(x)\min(1-\tau,\tau)}\right)^2 \\ \geq&\ \log(2T^2/\eta)\left(\left(\frac{2\sqrt{2}}{\hat{\beta}\,\delta(h)^{\hat{\gamma}}\min_{x\in\mathcal{X}}\bar{f}(x)}\right)^2 + \left(\frac{2}{\min(1-\tau,\tau)}\right)^2\right) \\ =&\ n'_{\eta,h} + M_\tau. \end{aligned}$$

To conclude the whole proof, since $\mathcal{C}_\eta \subset \mathcal{A}_\eta \cap \mathcal{B}_\eta$, we obtain $\mathbb{P}(\mathcal{A}_\eta \cap \mathcal{B}_\eta) \geq 1-\eta$. ■

Proposition 1 *For any $\eta > 0$, for all $1 \leq t \leq T$, $1 \leq h \leq t$ and $1 \leq j \leq K^h$, define*

$$U^\eta_{h,j}(t) = \begin{cases} \min\left\{q,\ \hat{F}^t_{h,j}(q) \geq \tau + \epsilon^{\eta,T}_{N_{h,j}(t)}\right\} & \text{if } \tau + \epsilon^{\eta,T}_{N_{h,j}(t)} < 1 \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$L^\eta_{h,j}(t) = \begin{cases} \max\left\{q,\ \hat{F}^t_{h,j}(q) \geq \tau - \epsilon^{\eta,T}_{N_{h,j}(t)}\right\} & \text{if } \tau - \epsilon^{\eta,T}_{N_{h,j}(t)} > 0 \\ -\infty & \text{otherwise,} \end{cases}$$

with

$$\epsilon^{\eta,T}_{N_{h,j}(t)} = \frac{\log(2T^2/\eta)}{3N_{h,j}(t)}\left(1 + \sqrt{1 + \frac{18N_{h,j}(t)\tau(1-\tau)}{\log(2T^2/\eta)}}\right).$$

If g is the conditional quantile of order τ then the event $\mathcal{A}_\eta$ has probability at least $1-\eta$.

Proof

Let $Y_1, \cdots, Y_n$ be n *i.i.d.* random variables bounded by the interval $[0,1]$. Define $\hat{F}^n(q(\tau)) = \frac{1}{n}\sum_{i=1}^n \mathbb{1}_{Y_i \leq q(\tau)}$. For $x > \tau$ the Bernstein's inequality gives

$$\mathbb{P}(|\hat{F}^n(q(\tau)) - \tau| > \epsilon) \leq 2\exp\left(\frac{n\epsilon^2}{2\tau(1-\tau) + 2\epsilon/3}\right).$$

Let us consider the event

$$\begin{aligned} \xi_\eta \ =&\ \ \{\forall\ h \geq 0, \forall\ 0 \leq j \leq K^h, \forall\ 1 \leq t \leq T, \\ &\ \ \hat{F}^t_{h,j}\Big(q_{h,j}(\tau)\Big) \geq \tau + \epsilon^{\eta,T}_{N_{h,j}(t)} \text{ or } \hat{F}^t_{h,j}\Big(q_{h,j}(\tau)\Big) < \tau - \epsilon^{\eta,T}_{N_{h,j}(t)}\}. \end{aligned}$$

Using the same lines as in the proof of Proposition 8 we have

$$\mathbb{P}(\xi_\eta) \leq \sum_{w=1}^{T}\sum_{u=1}^{T}\mathbb{P}\Big(|\frac{1}{u}\sum_{s=1}^{u}\mathbb{1}_{Y_w(\zeta_w^s)\leq q_w(\tau)} - \tau| > \epsilon_u^{\eta,T}\Big)$$

then applying the Bernstein's inequality we obtain

$$\leq \sum_{w=1}^{T}\sum_{u=1}^{T} 2\exp\Bigg(-\frac{u{\epsilon_{N_{h,j}(t)}^{\eta,T}}^2}{2\tau(1-\tau)+2\epsilon_{N_{h,j}(t)}^{\eta,T}/3}\Bigg). \tag{2}$$

By now the goal is to find $\epsilon_{N_{h,j}(t)}^{\eta,T} > 0$ such that

$$\frac{u{\epsilon_{N_{h,j}(t)}^{\eta,T}}^2}{2\tau(1-\tau)+2\epsilon_{N_{h,j}(t)}^{\eta,T}/3} = \log(2T^2/\eta).$$

Finding such $\epsilon_{N_{h,j}(t)}^{\eta,T}$ can be easily done because it is a square of a second order polynomial. The result is

$$\epsilon_{N_{h,j}(t)}^{\eta,T} = \frac{\log(2T^2/\eta)}{3u}\Bigg(1+\sqrt{1+\frac{18u\tau(1-\tau)}{\log(2T^2/\eta)}}\Bigg).$$

Plugging the value of $\epsilon_{N_{h,j}(t)}^{\eta,T}$ inside (2) concludes the proof. ■

Proof of Proposition 11

Step 1: bounds on $\hat{F}^n(q(\tau))$ for a i.i.d sample

Let $Y_1, \cdots, Y_n$ be n *i.i.d.* random variables bounded by the interval $[0,1]$. Define $\hat{F}^n(q) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{Y_i\leq q}$. For $x > \tau$ the Chernoff's inequality gives

$$\mathbb{P}(\hat{F}^n(q(\tau)) \geq x) \leq \exp(-n\,\mathrm{kl}(x,\tau)).$$

Let $\tau^+ > \tau$ be the value such that $\mathrm{kl}(\tau^+,\tau) = \frac{\log(2/\eta)}{n}$, then for all $x \geq \tau^+$:

$$\mathbb{P}(\hat{F}^n(q(\tau)) \geq x) \leq \mathbb{P}(\hat{F}^n(q(\tau)) \geq \tau^+) \leq \exp(n\frac{\log(2/\eta)}{n}) = \frac{\eta}{2}.$$

Now let us define the candidate for the UCB of a i.i.d sample:

$$U(n) = \min\big\{q,\ \ \hat{F}^n(q) \geq \tau \text{ and } n\,\mathrm{kl}(\hat{F}^n(q),\tau) \geq \log(2/\eta)\big\},$$

and let us remark that

$$\hat{F}^n(U(n)) \leq \hat{F}^n(q(\tau)) \Leftrightarrow \tau \leq \hat{F}^n(q(\tau)) \ \text{ and }\ \mathrm{kl}(\hat{F}^n(q(\tau)),\tau) \geq \frac{\log(2/\eta)}{n}, \tag{3}$$

thus

$$\mathbb{P}(\hat{F}^n(U(n)) \leq \hat{F}^n(q(\tau))) = \mathbb{P}(\tau \leq \hat{F}^n(q(\tau)) \ \text{ and }\ \mathrm{kl}(\hat{F}^n(q(\tau)),\tau) \geq \frac{\log(2/\eta)}{n})$$

$$\leq \mathbb{P}(\hat{F}^n(q(\tau)) \geq \tau^+) \leq \frac{\eta}{2}.$$

For $x < \tau$ let us introduce

$$L(n) = \max\left\{q, \quad \hat{F}^n(q) \leq \tau \text{ and } n\,\mathrm{kl}(\hat{F}^n(q), \tau) \geq \log(2/\eta)\right\},$$

one proves in the same way

$$\mathbb{P}(\hat{F}^n(L(n)) > \hat{F}^n(q(\tau))) \leq \frac{\eta}{2}.$$

Step 2: Double union bound

Let us consider the event

$$\begin{aligned}\xi_\eta \quad = \quad & \Big\{\forall\ h \geq 0, \forall\ 0 \leq j \leq K^h, \forall\ 1 \leq t \leq T, \\ & \hat{F}^t_{h,j}\big(q_{h,j}(\tau)\big) \geq \hat{F}^t_{h,j}(U^\eta_{h,j}) \text{ or } \hat{F}^t_{h,j}\big(q_{h,j}(\tau)\big) < \hat{F}^t_{h,j}(L^\eta_{h,j})\Big\}.\end{aligned}$$

$$\begin{aligned}\mathbb{P}\big(\xi_\eta\big) \leq \quad & \mathbb{P}\Big(\forall h \leq 0, \forall 0 \leq j \leq K^h, \forall 1 \leq t \leq T,\ \hat{F}^t_{h,j}\big(q_{h,j}(\tau)\big) \geq \hat{F}^t_{h,j}(U^\eta_{h,j})\Big) \\ & + \mathbb{P}\Big(\forall h \geq 0, \forall 0 \leq j \leq K^h, \forall 1 \leq t \leq T,\ \hat{F}^t_{h,j}\big(q_{h,j}(\tau)\big) < \hat{F}^t_{h,j}(L^\eta_{h,j})\Big)\end{aligned}$$

Following the notation of the proof of Proposition 8 we have

$$\begin{aligned}& \mathbb{P}\Big(\forall h \geq 0, \forall 0 \leq j \leq K^h, \forall 1 \leq t \leq T,\ \hat{F}^t_{h,j}\big(q_{h,j}(\tau)\big) \geq \hat{F}^t_{h,j}(U^\eta_{h,j})\Big) \\ \leq \ & \mathbb{P}\Big(\exists\ 1 \leq w \leq T,\ \exists\ 1 \leq u \leq T,\ \sum_{s=1}^{u} \mathbb{1}_{Y_w(\zeta_w^s) \leq q_w(\tau)} \geq \sum_{s=1}^{u} \mathbb{1}_{Y_w(\zeta_w^s) \leq U_w^\eta}\Big) \\ \leq \ & \sum_{w=1}^{T}\sum_{u=1}^{T} \mathbb{P}\Big(\sum_{s=1}^{u} \mathbb{1}_{Y_w(\zeta_w^s) \leq q_w(\tau)} \geq \sum_{s=1}^{u} \mathbb{1}_{Y_w(\zeta_w^s) \leq U_w^\eta}\Big).\end{aligned}$$

Using the equivalence (3), the probability can be reformulated as

$$= \sum_{w=1}^{T}\sum_{u=1}^{T} \mathbb{P}\Big(\tau \leq \hat{F}^u(q(\tau)) \quad \text{and} \quad \mathrm{kl}(\hat{F}^u(q(\tau)), \tau) \geq \frac{\log(2T^2/\eta)}{u}\Big).$$

Now using Chernoff's inequality we obtain

$$\begin{aligned}& \mathbb{P}\Big(\forall h \geq 0, \forall 0 \leq j \leq K^h, \forall 1 \leq t \leq T,\ \hat{F}^t_{h,j}\big(q_{h,j}(\tau)\big) \geq \hat{F}^t_{h,j}(U^\eta_{h,j})\Big) \\ \leq \ & \sum_{w=1}^{T}\sum_{u=1}^{T} \exp(-u\frac{\log(2T^2/\eta)}{u}) = \eta/2.\end{aligned}$$

By equivalence (4) this implies that, $\forall h \geq 0, \forall 0 \leq j \leq K^h, \forall 1 \leq t \leq T$, with probability at least $\eta/2$, $U^{\eta}_{h,j}(t) \leq q_{h,j}(\tau)$. Using the same lines one can show

$$\mathbb{P}\Big(\forall h \geq 0, \forall 0 \leq j \leq K^h, \forall 1 \leq t \leq T,\ \hat{F}^t_{h,j}\Big(q_{h,j}(\tau)\Big) < \hat{F}^t_{h,j}(L)\Big) \leq \eta/2,$$

By equivalence (5) this implies that, $\forall h \geq 0, \forall 0 \leq j \leq K^h, \forall 1 \leq t \leq T$, $L^{\eta}_{h,j}(t) > q_{h,j}(\tau)$ with probability at least $\eta/2$. Putting this two probabilities together prove the result. ■

Proof of Proposition 12

Define

$$\tilde{S}^{\tau}_{h,j}(n) = \sum_{i=1}^{n} \mathbb{1}_{Y_{h,j}(i) \leq q_{h,j}(\tau)}.$$

Step 1: Martingale For every $\lambda \in \mathbb{R}$, let $\phi_\tau(\lambda) = \log \mathbb{E}[\exp(\lambda \mathbb{1}_{Y_{h,j}(1) \leq q_{h,j}(\tau))}]$. Let $W^{\lambda}_0 = 1$ and for $n \geq 1$,

$$W^{\lambda}_n = \exp(\lambda \tilde{S}^{\tau}_{h,j}(n) - n\phi_\tau(\lambda)).$$

$(W^{\lambda}_n)_{n \geq 0}$ is a martingale relative to $(\mathcal{F}_n)_{n \geq 0}$. In fact,

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(\lambda\{\tilde{S}^{\tau}_{h,j}(n+1) - \tilde{S}^{\tau}_{h,j}(n)\}\Big)|\mathcal{F}_n\Big] =& \mathbb{E}\Big[\exp(\lambda X_{n+1})|\mathcal{F}_n\Big] \\ =& \exp\Big(\log \mathbb{E}[\exp(\lambda X_1]\Big) \\ =& \exp\Big(\{(n+1) - n\}\phi_\mu(\lambda)\Big)\end{aligned}$$

That is equivalent to

$$\mathbb{E}\Big[\exp\Big(\lambda\{\tilde{S}^{\tau}_{h,j}(n+1) - \tilde{S}^{\tau}_{h,j}(n)\}\Big)|\mathcal{F}_n\Big] = \exp\Big(\lambda S_n - n\phi_\mu(\lambda)\Big).$$

Step 2: Peeling Let us devide the interval $\{1, \cdots, T\}$ into *slices* $\{t_{k-1}+1, \cdots, t_k\}$ of geometric increasing size. We may assume that $\delta > 1$, since otherwise the bound is trivial. Take $\xi = 1/(1 - \delta_\eta(T))$, let $t_0 = 0$ and for all $k \in \mathbb{N}^*$, let $t_k = \lfloor (1+\xi)^k \rfloor$.

$$\begin{aligned}&\mathbb{P}\Big(\forall h \geq 0, \forall 0 \leq j \leq K^h, \forall\ 1 \leq t \leq T,\ U^{\eta}_{h,j}(t) \leq q_{h,j}(\tau)\Big) \\ \leq\ &\mathbb{P}\Big(\exists\ h \geq 0, \exists\ 0 \leq j \leq K^h,\ \exists\ 1 \leq t \leq T,\ U^{\eta}_{h,j}(t) \leq q_{h,j}(\tau)\Big).\end{aligned}$$

Define $m \leq T$ the number of nodes expanded throughout the algorithm, thus for $1 \leq w \leq m$, it is possible to rewrite the last probability as

$$\begin{aligned}&\mathbb{P}\Big(\exists\ 1 \leq w \leq T,\ \exists\ 1 \leq n \leq T,\ U^{\eta}_w(n) \leq q_w(\tau)\Big) \\ \leq\ &\sum_{w=1}^{T} \mathbb{P}\Big(\ \exists\ 1 \leq k \leq D,\ \exists\ t_{k-1} < n \leq t_k \ \text{ and }\ U^{\eta}_w(n) \leq q_w(\tau)\Big) \quad \text{with } D = \frac{\log(T)}{\log(1+\eta)}\end{aligned}$$

$$\leq \sum_{w=1}^{T}\sum_{k=1}^{D}\mathbb{P}\Big(A_k\Big),$$

with

$$A_k = \Big\{\ \exists\ t_{k-1} < n \leq t_k \ \text{ and } \ U_w^{\eta}(n) \leq q_w(\tau)\Big\}.$$

Observe that $U_w^{\eta}(n) \leq q_w(\tau)$ if and only if $\frac{1}{n}\sum_{s=1}^{u} \mathbb{1}_{Y_w(\zeta_w^s)\leq U_w^{\eta}} \leq \frac{1}{n}\tilde{S}_w^{\tau}(n)$ and

$$\frac{1}{n}\sum_{s=1}^{u} \mathbb{1}_{Y_w(\zeta_w^s)\leq U_w^{\eta}} \leq \frac{\tilde{S}_w^{\tau}(n)}{n} \Leftrightarrow \tau \leq \frac{\tilde{S}_w^{\tau}(n)}{n} \quad \text{and} \quad \mathrm{kl}(\frac{\tilde{S}_w^{\tau}(n)}{n}, \tau) \geq \delta_\eta(T) + \frac{1}{n}.$$

Define $\delta = \delta_\eta(T) + 1/n$, let s be the smallest integer such that $\delta/(s+1) \leq \mathrm{kl}(1,\tau)$; if $n \leq s$, then $n\,\mathrm{kl}(\frac{\tilde{S}_w^{\tau}(n)}{n}, \tau) \leq s\,\mathrm{kl}(\frac{\tilde{S}_w^{\tau}(n)}{n}, \tau) \leq s\,\mathrm{kl}(1,\tau) < \delta$ thus $\mathbb{P}(U(n) < q(\tau)) = 0$. Thus for all k such that $t_k \geq s$, we obtain $\mathbb{P}(A_k = 0)$. For k such that $t_k > s$, let $\tilde{t}_{k-1} = \max\{t_{k-1}, s\}$. Let $x \in]\tau, 1[$ be such that $\mathrm{kl}(x,\tau) = \delta/n$ and let $\lambda(x) = \log(x(1-\tau)) - \log(\tau(1-x)) > 0$, so that $\mathrm{kl}(x,\tau) = \lambda(x)x - (1-\tau+\tau\exp(\lambda(x)))$. Consider z such that $z > \tau$ and $\mathrm{kl}(z,\tau) = \delta/(1+\xi)^k$.

Observe that

- if $n > \tilde{t}_{k-1}$, then
$$\mathrm{kl}(z,\tau) = \frac{\delta}{(1+\xi)^k} \geq \frac{\delta}{(1+\xi)n};$$
- if $n \leq t_k$, then as
$$\mathrm{kl}\,\big(\frac{\tilde{S}_w^{\tau}(n)}{n}, \tau\big) > \frac{\delta}{n} > \frac{\delta}{(1+\xi)^k} = \mathrm{kl}(z,\tau),$$

it holds that:

$$\tau \leq \frac{\tilde{S}_w^{\tau}(n)}{n} \quad \text{and} \quad \mathrm{kl}(\frac{\tilde{S}_w^{\tau}(n)}{n}, \tau) \geq \frac{\delta}{n} \Rightarrow \frac{\tilde{S}_w^{\tau}(n)}{n} \geq z.$$

Hence on the event $\{\tilde{t}_{k-1} < n < t_k\} \cap \{\tau \leq \frac{\tilde{S}_w^{\tau}(n)}{n}\} \cap \{\mathrm{kl}(\frac{\tilde{S}_w^{\tau}(n)}{n}, \tau) \geq \frac{\delta}{n}\}$ it holds that

$$\lambda(z)\frac{\tilde{S}_w^{\tau}(n)}{n} \geq \lambda(z)z - \phi_\tau(\lambda(z)) = \mathrm{kl}(z,\tau) \geq \frac{\delta}{(1+\xi)n}.$$

Step 3: Putting everything together

$$\{\tilde{t}_{k-1} < n < t_k\} \cap \{\tau \leq \frac{\tilde{S}_w^{\tau}(n)}{n}\} \cap \{\mathrm{kl}(\frac{\tilde{S}_w^{\tau}(n)}{n}, \tau) \geq \frac{\delta}{n}\}$$

$$\begin{aligned}
&\subset\{\lambda(z)\frac{\tilde{S}_w^{\tau}(n)}{n} - \phi_\tau(\lambda(z)) \geq \frac{\delta}{n(1+\xi)}\} \\
&\subset\{\lambda(z)S_w(n) - n\phi_\tau(\lambda(z)) \geq \frac{\delta_\eta(T)}{(1+\xi)}\} \\
&\subset\{W_n^{\lambda(z)} > \exp(\frac{\delta_\eta(T)}{(1+\xi)})\}.
\end{aligned}$$

As $(W_n^{\lambda})_{n\geq 0}$ is a martingale, $\mathbb{E}[W_n^{\lambda(z)}] \leq \mathbb{E}[W_0^{\lambda(z)}] = 1$. Thus the Doob's inequality for martingales provides:

$$\mathbb{P}\bigg(\sup_{\tilde{t}_{k-1}<n<t_k} W_n^{\lambda(z)} > \exp\Big(\frac{\delta_\eta(T)}{1+\xi}\Big)\bigg) \leq \exp\Big(-\frac{\delta_\eta(T)}{1+\xi}\Big)$$

Finally

$$\sum_{w=1}^{T}\sum_{k=1}^{D}\mathbb{P}\Big(\ \exists\ t_{k-1} < n \leq t_k \ \text{ and }\ U_w^\eta(n) \leq q_w(\tau)\Big) \leq TD\exp(-\frac{\delta_\eta(T)}{(1+\xi)}).$$

But as $\xi = 1/(\delta_\eta(T)-1)$, $D = \Big\lceil \dfrac{\log(T)}{\log(1+1/(\delta_\eta(T)+1))}\Big\rceil$ and as long as

$$\log(1+1/(\delta_\eta(T)-1)) \geq 1/\delta_\eta(T),$$

we obtain:

$$\mathbb{P}(\mathcal{A}^c) \leq T\Big\lceil \frac{\log(T)}{\log(1+1/(\delta_\eta(T)+1))}\Big\rceil \exp(-\delta_\eta(T)+1) \leq Te\lceil \delta_\eta(T)\log(T)\rceil \exp(-\delta_\eta(T)) \leq \eta/2.$$

Using the same lines for the LCB concludes the proof. ■

4. Optimizing CVaR

Proof of Proposition 13

Let us consider the event

$$\begin{aligned}\xi_\eta \;=\; & \Big\{\forall\ h \geq 0, \forall\ 0 \leq j \leq K^h, \forall\ 1 \leq t \leq T, \\ & \widehat{\text{CVaR}_\tau}^t(Y_{x_{h,j}}) \geq \text{CVaR}_\tau(Y_{x_{h,j}}) + \tilde{\epsilon}^\eta_{N_{h,j}(t)} \text{ or } \widehat{\text{CVaR}_\tau}^t(Y_{x_{h,j}}) \leq \text{CVaR}_\tau(Y_{x_{h,j}}) - \epsilon^\eta_{N_{h,j}(t)}\Big\}.\end{aligned}$$

$$\begin{aligned}\mathbb{P}\big(\xi_\eta\big) = \mathbb{P}\Big(&\forall h \geq 0, \forall 0 \leq j \leq K^h, \forall 1 \leq t \leq T,\ \widehat{\text{CVaR}_\tau}^t(Y_{x_{h,j}}) \geq \text{CVaR}_\tau(Y_{x_{h,j}}) + \tilde{\epsilon}^\eta_{N_{h,j}(t)} \text{ or },\\ & \widehat{\text{CVaR}_\tau}^t(Y_{x_{h,j}}) \leq \text{CVaR}_\tau(Y_{x_{h,j}}) - \epsilon^\eta_{N_{h,j}(t)}\Big)\\ \leq\ & \mathbb{P}\Big(\forall h \leq 0, \forall 0 \leq j \leq K^h, \forall 1 \leq t \leq T,\ \widehat{\text{CVaR}_\tau}^t(Y_{x_{h,j}}) \geq \text{CVaR}_\tau(Y_{x_{h,j}}) + \tilde{\epsilon}^\eta_{N_{h,j}(t)}\Big) \quad (4)\\ & + \mathbb{P}\Big(\forall h \geq 0, \forall 0 \leq j \leq K^h, \forall 1 \leq t \leq T,\ \widehat{\text{CVaR}_\tau}^t(Y_{x_{h,j}}) \leq \text{CVaR}_\tau(Y_{x_{h,j}}) - \epsilon^\eta_{N_{h,j}(t)}\Big) \quad (5)\end{aligned}$$

First let us consider (4):

$$\mathbb{P}\Big(\forall h \geq 0, \forall 0 \leq j \leq K^h, \forall 1 \leq t \leq T,\ \widehat{\text{CVaR}_\tau}^t(Y_{x_{h,j}}) \geq \text{CVaR}_\tau(Y_{x_{h,j}}) + \tilde{\epsilon}^\eta_{N_{h,j}(t)}\Big)$$

$$\leq \mathbb{P}\Big(\exists\, 1 \leq w \leq T,\ \exists\, 1 \leq u \leq T,\ \inf_{z\in\mathbb{R}}\{z + \frac{1}{u(1-\tau)}\sum_{s=1}^{u}(Y_w(\zeta_w^s) - z)^+\} \geq \text{CVaR}_\tau(Y_{x_w}) + \tilde{\epsilon}_u^\eta\Big)$$

$$\leq \sum_{w=1}^{T}\sum_{u=1}^{T}\mathbb{P}\Big(\inf_{z\in\mathbb{R}}\{z + \frac{1}{u(1-\tau)}\sum_{s=1}^{u}(Y_w(\zeta_w^s) - z)^+\} \geq \text{CVaR}_\tau(Y_{x_w}) + \tilde{\epsilon}_u^\eta\Big).$$

Thus by Brown's inequality

$$(4) < \sum_{w=1}^{T}\sum_{u=1}^{T}\exp(-2(\tau\tilde{\epsilon}_u^\eta/(b-a))^2 u).$$

Taking

$$\tilde{\epsilon}_u^\eta = \frac{(b-a)}{\tau}\sqrt{\frac{\log(2T^2/\eta)}{2u}}$$

provides the first part, *i.e* (4)$< \dfrac{\eta}{2}$.

We use the same scheme of proof to bound (5), the only difference comes from the fact that the inequality of deviation is different:

$$\mathbb{P}\Bigg(\forall h \geq 0, \forall 0 \leq j \leq K^h, \forall 1 \leq t \leq T,\ \widehat{\text{CVaR}_\tau}^{\,t}(Y_{x_{h,j}}) \leq \text{CVaR}_\tau(Y_{x_{h,j}}) - \epsilon^\eta_{N_{h,j}(t)}\Bigg)$$

$$\leq \mathbb{P}\Big(\exists\, 1 \leq w \leq T,\ \exists\, 1 \leq u \leq T,\ \inf_{z\in\mathbb{R}}\{z + \frac{1}{u(1-\tau)}\sum_{s=1}^{u}(Y_w(\zeta_w^s) - z)^+\} \leq \text{CVaR}_\tau(Y_{x_w}) - \epsilon_u^\eta\Big)$$

$$\leq \sum_{w=1}^{T}\sum_{u=1}^{T}\mathbb{P}\Big(\inf_{z\in\mathbb{R}}\{z + \frac{1}{u(1-\tau)}\sum_{s=1}^{u}(Y_w(\zeta_w^s) - z)^+\} \leq \text{CVaR}_\tau(x_w) - \epsilon_u^\eta\Big).$$

By Brown's inequality

$$(5) < \sum_{w=1}^{T}\sum_{u=1}^{T} 3\exp\left(-\frac{\tau}{5}\Big(\frac{\epsilon_u^\eta}{b-a}\Big)^2 u\right)$$

Taking

$$\tilde{\epsilon}_u^\eta = (b-a)\sqrt{\frac{5\log(6T^2/\eta)}{\tau u}}$$

provides (5) $< \dfrac{\eta}{2}$.

Finally putting (4) and (5) together provides $\mathbb{P}\big(\xi_\eta\big) < \eta$ and hence $\mathbb{P}(\xi_\eta^c) = \mathbb{P}(\mathcal{A}_\eta) = 1 - \eta$.

■

Proposition 2 *Assume for all $x \in \mathcal{X}$, Y_x is bounded by $(a, b) \in \mathbb{R}^2$. For any $\eta \in (0, 0.5]$, for all $1 \leq t \leq T$, $1 \leq h \leq t$ and $1 \leq j \leq K^h$, define*

$$L^\eta_{h,j}(t) = \frac{1}{1-\tau}\sum_{i=1}^{n}(Y_{i+1} - Y_i)\Big(\frac{i}{n} - \sqrt{\frac{\log(2T^2/\eta)}{2N_{h,j}(t)}} - \tau\Big)^+ - Y_{n+1}$$

and

$$U^{\eta}_{h,j}(t) = \frac{1}{1-\tau}\sum_{i=0}^{n-1}(Y_{i+1}-Y_i)\Big(\min\{1, \frac{i}{n} + \sqrt{\frac{\log(2T^2/\eta)}{2N_{h,j}(t)}}\} - \tau\Big)^+ - Y_n,$$

with $Y_0 = a$ *and* $Y_{n+1} = b$. *Then if* $g = -\,\mathrm{CVaR}_\tau$, *the event* $\mathcal{A}_\eta$ *has probability at least* $1-\eta$.

Proof If $Y_1 \cdots, Y_n$ are i.i.d random variables bounded by (a, b) then Thomas-Learned-Miller's inequalities provide

$$\mathbb{P}\Bigg(-\,\mathrm{CVaR}_\tau < \frac{1}{1-\tau}\sum_{i=1}^{n}(Y_{i+1}-Y_i)\Big(\frac{i}{n} - \sqrt{\frac{\log(1/\eta)}{2n}} - \tau\Big)^+ - Y_{n+1}\Bigg) < \eta$$

and

$$\mathbb{P}\Bigg(-\,\mathrm{CVaR}_\tau > \frac{1}{1-\tau}\sum_{i=0}^{n-1}(Y_{i+1}-Y_i)\Big(\min\{1, \frac{i}{n} + \sqrt{\frac{\log(2T^2/\eta)}{2N_{h,j}(t)}}\} - \tau\Big)^+ - Y_n\Bigg) < \eta.$$

Define

$$\xi_{\eta,1} \;=\; \{\forall\, h \geq 0, \forall\, 0 \leq j \leq K^h, \forall\, 1 \leq t \leq T, -\,\mathrm{CVaR}_\tau(Y_{h,j}) < U^{\eta}_{N_{h,j}(t)}\},$$

and

$$\xi_{\eta,2} \;=\; \{\forall\, h \geq 0, \forall\, 0 \leq j \leq K^h, \forall\, 1 \leq t \leq T, -\,\mathrm{CVaR}_\tau(Y_{h,j}) > L^{\eta}_{N_{h,j}(t)}\},$$

To treat the sequential point of view, here we use a double union bound as it is done in the proof of Proposition 13, then it can be shown that

$$\mathbb{P}(\xi_{\eta,1}) < \sum_{w=1}^{T}\sum_{u=1}^{T}\mathbb{P}\Big(-\,\mathrm{CVaR}_\tau(Y^u_w) < U^{\eta}_u\Big).$$

Thus by defining

$$U^{\eta}_u = \frac{1}{1-\tau}\sum_{i=0}^{u-1}(Y_{i+1}-Y_i)\Big(\min\{1, \frac{i}{u} + \sqrt{\frac{\log(2T^2/\eta)}{2u}}\} - \tau\Big)^+ - Y_u$$

we obtain

$$\mathbb{P}(\xi_{\eta,1}) < \sum_{w=1}^{T}\sum_{u=1}^{T}\frac{\eta}{2T^2} = \frac{\eta}{2}.$$

Using the same scheme of proof with

$$L^{\eta}_u = \frac{1}{1-\tau}\sum_{i=1}^{u}(Y_{i+1}-Y_i)\Big(\frac{i}{u} - \sqrt{\frac{\log(2T^2/\eta)}{2u}} - \tau\Big)^+ - Y_{u+1}$$

provides

$$\mathbb{P}(\xi_{\eta,2}) < \frac{\eta}{2}.$$

Finally

$$\mathbb{P}(\xi_{\eta,1} \cup \xi_{\eta,1}) < \eta,$$

and hence $\mathbb{P}\Big((\xi_{\eta,1} \cup \xi_{\eta,1})^c\Big) = \mathbb{P}(\mathcal{A}_\eta) = 1-\eta$. ■