Supplement to: Fast Markov chain Monte Carlo algorithms via Lie groups

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1 Proofs

Proof of Lemma 1. $e_{(j,k)}e_{(\ell,m)} = (\delta_{k\ell} - \delta_{n\ell})e_{(j,m)}$. Considering $j \leftrightarrow \ell, k \leftrightarrow m$, we are done. □

Proof of Lemma 2. Using the rightmost expression in (5) and using $j, k, \ell, m \neq n$ to simplify the product of the innermost two factors, we have that

$$e^{(p)}_{(j,k)}e^{(p)}_{(\ell,m)} = (\delta_{k\ell} + r_\ell)\, e^{(p)}_{(j,m)}.$$

Taking $j = \ell$ and $k = m$ establishes the result for $i \leq 2$. The general case follows by induction on i. □

Proof of Theorem 1. Note that

$$pe^{(p)}_{(j,k)} = (p_j - r_j p_n)\left(e^T_k - e^T_n\right) \equiv 0.$$

Furthermore, linear independence and the commutation relations are obvious, so it suffices to show that $\exp te^{(p)}_{(j,k)} \in \langle p \rangle$ for all $t \in \mathbb{R}$. By Lemma 2,

$$\begin{aligned}\exp te^{(p)}_{(j,k)} &= I + e^{(p)}_{(j,k)} \sum_{i=1}^{\infty} \frac{t^i \left(\delta_{jk} + r_j\right)^{i-1}}{i!} \\ &= I + \frac{e^{t(\delta_{jk}+r_j)} - 1}{\delta_{jk} + r_j} e^{(p)}_{(j,k)}.\end{aligned}$$

□

Proof of Lemma 3. By hypothesis and (5), $-\sum_j t_j e^{(p)}_{(j,j)}$ has nonpositive diagonal entries and nonnegative off-diagonal entries (i.e., it is a generator matrix for a continuous-time Markov process); the result follows. □

Proof of Lemma 4.

$$\begin{aligned}\alpha^{(p)}_{(\mathcal{J})}\beta^{(p)}_{(\mathcal{J})} &= \sum_{u,v,w,x} \alpha_{j_u j_v}\beta_{j_w j_x} e^{(p)}_{(j_u,j_v)} e^{(p)}_{(j_w,j_x)} \\ &= \sum_{u,v,w,x} \alpha_{j_u j_v}\left(\delta_{j_v j_w} + r_{j_w}\right)\beta_{j_w j_x} e^{(p)}_{(j_u,j_x)} \\ &= \sum_{u,x} \left(\alpha_{(\mathcal{J})}(I + 1r_{(\mathcal{J})})\beta_{(\mathcal{J})}\right)_{ux} e^{(p)}_{(j_u,j_x)}.\end{aligned}$$

where the second equality follows from (1) and the third from bookkeeping. □

Proof of Theorem 2. The Sherman-Morrison formula (see Horn and Johnson (2013)) gives that

$$\omega(I + 1r_{(\mathcal{J})})^{-1} = \omega\left(I - \frac{1}{1 + r_{(\mathcal{J})}1} 1r_{(\mathcal{J})}\right)$$

and the elements of this matrix are precisely the coefficients in (13). Using the notation of Lemma 4, we can therefore rewrite (13) as

$$A^{(p;\omega)}_{(\mathcal{J})} = \left(\omega(I + 1r_{(\mathcal{J})})^{-1}\right)^{(p)}_{(\mathcal{J})},$$

whereupon invoking the lemma itself yields $\left(A^{(p;\omega)}_{(\mathcal{J})}\right)^{i+1} = \omega^i A^{(p;\omega)}_{(\mathcal{J})}$ for $i \in \mathbb{N}$. The result now follows similarly to Theorem 1. □

Proof of Lemma 5. Writing $A \equiv A^{(p;\omega)}_{(\mathcal{J})}$ here for clarity, the result follows from three elementary observations: $\Delta(A) \geq 0$, $\max \Delta(A) > 0$, and $A - \Delta(\Delta(A)) \leq 0$. □

References

R. A. Horn and C. R. Johnson. *Matrix Analysis, 2nd. ed.* Cambridge, 2013.