Appendix A. Proofs

Proof [Proof of Lemma 7] For the sake of brevity, we denote

$$\begin{aligned}&K^1_{\mathrm{H,T}}\\&:=\{A\in\mathscr{Q}\colon((\exists h_1,h_2\in A)(h_1(\mathrm{T})>0\text{ and }h_2(\mathrm{H})>0))\\&\qquad\qquad\text{or }A\cap\mathscr{L}^{\mathrm{s}}_{>0}\neq\emptyset\},\\&K^2_{\mathrm{H,T}}\\&:=\mathrm{Rs}(\{\{h_1,h_2\}\colon h_1,h_2\in\mathscr{L}_{\not\leq 0},(h_1(\mathrm{T}),h_2(\mathrm{H}))>0\}).\end{aligned}$$

We will show (i) that $\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0}\cup\mathscr{A})\subseteq K^1_{\mathrm{H,T}}$, (ii) that $K^1_{\mathrm{H,T}}\subseteq K^2_{\mathrm{H,T}}$, and (iii) that $K^2_{\mathrm{H,T}}\subseteq\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0}\cup\mathscr{A})$.

For (i)—to show that $\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0}\cup\mathscr{A})\subseteq K^1_{\mathrm{H,T}}$—consider any gamble set A in $\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0}\cup\mathscr{A})$. This means that there are n in $\mathbb{N}$, A_1, ..., A_n in $\mathscr{L}^{\mathrm{s}}_{>0}\cup\mathscr{A}$, and, for all $f_{1:n}$ in $\times_{k=1}^n A_k$, coefficients $\lambda^{f_{1:n}}_{1:n}>0$, such that $A=\left\{\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k\colon f_{1:n}\in\times_{k=1}^n A_k\right\}$. Without loss of generality, assume that $A_1,\dots,A_\ell\in\mathscr{A}$ and $A_{\ell+1},\dots,A_n\in\mathscr{L}^{\mathrm{s}}_{>0}$ for some ℓ in $\{0,\dots,n\}$. Therefore, we may denote, also without loss of generality, $A_1=\{-\mathbb{I}_{\{\mathrm{H}\}}+\varepsilon_1,-\mathbb{I}_{\{\mathrm{T}\}}+\delta_1\}$, ..., $A_\ell=\{-\mathbb{I}_{\{\mathrm{H}\}}+\varepsilon_\ell,-\mathbb{I}_{\{\mathrm{T}\}}+\delta_\ell\}$, $A_{\ell+1}=\{g_{\ell+1}\}$, ..., $A_n=\{g_n\}$, where ε_1, δ_1, ..., ε_ℓ, δ_ℓ are elements of $\mathbb{R}_{>0}$ and $g_{\ell+1}$, ..., g_n elements of $\mathscr{L}_{>0}$. If $\ell=0$ or $\lambda^{f_{1:n}}_{1:\ell}=0$ for some $f_{1:n}$ in $\times_{k=1}^n A_k$—and therefore necessarily $\lambda^{f_{1:n}}_{\ell+1:n}>0$—then we have that $\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k=\sum_{k=\ell+1}^n\lambda_k^{f_{1:n}}g_k$ is a gamble in $\mathscr{L}_{>0}$, so we find that $\left\{\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k\colon f_{1:n}\in\times_{k=1}^n A_k\right\}=A\cap\mathscr{L}^{\mathrm{s}}_{>0}\neq\emptyset$. If, on the other hand, $\ell\geq 1$ and $\lambda^{f_{1:n}}_{1:\ell}\neq 0$—and hence $\lambda^{f_{1:n}}_{1:\ell}>0$—for every $f_{1:n}$ in $\times_{k=1}^n A_k$, then for the two sequences of gambles $f^{\mathrm{H}}_{1:n}=(f^{\mathrm{H}}_1,\dots,f^{\mathrm{H}}_n):=(-\mathbb{I}_{\{\mathrm{H}\}}+\varepsilon_1,\dots,-\mathbb{I}_{\{\mathrm{H}\}}+\varepsilon_\ell,g_{\ell+1},\dots,g_n)$ and $f^{\mathrm{T}}_{1:n}=(f^{\mathrm{T}}_1,\dots,f^{\mathrm{T}}_n):=(-\mathbb{I}_{\{\mathrm{T}\}}+\delta_1,\dots,-\mathbb{I}_{\{\mathrm{T}\}}+\delta_\ell,g_{\ell+1},\dots,g_n)$ in $\times_{k=1}^n A_k$ we have that

$$\begin{aligned}h_1:=\sum_{k=1}^n\lambda_k^{f^{\mathrm{H}}_{1:n}}f^{\mathrm{H}}_k&=\sum_{k=1}^\ell\lambda_k^{f^{\mathrm{H}}_{1:n}}f^{\mathrm{H}}_k+\sum_{k=\ell+1}^n\lambda_k^{f^{\mathrm{H}}_{1:n}}g_k\\&\geq\sum_{k=1}^\ell\lambda_k^{f^{\mathrm{H}}_{1:n}}f^{\mathrm{H}}_k\\&=-\mathbb{I}_{\{\mathrm{H}\}}\sum_{k=1}^\ell\lambda_k^{f^{\mathrm{H}}_{1:n}}+\sum_{k=1}^\ell\lambda_k^{f^{\mathrm{H}}_{1:n}}\varepsilon_k\end{aligned}$$

and, similarly,

$$h_2:=\sum_{k=1}^n\lambda_k^{f^{\mathrm{T}}_{1:n}}f^{\mathrm{T}}_k\geq-\mathbb{I}_{\{\mathrm{T}\}}\sum_{k=1}^\ell\lambda_k^{f^{\mathrm{T}}_{1:n}}+\sum_{k=1}^\ell\lambda_k^{f^{\mathrm{T}}_{1:n}}\delta_k,$$

so $h_1(\mathrm{T})\geq\sum_{k=1}^\ell\lambda_k^{f^{\mathrm{H}}_{1:n}}\varepsilon_k>0$ and $h_2(\mathrm{H})\geq\sum_{k=1}^\ell\lambda_k^{f^{\mathrm{T}}_{1:n}}\delta_k>0$. Note that both h_1 and h_2 belong to A, so we find that $(\exists h_1,h_2\in A)(h_1(\mathrm{T})>0\text{ and }h_2(\mathrm{H})>0)$. Therefore indeed $\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0}\cup\mathscr{A})\subseteq K^1_{\mathrm{H,T}}$.

For (ii)—to show that $K^1_{\mathrm{H,T}}\subseteq K^2_{\mathrm{H,T}}$—consider any gamble set A in $K^1_{\mathrm{H,T}}$. Then (a) $h_1(\mathrm{T})>0$ and $h_2(\mathrm{H})>0$ for some h_1 and h_2 in A, or (b) $A\cap\mathscr{L}^{\mathrm{s}}_{>0}\neq\emptyset$. If (a), then $h_1,h_2\in\mathscr{L}_{\not\leq 0}$, and $(h_1(\mathrm{T}),h_2(\mathrm{H}))>0$, so $A\in K^2_{\mathrm{H,T}}$. If (b), then $h>0$ for some h in A, so for $h_1:=h_2:=h$ trivially $h_1,h_2\in\mathscr{L}_{\not\leq 0}$, and $(h_1(\mathrm{T}),h_2(\mathrm{H}))=(h(\mathrm{T}),h(\mathrm{H}))>0$, whence $A\in K^2_{\mathrm{H,T}}$. We conclude that indeed $K^1_{\mathrm{H,T}}\subseteq K^2_{\mathrm{H,T}}$.

For (iii)—to show that $K^2_{\mathrm{H,T}}\subseteq\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0}\cup\mathscr{A})$—consider any gamble set A in $K^2_{\mathrm{H,T}}$. Then $A\supseteq\{h_1,h_2\}\setminus\mathscr{L}_{\leq 0}=\{h_1,h_2\}$ for some h_1 and h_2 in $\mathscr{L}_{\not\leq 0}$ such that $(h_1(\mathrm{T}),h_2(\mathrm{H}))>0$. Without loss of generality, rename the gambles in

$$A=\{f^{\mathrm{I}}_1,\dots,f^{\mathrm{I}}_{n_{\mathrm{I}}},f^{\mathrm{II}}_1,\dots,f^{\mathrm{II}}_{n_{\mathrm{II}}},f^{\mathrm{III}}_1,\dots,f^{\mathrm{III}}_{n_{\mathrm{III}}},f^{\mathrm{IV}}_1,\dots,f^{\mathrm{IV}}_{n_{\mathrm{IV}}}\},$$

with n_{I}, n_{II}, n_{III} and n_{IV} in $\{0\}\cup\mathbb{N}$ such that $n:=2n_{\mathrm{I}}+n_{\mathrm{II}}+2n_{\mathrm{III}}+n_{\mathrm{IV}}\geq 1$, gambles f^{I}_1, ..., $f^{\mathrm{I}}_{n_{\mathrm{I}}}$ in the positive quadrant $\mathscr{L}_{>0}$, gambles f^{II}_1, ..., $f^{\mathrm{II}}_{n_{\mathrm{II}}}$ in the second quadrant $\mathscr{L}_{\mathrm{II}}:=\{f\in\mathscr{L}\colon f(\mathrm{H})<0<f(\mathrm{T})\}$, gambles f^{III}_1, ..., $f^{\mathrm{III}}_{n_{\mathrm{III}}}$ in the negative quadrant $\mathscr{L}_{\leq 0}$, and gambles f^{IV}_1, ..., $f^{\mathrm{IV}}_{n_{\mathrm{IV}}}$ in the fourth quadrant $\mathscr{L}_{\mathrm{IV}}:=\{f\in\mathscr{L}\colon f(\mathrm{T})<0<f(\mathrm{H})\}$. We must show that A belongs to $\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0}\cup\mathscr{A})$. To this end, we will construct n gamble sets A_1, ..., A_n and, for every $f_{1:n}$ in $\times_{k=1}^n A_k$, coefficients $\lambda^{f_{1:n}}_{1:n}>0$ such that $A=\left\{\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k\colon f_{1:n}\in\times_{k=1}^n A_k\right\}$.

Let $A_1:=\{g_1\}\in\mathscr{L}^{\mathrm{s}}_{>0}$, ..., $A_{n_{\mathrm{I}}}:=\{g_{n_{\mathrm{I}}}\}\in\mathscr{L}^{\mathrm{s}}_{>0}$. We consider the additional n_{I} gamble sets $A_{n_{\mathrm{I}}+1}:=\ldots:=A_{2n_{\mathrm{I}}}:=\{-\mathbb{I}_{\{\mathrm{H}\}}+1,-\mathbb{I}_{\{\mathrm{T}\}}+1\}\in\mathscr{A}$, in order to have enough freedom in selecting the coefficients $\lambda^{f_{1:n}}_{1:n}>0$ later on. For every i in $\{1,\dots,n_{\mathrm{II}}\}$, let $A_{2n_{\mathrm{I}}+i}:=\{-\mathbb{I}_{\{\mathrm{H}\}}+\varepsilon_i,-\mathbb{I}_{\{\mathrm{T}\}}+\delta\}\in\mathscr{A}$ with $\varepsilon_i:=\frac{f^{\mathrm{II}}_i(\mathrm{T})}{f^{\mathrm{II}}_i(\mathrm{T})-f^{\mathrm{II}}_i(\mathrm{H})}>0$ and $\delta:=\frac{f^{\mathrm{IV}}_1(\mathrm{H})}{f^{\mathrm{IV}}_1(\mathrm{H})-f^{\mathrm{IV}}_1(\mathrm{T})}>0$ if $n_{\mathrm{IV}}\geq 1$, otherwise $\delta:=1$. For every i in $\{1,\dots,n_{\mathrm{III}}\}$, if $f^{\mathrm{III}}_i\neq 0$, let $A_{2n_{\mathrm{I}}+n_{\mathrm{II}}+i}:=\{-\mathbb{I}_{\{\mathrm{H}\}}+\frac{1}{4},-\mathbb{I}_{\{\mathrm{T}\}}+1\}\in\mathscr{A}$ and $A_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+i}:=\{-\mathbb{I}_{\{\mathrm{H}\}}+1,-\mathbb{I}_{\{\mathrm{T}\}}+\frac{1}{4}\}\in\mathscr{A}$; if $f^{\mathrm{III}}_i=0$, let $A_{2n_{\mathrm{I}}+n_{\mathrm{II}}+i}:=A_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+i}:=\{-\mathbb{I}_{\{\mathrm{H}\}}+\frac{1}{2},-\mathbb{I}_{\{\mathrm{T}\}}+\frac{1}{2}\}\in\mathscr{A}$. For every i in $\{1,\dots,n_{\mathrm{IV}}\}$, let $A_{2n_{\mathrm{I}}+n_{\mathrm{II}}+2n_{\mathrm{III}}+i}:=\{-\mathbb{I}_{\{\mathrm{H}\}}+1,-\mathbb{I}_{\{\mathrm{T}\}}+\delta_i\}\in\mathscr{A}$ with $\delta_i:=\frac{f^{\mathrm{IV}}_i(\mathrm{H})}{f^{\mathrm{IV}}_i(\mathrm{H})-f^{\mathrm{IV}}_i(\mathrm{T})}>0$.

The set $\times_{k=1}^n A_k$ contains $2^{n-n_{\mathrm{I}}}=2^{n_{\mathrm{I}}+n_{\mathrm{II}}+2n_{\mathrm{III}}+n_{\mathrm{IV}}}$ sequences. Each such sequence $f_{1:n}$ is characterised by a choice of f_i in the binary set A_i—which we will denote by $\{g^{\mathrm{H}}_i,g^{\mathrm{T}}_i\}$, where g^{H}_i is the gamble in A_i of the form $-\mathbb{I}_{\{\mathrm{H}\}}+\varepsilon$ and g^{T}_i the gamble in A_i of the form $-\mathbb{I}_{\{\mathrm{T}\}}+\delta$—, for every i in $\{n_{\mathrm{I}}+1,\dots,n\}$. For the first n_{I} entries $f_{1:n_{\mathrm{I}}}$ of $f_{1:n}$ we have no choice but to chose $f_{1:n_{\mathrm{I}}}=g_{1:n_{\mathrm{I}}}$, since $\times_{k=1}^{n_{\mathrm{I}}}A_k$ is the singleton $\{g_{1:n_{\mathrm{I}}}\}$.

For any sequence $f_{1:n}$ in $\times_{k=1}^n A_k$, define n real coefficients $\lambda^{f_{1:n}}_{1:n}$ as follows:

- Situation (a): If there is an i in $\{2n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}\}$ such that

$$(f_{2n_{\mathrm{I}}+1},\dots,f_{i-1},f_i,f_{i+1},\dots,f_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}},$$

$$f_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1},\dots,f_n) = (g^{\mathrm{T}}_{2n_{\mathrm{I}}+1},\dots,g^{\mathrm{T}}_{i-1},g^{\mathrm{H}}_i,g^{\mathrm{T}}_{i+1},\dots,g^{\mathrm{T}}_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}}, g^{\mathrm{H}}_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1},\dots,g^{\mathrm{H}}_n)$$

or, in other words, such that $f_i = g^{\mathrm{H}}_i$, $(\forall k \in \{2n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}\}\setminus\{i\})f_k = g^{\mathrm{T}}_k$, and $(\forall k \in \{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1,\dots,n\})f_k = g^{\mathrm{H}}_k$, then let

$$\lambda_i^{f_{1:n}} := f^{\mathrm{II}}_j(\mathrm{T}) - f^{\mathrm{II}}_j(\mathrm{H}) > 0 \text{ for } j := i-2n_{\mathrm{I}},$$
$$\lambda_k^{f_{1:n}} := 0 \text{ for all } k \text{ in } \{1,\dots,n\}\setminus\{i\}.$$

- Situation (b): If there is an i in $\{2n_{\mathrm{I}}+n_{\mathrm{II}}+2n_{\mathrm{III}}+1,\dots,n\}$ such that

$$(f_{2n_{\mathrm{I}}+1},\dots,f_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}}, f_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1},\dots,f_{i-1},f_i,f_{i+1},\dots,f_n) = (g^{\mathrm{T}}_{2n_{\mathrm{I}}+1},\dots,g^{\mathrm{T}}_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}}, g^{\mathrm{H}}_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1},\dots,g^{\mathrm{H}}_{i-1},g^{\mathrm{T}}_i,g^{\mathrm{H}}_{i+1},\dots,g^{\mathrm{H}}_n),$$

or, in other words, such that $f_i = g^{\mathrm{T}}_i$, $(\forall k \in \{2n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}\})f_k = g^{\mathrm{T}}_k$, and $(\forall k \in \{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1,\dots,n\}\setminus\{i\})f_k = g^{\mathrm{H}}_k$, then let

$$\lambda_i^{f_{1:n}} := f^{\mathrm{IV}}_j(\mathrm{H}) - f^{\mathrm{IV}}_j(\mathrm{T}) > 0 \quad \text{for } j := i-2n_{\mathrm{I}}-n_{\mathrm{II}}-2n_{\mathrm{III}},$$
$$\lambda_k^{f_{1:n}} := 0 \text{ for all } k \text{ in } \{1,\dots,n\}\setminus\{i\}.$$

- Situation (c): If there is an i in $\{2n_{\mathrm{I}}+n_{\mathrm{II}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}\}$ such that

$$(f_{2n_{\mathrm{I}}+1},\dots,f_{i-1},f_i,f_{i+1},\dots,f_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}}, f_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1},\dots,f_{n_{\mathrm{III}}+i-1},f_{n_{\mathrm{III}}+i},f_{n_{\mathrm{III}}+i+1},\dots,f_n) = (g^{\mathrm{T}}_{2n_{\mathrm{I}}+1},\dots,g^{\mathrm{T}}_{i-1},g^{\mathrm{H}}_i,g^{\mathrm{T}}_{i+1},\dots,g^{\mathrm{T}}_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}}, g^{\mathrm{H}}_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1},\dots,g^{\mathrm{H}}_{n_{\mathrm{III}}+i-1},g^{\mathrm{T}}_{n_{\mathrm{III}}+i},g^{\mathrm{H}}_{n_{\mathrm{III}}+i+1},\dots,g^{\mathrm{H}}_n),$$

or, in other words, such that $f_i = g^{\mathrm{H}}_i$, $f_{n_{\mathrm{III}}+i} = g^{\mathrm{T}}_{n_{\mathrm{III}}+i}$, $(\forall k \in \{2n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}\}\setminus\{i\})f_k = g^{\mathrm{T}}_k$ and $(\forall k \in \{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1,\dots,n\}\setminus\{n_{\mathrm{III}}+i\})f_k = g^{\mathrm{H}}_k$, then let

$$\lambda_i^{f_{1:n}} := \lambda_{n_{\mathrm{III}}+i}^{f_{1:n}} := 1 \text{ if } f^{\mathrm{III}}_{i-2n_{\mathrm{I}}+n_{\mathrm{II}}} = 0,$$
$$\lambda_i^{f_{1:n}} := -\frac{1}{2}(3f^{\mathrm{III}}_j(\mathrm{H}) + f^{\mathrm{III}}_j(\mathrm{T})) > 0 \text{ and}$$
$$\lambda_{n_{\mathrm{III}}+i}^{f_{1:n}} := -\frac{1}{2}(f^{\mathrm{III}}_j(\mathrm{H}) + 3f^{\mathrm{III}}_j(\mathrm{T})) > 0 \quad \text{for } j := i-2n_{\mathrm{I}}-n_{\mathrm{II}} \text{ and if } f^{\mathrm{III}}_j \neq 0,$$
$$\lambda_k^{f_{1:n}} := 0 \text{ for all } k \text{ in } \{1,\dots,n\}\setminus\{i,n_{\mathrm{III}}+i\}.$$

- Situation (d): If none of the Situations (a), (b) nor (c) apply, and if there is an i in $\{n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}\}$ such that

$$(f_{n_{\mathrm{I}}+1},\dots,f_{i-1},f_i,f_{i+1},\dots,f_{2n_{\mathrm{I}}}) = (g^{\mathrm{T}}_{n_{\mathrm{I}}+1},\dots,g^{\mathrm{T}}_{i-1},g^{\mathrm{H}}_i,g^{\mathrm{T}}_{i+1},\dots,g^{\mathrm{T}}_{2n_{\mathrm{I}}}),$$

or, in other words, such that $f_i = g^{\mathrm{H}}_i$ and $(\forall k \in \{n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}\}\setminus\{i\})f_k = g^{\mathrm{T}}_k$, then let

$$\lambda_{i-n_{\mathrm{I}}}^{f_{1:n}} := 1,$$
$$\lambda_k^{f_{1:n}} := 0 \text{ for all } k \text{ in } \{1,\dots,n\}\setminus\{i-n_{\mathrm{I}}\}.$$

- Situation (e1): If $A \cap \mathscr{L}_{>0} \neq \emptyset$—so $n_{\mathrm{I}} \geq 1$—and none of the Situations (a), (b), (c) nor (d) apply, then let

$$\lambda_1^{f_{1:n}} := 1 \text{ and } \lambda_{2:n}^{f_{1:n}} := 0.$$

- Situation (e2): If $A \cap \mathscr{L}_{>0} = \emptyset$—so $n_{\mathrm{II}} \geq 1$ and $n_{\mathrm{IV}} \geq 1$ because $(h_1(\mathrm{T}), h_2(\mathrm{H})) > 0$—and none of the Situations (a), (b), (c) nor (d) apply, then let, with $i := 2n_{\mathrm{I}}+1$,

$$\lambda_i^{f_{1:n}} := f^{\mathrm{II}}_1(\mathrm{T}) - f^{\mathrm{II}}_1(\mathrm{H}) > 0 \text{ if } f_i = g^{\mathrm{H}}_i,$$
$$\lambda_i^{f_{1:n}} := f^{\mathrm{IV}}_1(\mathrm{H}) - f^{\mathrm{IV}}_1(\mathrm{T}) > 0 \text{ if } f_i = g^{\mathrm{T}}_i,$$
$$\lambda_k^{f_{1:n}} := 0 \text{ or all } k \text{ in } \{1,\dots,n\}\setminus\{i\}.$$

In this way, we have defined coefficients $\lambda_{1:n}^{f_{1:n}} > 0$ for every $f_{1:n}$ in $\bigtimes_{k=1}^n A_k$. It only remains to show, with our choices of $\lambda_{1:n}^{f_{1:n}} > 0$, that $A = \big\{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \bigtimes_{k=1}^n A_k\big\}$.

We first prove that $A \subseteq \big\{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \bigtimes_{k=1}^n A_k\big\}$. To show that $f^{\mathrm{II}}_j \in \big\{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \bigtimes_{k=1}^n A_k\big\}$ for every j in $\{1,\dots,n_{\mathrm{II}}\}$, consider any $f_{1:n}$ in $\bigtimes_{k=1}^n A_k$ such that $f_{1:n}$ satisfies the conditions of Situation (a) for $i := j+2n_{\mathrm{I}}$, which is then an element of $\{2n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}\}$. Then $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k = (f^{\mathrm{II}}_j(\mathrm{T}) - f^{\mathrm{II}}_j(\mathrm{H}))g^{\mathrm{H}}_i = (f^{\mathrm{II}}_j(\mathrm{T}) - f^{\mathrm{II}}_j(\mathrm{H}))\big(-\mathbb{I}_{\{\mathrm{H}\}} + \frac{f^{\mathrm{II}}_i(\mathrm{T})}{f^{\mathrm{II}}_i(\mathrm{T})-f^{\mathrm{II}}_i(\mathrm{H})}\big) = (f^{\mathrm{II}}_j(\mathrm{H}) - f^{\mathrm{II}}_j(\mathrm{T}))\mathbb{I}_{\{\mathrm{H}\}} + f^{\mathrm{II}}_j(\mathrm{T}) = f^{\mathrm{II}}_j$, so indeed $f^{\mathrm{II}}_j \in \big\{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \bigtimes_{k=1}^n A_k\big\}$.

To show that $f^{\mathrm{IV}}_j \in \big\{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \bigtimes_{k=1}^n A_k\big\}$ for every j in $\{1,\dots,n_{\mathrm{IV}}\}$, consider any $f_{1:n}$ in $\bigtimes_{k=1}^n A_k$ such that $f_{1:n}$ satisfies the conditions of Situation (b) for $i := j+2n_{\mathrm{I}}+n_{\mathrm{II}}+2n_{\mathrm{III}}$, which is then an element of $\{2n_{\mathrm{I}}+n_{\mathrm{II}}+2n_{\mathrm{III}}+1,\dots,n\}$. Then $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k = (f^{\mathrm{IV}}_j(\mathrm{H}) - f^{\mathrm{IV}}_j(\mathrm{T}))g^{\mathrm{T}}_i = (f^{\mathrm{IV}}_j(\mathrm{H}) - f^{\mathrm{IV}}_j(\mathrm{T}))\big(-\mathbb{I}_{\{\mathrm{T}\}} + \frac{f^{\mathrm{IV}}_i(\mathrm{H})}{f^{\mathrm{IV}}_i(\mathrm{H})-f^{\mathrm{IV}}_i(\mathrm{T})}\big) = (f^{\mathrm{IV}}_j(\mathrm{T}) - f^{\mathrm{IV}}_j(\mathrm{H}))\mathbb{I}_{\{\mathrm{T}\}} + f^{\mathrm{IV}}_j(\mathrm{H}) = f^{\mathrm{IV}}_j$, so indeed $f^{\mathrm{IV}}_j \in \big\{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \bigtimes_{k=1}^n A_k\big\}$.

To show that $f^{\mathrm{III}}_j \in \big\{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \bigtimes_{k=1}^n A_k\big\}$ for every j in $\{1,\dots,n_{\mathrm{III}}\}$, consider any $f_{1:n}$ in $\bigtimes_{k=1}^n A_k$ such that $f_{1:n}$ satisfies the conditions of Situation (c) for $i := j+2n_{\mathrm{I}}+n_{\mathrm{II}}$, which is then an element of $\{2n_{\mathrm{I}}+n_{\mathrm{II}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}\}$. Then $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k = -\frac{1}{2}(3f^{\mathrm{III}}_j(\mathrm{H}) + f^{\mathrm{III}}_j(\mathrm{T}))g^{\mathrm{H}}_i - \frac{1}{2}(f^{\mathrm{III}}_j(\mathrm{H}) + 3f^{\mathrm{III}}_j(\mathrm{T}))g^{\mathrm{T}}_i = -\frac{1}{2}(3f^{\mathrm{III}}_j(\mathrm{H}) + f^{\mathrm{III}}_j(\mathrm{T}))\big(-\mathbb{I}_{\{\mathrm{H}\}} + \frac{1}{4}\big) - \frac{1}{2}(f^{\mathrm{III}}_j(\mathrm{H}) +$

$3f_j^{\mathrm{III}}(\mathrm{T}))\left(-\mathbb{I}_{\{\mathrm{T}\}}+\frac{1}{4}\right)=f_j^{\mathrm{III}}(\mathrm{H})\mathbb{I}_{\{\mathrm{H}\}}+f_j^{\mathrm{III}}(\mathrm{T})\mathbb{I}_{\{\mathrm{T}\}}=f_j^{\mathrm{III}}$ if $f_j^{\mathrm{III}}\neq 0$, and $\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k=g_i^{\mathrm{H}}+g_i^{\mathrm{T}}=-\mathbb{I}_{\{\mathrm{H}\}}+\frac{1}{2}-\mathbb{I}_{\{\mathrm{T}\}}+\frac{1}{2}=0=f_j^{\mathrm{III}}$ if $f_j^{\mathrm{III}}=0$, so indeed $f_j^{\mathrm{III}}\in\{\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k\colon f_{1:n}\in\times_{k=1}^nA_k\}$.

To show that $f_j^{\mathrm{I}}\in\{\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k\colon f_{1:n}\in\times_{k=1}^nA_k\}$ for every j in $\{1,\dots,n_{\mathrm{I}}\}$, consider any $f_{1:n}$ in $\times_{k=1}^nA_k$ such that $f_{1:n}$ satisfies the conditions of Situation (d) for $i:=j+n_{\mathrm{I}}$, which is then an element of $\{n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}\}$. Then $\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k=g_j=f_j^{\mathrm{I}}$, so indeed $f_j^{\mathrm{I}}\in\{\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k\colon f_{1:n}\in\times_{k=1}^nA_k\}$.

We finally show, conversely, that $A\supseteq\{\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k\colon f_{1:n}\in\times_{k=1}^nA_k\}$. Consider any f in $\{\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k\colon f_{1:n}\in\times_{k=1}^nA_k\}$. Then $f=\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k$ for some $f_{1:n}$ in $\times_{k=1}^nA_k$. If this $f_{1:n}$ satisfies the conditions of Situation (a) for some i in $\{2n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}\}$, then $\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k=f_j^{\mathrm{II}}$ for $j:=i-2n_{\mathrm{I}}$, as shown above, so $f\in A$. If $f_{1:n}$ satisfies the conditions of Situation (b) for some i in $\{2n_{\mathrm{I}}+n_{\mathrm{II}}+2n_{\mathrm{III}}+1,\dots,n\}$, then $\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k=f_j^{\mathrm{IV}}$ for $j:=i-2n_{\mathrm{I}}-n_{\mathrm{II}}-2n_{\mathrm{III}}$, as shown above, so $f\in A$. If $f_{1:n}$ satisfies the conditions of Situation (c) for some i in $\{2n_{\mathrm{I}}+n_{\mathrm{II}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}\}$, then $\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k=f_j^{\mathrm{III}}$ for $j:=i-2n_{\mathrm{I}}-n_{\mathrm{II}}$, as shown above, so $f\in A$. If $f_{1:n}$ satisfies the conditions of Situation (d) for some i in $\{n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}\}$, then $\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k=f_j^{\mathrm{I}}$ for $j:=i-n_{\mathrm{I}}$, as shown above, so $f\in A$. The only other possibility is that $f_{1:n}$ satisfies the conditions of Situation (e1) or (e2), depending on whether or not $A\cap\mathscr{L}_{>0}\neq\emptyset$. If $A\cap\mathscr{L}_{>0}\neq\emptyset$ (so Situation (e1)), then $\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k=f_1^{\mathrm{I}}$, which is an element of A since $n_{\mathrm{I}}\geq 1$, so $f\in A$. If $A\cap\mathscr{L}_{>0}=\emptyset$ (so Situation (e2)), then $\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k=(f_1^{\mathrm{II}}(\mathrm{T})-f_1^{\mathrm{II}}(\mathrm{H}))\left(-\mathbb{I}_{\{\mathrm{H}\}}+\frac{f_1^{\mathrm{II}}(\mathrm{T})}{f_1^{\mathrm{II}}(\mathrm{T})-f_i^{\mathrm{II}}(\mathrm{H})}\right)=f_1^{\mathrm{II}}$ or $\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k=(f_1^{\mathrm{IV}}(\mathrm{H})-f_1^{\mathrm{IV}}(\mathrm{T}))\left(-\mathbb{I}_{\{\mathrm{H}\}}+\frac{f_1^{\mathrm{IV}}(\mathrm{H})}{f_1^{\mathrm{IV}}(\mathrm{H})-f_i^{\mathrm{IV}}(\mathrm{T})}\right)=f_1^{\mathrm{IV}}$, which both belong to A since $n_{\mathrm{II}}\geq 1$ and $n_{\mathrm{IV}}\geq 1$, so $f\in A$. There are no other possibilities, so we conclude that indeed $A\supseteq\{\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k\colon f_{1:n}\in\times_{k=1}^nA_k\}$. ■

Proof [Proof of Proposition 10] By definition, the least informative coherent set of desirable gamble sets that includes $\{\{f\}\colon f\in D\}$ is the natural extension $\mathrm{cl}_{\overline{\mathbf{K}}}(\mathscr{A}_D)$ of the assessment $\mathscr{A}_D:=\{\{f\}\colon f\in D\}$.

Let us first show that $\mathscr{A}_D$ is consistent. By Theorem 6, we need to show that $\emptyset\notin\mathscr{A}_D$ and $\{0\}\notin\mathrm{Posi}(\mathscr{L}_{>0}^{\mathrm{s}}\cup\mathscr{A}_D)=\mathrm{Posi}(\mathscr{A}_D)$, where the equality follows from the fact that $\mathscr{L}_{>0}^{\mathrm{s}}\subseteq\mathscr{A}_D$ by Axiom D_2. By definition, $\emptyset\notin\mathscr{A}_D$, so it remains to prove that $\{0\}\notin\mathrm{Posi}(\mathscr{A}_D)$. To this end, consider any singleton $\{g\}$ in $\mathrm{Posi}(\mathscr{A}_D)$. There are n in $\mathbb{N}$, $A_1,\dots,A_n$ in $\mathscr{A}_D$, and, for all $f_{1:n}$ in $\times_{k=1}^nA_k$, coefficients $\lambda_{1:n}^{f_{1:n}}>0$, such that $\left\{\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k\colon f_{1:n}\in\times_{k=1}^nA_k\right\}=\{g\}$. Since the entries of any sequence $f_{1:n}$ in $\times_{k=1}^nA_k$ belong to $\mathscr{L}_{>0}\cup D$, so does $\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k$, by repeated application of Axiom D_3. So $g\in D\cup\mathscr{L}_{>0}$. By Axiom D_1, $0\notin D$, whence indeed $g\neq 0$.

We now know that $\mathscr{A}_D$ is consistent, so by Theorem 6, its natural extension $\mathscr{A}_D$ is equal to $\mathrm{Rs}(\mathrm{Posi}(\mathscr{A}_D))$, since we already know that $\mathscr{L}_{>0}^{\mathrm{s}}\subseteq\mathscr{A}_D$. Let us show that $K_D=\mathrm{Rs}(\mathrm{Posi}(\mathscr{A}_D))$; we prove (i) $K_D\subseteq\mathrm{Rs}(\mathrm{Posi}(\mathscr{A}_D))$ and (ii) $K_D\supseteq\mathrm{Rs}(\mathrm{Posi}(\mathscr{A}_D))$. For (i), consider any A in K_D, so $A\cap D\neq\emptyset$, and therefore $f\in A$ for some f in D. This tells us that $\{f\}\in\mathscr{A}_D$. Since $K\subseteq\mathrm{Posi}(K)$ for any K in $\mathbf{K}$, we find that $\{f\}\in\mathrm{Posi}(\mathscr{A}_D)$. Therefore, any superset of $\{f\}$—and in particular indeed the set A—will belong to $\mathrm{Rs}(\mathrm{Posi}(\mathscr{A}_D))$.

Let us now show that (ii) $K_D\supseteq\mathrm{Rs}(\mathrm{Posi}(\mathscr{A}_D))$. To this end, consider any A in $\mathrm{Rs}(\mathrm{Posi}(\mathscr{A}_D))$. Then, by the definition of the Rs operator, there is some B in $\mathrm{Posi}(\mathscr{A}_D)$ such that $B\setminus\mathscr{L}_{\leq 0}\subseteq A$. This means that there are n in $\mathbb{N}$, $A_1,\dots,A_n$ in $\mathscr{L}_{>0}^{\mathrm{s}}\cup\mathscr{A}_D$, and, for all $f_{1:n}$ in $\times_{k=1}^nA_k$, coefficients $\lambda_{1:n}^{f_{1:n}}>0$, such that $\left\{\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k\colon f_{1:n}\in\times_{k=1}^nA_k\right\}=B$. Since the entries of any sequence $f_{1:n}$ in $\times_{k=1}^nA_k$ belong to $\mathscr{L}_{>0}\cup D$, so does $\sum_{k=1}^n\lambda_k^{f_{1:n}}f_k$, by repeated application of Axiom D_3. So we find $B\subseteq D\cup\mathscr{L}_{>0}=D$, where the equality is a consequence of Axiom D_2, and hence $B\setminus\mathscr{L}_{\leq 0}\subseteq D$. So $A\cap D\neq\emptyset$, and therefore indeed $A\in\mathbf{K}_D$.

We finish the proof by showing that K_D is indeed compatible with D, or, in other words, that $D_{K_D}=D$. Indeed, infer that $D_{K_D}=\{f\in\mathscr{L}\colon\{f\}\in K_D\}=\{f\in\mathscr{L}\colon\{f\}\cap D\neq\emptyset\}=D$. ■

Proof [Proof of Proposition 14] For Axiom K_0, consider any A in $K\rfloor E$. Then $\mathbb{I}_EA\in K$, whence $\mathbb{I}_EA\neq\emptyset$ since K satisfies Axiom K_0. Therefore indeed $A\neq\emptyset$.

For Axiom K_1, consider any A in $K\rfloor E$. Then $\mathbb{I}_EA\in K$, whence $\mathbb{I}_EA\setminus\{0\}\in K$ since K satisfies Axiom K_1. Since $\mathbb{I}_Ef\neq 0\Leftrightarrow f\neq 0$ for any gamble f on E, we find that $\mathbb{I}_E(A\setminus\{0\})\in K$, whence indeed $A\setminus\{0\}\in K\rfloor E$.

For Axiom K_2, consider any f in $\mathscr{L}(E)_{>0}$. Then $\mathbb{I}_Ef\in\mathscr{L}(\mathscr{X})_{>0}$, whence by Axiom K_2 $\{\mathbb{I}_Ef\}\in K$. Therefore indeed $\{f\}\in K\rfloor E$.

For Axiom K_3, consider any A_1 and A_2 in $K\rfloor E$, and, for any f in A_1 and g in A_2, any $(\lambda_{f,g},\mu_{f,g})>0$. Then $\mathbb{I}_EA_1\in K$ and $\mathbb{I}_EA_2\in K$, whence by Axiom K_3 $\{\lambda_{f,g}f+\mu_{f,g}g\colon f\in\mathbb{I}_EA_1,g\in\mathbb{I}_EA_2\}=\{\lambda_{f,g}\mathbb{I}_Ef+\mu_{f,g}\mathbb{I}_Eg\colon f\in A_1,g\in A_2\}=\mathbb{I}_E\{\lambda_{f,g}f+\mu_{f,g}g\colon f\in A_1,g\in A_2\}\in K$, where we identified $(\lambda_{\mathbb{I}_Ef,\mathbb{I}_Eg},\mu_{\mathbb{I}_Ef,\mathbb{I}_Eg})$ with $(\lambda_{f,g},\mu_{f,g})$, for any f in A_1 and g in A_2. Therefore indeed $\{\lambda_{f,g}f+\mu_{f,g}g\colon f\in A_1,g\in A_2\}\in K\rfloor E$.

For Axiom K_4, consider any A_1 in $K\rfloor E$ and any A_2 in $\mathscr{Q}$ such that $A_1\subseteq A_2$. Then $\mathbb{I}_EA_1\in K$ and $\mathbb{I}_EA_1\subseteq\mathbb{I}_EA_2$, whence by Axiom K_4 $\mathbb{I}_EA_2\in K$. Therefore indeed $A_2\in R\rfloor E$. ■

Proof [Proof of Proposition 15] For the first statement, consider any f in $\mathscr{L}(E)$, and infer the following chain of equivalences:

$$f \in D_K\rfloor E \Leftrightarrow \mathbb{I}_E f \in D_K \Leftrightarrow \{\mathbb{I}_E f\} \in K \Leftrightarrow \{f\} \in K\rfloor E \Leftrightarrow f \in D_{K\rfloor E},$$

where the first equivalence follows from Definition 12, the second one and the last one are due to Equation (3), and the third one follows from Definition 13.

For the second statement, consider any A in $\mathscr{Q}(\mathscr{L}(E))$ and the following chain of equivalences:

$$\begin{aligned} A \in K_D\rfloor E &\Leftrightarrow \mathbb{I}_E A \in K_D \Leftrightarrow \mathbb{I}_E A \cap D \neq \emptyset \\ &\Leftrightarrow (\exists f \in A)\mathbb{I}_E f \in D \\ &\Leftrightarrow A \cap D\rfloor E \neq \emptyset \Leftrightarrow A \in K_{D\rfloor E}, \end{aligned}$$

where the first equivalence follows from Definition 13, the second one and the last one are due to Proposition 10, and the fourth one follows from Definition 12.

We now turn to the last statement. By Theorem 11 we have that $K = \bigcap\{K_D : D \in \overline{\mathbf{D}}(K)\}$, implying that $A \in K \Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))A \in K_D$, for any A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}))$. Therefore in particular, for any A in $\mathscr{Q}(\mathscr{L}(E))$,

$$\begin{aligned} A \in K\rfloor E \Leftrightarrow \mathbb{I}_E A \in K &\Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))\mathbb{I}_E A \in K_D \\ &\Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))A \in K_D\rfloor E \\ &\Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))A \in K_{D\rfloor E} \\ &\Leftrightarrow A \in \bigcap\{K_{D\rfloor E} : D \in \overline{\mathbf{D}}(K)\}, \end{aligned}$$

where the first and third equivalences follow from Definition 13, and the fourth one follows from the already established second statement of this proposition. Therefore indeed $K\rfloor E = \bigcap\{K_{D\rfloor E} : D \in \overline{\mathbf{D}}(K)\}$. ■

Proof [Proof of Proposition 19] The result follows immediately, once we realise that $A_1 \neq \emptyset \Leftrightarrow A_1^* \neq \emptyset$, $f > 0 \Leftrightarrow f^* > 0$, $\lambda f + \mu g \in A_1 \Leftrightarrow \lambda f^* + \mu g^* \in A_1^*$, and $A_1 \subseteq A_2 \Leftrightarrow A_1^* \subseteq A_2^*$, for all f in $\mathscr{L}(\mathscr{X}_O)$ whose cylindrical extension is f^*, all A_1 and A_2 in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$ whose cylindrical extensions are A_1^* and A_2^*, and all λ in μ in $\mathbb{R}$ such that $(\lambda,\mu) > 0$. ■

Proof [Proof of Proposition 20] For the first statement, observe that indeed

$$\begin{aligned} \mathrm{marg}_O D_K &= \{f \in \mathscr{L}(\mathscr{X}_O) : f \in D_K\} \\ &= \{f \in \mathscr{L}(\mathscr{X}_O) : \{f\} \in K\} \\ &= \{f \in \mathscr{L}(\mathscr{X}_O) : \{f\} \in \mathrm{marg}_O K\} = D_{\mathrm{marg}_O K}, \end{aligned}$$

where the second and last equalities follow from Equation (3), and the third one follows from Definition 18.

For the second statement, observe that

$$\begin{aligned} &\mathrm{marg}_O K_D \\ &= \{A \in \mathscr{Q}(\mathscr{L}(\mathscr{X}_O)) : A \in K_D\} \\ &= \{A \in \mathscr{Q}(\mathscr{L}(\mathscr{X}_O)) : A \cap D \neq \emptyset\} \\ &= \{A \in \mathscr{Q}(\mathscr{L}(\mathscr{X}_O)) : A \cap \mathrm{marg}_O D \neq \emptyset\} \\ &= \{A \in \mathscr{Q}(\mathscr{L}(\mathscr{X}_O)) : A \in K_{\mathrm{marg}_O D}\} = K_{\mathrm{marg}_O D}, \end{aligned}$$

where the first equality follows from Definition 18 and the second and penultimate equalities follow from Proposition 10.

We now turn to the last statement. By Theorem 11 we have that $K = \bigcap\{K_D : D \in \overline{\mathbf{D}}(K)\}$, implying that $A \in K \Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))A \in K_D$, for any A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_{1:n}))$. Therefore in particular, for any A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$,

$$\begin{aligned} A \in \mathrm{marg}_O K \Leftrightarrow A \in K &\Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))A \in K_D \\ &\Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))A \in \mathrm{marg}_O K_D \\ &\Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))A \in K_{\mathrm{marg}_O D} \\ &\Leftrightarrow A \in \bigcap\{K_{\mathrm{marg}_O D} : D \in \overline{\mathbf{D}}(K)\}, \end{aligned}$$

where the first and third equivalences follow from Definition 18, and the fourth one follows from the already established second statement of this proposition. Therefore indeed $\mathrm{marg}_O K = \bigcap\{K_{\mathrm{marg}_O D} : D \in \overline{\mathbf{D}}(K)\}$. ■

Proof [Proof of Proposition 21] We will first show that any coherent set of desirable gamble sets K' on $\mathscr{L}(\mathscr{X}_{1:n})$ that marginalises to K_O must be at least as informative as $\mathrm{ext}_{1:n}(K_O)$. To establish this, since K' marginalises to K_O, note that $A \in K_O \Leftrightarrow A \in K'$, for all A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$. Therefore, in particular, $A \in K_O \Rightarrow A \in K'$ for all A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$, so $K_O \subseteq K'$. This implies that indeed $\mathrm{ext}_{1:n}(K_O) = \mathrm{Rs}(\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}(\mathscr{X}_{1:n})_{>0} \cup K_O)) \subseteq \mathrm{Rs}(\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}(\mathscr{X}_{1:n})_{>0} \cup K')) = K'$, where the final equality holds because K' is coherent.

So we already know that any coherent set of desirable gamble sets that marginalises to K_O must be at least as informative as $\mathrm{ext}_{1:n}(K_O)$. It therefore suffices to prove that $\mathrm{ext}_{1:n}(K_O)$ is coherent and that it marginalises to K_O. To show that $\mathrm{ext}_{1:n}(K_O) = \mathrm{Rs}(\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}(\mathscr{X}_{1:n})_{>0} \cup \mathscr{A}_{K_O}^{1:n}))$ is coherent, by Theorem 6 it suffices to show that K_O is a consistent assessment—that is, to show that $\emptyset \notin \mathscr{A}_{K_O}^{1:n}$ and $\{0\} \notin \mathrm{Posi}(\mathscr{L}^{\mathrm{s}}(\mathscr{X}_{1:n})_{>0} \cup \mathscr{A}_{K_O}^{1:n})$. That this is indeed the case follows from the coherence of $K_O = \mathscr{A}_{K_O}^{1:n}$.

The proof is therefore complete if we can show that $\mathrm{marg}_O(\mathrm{ext}_{1:n}(K_O)) = K_O$. Since for any A in K_O it is obvious that both $A \in \mathrm{ext}_{1:n}(K_O)$ and $A \in \mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$, we see immediately that $K_O \subseteq \mathrm{marg}_O(\mathrm{ext}_{1:n}(K_O))$, so we concentrate on proving the converse inclusion. Consider any A in $\mathrm{marg}_O(\mathrm{ext}_{1:n}(K_O))$, meaning that both $A \in \mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$ and $A \in \mathrm{ext}_{1:n}(K_O)$. That $A \in \mathrm{ext}_{1:n}(K_O)$ implies that $B \setminus \mathscr{L}_{\leq 0} \subseteq A$ for some B in $\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup K_O)$. Then there are m in $\mathbb{N}$, $A_1, \dots, A_m$ in $\mathscr{L}^{\mathrm{s}}_{>0} \cup K_O$, and coefficients $\lambda_{1:m}^{f_{1:m}} > 0$ for all $f_{1:m}$ in $\times_{k=1}^m A_k$ such that $B = \left\{\sum_{k=1}^m \lambda_k^{f_{1:m}} f_k : f_{1:m} \in \times_{k=1}^m A_k\right\}$. Without loss of generality, assume that $A_1, \dots, A_\ell \in K_O$ and $A_{\ell+1}, \dots, A_m \in \mathscr{L}^{\mathrm{s}}_{>0}$ for some ℓ in $\{0, \dots, m\}$. Consider the special subset $P :=$

$\left\{f_{1:m} \in \times_{k=1}^{m} A_k \colon \lambda_{1:\ell}^{f_{1:m}} = 0\right\}$ of $\times_{k=1}^{m} A_k$. If $P \neq \emptyset$, then for every element $g_{1:m}$ of P we have that $\sum_{k=1}^{m} \lambda_k^{g_{1:m}} g_k > 0$, so $B \cap \mathcal{L}_{>0} \neq \emptyset$. Since $B \setminus \mathcal{L}_{\leq 0} \subseteq A$, also $A \cap \mathcal{L}_{>0}(\mathcal{X}_O) \neq \emptyset$, whence $A \in K_O$ by coherence [more specifically, by Axioms K_2 and K_4]. Therefore, assume that $P = \emptyset$, and define the coefficients

$$\mu_k^{f_{1:m}} := \begin{cases} \lambda_k^{f_{1:m}} & \text{if } k \leq \ell \\ 0 & \text{if } k \geq \ell + 1 \end{cases}$$

for all $f_{1:m}$ in $\times_{k=1}^{m} A_k$ and k in $\{1, \ldots, m\}$. Because $P = \emptyset$, for every $f_{1:m}$ in $\times_{k=1}^{m} A_k$ we have that $\mu_{1:\ell}^{f_{1:m}} = \lambda_{1:\ell}^{f_{1:m}} > 0$. Also, for every $f_{1:m}$ in $\times_{k=1}^{m} A_k$ and $k \geq \ell + 1$, the coefficient $\mu_k^{f_{1:m}}$ equals 0, so we identify $\mu_{1:\ell}^{f_{1:\ell}}$ with $\mu_{1:\ell}^{f_{1:m}}$. Then every element of $\left\{\sum_{k=1}^{m} \mu_k^{f_{1:m}} f_k \colon f_{1:m} \in \times_{k=1}^{m} A_k\right\} = \left\{\sum_{k=1}^{\ell} \mu_k^{f_{1:\ell}} f_k \colon f_{1:\ell} \in \times_{k=\ell+1}^{m} A_k\right\} \in \mathrm{Posi}(K_O) = K_O$ is dominated by an element of B. Therefore, by Lemma 27 below $B \in K_O$, whence by coherence, indeed also $A \in K_O$. ■

Lemma 27 *Consider any coherent set of desirable gamble sets K and any gamble sets A and B in $\mathcal{Q}$. If $A \in K$ and $(\forall f \in A)(\exists g \in B) f \leq g$, then $B \in K$.*

Proof Let $A := \{f_1, \ldots, f_m\}$ for some m in $\mathbb{N}$, and denote the finite possibility space $\mathcal{X} = \{x_1, \ldots, x_\ell\}$ for some ℓ in $\mathbb{N}$. Since $(\forall f \in A)(\exists g \in B) f \leq g$, we have that B is a superset of

$$\begin{aligned} B' &:= \left\{f_1 + \sum_{k=1}^{\ell} \mu_{k,1} \mathbb{I}_{\{x_k\}}, \ldots, f_m + \sum_{k=1}^{\ell} \mu_{k,m} \mathbb{I}_{\{x_k\}}\right\} \\ &= \left\{f_j + \sum_{k=1}^{\ell} \mu_{k,j} \mathbb{I}_{\{x_k\}} \colon j \in \{1, \ldots, m\}\right\} \end{aligned}$$

for some $\mu_{k,j} \geq 0$ for all k in $\{1, \ldots, \ell\}$ and j in $\{1, \ldots, m\}$. Use the definition of the Posi operator, with $A_1 := \{\mathbb{I}_{\{x_1\}}\} \in K$, ..., $A_\ell := \{\mathbb{I}_{\{x_\ell\}}\} \in K$, $A_{\ell+1} := A \in K$, and for all $f_{1:\ell+1}^j := (\mathbb{I}_{\{x_1\}}, \mathbb{I}_{\{x_2\}}, \ldots, \mathbb{I}_{\{x_\ell\}}, f_j) \in \times_{k=1}^{\ell+1} A_k$, let $\lambda_{1:\ell+1}^{f_{1:\ell+1}^j} := (\mu_{1,j}, \mu_{2,j}, \ldots, \mu_{\ell,j}, 1) > 0$, to infer that

$$\begin{aligned} &\left\{\sum_{k=1}^{\ell+1} \lambda_k^{f_{1:\ell+1}^j} f_k^j \colon f_{1:\ell+1}^j \in \times_{k=1}^{\ell+1} A_k\right\} \\ &\qquad = \left\{f_j + \sum_{k=1}^{\ell} \mu_{k,j} \mathbb{I}_{\{x_k\}} \colon j \in \{1, \ldots, m\}\right\} = B' \end{aligned}$$

belongs to $\mathrm{Posi}(K)$. Because $B \supseteq B'$, we have that $B \in \mathrm{Rs}(\mathrm{Posi}(K))$. But since $K = \mathrm{Rs}(\mathrm{Posi}(K))$ by coherence, we infer that indeed $B \in K$. ■

Proof [Proof of Proposition 22] By Theorem 11 we have that $K_O = \bigcap\{K_{D_O} \colon D_O \in \overline{\mathbf{D}}(K_O)\}$, implying that $A \in K_O \Leftrightarrow (\forall D_O \in \overline{\mathbf{D}}(K_O)) A \in K_{D_O}$, for any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O))$. Therefore, for any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O))$,

$$\begin{aligned} A \in \mathrm{ext}_{1:n}(K_O) &\Leftrightarrow A \in K_O \\ &\Leftrightarrow (\forall D_O \in \overline{\mathbf{D}}(K_O)) A \in K_D \\ &\Leftrightarrow (\forall D_O \in \overline{\mathbf{D}}(K_O)) A \cap D_O \neq \emptyset \\ &\Leftrightarrow (\forall D_O \in \overline{\mathbf{D}}(K_O)) A \cap \mathrm{ext}_{1:n}^{\mathbf{D}}(D_O) \neq \emptyset \\ &\Leftrightarrow A \in \bigcap\{K_{\mathrm{ext}_{1:n}^{\mathbf{D}}(D_O)} \colon D \in \overline{\mathbf{D}}(K_O)\}, \end{aligned}$$

where the first equivalence holds because $\mathrm{ext}_{1:n}(K_O)$ marginalises to K_O, the third one because of Proposition 10, and the fourth one because $\mathrm{ext}_{1:n}^{\mathbf{D}}(D_O)$ marginalises to D_O. So indeed $\mathrm{ext}_{1:n}(K_O) = \bigcap\{K_{\mathrm{ext}_{1:n}^{\mathbf{D}}(D_O)} \colon D_O \in \overline{\mathbf{D}}(K_O)\}$. ■

Proof [Proof of Proposition 23] Consider the following chain of equalities:

$$\begin{aligned} &\mathrm{marg}_O(K_n \rfloor E_I) \\ &\quad = \{A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O)) \colon A \in K_n \rfloor E_I\} \\ &\quad = \{A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O)) \colon \mathbb{I}_{E_I} A \in K_n\} \\ &\quad = \{A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O)) \colon \mathbb{I}_{E_I} A \in \mathrm{marg}_{I \cup O} K_n\} \\ &\quad = \{A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O)) \colon A \in (\mathrm{marg}_{I \cup O} K_n) \rfloor E_I\} \\ &\quad = \mathrm{marg}_O((\mathrm{marg}_{I \cup O} K_n) \rfloor E_I), \end{aligned}$$

where the third equality holds because $\mathbb{I}_{E_I} A$ is a set of gambles on $\mathcal{X}_{I \cup O}$. ■

Proof [Proof of Proposition 25] To show that (i) implies (ii), consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O))$ and E_I in $\mathcal{P}_{\overline{\emptyset}}(\mathcal{X}_I)$, and recall the following equivalences:

$$\begin{aligned} A \in K_n &\Leftrightarrow A \in \mathrm{marg}_O(K_n \rfloor E_I) && \text{by Definition 18 and (i)} \\ &\Leftrightarrow A \in K_n \rfloor E_I && \text{by Definition 18} \\ &\Leftrightarrow \mathbb{I}_{E_I} A \in K_n && \text{by Definition 13.} \end{aligned}$$

To show that (ii) implies (i), consider any E_I in $\mathcal{P}_{\overline{\emptyset}}(\mathcal{X}_I)$, and recall the following equalities:

$$\begin{aligned} \mathrm{marg}_O(K_n \rfloor E_I) &= \{A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O)) \colon A \in K_n \rfloor E_I\} \\ &= \{A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O)) \colon \mathbb{I}_{E_I} A \in K_n\} \\ &= \{A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O)) \colon A \in K_n\} \\ &= \mathrm{marg}_O K_n, \end{aligned}$$

where the first and last equalities follow from Definition 18, the second one from Definition 13, and the third one from (ii). ■

Proof [Proof of Theorem 26] We will first show that any coherent set of desirable gamble sets K' on $\mathcal{L}(\mathcal{X}_{1:n})$ that marginalises to K_O and that satisfies epistemic irrelevance of X_I to X_O must be at least as informative as $\mathrm{ext}_{1:n}^{\mathrm{irr}}(K_O)$. To this end, consider any B in $\mathcal{A}_{I \to O}^{\mathrm{irr}}$.

Then $B = \mathbb{I}_{E_I}A$ for some E_I in $\mathscr{P}_{\bar{\emptyset}}(\mathscr{X}_I)$ and A in K_O. Since K' marginalises to K_O, infer that $A \in K'$. Furthermore, since K' satisfies epistemic irrelevance of X_I to X_O, by Proposition 25 also $B = \mathbb{I}_{E_I}A \in K'$. We conclude that $B \in \mathscr{A}^{\mathrm{irr}}_{I\to O} \Rightarrow B \in K' \Leftrightarrow B \in \mathrm{marg}_{I\cup O}K'$, for every B in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_{I\cup O}))$, so $\mathscr{A}^{\mathrm{irr}}_{I\to O} \subseteq \mathrm{marg}_{I\cup O}K'$. This implies that $\mathrm{Rs}(\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}(\mathscr{X}_{I\cup O})_{>0} \cup \mathscr{A}^{\mathrm{irr}}_{I\to O})) \subseteq \mathrm{Rs}(\mathrm{Posi}(\mathrm{marg}_{I\cup O}K')) = \mathrm{marg}_{I\cup O}K'$, where the equality follows from the fact that $\mathrm{marg}_{I\cup O}K'$ is coherent by Proposition 19. Then $\mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O) = \mathrm{ext}_{1:n}(\mathrm{Rs}(\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}(\mathscr{X}_{I\cup O})_{>0} \cup \mathscr{A}^{\mathrm{irr}}_{I\to O}))) \subseteq \mathrm{ext}_{1:n}(\mathrm{marg}_{I\cup O}K')$ and since by Proposition 21 $\mathrm{ext}_{1:n}(\mathrm{marg}_{I\cup O}K')$ is the least informative coherent set of desirable gamble sets on $\mathscr{L}(\mathscr{X}_{1:n})$ that marginalises to $\mathrm{marg}_{I\cup O}K'$, we have that $\mathrm{ext}_{1:n}(\mathrm{Rs}(\mathrm{Posi}(\mathrm{marg}_{I\cup O}K'))) \subseteq K'$. Therefore indeed $\mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O) \subseteq K'$.

The proof of the first statement is therefore complete if we could show that $\mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O)$ (i) is coherent, (ii) marginalises to K_O, and (iii) satisfies epistemic irrelevance of X_I to X_O.

For (i), it suffices to show that $\emptyset \notin \mathscr{A}^{\mathrm{irr}}_{I\to O}$ and $\{0\} \notin K^{\mathrm{irr}}_{I\cup O} = \mathrm{Rs}(\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}(\mathscr{X}_{I\cup O})_{>0} \cup \mathscr{A}^{\mathrm{irr}}_{I\to O}))$:[6] indeed, if this is the case, then by Theorem 6 $K^{\mathrm{irr}}_{I\cup O}$ is a coherent set of desirable gamble sets on $\mathscr{L}(\mathscr{X}_{I\cup O})$, and then by Proposition 21 $\mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O)$ is a coherent set of desirable gamble sets on $\mathscr{L}(\mathscr{X}_{1:n})$. So we will show that $\emptyset \notin \mathscr{A}^{\mathrm{irr}}_{I\to O}$ and $\{0\} \notin K^{\mathrm{irr}}_{I\cup O}$. That $\emptyset \notin \mathscr{A}^{\mathrm{irr}}_{I\to O}$ is clear from Equation (6) because K_O is coherent. So we focus on proving that $\{0\} \notin K^{\mathrm{irr}}_{I\cup O}$. Assume *ex absurdo* that $\{0\} \in K^{\mathrm{irr}}_{I\cup O}$. By Lemma 28 below we would then infer that $\{\sum_{x_I\in\mathscr{X}_I} h(x_I,\bullet) \colon h \in \{0\}\} = \{0\} \in K_O$, contradicting the coherence of K_O. Therefore indeed $\{0\} \notin K^{\mathrm{irr}}_{I\cup O}$.

For (ii), we need to show that $A \in \mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O) \Leftrightarrow A \in K_O$ for any A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$. For necessity, consider any A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$ and assume that $A \in \mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O)$. By Lemma 28 then $\{\sum_{x_I\in\mathscr{X}_I} h(x_I,\bullet) \colon h \in A\} \in K_O$. Since A is a set of gambles on $\mathscr{X}_O$, we infer $\{\sum_{x_I\in\mathscr{X}_I} h(x_I,\bullet) \colon h \in A\} = \{\sum_{x_I\in\mathscr{X}_I} h \colon h \in A\} = \{|\mathscr{X}_I|h \colon h \in A\} = |\mathscr{X}_I|A$, whence by coherence, indeed $A \in K_O$. For sufficiency, consider any A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$ and assume that $A \in K_O$. Then $A = \mathbb{I}_{\mathscr{X}_I}A$ and $\mathscr{X}_I \in \mathscr{P}_{\bar{\emptyset}}(\mathscr{X}_I)$, so $A \in \mathscr{A}^{\mathrm{irr}}_{I\to O}$. Therefore indeed $A \in \mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O)$.

For (iii), by Proposition 25 it suffices to show that $A \in \mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O) \Leftrightarrow \mathbb{I}_{E_I}A \in \mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O)$, for all A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$ and E_I in $\mathscr{P}_{\bar{\emptyset}}(\mathscr{X}_I)$. For necessity, consider any A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$ and any E_I in $\mathscr{P}_{\bar{\emptyset}}(\mathscr{X}_I)$, and assume that $A \in \mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O)$. Since we just have shown that $\mathrm{marg}_O\mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O) = K_O$, this implies that $A \in K_O$, whence indeed $\mathbb{I}_{E_I}A \in \mathscr{A}^{\mathrm{irr}}_{I\to O} \subseteq \mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O)$. For sufficiency, consider any A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$ and any E_I in $\mathscr{P}_{\bar{\emptyset}}(\mathscr{X}_I)$, and assume that $\mathbb{I}_{E_I}A \in \mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O)$. Since by Proposition 21 $\mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O)$ marginalises to $K^{\mathrm{irr}}_{I\cup O}$, this implies that $\mathbb{I}_{E_I}A \in K^{\mathrm{irr}}_{I\cup O}$. Use Lemma 28 to infer that then $\{\sum_{x_I\in\mathscr{X}_I} h(x_I,\bullet) \colon h \in \mathbb{I}_{E_I}A\} = \{\sum_{x_I\in\mathscr{X}_I} \mathbb{I}_{E_I}h(x_I,\bullet) \colon h \in A\} = \{|E_I|h \colon h \in A\} = |E_I|A \in K_O$, whence by coherence indeed $A \in K_O$.

The second statement is a direct application of Proposition 22. ∎

6. These two conditions are equivalent to $\emptyset \notin \mathscr{A}^{\mathrm{irr}}_{I\to O}$ and $\{0\} \notin \mathrm{Posi}(\mathscr{L}^{\mathrm{s}}(\mathscr{X}_{I\cup O})_{>0} \cup \mathscr{A}^{\mathrm{irr}}_{I\to O})$.

Lemma 28 *Consider any disjoint and non-empty subsets I and O of $\{1,\dots,n\}$, and any coherent set of desirable gamble sets K_O on $\mathscr{L}(\mathscr{X}_O)$. Then*

$$A \in \mathrm{Rs}(\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}(\mathscr{X}_{I\cup O})_{>0} \cup \mathscr{A}^{\mathrm{irr}}_{I\to O})) \Rightarrow \left\{\sum_{x_I\in\mathscr{X}_I} h(x_I,\bullet) \colon h \in A\right\} \in K_O,$$

for all A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_{I\cup O}))$.

Proof Consider any A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_{I\cup O}))$ and assume that $A \in \mathrm{Rs}(\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}(\mathscr{X}_{I\cup O})_{>0} \cup \mathscr{A}^{\mathrm{irr}}_{I\to O}))$. Then $B \setminus \mathscr{L}_{\leq 0} \subseteq A$ for some B in $\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}(\mathscr{X}_{I\cup O})_{>0} \cup \mathscr{A}^{\mathrm{irr}}_{I\to O})$, implying that $B = \{\sum_{k=1}^m \lambda_k^{f_{1:m}} f_k \colon f_{1:m} \in \times_{k=1}^m A_k\}$ for some m in $\mathbb{N}$, $A_1, \dots, A_m$ in $\mathscr{L}^{\mathrm{s}}(\mathscr{X}_{I\cup O})_{>0} \cup \mathscr{A}^{\mathrm{irr}}_{I\to O}$, and coefficients $\lambda_{1:m}^{f_{1:m}} > 0$ for all $f_{1:m}$ in $\times_{k=1}^m A_k$. Without loss of generality, assume that $A_1,\dots,A_\ell \in \mathscr{A}^{\mathrm{irr}}_{I\to O}$ and $A_{\ell+1},\dots,A_m \in \mathscr{L}^{\mathrm{s}}(\mathscr{X}_{I\cup O})_{>0}$ for some ℓ in $\{0,\dots,m\}$. Consider the special subset $P := \{f_{1:m} \in \times_{k=1}^m A_k \colon \lambda_{1:\ell}^{f_{1:m}} = 0\}$ of $\times_{k=1}^m A_k$. If $P \neq \emptyset$, then for every element $g_{1:m}$ of P we have that $\sum_{k=1}^m \lambda_k^{g_{1:m}} g_k > 0$, so $B \cap \mathscr{L}(\mathscr{X}_{I\cup O})_{>0} \neq \emptyset$, and therefore also $A \cap \mathscr{L}(\mathscr{X}_{I\cup O})_{>0} \neq \emptyset$, whence $\{\sum_{x_I\in\mathscr{X}_I} h(x_I,\bullet) \colon h \in A\} \in K_O$ by the coherence of K_O [more specifically, by Axioms K_2 and K_4]. So assume that $P = \emptyset$, and define the coefficients

$$\mu_k^{f_{1:m}} := \begin{cases} \lambda_k^{f_{1:m}} & \text{if } k \leq \ell \\ 0 & \text{if } k \geq \ell+1 \end{cases}$$

for all $f_{1:m}$ in $\times_{k=1}^m A_k$ and k in $\{1,\dots,m\}$. Because $P = \emptyset$, for every $f_{1:m}$ in $\times_{k=1}^m A_k$ we have that $\mu_{1:\ell}^{f_{1:m}} = \lambda_{1:\ell}^{f_{1:m}} > 0$. Also, for every $f_{1:m}$ in $\times_{k=1}^m A_k$ and $k \geq \ell+1$, the coefficient $\mu_k^{f_{1:m}}$ equals 0, so we identify $\mu_{1:\ell}^{f_{1:m}}$ with $\mu_{1:\ell}^{f_{1:\ell}}$. Then every element of $B' := \{\sum_{k=1}^m \mu_k^{f_{1:m}} f_k \colon f_{1:m} \in \times_{k=1}^m A_k\} = \{\sum_{k=1}^\ell \mu_k^{f_{1:\ell}} f_k \colon f_{1:\ell} \in \times_{k=1}^\ell A_k\}$ is dominated by an element of B. For every k in $\{1,\dots,\ell\}$ the gamble set A_k belongs to $\mathscr{A}^{\mathrm{irr}}_{I\to O}$, so we may write $A_k = \mathbb{I}_{E_k}A_{O,k}$ with $E_k \in \mathscr{P}_{\bar{\emptyset}}(\mathscr{X}_I)$ and $A_{O,k} \in K_O$. Therefore $|A_k| = |A_{O,k}|$, and every f_k in A_k can be uniquely written as $f_k = \mathbb{I}_{E_k}g_k$ with g_k in $A_{O,k}$. So for every $f_{1:\ell}$ in $\times_{k=1}^\ell A_k$ there is a unique $g_{1:\ell}$ in $\times_{k=1}^\ell A_{O,k}$ such that $f_k = \mathbb{I}_{E_k}g_k$ for every k in

$\{1,\dots,\ell\}$. For every $f_{1:\ell}$ in $\bigtimes_{k=1}^{\ell} A_k$ and its corresponding unique $g_{1:\ell}$ in $\bigtimes_{k=1}^{\ell} A_{O,k}$, we define $\mu_{1:\ell}^{g_{1:\ell}} := \mu_{1:\ell}^{f_{1:\ell}}$. Therefore $B' = \left\{\sum_{k=1}^{\ell} \mu_k^{g_{1:\ell}} \mathbb{I}_{E_k} g_k \colon g_{1:\ell} \in \bigtimes_{k=1}^{\ell} A_{O,k}\right\}$, and hence

$$\left\{\sum_{x_I \in \mathscr{X}_I} h(x_I, \bullet) \colon h \in B'\right\}$$
$$= \left\{\sum_{x_I \in \mathscr{X}_I} \sum_{k=1}^{\ell} \mu_k^{g_{1:\ell}} \mathbb{I}_{E_k} g_k(x_I, \bullet) \colon g_{1:\ell} \in \bigtimes_{k=1}^{\ell} A_{O,k}\right\}$$
$$= \left\{\sum_{k=1}^{\ell} \mu_k^{g_{1:\ell}} |E_k| g_k(x_I, \bullet) \colon g_{1:\ell} \in \bigtimes_{k=1}^{\ell} A_{O,k}\right\}$$

belongs to $\mathrm{Posi}(K_O) = K_O$. Since every element of B' is dominated by an element of B, we have that every element of $\left\{\sum_{x_I \in \mathscr{X}_I} h(x_I, \bullet) \colon h \in B'\right\}$ is dominated by an element of $\left\{\sum_{x_I \in \mathscr{X}_I} h(x_I, \bullet) \colon h \in B\right\}$, so by Lemma 27 $\left\{\sum_{x_I \in \mathscr{X}_I} h(x_I, \bullet) \colon h \in B\right\} \in K_O$. By K_4 we have that also indeed $\left\{\sum_{x_I \in \mathscr{X}_I} h(x_I, \bullet) \colon h \in A\right\} \in K_O$. ■