Supplementary Material

Derivation of kernel convolution

We provided detailed derivations of the causal convolutional kernel that proposed in the main text. First, we started from the setup of assuming the latent force functions are drawn from a Gaussian process (GP) with a causal kernel modified from Cunningham et al. (2012):

$$f(t) \sim \mathcal{GP}(0, k_{f,f'}(t, t'; t_m)) \tag{1}$$

$$k_{f,f'}(t, t'; t_m) = \exp\left\{-\frac{[h(t-t_m) - h(t'-t_m)]^2}{\ell_m^2}\right\}, \tag{2}$$

where $h(t) = t\mathcal{I}(t > 0)$. Considering the case of one output $y(t)$ driven by one $f(t)$ following the first-order differential equation:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + D \cdot y(t) = B + S \cdot f(t), \tag{3}$$

From Alvarez et al. (2009), we can derive the closed-form cross-covariance function $k_{y,f}(t, t; t_m)$ and the kernel for the output process $k_{y,y}(t, t; t_m)$ through considering four cases separately in the following sections.

Cross-covariance function between one latent force and one output

For the cross-covariance function $k_{y,f}(t, t; t_m)$, we started from

$$\begin{aligned} k_{y,f}(t, t'; t_m) &= S\exp(-Dt)\int_0^t \exp(Du) k_{f,f}(u, t')du \\ &= S\exp(-Dt)\int_0^t \exp(Du)\exp\left\{-\frac{[h(u-t_m) - h(t'-t_m)]^2}{\ell^2}\right\}du. \end{aligned} \tag{4}$$

We first work on the case for $t > t_m$ and $t' > t_m$:

$$
\begin{aligned}
k_{y,f}(t,t';t_m) &= S\exp(-Dt)\int_0^t \exp(Du)k_{f,f}(u,t')du \\
&= S\exp(-Dt)\int_0^t \exp(Du)\exp\left\{-\frac{[h(u-t_m)-h(t'-t_m)]^2}{\ell^2}\right\}du \\
&= S\exp(-Dt)\int_0^{t_m} \exp(Du)\exp\left[-\left(\frac{t'-t_m}{\ell}\right)^2\right]du \\
&+S\exp(-Dt)\int_{t_m}^t \exp(Du)\exp\left\{-\frac{[(u-t_m)-(t'-t_m)]^2}{\ell^2}\right\}du \\
&= S\exp(-Dt)\exp\left[-\left(\frac{t'-t_m}{\ell}\right)^2\right]\cdot\left[\frac{1}{D}\exp(Du)\right]_{u=0}^{u=t_m} \\
&+S\exp(-Dt)\int_{t_m}^t \exp\left\{-\frac{(u-t_m)^2+(t'-t_m)^2-2(u-t_m)(t'-t_m)-\ell^2 Du}{\ell^2}\right\}du \\
&= S\exp(-Dt)\exp\left[-\left(\frac{t'-t_m}{\ell}\right)^2\right]\cdot\frac{1}{D}\left[\exp(Dt_m)-1\right] \\
&+S\exp(-Dt)\int_{t_m}^t \exp\left(-\frac{u^2+t'^2-2ut'-\ell^2 Du}{\ell^2}\right)du.
\end{aligned} \tag{5}
$$

Continuing working on the second term,

$$
\begin{aligned}
&S\exp(-Dt)\int_{t_m}^t \exp\left(-\frac{u^2+t'^2-2ut'-\ell^2 Du}{\ell^2}\right)du \\
&= S\exp(-Dt)\int_{t_m}^t \exp\left(-\frac{[u-(t'+{}^{\ell^2 D}/_2)]^2-(t'+{}^{\ell^2 D}/_2)^2+t'^2}{\ell^2}\right)du \\
&= S\exp(-Dt)\int_{t_m}^t \exp\left(-\frac{[u-(t'+{}^{\ell^2 D}/_2)]^2}{\ell^2}\right)\exp\left(\frac{(t'+{}^{\ell^2 D}/_2)^2-t'^2}{\ell^2}\right)du \\
&= S\exp(-Dt)\int_{t_m}^t \exp\left(-\frac{[u-(t'+{}^{\ell^2 D}/_2)]^2}{\ell^2}\right)\exp\left(Dt'+{}^{\ell^2 D^2}/_4\right)du \\
&= S\exp[-D(t-t')]\exp({}^{\ell^2 D^2}/_4)\int_{t_m}^t \exp\left(-\frac{[u-(t'+{}^{\ell^2 D}/_2)]^2}{\ell^2}\right)du \qquad \text{let } \nu = {}^{\ell D}/_2 \\
&= S\exp[-D(t-t')]\exp(\nu^2)\int_{t_m}^t \exp\left\{-\left[\frac{u}{\ell}-\left(\frac{t'}{\ell}+\nu\right)\right]^2\right\}du \qquad \text{let } z = \tfrac{u}{\ell}-\left(\tfrac{t'}{\ell}+\nu\right) \\
&= S\exp[-D(t-t')]\exp(\nu^2)\int_{\frac{t_m}{\ell}-\left(\frac{t'}{\ell}+\nu\right)}^{\frac{t}{\ell}-\left(\frac{t'}{\ell}+\nu\right)} \exp(-z^2)dz\cdot\frac{du}{dz} \\
&= \frac{\sqrt{\pi}\ell}{2}S\exp[-D(t-t')]\exp(\nu^2)\cdot\left[\operatorname{erf}\left(\frac{t-t'}{\ell}-\nu\right)-\operatorname{erf}\left(\frac{t_m-t'}{\ell}-\nu\right)\right] \\
&= \frac{\sqrt{\pi}\ell}{2}S\exp[-D(t-t')]\exp(\nu^2)\cdot\left[\operatorname{erf}\left(\frac{t-t'}{\ell}-\nu\right)+\operatorname{erf}\left(\frac{t'-t_m}{\ell}+\nu\right)\right].
\end{aligned} \tag{6}
$$

Therefore,

$$
\begin{aligned}
k_{y,f}(t,t';t_m) &= S\exp(-Dt)\exp\left[-\left(\frac{t'-t_m}{\ell}\right)^2\right]\cdot\frac{1}{D}\left[\exp(Dt_m)-1\right] \\
&+\frac{\sqrt{\pi}\ell}{2}S\exp\left[-D(t-t')\right]\exp\left(\nu^2\right)\cdot\left[\operatorname{erf}\left(\frac{t-t'}{\ell}-\nu\right)+\operatorname{erf}\left(\frac{t'-t_m}{\ell}+\nu\right)\right].
\end{aligned} \tag{7}
$$

Intuitively, for the case $t \leq t_m$ and $t' > t_m$:

$$
\begin{aligned}
k_{y,f}(t,t';t_m) &= S\exp(-Dt)\int_0^t \exp(Du)\exp\left\{-\frac{[h(u-t_m)-h(t'-t_m)]^2}{\ell^2}\right\}du \\
&= S\exp(-Dt)\int_0^t \exp(Du)\exp\left[-\left(\frac{t'-t_m}{\ell}\right)^2\right]du \\
&= S\exp(-Dt)\exp\left[-\left(\frac{t'-t_m}{\ell}\right)^2\right]\cdot\frac{1}{D}\left[\exp(Dt)-1\right].
\end{aligned} \tag{8}
$$

Considering $t > t_m$ and $t' \leq t_m$:

$$
\begin{aligned}
k_{y,f}(t,t';t_m) &= S\exp(-Dt)\int_0^t \exp(Du)\exp\left\{-\frac{[h(u-t_m)-h(t'-t_m)]^2}{\ell^2}\right\}du \\
&= S\exp(-Dt)\int_0^{t_m}\exp(Du)du \\
&+S\exp(-Dt)\int_{t_m}^t \exp(Du)\exp\left[-\frac{(u-t_m)^2}{\ell^2}\right]du \\
&= S\exp(-Dt)\cdot\left[\frac{1}{D}\exp(Du)\right]_{u=0}^{u=t_m} \\
&+S\exp(-Dt)\int_{t_m}^t \exp\left(-\frac{u^2-2ut_m+t_m^2-\ell^2Du}{\ell^2}\right)du \\
&= S\exp(-Dt)\cdot\frac{1}{D}\left[\exp(Dt_m)-1\right] \\
&+S\exp(-Dt)\int_{t_m}^t \exp\left[-\frac{[u-(t_m+\ell^2D/2)]^2-(t_m+\ell^2D/2)^2+t_m^2}{\ell^2}\right]du.
\end{aligned} \tag{9}
$$

Continuing working on the second term:

$$
\begin{aligned}
& S\exp(-Dt)\int_{t_m}^{t}\exp\left[-\frac{[u-(t_m+\ell^2D/2)]^2-(t_m+\ell^2D/2)^2+t_m^2}{\ell^2}\right]du \\
&= S\exp[-D(t-t_m)]\exp(\ell^2D^2/4)\int_{t_m}^{t}\exp\left\{-\frac{[u-(t_m+\ell^2D/2)\}^2}{\ell^2}\right]du \quad \text{let } \nu = \ell D/2 \\
&= S\exp[-D(t-t_m)]\exp(\nu^2)\int_{t_m}^{t}\exp\left\{-\left[\frac{u}{\ell}-\left(\frac{t_m}{\ell}+\nu\right)\right]^2\right\}du \qquad \text{let } z = \tfrac{u}{\ell}-\left(\tfrac{t_m}{\ell}+\nu\right) \\
&= S\exp[-D(t-t_m)]\exp(\nu^2)\int_{\frac{t_m}{\ell}-\left(\frac{t_m}{\ell}+\nu\right)}^{\frac{t}{\ell}-\left(\frac{t_m}{\ell}+\nu\right)}\exp(-z)^2dz\cdot\frac{du}{dz} \\
&= \frac{\sqrt{\pi}\ell}{2}S\exp[-D(t-t_m)]\exp(\nu^2)\cdot\left[\operatorname{erf}\left(\frac{t-t_m}{\ell}-\nu\right)+\operatorname{erf}(\nu)\right].
\end{aligned} \tag{10}
$$

Therefore,

$$
\begin{aligned}
k_{y,f}(t,t';t_m) &= S\exp(-Dt)\cdot\frac{1}{D}[\exp(Dt_m)-1] \\
&+\frac{\sqrt{\pi}\ell}{2}S\exp[-D(t-t_m)]\exp(\nu^2)\cdot\left[\operatorname{erf}\left(\frac{t-t_m}{\ell}-\nu\right)+\operatorname{erf}(\nu)\right].
\end{aligned} \tag{11}
$$

Similarly, if $t \le t_m$ and $t' \le t_m$

$$
\begin{aligned}
k_{y,f}(t,t';t_m) &= S\exp(-Dt)\int_0^t\exp(Du)\exp\left\{-\frac{[h(u-t_m)-h(t'-t_m)]^2}{\ell^2}\right\}du \\
&= S\exp(-Dt)\int_0^t\exp(Du)du \\
&= S\exp(-Dt)\cdot\frac{1}{D}[\exp(Du)]_0^t \\
&= S\exp(-Dt)\cdot\frac{1}{D}[\exp(Dt)-1].
\end{aligned} \tag{12}
$$

Covariance function between multiple outputs

If we assume one latent force is driving multiple outputs, we can derive the cross-covariance function between these outputs. Assuming each output could have different decays and scaling factors:

$$\frac{dy_j(t)}{dt}+D_jy_j(t)=B_j+S_j\cdot f(t), \tag{13}$$

$$\frac{dy_k(t)}{dt}+D_ky_k(t)=B_k+S_k\cdot f(t), \tag{14}$$

From Alvarez et al. (2009), we have

$$k_{y_j,y_k}(t,t';t_m)=S_jS_k\exp(-D_jt-D_kt')\int_0^t\exp(D_ju)\int_0^{t'}\exp(D_ku')k_{f,f}(u,u')du'du. \tag{15}$$

Similar with deriving $k_{y,f}(t,t';t_m)$, we look at four cases separately.

(i) When $t \leq t_m$ and $t' \leq t_m$:

$$
\begin{aligned}
k_{y_j,y_k}(t,t';t_m) &= S_j S_k \exp(-D_j t - D_k t') \int_0^t \exp(D_j u) \int_0^{t'} \exp(D_k u') \cdot 1 \cdot du' du \\
&= S_j S_k \exp(-D_j t - D_k t') \int_0^t \exp(D_j u) \frac{1}{D_k} \left[\exp\left(D_k u'\right)\right]_0^{t'} du \\
&= \frac{S_j S_k}{D_k} \exp(-D_j t - D_k t') \left[\exp\left(D_k t'\right) - 1\right] \int_0^t \exp(D_j u) du \\
&= \frac{S_j S_k}{D_j D_k} \exp(-D_j t - D_k t') \left[\exp\left(D_k t'\right) - 1\right] \left[\exp\left(D_j t\right) - 1\right].
\end{aligned}
\tag{16}
$$

(ii) When $t > t_m$ and $t' \leq t_m$:

$$
\begin{aligned}
k_{y_j,y_k}(t,t';t_m) &= S_j S_k \exp(-D_j t - D_k t') \int_0^{t_m} \exp(D_j u) \int_0^{t'} \exp(D_k u') du' du \\
&+ S_j S_k \exp(-D_j t - D_k t') \int_{t_m}^t \exp(D_j u) \int_0^{t'} \exp(D_k u') \exp\left[-\left(\frac{u - t_m}{\ell}\right)^2\right] du' du \\
&= \frac{S_j S_k}{D_j D_k} \exp(-D_j t - D_k t') \left[\exp\left(D_k t'\right) - 1\right] \left[\exp\left(D_j t_m\right) - 1\right] \\
&+ \frac{S_j S_k}{D_k} \exp(-D_j t - D_k t') \left[\exp\left(D_k t'\right) - 1\right] \int_{t_m}^t \exp(D_j u) \exp\left[-\left(\frac{u - t_m}{\ell}\right)^2\right] du \\
&= \frac{S_j S_k}{D_k} \exp(-D_j t - D_k t') \left[\exp\left(D_k t'\right) - 1\right] \\
&\cdot \left\{ \frac{1}{D_j} \left[\exp\left(D_j t_m\right) - 1\right] + \frac{\sqrt{\pi}\ell}{2} \exp\left(D_j t_m\right) \exp\left(\nu_j^2\right) \cdot \left[\operatorname{erf}\left(\frac{t - t_m}{\ell} - \nu_j\right) + \operatorname{erf}\left(\nu_j\right)\right] \right\}, \\
&\text{let } \nu_j = {}^{\ell D_j}/_2.
\end{aligned}
\tag{17}
$$

(iii) When $t \leq t_m$ and $t' > t_m$:

$$
\begin{aligned}
k_{y_j,y_k}(t,t';t_m) &= S_j S_k \exp(-D_j t - D_k t') \int_0^t \exp(D_j u) \int_0^{t_m} \exp(D_k u') du' du \\
&+ S_j S_k \exp(-D_j t - D_k t') \int_0^t \exp(D_j u) \int_{t_m}^{t'} \exp(D_k u') \exp\left[-\left(\frac{u' - t_m}{\ell}\right)^2\right] du' du \\
&= \frac{S_j S_k}{D_j} \exp(-D_j t - D_k t') \left[\exp\left(D_j t\right) - 1\right] \\
&\cdot \left\{ \frac{1}{D_k} \left[\exp\left(D_k t_m\right) - 1\right] + \frac{\sqrt{\pi}\ell}{2} \exp\left(D_k t_m\right) \exp\left(\nu_k^2\right) \cdot \left[\operatorname{erf}\left(\frac{t' - t_m}{\ell} - \nu_k\right) + \operatorname{erf}\left(\nu_k\right)\right] \right\}, \\
&\text{let } \nu_k = {}^{\ell D_k}/_2.
\end{aligned}
\tag{18}
$$

(iv) When $t > t_m$ and $t' > t_m$:

$$
\begin{aligned}
k_{y_j,y_k}(t,t';t_m) \quad &= S_jS_k\exp(-D_jt-D_kt') \\
&\cdot\left\{\int_0^{t_m}\exp(D_ju)\int_0^{t_m}\exp(D_ku')du'du\right. \\
&+\int_0^{t_m}\exp(D_ju)\int_{t_m}^{t'}\exp(D_ku')\exp\left[-\left(\frac{u'-t_m}{\ell}\right)^2\right]du'du \\
&+\int_{t_m}^{t}\exp(D_ju)\int_0^{t_m}\exp(D_ku')\exp\left[-\left(\frac{u-t_m}{\ell}\right)^2\right]du'du \\
&\left.+\int_{t_m}^{t}\exp(D_ju)\int_{t_m}^{t'}\exp(D_ku')\exp\left[-\left(\frac{u-u'}{\ell}\right)^2\right]du'du\right\}.
\end{aligned}
\tag{19}
$$

The first three terms have been computed in (i), (ii) and (iii) above, so we just need to compute the fourth term.

$$
\begin{aligned}
&\int_{t_m}^{t}\exp(D_ju)\int_{t_m}^{t'}\exp(D_ku')\exp\left[-\left(\frac{u-u'}{\ell}\right)^2\right]du'du \\
&=\int_{t_m}^{t}\exp(D_ju)\int_{t_m}^{t'}\exp\left[-\frac{(u'^2-2u'u+u^2-\ell^2D_ku')}{\ell^2}\right]du'du \\
&=\frac{\sqrt{\pi}\ell}{2}\exp\left(\nu_k^2\right)\int_{t_m}^{t}\exp(D_ju)\exp(D_ku)\left[\operatorname{erf}\left(\frac{t'-u}{\ell}-\nu_k\right)+\operatorname{erf}\left(\frac{u-t_m}{\ell}+\nu_k\right)\right]du \\
&=\frac{\sqrt{\pi}\ell}{2}\exp\left(\nu_k^2\right)\int_{t_m}^{t}\exp\left[(D_j+D_k)u\right]\left[\operatorname{erf}\left(\frac{t'-u}{\ell}-\nu_k\right)+\operatorname{erf}\left(\frac{u-t_m}{\ell}+\nu_k\right)\right]du.
\end{aligned}
\tag{20}
$$

We can use integration by parts to compute the two terms above:

$$
\begin{aligned}
&\int_{t_m}^{t}\exp\left[(D_j+D_k)u\right]\cdot\operatorname{erf}\left(\frac{t'-u}{\ell}-\nu_k\right)du \\
&=\left[\frac{\exp\left[(D_j+D_k)u\right]}{D_j+D_k}\cdot\operatorname{erf}\left(\frac{t'-u}{\ell}-\nu_k\right)\right]_{t_m}^{t} \\
&-\frac{2}{\sqrt{\pi}}\int_{t_m}^{t}\frac{\exp\left[(D_j+D_k)u\right]}{D_j+D_k}\cdot\exp\left[-\left(\frac{t'-u}{\ell}-\nu_k\right)^2\right]\left(-\frac{1}{\ell}\right)du \\
&=\frac{1}{D_j+D_k}\left\{\exp\left[(D_j+D_k)t\right]\cdot\operatorname{erf}\left(\frac{t'-t}{\ell}-\nu_k\right)-\exp\left[(D_j+D_k)t_m\right]\cdot\operatorname{erf}\left(\frac{t'-t_m}{\ell}-\nu_k\right)\right. \\
&\left.+\frac{2}{\sqrt{\pi}\ell}\int_{t_m}^{t}\exp\left[-\left(\frac{t'^2-2ut'+u^2-2\ell\nu_k(t'-u)+\ell^2\nu_k^2-\ell^2u(D_j+D_k)}{\ell^2}\right)\right]du\right\} \\
&=\frac{1}{D_j+D_k}\left\{\exp\left[(D_j+D_k)t\right]\cdot\operatorname{erf}\left(\frac{t'-t}{\ell}-\nu_k\right)-\exp\left[(D_j+D_k)t_m\right]\cdot\operatorname{erf}\left(\frac{t'-t_m}{\ell}-\nu_k\right)\right. \\
&\left.+\frac{2}{\sqrt{\pi}\ell}\exp\left(D_jt'+D_kt'\right)\frac{\exp\left(\nu_j^2\right)}{\exp\left(\nu_k^2\right)}\int_{t_m}^{t}\exp\left[-\left(\frac{u-t'}{\ell}-\nu_j\right)^2\right]du\right\} \\
&=\frac{1}{D_j+D_k}\left\{\exp\left[(D_j+D_k)t\right]\cdot\operatorname{erf}\left(\frac{t'-t}{\ell}-\nu_k\right)-\exp\left[(D_j+D_k)t_m\right]\cdot\operatorname{erf}\left(\frac{t'-t_m}{\ell}-\nu_k\right)\right. \\
&\left.+\exp\left[(D_j+D_k)t'\right]\left[\frac{\exp\left(\nu_j^2\right)}{\exp\left(\nu_k^2\right)}\right]\left[\operatorname{erf}\left(\frac{t-t'}{\ell}-\nu_j\right)+\operatorname{erf}\left(\frac{t'-t_m}{\ell}+\nu_j\right)\right]\right\}.
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
&\int_{t_m}^{t} \exp\left[(D_j + D_k)u\right] \cdot \operatorname{erf}\left(\frac{u - t_m}{\ell} + \nu_k\right) du \\
&= \left[\frac{\exp\left[(D_j + D_k)u\right]}{D_j + D_k} \cdot \operatorname{erf}\left(\frac{u - t_m}{\ell} + \nu_k\right)\right]_{t_m}^{t} \\
&- \frac{2}{\sqrt{\pi}\ell} \int_{t_m}^{t} \frac{\exp\left[(D_j + D_k)u\right]}{D_j + D_k} \cdot \exp\left[-\left(\frac{u - t_m}{\ell} + \nu_k\right)^2\right] du \\
&= \frac{1}{D_j + D_k} \left\{\exp\left[(D_j + D_k)t\right] \cdot \operatorname{erf}\left(\frac{t - t_m}{\ell} + \nu_k\right) - \exp\left[(D_j + D_k)t_m\right] \cdot \operatorname{erf}(\nu_k)\right. \\
&\left. - \exp\left[(D_j + D_k)t_m\right] \left[\frac{\exp(\nu_j^2)}{\exp(\nu_k^2)}\right] \left[\operatorname{erf}\left(\frac{t - t_m}{\ell} - \nu_j\right) + \operatorname{erf}(\nu_j)\right]\right\}.
\end{aligned} \tag{22}
$$

Summarizing the equations above, we have

$$
k_{y_j, y_k}(t, t'; t_m) = g_1(t, t'; t_m) + g_2(t, t'; t_m) + g_3(t, t'; t_m) + g_4(t, t'; t_m), \tag{23}
$$

where

$$
g_1(t, t'; t_m) = \frac{S_j S_k}{D_j D_k} \exp(-D_j t - D_k t') \left[\exp(D_k t_m) - 1\right] \left[\exp(D_j t_m) - 1\right], \tag{24}
$$

$$
g_2(t, t'; t_m) = \frac{S_j S_k \sqrt{\pi}\ell}{2D_j} \exp(-D_j t - D_k t') \left[\exp(D_j t_m) - 1\right] \exp(D_k t_m) \exp(\nu_k^2) \cdot \left[\operatorname{erf}\left(\frac{t' - t_m}{\ell} - \nu_k\right) + \operatorname{erf}(\nu_k)\right], \tag{25}
$$

$$
g_3(t, t'; t_m) = \frac{S_j S_k \sqrt{\pi}\ell}{2D_k} \exp(-D_j t - D_k t') \left[\exp(D_k t_m) - 1\right] \exp(D_j t_m) \exp(\nu_j^2) \cdot \left[\operatorname{erf}\left(\frac{t - t_m}{\ell} - \nu_j\right) + \operatorname{erf}(\nu_j)\right], \tag{26}
$$

$$
g_4(t, t'; t_m) = \frac{S_j S_k \sqrt{\pi}\ell}{2} \left[h_{jk}(t, t') + h_{kj}(t', t)\right], \tag{27}
$$

$$
\begin{aligned}
h_{jk}(t, t') &= \frac{\exp(\nu_j^2)}{D_j + D_k} \exp(-D_j t) \left\{\exp(D_j t') \left[\operatorname{erf}\left(\frac{t - t'}{\ell} - \nu_j\right) + \operatorname{erf}\left(\frac{t' - t_m}{\ell} + \nu_j\right)\right]\right. \\
&\left. - \exp(D_j t_m) \exp(D_k t_m - D_k t') \left[\operatorname{erf}\left(\frac{t - t_m}{\ell} - \nu_j\right) + \operatorname{erf}(\nu_j)\right]\right\},
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
h_{kj}(t', t) &= \frac{\exp(\nu_k^2)}{D_k + D_j} \exp(-D_k t') \left\{\exp(D_k t) \left[\operatorname{erf}\left(\frac{t' - t}{\ell} - \nu_k\right) + \operatorname{erf}\left(\frac{t - t_m}{\ell} + \nu_k\right)\right]\right. \\
&\left. - \exp(D_k t_m) \exp(D_j t_m - D_j t) \left[\operatorname{erf}\left(\frac{t' - t_m}{\ell} - \nu_k\right) + \operatorname{erf}(\nu_k)\right]\right\}.
\end{aligned} \tag{29}
$$

References

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John Cunningham, Zoubin Ghahramani, and Carl Rasmussen. Gaussian processes for time-marked time-series data. In Neil D. Lawrence and Mark Girolami, editors, *Proceedings of the Fifteenth International Conference on Artificial Intelligence and Statistics*, volume 22 of *Proceedings of Machine Learning Research*, pages 255–263, La Palma, Canary Islands, 21–23 Apr 2012. PMLR.