A Proofs of simple regret for the uniform strategies

Results in the deterministic and stochastic cases with known smoothness parameters were also reported in Hren and Munos (2008) and Bubeck and Munos (2010).

A.1 Deterministic case

Deterministic feedback Let us consider the uniform exploration that evaluates all the cells at the deepest possible depth H with a budget of n and recommends $x(n)$ the x with the highest observed $f(x)$. We have H the largest value such that $K^H \leq n$. Therefore $H = \lfloor \log_K(n) \rfloor$. Because of Assumption 1 we have $r_n \leq \nu\rho^H$. Therefore $r_n = \mathcal{O}\Big((K/n)^{\frac{\log 1/\rho}{\log K}}\Big)$.

Proof. Consider one global optimum $x^\star$. For all $i \in [K^H]$, let $x_{H,i}$ be the element selected for evaluation by the uniform exploration in $\mathcal{P}_{H,i}$. Then,

$$f(x(n)) \overset{\textbf{(a)}}{\geq} f(x_{H,i^\star_H}) \overset{\textbf{(b)}}{\geq} f(x^\star) - \nu\rho^H.$$

where **(a)** is because uniform has opened all the cells at depth H and $x(n) = \arg\max_{\mathcal{P}_{H,i} \in \mathcal{T}} f(x_{H,i})$, and **(b)** is by Assumption 1. Therefore $r_n = f(x^\star) - f(x(n)) \leq \nu\rho^H = \nu\rho^{\lfloor \log_K(n) \rfloor} = \nu\rho^{\lfloor \log_K(n/K)+1 \rfloor} \leq \nu\rho^{\log_K(n/K)} = \nu\Big((K/n)^{\frac{\log 1/\rho}{\log K}}\Big)$. □

A.2 Stochastic case without knowledge of the smoothness parameters ν, ρ

Proof. Consider one global optimum $x^\star$. For all $i \in [K^H]$, let us fix $x_{h,i}$ be the element selected for evaluation by the uniform exploration in $\mathcal{P}_{h,i}$ each of the $\lfloor \frac{n}{K^H} \rfloor$ times this cell is selected. We define and consider event ξ_δ and prove it holds with high probability.

Let ξ_δ be the event under which all average estimates in the cells receiving at least one evaluation from uniform are within their classical confidence interval, then $P(\xi_\delta) \geq 1 - \delta$, where

$$\xi_\delta \triangleq \left\{ \forall i \in [K^H], \left| \widehat{f}_{H,i} - f(x_{H,i}) \right| \leq b\sqrt{\frac{\log(2n/\delta)}{n/K^H}} \right\}.$$

We have $P(\xi_\delta) \geq 1 - \delta$, using Chernoff-Hoeffding's inequality taking a union bound on all opened cells. On ξ_δ we have,

$$\begin{aligned} f(x(n)) &\overset{\textbf{(a)}}{\geq} \widehat{f}(x(n)) - b\sqrt{\frac{\log(2n/\delta)}{n/K^H}} \\ &\overset{\textbf{(b)}}{\geq} \widehat{f}_{H,i^\star} - \sqrt{\frac{\log(2n/\delta)}{n/K^H}} \\ &\overset{\textbf{(a)}}{\geq} f(x_{H,i^\star}) - 2b\sqrt{\frac{\log(2n/\delta)}{n/K^H}} \\ &\overset{\textbf{(c)}}{\geq} f(x^\star) - \nu\rho^H - 2b\sqrt{\frac{\log(2n/\delta)}{n/K^H}}. \end{aligned}$$

where **(a)** is because ξ_δ holds and **(b)** is because uniform has opened all the cells at depth H and $x(n) = \arg\max_{\mathcal{P}_{h,i} \in \mathcal{T}} \widehat{f}(x_{h,i})$, and **(c)** is by Assumption 1. We have $\nu\rho^H \leq \nu\left(\left(\frac{K}{n\rho^2}\right)^{\frac{1}{\frac{\log K}{\log 1/\rho}+2}}\right)$ and $\sqrt{\frac{\log(2n/\delta)}{n/K^H}} \leq \sqrt{\log(2n/\delta)}\left(\left(\frac{K}{n\rho^2}\right)^{\frac{1}{\frac{\log K}{\log 1/\rho}+2}}\right)$.

Therefore $r_n = f(x^\star) - f(x(n)) \leq \nu\rho^H - 2b\sqrt{\frac{\log(2n/\delta)}{n/K^H}}$.
$r_n = \widetilde{\mathcal{O}}\left(\log(1/\delta)\nu\left(\left(\frac{K}{n\rho^2}\right)^{\frac{1}{\frac{\log K}{\log 1/\rho}+2}}\right)\right)$ □

A.3 The non-stochastic case

Theorem 1 (Upper bounds for ROBUNI)**.** *Consider any sequence of functions $f_1, \dots, f_n$ such that $|f_t(x)| \leq f_{max}$ for all $x \in \mathcal{X}$ and $t \in [n]$. Let $f = \frac{1}{n}\sum_{t=1}^n f_t$, and $x^\star$ be one of the global optima of f with associated (ν, ρ). Then after n rounds, the simple regret of* ROBUNI *is bounded as:*

$$\mathbb{E}[r_n] = \mathcal{O}\left(\log(n/\delta)\left(\frac{K}{n\rho^2}\right)^{\frac{1}{\frac{\log K}{\log 1/\rho}+2}}\right)$$

Proof. Let us fix some depth H and consider a collection of functions $f_1, \dots, f_n$. Given $f_1, \dots, f_n$, after n rounds the random variables $\widetilde{f}_{H,i}(t)$ are conditionally independent from each other for all i at depth H and for all $t \in [n]$ as we have $\mathbb{P}(x_t \in \mathcal{P}_{H,i} \cap h \geq 0) = \mathbb{P}(x_t \in \mathcal{P}_{H,i}) \geq 1/K^H$ are fixed for all i at depth H and $t \in [n]$.

The variance of $\widetilde{f}_{H,i}(t)$ is the variance of a scaled Bernoulli random variable with parameter $\mathbb{P}(x_t \in \mathcal{P}_{H,i}) \geq 1/K^H$ and range $\left[0, K^H E_{x\sim U(\mathcal{P}_{h,i})}[f_t(x)]\right]$, therefore we have $|\widetilde{f}_{H,i}(t) - E_{x\sim U(\mathcal{P}_{h,i})}[f_t(x)]| \leq K^H$, and $\sigma^2_{\widetilde{f}_{H,i}(t) - E_{x\sim U(\mathcal{P}_{h,i})}[f_t(x)]} = \sigma^2_{\widetilde{f}_{H,i}(t)} \leq 1/K^H(1 - 1/K^H)K^{2H}\widetilde{f}^2_{H,i}(t) \leq K^H f^2_{max}$.

We define and consider event ξ_δ and prove it holds with high probability. Let ξ_δ be the event under which all average estimates in all the cells at depth H are within their classical confidence interval, then $P(\xi_\delta) \geq 1-\delta$, where

$$\xi_\delta \triangleq \Big\{\forall \mathcal{P}_{H,i},\ \Big|\widetilde{F}_{H,i}(n) - F_{H,i}(n)\Big| \leq \sqrt{2nf_{max}^2 K^H \log(n^2/\delta)} + \frac{f_{max}^2}{3}K^H\log(n^2/\delta)\Big\}.$$

We have $P(\xi_\delta) \geq 1-\delta$, using Bennett's inequality from Theorem 3 in Maurer and Pontil (2009) and from taking a union bound on all opened cells. We denote $B_h = \sqrt{2nK^h\log(n^2/\delta)} + bK^h\log(n^2/\delta)$ and we denote by $h(n)$ the depth of $x(n)$. On ξ_δ we have, for any $H \in [\lfloor \log_K(n)\rfloor]$,

$$\begin{aligned}\mathbb{E}[f(x(n))] &\overset{\textbf{(a)}}{\geq} \frac{1}{n}\Big(\widetilde{F}(x(n)) - B_{h(n)}\Big) \overset{\textbf{(b)}}{\geq} \frac{1}{n}\Big(\widetilde{F}_{H,i^\star} - B_H\Big)\\ &\overset{\textbf{(a)}}{\geq} \frac{1}{n}(F_{H,i^\star} - 2B_H)\\ &\overset{\textbf{(c)}}{\geq} f(x^\star) - \nu\rho^H - 2B_H/n. \end{aligned} \quad (5)$$

where **(a)** is because ξ_δ holds **(b)** is by definition of $x(n)$ as $x(n) \leftarrow \arg\max_{x_{h,i}} \widetilde{F}_{h,i}(n) - B_h$, and **(c)** is by Assumption 1.

In order to maximize the lower bound in 5 we set $H = \Big\lfloor \log_{K/\rho^2}(n)\Big\rfloor$. We have $\nu\rho^H \leq \nu\left(\left(\frac{K}{n\rho^2}\right)^{\frac{1}{\frac{\log K}{\log 1/\rho}+2}}\right)$ and $\sqrt{\log(n^2/\delta)K^H/n} \leq \sqrt{\log(n^2/\delta)}\left(\left(\frac{K}{n\rho^2}\right)^{\frac{1}{\frac{\log K}{\log 1/\rho}+2}}\right)$ and $bK^H/n\log(n^2/\delta) = \mathcal{O}\left(\sqrt{\log(n^2/\delta)}\left(\left(\frac{K}{n\rho^2}\right)^{\frac{2}{\frac{\log K}{\log 1/\rho}+2}}\right)\right)$.

Therefore $\mathbb{E}_{\nu_n}[r_n] = f(x^\star) - \mathbb{E}[f(x(n))] \leq \nu\rho^{H+1} - 2B/n$. $r_n = \mathcal{O}\left(\log(n/\delta)\left(\frac{K}{n\rho^2}\right)^{\frac{1}{\frac{\log K}{\log 1/\rho}+2}}\right)$

□

B Proofs of simple regret for VROOM

The non-stochastic feedback case

Proof. Let us fix some depth H and consider a collection of functions $f_1,\dots,f_n$. Given $f_1,\dots,f_n$, after n rounds the random variables $\widetilde{f}_{H,i}(t)$ can be dependent of each other for all $h \geq 0$ and $i \in [K^H]$ and $t \in [n]$ as $\boldsymbol{p}_{h,i,t}$ depends on previous observations at previous rounds. Therefore, we use the Bernstein inequality for martingale differences by Freedman (1975).

The variance of $\widetilde{f}_{H,i}(t)$ is the variance of a scaled Bernoulli random variable with parameter $\mathbb{P}(x_t \in \mathcal{P}_{H,i}) \geq 1/K^H\log_K^2(n)$ and range $\big[0, K^H E_{x\sim U(\mathcal{P}_{h,i})}[f_t(x))\log^2(n)\big]$, therefore we have $|\widetilde{f}_{H,i}(t) - E_{x\sim U(\mathcal{P}_{h,i})}[f_t(x)]| \leq K^H\log_K^2(n)f_{max}$, and $\sigma^2_{\widetilde{f}_{H,i}(t)-E_{x\sim U(\mathcal{P}_{h,i})}[f_t(x)]} = \sigma^2_{\widetilde{f}_{H,i}(t)} \leq 1/K^H(1-1/K^H)K^{2H}\widetilde{f}^2_{H,i}(t) \leq K^H f^2_{max}$.

Then, following the same reasoning as in the proof of Theorem 1, but replacing the Bernstein inequality by the Bernstein inequality for martingale differences of Freedman (1975) applied to the martingale differences $\widetilde{f}_{k,t} - \widetilde{f}_{k,t}$, we obtain the claimed result for the adversarial case. □

The i.i.d. stochastic feedback case

Proof. Note that as the regret guaranties proved in the non-stochastic case also hold in the stochastic case. So we are left to prove $\mathbb{E}[r_n] = \widetilde{\mathcal{O}}\big(\frac{1}{n}\big)^{\frac{1}{d+3}}$.

We place ourselves in the i.i.d. stochastic setting described in Section 1. Let us consider a fixed depth H which value will be chosen towards the end of the proof in order to minimize the simple regret with respect to this H.

We consider one global optimum $x^\star$ of $\overline{f}$ with associated (ν,ρ), $C > 1$, and near-optimality dimension $d = d(\nu, C, \rho)$.

We define $n_\alpha \in [n]$ and will analyze how VROOM explore the depth $h \leq \lfloor \log_K(n_\alpha)\rfloor$.

First, we define the rounds used for comparisons.

We define the times $n_h = \beta n_\alpha \frac{\sum_{h'=1}^{h}\frac{1}{h'}}{\log_K(n_\alpha)}$ for $h \in \lfloor \log_K(n_\alpha)\rfloor$ and where $\beta > 1$ is a constant that we will fix later such that $n_h \leq n$. To ease the notation and without loss of generality, for each depth h, we assume that the cells are sorted by their means so that cell 1 is the best, $\overline{f}_{h,1} \geq \overline{f}_{h,2} \geq \ldots \geq \overline{f}_{h,K^h}$.

We define and consider event ξ_δ and prove it holds with high probability.

Let ξ_δ be the event under which all average estimates in all the cells at depth H are within their classical confidence interval, then $P(\xi_\delta) \geq 1-\delta$, where ξ_δ is decomposed in three sub-events $\xi_\delta = \xi_\delta^1 \cap \xi_\delta^2 \cap \xi_\delta^3$ where

$$\begin{aligned}\xi_\delta^1 \triangleq &\{\forall \mathcal{P}_{h,i}, h \leq \lfloor \log_K(n)\rfloor :\\ &\Big|\widetilde{F}_{h,i}(n) - n\overline{f}_{h,i}\Big| \leq B_{h,i}^{adv}(n)\\ &\text{and } \Big|\widetilde{F}_{h,i}(n) - F_{h,i}(n)\Big| \leq B_{h,i}^{adv}(n)\Big\},\\ \xi_\delta^2 \triangleq &\{\forall \mathcal{P}_{h,i}, h \leq \lfloor \log_K(n)\rfloor, \forall t \in [n],\\ &\Big|\widehat{f}_{h,i}(t) - \overline{f}_{h,i}\Big| \leq B_{h,i}^{iid}(t)\Big\},\\ \xi_\delta^3 \triangleq &\{\forall h \leq \lfloor \log_K(n_\alpha)\rfloor,\\ &\forall t \geq 8n_\alpha\log^3(n), T_{h,i^\star}(t) \geq \mathbb{E}\Big[\frac{T_{h,i^\star}(t)}{2}\Big]\Big\}.\end{aligned}$$

We have $P(\xi_\delta^1) \geq 1-\delta/2$. Indeed to bound

$\left|\widetilde{F}_{h,i}(n) - n\overline{f}_{h,i}\right|$ we use the Bernstein inequality for martingale differences of Freedman (1975) applied to the martingale differences $\widetilde{f}_{k,t} - \overline{f}_{k,t}$ and from taking a union bound on all cells at depth $h \leq \lfloor \log_K(n) \rfloor$. We have $P(\xi_\delta^1) \geq 1 - \delta/2$. Indeed, to bound $\left|\widehat{f}_{h,i}(t) - \overline{f}_{h,i}\right|$ we use the Chernoff-Hoeffding inequality and take a union bound on all cells at depth $h \leq \lfloor \log_K(n) \rfloor$. Finally we have $P(\xi_\delta^3) \geq 1 - \log(n)/n$. Indeed, using a Chernoff bound we have for $\forall h \leq \lfloor \log_K(n_\alpha) \rfloor, \forall t \geq 8 n_\alpha \log^3(n)$,

$$
\begin{aligned}
&\mathbb{P}\left(T_{h,i^\star}(t) \leq \mathbb{E}\left[\frac{T_{h,i^\star}(t)}{2}\right]\right) \\
&\leq \exp\left(-\frac{1}{8}\mathbb{E}\left[\frac{T_{h,i^\star}(t)}{2}\right]\right) \\
&\leq \exp\left(-\frac{1}{8}\mathbb{E}\left[\sum_{s=1}^{t-1}\mathbb{P}(x_s \in \mathcal{P}_{h,i^\star})\right]\right) \\
&\overset{\textbf{(a)}}{\leq} \exp\left(-\frac{1}{8}\sum_{s=1}^{t-1}\frac{1}{n_\alpha \log^2(n)}\right) \\
&\leq \exp\left(-\frac{1}{8}\frac{8 n_\alpha \log^3(n)}{n_\alpha \log^2(n)}\right) = \exp(-\log(n)) = \frac{1}{n}
\end{aligned}
$$

where **(a)** is because $\mathbb{P}(x_t \in \mathcal{P}_{h,i^\star}) \geq \boldsymbol{p}_{h,i^\star,t} = \frac{1}{h\widehat{\langle i^\star \rangle}_{h,t}\overline{\log}_K(n)} \geq \frac{1}{\log_K n_\alpha K^h \overline{\log}_K(n)} \geq \frac{1}{\log_K n_\alpha n_\alpha \overline{\log}_K(n)} \geq \frac{1}{n_\alpha \log_K^2(n)} \geq \frac{1}{n_\alpha \log^2(n)}$.

We can therefore decompose the regret r_n as

$$
\begin{aligned}
\mathbb{E}[r_n] &= \left(\delta + \frac{\log(n)}{n}\right)\mathbb{E}[r_n|\xi_\delta^c] + \left(1 - \delta - \frac{\log(n)}{n}\right)\mathbb{E}[r_n|\xi_\delta] \\
&\leq \left(\delta + \frac{\log(n)}{n}\right) f_{\max} + \mathbb{E}[r_n|\xi_\delta]. \qquad (6)
\end{aligned}
$$

As we will set $\delta = \frac{4b}{f_{\max}\sqrt{n}}$ the first term of Inequality 6 is already smaller than the claimed result of the Theorem so we now focus on bounding the second term.

For any $x^\star$, we write

$$
\bot_h = \left\{h' \geq 0 : \forall t \geq n_h, \widehat{\langle \mathcal{P}_{h',i^\star} \rangle}_{h',t} \leq C\rho^{-dh'}\right\}
$$

that contains all the depth h such that for all time $t \geq n_h$ the cell containing $x^\star$ at depth h is ranked with a smaller index than $C\rho^{-dh}$ by `VROOM`. As explained above we are trying here to introduce tools that will help us to upper bound the ranking of the best arm to be able then to upper bound the variance of its estimates.

On ξ_δ we have, for all $H \in [\lfloor \log_K(n) \rfloor]$

$$
\begin{aligned}
\mathbb{E}[f(x(n))] &\overset{\textbf{(a)}}{\geq} \frac{1}{n}\left(\widetilde{F}_{h(n),i(n)} - B_{h(n),i(n)}(n)\right) \\
&\overset{\textbf{(b)}}{\geq} \frac{1}{n}\left(\widetilde{F}_{\bot_H,1} - B_{\bot_H,1}(n)\right) \\
&\overset{\textbf{(a)}}{\geq} \frac{1}{n}\left(n\overline{f}_{\bot_H,1} - 2B_{\bot_H,1}(n)\right) \\
&\overset{\textbf{(c)}}{\geq} \overline{f}(x^\star) - \nu\rho^{\bot_H} - 2B_{\bot_H,1}(n)/n. \qquad (7)
\end{aligned}
$$

where **(a)** is because ξ_δ holds **(b)** is by definition of $x(n)$ as $x(n) \leftarrow \arg\max_{x_{h,i}} \widetilde{F}_{h,i}(n) - B_{h,i}(n)$, and **(c)** is by Assumption 1.

We now need to bound $\bot_H$ and bound $B_{\bot_H,1}(n)$ for some $H \in [\lfloor \log_K(n_\alpha) \rfloor]$. To obtain a tight bound we try to have $\nu\rho^{\bot_H}$ and $B_{\bot_H,1}(n)$ of the same order.

We use for that Lemma 2 that provide sufficient condition in Equation 17 to lower bound $\bot_H$. We now define the quantity $\widetilde{h}$ that verify this condition. $\widetilde{h}$ is so that the $\nu\rho^{\bot_H}$ and $B^{iid}_{\bot_H,1}(n)$ are equal. We denote $\widetilde{h}$ the real number satisfying

$$
\frac{n_\alpha \nu^2 \rho^{2\widetilde{h}}}{K\widetilde{h}b^2\log^2(2n^2/\delta)} = C\rho^{-d\widetilde{h}}. \qquad (8)
$$

Our approach is to solve Equation 8 and then verify that it gives a valid indication of the behavior of our algorithm in term of its optimal h. We have

$$
\widetilde{h} = \frac{1}{(d+2)\log(1/\rho)} W\left(\frac{\nu^2 n_\alpha (d+2)\log(1/\rho)}{KCb^2\log^2(2n^2/\delta)}\right)
$$

where standard W is the Lambert W function.

Using standard properties of the $\lfloor \cdot \rfloor$ function, we have

$$
\begin{aligned}
&\frac{n_\alpha \nu^2 \rho^{2\lfloor \widetilde{h} \rfloor}}{K\left\lfloor \widetilde{h} \right\rfloor b^2 \log^2(2n^2/\delta)} \geq \frac{n_\alpha \nu^2 \rho^{2\widetilde{h}}}{K\widetilde{h}b^2\log^2(2n^2/\delta)} \\
&= C\rho^{-d\widetilde{h}} \geq C\rho^{-d\lfloor \widetilde{h} \rfloor}.
\end{aligned}
$$

From the previous inequality we also have, as $d \leq \log(K)/\log(1/\rho)$,

$$
\begin{aligned}
&n_\alpha \geq \\
&\frac{n_\alpha \nu^2 \rho^{2\lfloor \widetilde{h} \rfloor}}{K\left\lfloor \widetilde{h} \right\rfloor b^2 \log^2(2n^2/\delta)} \geq C\rho^{-d\lfloor \widetilde{h} \rfloor} \geq K^{\lfloor \widetilde{h} \rfloor}.
\end{aligned}
$$

which leads to $\left\lfloor \widetilde{h} \right\rfloor \leq \lfloor \log_K(n_\alpha) \rfloor$. Having $\left\lfloor \widetilde{h} \right\rfloor \in [\lfloor \log_K(n_\alpha) \rfloor]$ and using Lemma 2 we have that if $\beta \geq 8\log^3(n)\lfloor \log_K(n_\alpha) \rfloor$ then $\bot_{\lfloor \widetilde{h} \rfloor} \geq \left\lfloor \widetilde{h} \right\rfloor$.

To bound $B_{\bot_H,1}(n)$ we use Lemma 1. Therefore, choosing

$H = \left\lfloor \widetilde{h} \right\rfloor$, we get to rewrite Equation 7 as

$$\begin{aligned}\mathbb{E}[f(x(n))] &\geq \overline{f}(x^\star) - \nu\rho^{\perp_{\lfloor \tilde{h} \rfloor}} - 2B_{\perp_{\lfloor \tilde{h} \rfloor},1}(n)/n \\ &\geq \overline{f}(x^\star) - \nu\rho^{\lfloor \widetilde{h} \rfloor} \\ &\quad - 4f_{\max}\sqrt{\log^3(2n^2/\delta)\left(\frac{n_\alpha^2}{n^2} + \frac{C\rho^{-d\perp_{\lfloor \tilde{h} \rfloor}}}{n}\right)}\end{aligned} \tag{9}$$

Moreover, as proved by Hoorfar and Hassani (2008), the Lambert $W(x)$ function verifies for $x \geq e$, $W(x) \geq \log\left(\frac{x}{\log x}\right)$. Therefore, if $\frac{\nu^2 n_\alpha (d+2)\log(1/\rho)}{KCb^2\log(2n^2/\delta)} > e$ we have, we have the first term in Equation 9

$$\begin{aligned}\rho^{\lfloor \widetilde{h} \rfloor} &\leq \rho^{\frac{1}{(d+2)\log(1/\rho)} W\left(\frac{\nu^2 n_\alpha (d+2)\log(1/\rho)}{KCb^2\log^2(2n^2/\delta)}\right) - 1} \\ &\leq \rho^{\frac{1}{(d+2)\log(1/\rho)}\log\left(\frac{\frac{\nu^2 n_\alpha (d+2)\log(1/\rho)}{KCb^2\log^2(2n^2/\delta)}}{e\log^2\left(\frac{\nu^2 n_\alpha (d+2)\log(1/\rho)}{KCb^2\log^2(2n^2/\delta)}\right)}\right)} \\ &= \left(\frac{\frac{\nu^2 n_\alpha (d+2)\log(1/\rho)}{KCb^2\log^2(2n^2/\delta)}}{e\log\left(\frac{\nu^2 n_\alpha (d+2)\log(1/\rho)}{KCb^2\log^2(2n^2/\delta)}\right)}\right)^{\frac{-1}{(d+2)}}\end{aligned}$$

Then we have, from Equation 8,

$$\begin{aligned}\sqrt{\frac{C\rho^{-d\perp_{\lfloor \tilde{h} \rfloor}}}{n}} \leq \sqrt{\frac{C\rho^{-d\widetilde{h}}}{n}} &= \sqrt{\frac{n_\alpha \nu^2 \rho^{2\widetilde{h}}}{nK\widetilde{h}b^2\log^2(2n^2/\delta)}} \\ &\leq \frac{\nu\rho^{\widetilde{h}}}{\sqrt{K}b\log^2(2n^2/\delta)},\end{aligned}$$

which is bounded above.

Then in Equation 9, using that $\sqrt{a' + b'} \leq \sqrt{a'} + \sqrt{b'}$ for two non negative numbers (a', b'), we have three terms of the shape: $n_\alpha^{\frac{-1}{d+2}} + n_\alpha/n + n_\alpha^{\frac{-1}{d+2}}$. As explained in the sketch of proof we need to have n_α of order $n^{\frac{d+2}{d+3}}$ in order to minimize the previous sum.

More precisely we set $n_\alpha = n^{\frac{d+2}{d+3}}/(8\log^4(n))$ and set $\beta = 8\log^4(n)$ and $\delta = \frac{4b}{f_{\max}\sqrt{n}}$ and obtain the claimed result.

□

Lemma 1. *If $\beta \geq 8\log^4(n)\lfloor \log_K(n_\alpha) \rfloor$, for any global optimum $x^\star$ with associated (ν, ρ) from Assumption 1, any $C > 1$, for any $\delta \in (0,1)$, on event ξ_δ defined above, for any depth $h \in [\lfloor \log_K(n_\alpha) \rfloor]$, we have that if*

$$\frac{n_\alpha}{K}\nu^2\rho^{2h}/(b^2 h \log^2(2n^2/\delta)) \geq C\rho^{-d(\nu,C,\rho)h}, \tag{10}$$

that

$$B_{\perp_h,1}(n) \leq 2f_{\max}\sqrt{\log^3(2n^2/\delta)(n_\alpha^2 + nC\rho^{-d\perp_h})}.$$

Proof. The assumptions of Lemma 2 being verified we have $h \in \perp_h$. Also we have,

$$B_{\perp_h,1}(n) \tag{11}$$
$$= f_{\max}\sqrt{2\perp_h(n)\overline{\log}_K(n)\log 2n^2/\delta \sum_{s=1}^{n} \widehat{\langle 1 \rangle}_{h,s}} \tag{12}$$
$$+ f_{\max}\overline{\log}_K(n)\frac{\log 2n^2/\delta}{3}. \tag{13}$$

We bound the first term by having

$$\begin{aligned}\sum_{s=1}^{n}\widehat{\langle 1 \rangle}_{h,s} &= \sum_{h=0}^{\lfloor \log_K(n_\alpha) \rfloor - 1}\sum_{s=n_h+1}^{n_h}\widehat{\langle 1 \rangle}_{h,s} \\ &\quad + \sum_{s=n_{\lfloor \log_K(n_\alpha) \rfloor}+1}^{n}\widehat{\langle 1 \rangle}_{h,s} \\ &\overset{\textbf{(a)}}{\leq} \sum_{h=0}^{\lfloor \log_K(n_\alpha) \rfloor - 1}\sum_{s=n_h+1}^{n_h} K^{\lfloor \log(n_\alpha) \rfloor} \\ &\quad + \sum_{s=n_{\lfloor \log_K(n_\alpha) \rfloor}+1}^{n} C\rho^{-d\perp_h} \\ &\leq \sum_{h=0}^{\lfloor \log_K(n_\alpha) \rfloor - 1}\sum_{s=n_h+1}^{n_h} n_\alpha + nC\rho^{-d\perp_h} \\ &\leq n_\alpha^2 + nC\rho^{-d\perp_h}\end{aligned}$$

where **(a)** is because $h \in [\lfloor \log_K(n_\alpha) \rfloor]$ and $\perp_h \geq h$.

Because in Equation 11 the second term is smaller than the first, we have

$$B_{\perp_h,1}(n) \tag{14}$$
$$= 2f_{\max}\sqrt{2\log_K(n_\alpha)\overline{\log}_K(n)\log(2n^2/\delta)(n_\alpha^2 + nC\rho^{-d\perp_h})} \tag{15}$$
$$\leq 2f_{\max}\sqrt{2\log^3(2n^2/\delta)(n_\alpha^2 + nC\rho^{-d\perp_h})}. \tag{16}$$

□

Lemma 2. *If $\beta \geq 8\log^4(n)\lfloor \log_K(n_\alpha) \rfloor$, For any global optimum $x^\star$ with associated (ν, ρ) from Assumption 1, any $C > 1$, for any $\delta \in (0,1)$, on event ξ_δ defined above, for any depth $h \in [\lfloor \log_K(n_\alpha) \rfloor]$, we have that if*

$$\frac{n_\alpha}{K}\nu^2\rho^{2h}/(b^2 h \log^2(2n^2/\delta)) \geq C\rho^{-d(\nu,C,\rho)h}, \tag{17}$$

that $h \in \perp_h$.

Proof. To simplify notation we write $d(\nu, C, \rho)$ as d. We place ourselves on event ξ_δ defined above. We prove the statement of the lemma, given that event ξ_δ holds, by induction in the following sense. For a given h, we assume the hypotheses of the lemma for that h are true and we prove

by induction that $h' \in \perp_{h'}$ for $h' \in [h]$.
1° For $h' = 0$, we trivially have that $0 \in \perp_{h'}$.
2° Now consider $h' > 0$, and assume $h' - 1 \in \perp_{h'-1}$ with the objective to prove that $h' \in \perp_{h'}$. Therefore, for all $t \geq n_{h'-1}$, $\langle \widehat{\mathcal{P}_{h'-1,i^\star}} \rangle_{h'-1,t} \leq C\rho^{-d(h'-1)}$.

For the purpose of contradiction, let us assume that their exists $t \geq n_{h'}$, such that $\langle \widehat{\mathcal{P}_{h',i^\star}} \rangle_{h',t} > C\rho^{-dh'}$. This would mean that there exist at least $C\rho^{-dh'}$ cells from $\{\mathcal{P}_{h',i}\}$, distinct from $\mathcal{P}_{h',i_h^\star}$, satisfying $\widehat{f}^-_{h',i}(t) \geq \widehat{f}^-_{h',i^\star_{h'}}(t)$. This means that, for these cells we have

$$\begin{aligned}\overline{f}_{h',i} &\overset{\textbf{(b)}}{\geq} \widehat{f}^-_{h',i}(t) \geq \widehat{f}^-_{h',i^\star_{h'}}(t) \overset{\textbf{(b)}}{\geq} \overline{f}_{h',i^\star_{h'}}(t) - 2b\sqrt{\frac{\log(2n^2/\delta)}{2T_{h',i^\star_{h'}}(t)}} \\ &\overset{\textbf{(c)}}{\geq} \overline{f}_{h',i^\star_{h'}}(t) - 2b\sqrt{\frac{\log(2n^2/\delta)}{\frac{\beta n_\alpha}{h\lfloor \log_K(n_\alpha)\rfloor CK\rho^{-dh'-1}}}} \\ &\geq \overline{f}_{h',i^\star_{h'}}(t) - 2b\sqrt{\frac{\log(2n^2/\delta)}{\frac{n_\alpha}{h\lfloor \log_K(n_\alpha)\rfloor CK\rho^{-dh'}}}} \\ &\overset{\textbf{(d)}}{\geq} \overline{f}_{h',i^\star_{h'}} - 2\nu\rho^{h} \geq \overline{f}_{h',i^\star_{h'}} - 2\nu\rho^{h'},\end{aligned}$$

where **(b)** is because ξ_δ holds, **(d)** is because by assumption (Equation 17) of the lemma, for $h' \in [h]$, $\frac{n_\alpha}{K}\nu^2\rho^{2h'}/(b^2 h\log^2(2n^2/\delta)) \geq \frac{n_\alpha}{K}\nu^2\rho^{2h}/(b^2 h\log^2(2n^2/\delta)) \geq C\rho^{-dh} \geq C\rho^{-dh'}$. **(c)** is because on ξ_δ, as $\beta \geq 8\log^3(n)\lfloor\log_K(n_\alpha)\rfloor$ and $h \leq \lfloor\log_K(n_\alpha)\rfloor$, $\forall t \geq n_h = \beta n_\alpha \frac{\sum_{h'=1}^{h}\frac{1}{h'}}{\log_K(n_\alpha)} \geq 8n_\alpha\log^3(n)$, have

$$\begin{aligned}T_{h',i^\star_{h'}}(t) &\geq \mathbb{E}\left[\sum_{s=1}^{t-1}\frac{\mathbb{P}(x_s \in \mathcal{P}_{h',i^\star})}{2}\right] \\ &\geq \mathbb{E}\left[\sum_{s=n_{h'-1}}^{n_{h'}}\frac{\mathbb{P}(x_s \in \mathcal{P}_{h',i^\star})}{2}\right] \\ &\overset{\textbf{(e)}}{\geq} \sum_{s=n_{h'-1}}^{n_{h'}}\frac{1}{2CK\rho^{-dh'-1}} \\ &\geq \beta\frac{n_\alpha}{2h\lfloor\log_K(n_\alpha)\rfloor CK\rho^{-dh'-1}},\end{aligned}$$

where **(e)** is because we have $\langle \mathcal{P}_{h'-1,i^\star}\rangle_{h'-1,t} \leq C\rho^{-d(h'-1)}$ which gives $\mathbb{P}(x_t \in \mathcal{P}_{h,i}) \geq \frac{1}{KC\rho^{-d(h'-1)}}$ as $f_{h',i^\star_{h'}} \geq f(x^\star) - \nu\rho^{h'}$ by Assumption 1, it follows that $\mathcal{N}_{h'}(3\nu\rho^{h'}) > \left\lfloor C\rho^{-dh'}\right\rfloor$. This leads to having a contradiction with the function f being of near-optimality dimension d as defined in Definition 1. Indeed, the condition $\mathcal{N}_{h'}(3\nu\rho^{h'}) \leq C\rho^{-dh'}$ in Definition 1 is equivalent to the condition $\mathcal{N}_{h'}(3\nu\rho^{h'}) \leq \left\lfloor C\rho^{-dh'}\right\rfloor$ as $\mathcal{N}_{h'}(3\nu\rho^{h'})$ is an integer. Reaching the contradiction proves the claim of the lemma. □