A Proof of Myerson Lemma

Lemma 1 (Integrated version of the Myerson lemma)**.** *Let bidder i have value distribution F_i and call β_i her strategy, F_{B_i} the induced distribution of bids and ψ_{B_i} the corresponding virtual value function. Assume that F_{B_i} has a density and finite mean. Suppose that i's bids are independent of the bids of the other bidders and denote by G_i the cdf of the maximum of their bids. Suppose a lazy second price auction with reserve price denoted by r is run. Then the payment M_i of bidder i to the seller can be expressed as*

$$M(\beta_i) = \mathbb{E}_{B_i \sim F_{B_i}} \left(\psi_{B_i}(B_i) G_i(B_i) \mathbf{1}(B_i \geq r) \right) .$$

When the other bidders are bidding truthfully, G_i is the distribution of the maximum value of the other bidders.

Proof. The proof is similar to the original one Myerson (1981) (see Krishna (2009) for more details). It consists in using Fubini's theorem and integration by parts (this is why we need conditions on F_{B_i}) to transform the standard form of the seller revenue, i.e.

$$\mathbb{E}_{B_i \sim F_{B_i}, B_j \sim F_{B_j}} \left(\max_{j \neq i}(B_j, r) \mathbf{1}_{[B_i \geq \max_{j \neq i}(B_j, r)]} \right)$$

into the above equation. We consider a lazy second price auction. Call r the reserve price for bidder i. So the seller revenue from bidder i is

$$\mathbb{E} \left(\max_{j \neq i}(B_j, r) \mathbf{1}_{[B_i \geq \max_{j \neq i}(B_j, r)]} \right)$$

In general, we could just call $Y_i = \max_{j \neq i} B_j$ and say that the revenue from i, or i'th expected payment is

$$\mathbb{E} \left(\max(Y_i, r) \mathbf{1}_{[B_i \geq \max(Y_i, r)]} \right) .$$

Call G the cdf of Y_i and $\tilde{G}$ the cdf of $\max(Y_i, r)$. Note that $\tilde{G}(t) = \mathbf{1}_{[t \geq r]} G(t)$.

So if we note $B_i = t$, we have

$$\mathbb{E} \left(\max(Y_i, r) \mathbf{1}_{[B_i \geq \max(Y_i, r)]} | B_i = t \right) = \int_0^t u d\tilde{G}(u)$$

Integrating by parts we get

$$\begin{aligned} \int_0^t u d\tilde{G}(u) &= u\tilde{G}(u) \Big|_0^t - \int_0^t \tilde{G}(u) du \\ &= t\tilde{G}(t) - \int_0^t \tilde{G}(u) du = \mathbf{1}_{[t \geq r]} \left[tG(t) - \int_r^t G(u) du \right] \end{aligned}$$

Hence,

$$\mathbb{E} \left(\max(Y_i, r) \mathbf{1}_{[b_i \geq \max(Y_i, r)]} \right) = \int_0^\infty \left[\mathbf{1}_{[b_i \geq r]} b_i G(b_i) - \int_0^{b_i} \mathbf{1}_{[u \geq r]} G(u) du \right] f_{B_i}(b_i) db_i$$

The first term of this integral is simply

$$\int_0^\infty \mathbf{1}_{[b_i \geq r]} b_i G(b_i) f_{B_i}(b_i) db_i = \mathbb{E} \left(B_i G(B_i) \mathbf{1}_{[B_i \geq r]} \right) .$$

Note that to split the two terms of the integral we need to assume that $\mathbb{E}(B_i) < \infty$, hence the first moment assumption on F_i. The other part of the integral is

$$\int_0^\infty \left(\int_0^{b_i} \mathbf{1}_{[u \geq r]} G(u) du \right) f_{B_i}(b_i) db_i = \int_0^\infty \left(\int \mathbf{1}_{[b_i \geq u]} \mathbf{1}_{[u \geq r]} G(u) du \right) f_{B_i}(b_i) db_i \tag{2}$$

$$= \int \int \mathbf{1}_{[u \geq r]} G(u) \mathbf{1}_{[b_i \geq u]} f_{B_i}(b_i) db_i du = \int \mathbf{1}_{[u \geq r]} G(u) \left(\int \mathbf{1}_{[b_i \geq u]} f(b_i) db_i \right) du \tag{3}$$

$$= \int \mathbf{1}_{[u \geq r]} G(u) P(B_i \geq u) du = \int \mathbf{1}_{[u \geq r]} G(u) \frac{1 - F_{B_i}(u)}{f_{B_i}(u)} f_{B_i}(u) du \tag{4}$$

$$= \mathbb{E} \left(\mathbf{1}_{[B_i \geq r]} G(B_i) \frac{1 - F_{B_i}(B_i)}{f_{B_i}(B_i)} \right) \tag{5}$$

We used Fubini's theorem to change order of integrations, since all functions are non-negative. The result follows.

Of course, when $f_{B_i}(b_i) = 0$ somewhere we understand $f_{B_i}(b_i)/f_{B_i}(b_i) = 0/0$ as being equal to 1. To avoid this problem completely we can also simply write

$$M(\beta_i) = \int \left[b_i f_{B_i}(b_i) - (1 - F_{B_i}(b_i))\right] G(b_i) \mathbf{1}_{[b_i \geq r]} db_i = \int \frac{\partial [b_i(F_{B_i}(b_i) - 1)]}{\partial b_i} G(b_i) \mathbf{1}_{[b_i \geq x]} db_i \ .$$

If F_{B_i} is not differentiable but absolutely continuous, its Radon-Nikodym derivative is used when interpreting the differentiation of $[b_i(F_{B_i}(b_i) - 1)]$ with respect to b_i.

A.1 Technical lemmas

Lemma 6. *Suppose $B_i = \beta_i(X_i)$, where β_i is increasing and differentiable and X_i is a random variable with cdf F_i and pdf f_i. Then*

$$h_{\beta_i}(x_i) \triangleq \beta_i(x) - \beta_i'(x) \frac{1 - F_i(x)}{f_i(x)} = \psi_{F_{B_i}}(\beta_i(x)) \ . \tag{6}$$

Proof. $\psi_{F_{B_i}}(b) = b - \frac{1 - F_{B_i}(b)}{f_{B_i}(b)}$ with $F_{B_i}(b) = F_i(\beta_i^{-1}(b))$ and $f_{B_i}(b) = f_i(\beta_i^{-1}(b)/\beta_i'(\beta_i^{-1}(b)$.
Then, $h_{\beta_i}(x) = \psi_{B_i}(\beta_i(x)) = \beta_i(x) - \beta_i'(x) \frac{1 - F_i(x)}{f_i(x)}$. □

The second lemma shows that for any function g we can find a function β such that $h_\beta = g$.

Lemma 7. *Let X be a random variable with cdf F and pdf f. Assume that $f > 0$ on the support of X. Let x_0 in the support of X, $C \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. lf we note*

$$\beta_g(x) = \frac{C(1 - F(x_0)) - \int_{x_0}^{x} g(u) f(u) du}{1 - F(x)} \ . \tag{7}$$

Then, if $B = \beta_g(X)$,

$$h_{\beta_g}(x) = g(x) \text{ and } \beta_g(x_0) = C \ .$$

If $x_0 \leq t$ and g is non-decreasing on $[x_0, t]$, $\beta_g'(x) \geq (C - g(x))(1 - F(x_0)) f(x)/(1 - F(x))$ for $x \in [x_0, t]$. Hence β_g is increasing on $[x_0, t]$ if g is non-decreasing and $g < C$.

Proof. The result follows by simply differentiating the expression for β_g, and plugging-in the expression for h_{β_g} obtained in Lemma 6. The result on the derivative is simple algebra. □

□

B Optimal Classes of Strategies for two-step Process

Disclaimer: Our lines of proof here are very similar to the one in Tang and Zeng (2018), adapted to the two-stage process. The main difference is that in the two-stage process, the optimal reserve value is not always 0, which gives slightly different forms of optimal strategies for the regular case and make us introduce a quasi-regular case. Also, note that some of the proofs do not require assumptions as strong as we do in the paper on the continuity of F.

General two-stage process with commitment: A general two-stage process with commitment is a tuple of the form $P = (G, H, r, F)$ that is defined as follow:

1 – exploration The seller runs a lazy 2^{nd}-price auction without reserve price. The current bidder faces a competition G. In this step, the potential randomized or deterministic reserve price is denoted by the distribution H.

2 – exploitation The seller runs a lazy 2^{nd}-price auction with reserve price r. The current bidder faces the same competition G as in the first step.

The bidder has a value distribution F. In this process, the bidder commits to a strategy β corresponding to a bid distribution F_B that stays the same in both steps. We additionally assume that F_B is also not too fat-tailed – meaning $1 - F_B(b) = o(b^{-1})$ to ensure the seller's revenue supremum is reached for at least some finite values of reserve price.

The two-step process described in the main body of the paper corresponds to a such a tuple $P_\beta = (G_1, G_2, r_B^*, F)$ where r is chosen to maximize $R_B(r)$, the revenue of the seller defined as $R_B(r) = r\mathbb{P}_{B\sim F_B}(r \le B)$. We also assume the seller to be welfare-benevolent, meaning that $r_\beta^* = \min \operatorname{argmax}_r R_B(r)$.

Technical note on R_B. If F_B is not continuous, we need to make a difference between $F_B(b) = \mathbb{P}_{B\sim F_B}(B \le b)$ and $\tilde{F}_B(b) = \mathbb{P}_{B\sim F_B}(B < b)$. Technically, F_B is càdlàg as a repartition function, while $\tilde{F}_B$ is làdcàg. More practically, if the distribution contains a point mass, the value at which there is the point mass won't be on the same side of the discontinuity in F_B and in $\tilde{F}_B$. In terms of definition of the seller's revenue R_B, as beating the reserve price r is usually implemented as $\mathbf{1}_{[B\ge r]}$, then when doing the integration, we can indeed find the revenue is defined as $R_B(b) = b(1 - \tilde{F}_B(b))$.

First, we state and (re-)prove a well-known property that is central for our proofs: that when the reserve price does not depends on one bidder distribution, the closer her strategy to truthful, the better in utility. For the sake of simplicity, we consider in this statement the reserve price is part of the fixed competition.

Lemma 8. *Given a two-stage process $P = (G, H, r, F)$, for any β, ρ and any $\alpha \in [0, 1]$,*

$$\forall x \in [0, +\infty), \rho(x)(x - \beta(x)) \le 0 \quad \Rightarrow \quad U_\alpha(\beta + \rho) \le U_\alpha(\beta)$$

Proof. Denoting G_r is the cdf of $Y_r = \max(Y, r)$ and $G_\alpha(x) = \alpha G(x)H(x) + (1 - \alpha)G_r(x)$, we have

$$\begin{aligned} U_\alpha(\beta + \rho) - U_\alpha(\beta) &= \alpha \mathbb{E}_{\substack{X\sim F \\ Y\sim GH}} \left[(X - Y)\left(\mathbf{1}_{[\beta(X)+\rho(X)\ge Y]} - \mathbf{1}_{[\beta(X)\ge Y]}\right)\right] \\ &\quad + (1-\alpha)\mathbb{E}_{\substack{X\sim F \\ Y_r\sim G_r}} \left[(X - Y_r)\left(\mathbf{1}_{[\beta(X)+\rho(X)\ge Y_r]} - \mathbf{1}_{[\beta(X)\ge Y_r]}\right)\right] \\ &= \mathbb{E}_{\substack{X\sim F \\ Y\sim G_\alpha}} \left[(X - Y)\left(\mathbf{1}_{[\beta(X)+\rho(X)\ge Y>\beta(X)]} - \mathbf{1}_{[\beta(X)\ge Y>\beta(X)+\rho(X)]}\right)\right] \end{aligned}$$

- if $\forall x, \beta(x) + \rho(x) \ge \beta(x) \ge x$ then $\forall x, (X - Y)\mathbf{1}_{[\beta(X)+\rho(X)\ge Y>\beta(X)]} \le 0$ (and the other indicator is 0)
- if $\forall x, \beta(x) + \rho(x) \le \beta(x) \le x$ then $\forall x, (X - Y)\mathbf{1}_{[\beta(X)\ge Y>\beta(X)+\rho(X)]} \ge 0$ (and the other indicator is 0)

Both ways, $\forall x, \rho(x)(x - \beta(x)) \le 0$, then $U_\alpha(\beta + \rho) - U_\alpha(\beta) \le 0$. □

B.1 Seller Revenue is Lower than with Truthful Bidding

If the reader is intuitively convinced that there exists a best response strategy β with $B = \beta(X)$ such that the monopoly revenue is not higher than the one of the truthful strategy – i.e. $\sup_{r'} R_B(r') \le \sup_{r'} R_X(r')$ – you can safely skip this section. Only for this section, we are going to explicit the dependency of the utility on the two-stage process P by using the notation $U_\alpha^P(\beta)$.

Lemma 9. *Given a two-step process $P_\beta = (G, H, r_\beta^*, F)$ with $r_\beta^* = \inf \operatorname{argmax}_r R_B(r)$,*

$$\max_r R_B(r) > \max_r R_X(r) \;\Rightarrow\; \forall \alpha \in [0, 1],\, U_\alpha^{P_\beta}(\beta) < \max_\omega U_\alpha^{P_\omega}(\omega)$$

Proof. Assuming β is such that $\max_r R_B(r) > R^* = \max_r R_X(r)$, we can build ω such that for any x such that $\beta(x)(1 - \tilde{F}(x)) \le R^*$, $\omega(x) = \beta(x)$ and $\omega(x) = \frac{R^*}{1-\tilde{F}(x)}$ otherwise.

We can choose $\rho(x) = \beta(x) - \omega(x)$ for any x. For any x, either we have $x \ge \beta(x)$ and $\rho(x) \ge 0$ or $\rho(x) = 0$. Hence, for any x, $(x - \beta(x))\rho(x) \ge 0$. By Lemma 8, we have that in the process P_ω (the one with reserve price r_ω^* in the second step), $U_\alpha^{P_\omega}(\beta) \le U_\alpha^{P_\omega}(\omega)$.

Then, because $r_\omega^* \le r_\beta^*$, we have $U_\alpha^{P_\beta}(\beta) \le U_\alpha^{P_\omega}(\beta)$. □

B.2 Deriving the Best Quasi-Regular Response

Because the revenue curve generated by the best response Tang and Zeng (2018) has some limiting properties in practice, we propose to further constrain β by asking R_B to be quasi-concave when R_X is so.

Theorem 1. *Given a 2-step process (G, H, r^*_β, F) with $r^*_\beta = \inf \operatorname{argmax}_r R_B(r)$ and such that F is quasi-regular*[2]*, $\forall \alpha \in [0,1]$, $\exists 0 \le x_0 \le x_1$ such that the best quasi-regular response (maximizing $U_\alpha(\beta)$ with F_B regular) is*

$$\tilde{\beta}_{x_0,x_1}(x) = \mathbf{1}_{[x \le x_0]} x + \mathbf{1}_{[x_0 < x \le x_1]} \frac{R}{1 - F(x)} + \mathbf{1}_{[x > x_1]} x \qquad \text{where} \qquad R = x_1(1 - F(x_1)) \tag{8}$$

Moreover, we have $x_1 = \sup\{x : x(1-F(x)) \ge R\}$ and $x_0 \le \overline{x}_0 = \inf\{x : x(1-F(x)) \ge R\}$.

Proof. Thanks to Lemma 9 and to the Assumption that we have $1 - F_B(b) = o(b^{-1})$, we can restrict ourselves w.l.o.g. to β with a finite reserve price r^*_β. As R_B is làdcàg and upper semicontinuous (because $\tilde{F}_B$ is so), then $\exists x_r, r^*_\beta = \beta(x_r)$.

We denote $R = \sup_x (x \wedge \beta(x))(1 - \tilde{F}(x))$ an define

$$\omega(x) = \begin{cases} \frac{R}{1-\tilde{F}(x)} \text{ if } \beta(x)(1-\tilde{F}(x)) \le R \le x(1-\tilde{F}(x)) \\ \frac{R}{1-\tilde{F}(x)} \text{ if } \beta(x)(1-\tilde{F}(x)) \ge R \ge x(1-\tilde{F}(x)) \\ x \text{ otherwise} \end{cases}$$

By definition of ω, denoting $\rho(x) = \beta(x) - \omega(x)$, we have that for any x, $\rho(x)(x - \omega(x)) \le 0$. Thus, by Lemma 8, $U_\alpha^{(P_\beta)}(\omega) \le U_\alpha^{(P_\beta)}(\beta)$.

Moreover, $\omega(x_r)(1 - \tilde{F}(x_r)) = R$, thus $r^*_\omega \le r^*_\beta$ and $U_\alpha^{(P_\omega)}(\omega) \le U_\alpha^{(P_\beta)}(\omega)$. In the end, we have $U_\alpha^{(P_\omega)}(\omega) \le U_\alpha^{(P_\beta)}(\beta)$.

We can use Lemma 10 to show that $\{x : \omega(x) = \frac{R}{1-\tilde{F}(x)}\}$ is convex and the decrease rate of the right tail to say it is bounded, hence $\exists x_1 \ge x_0 \ge 0, \quad \{x : \omega(x) = \frac{R}{1-\tilde{F}(x)}\} = [x_0, x_1]$.

Finally, we need to check the conditions on x_0 and x_1. By definition of R and x_0, we have $x_0 \le \inf\{x : x(1-\tilde{F}(x)) \ge R\}$. Then, if $x_1 > \sup\{x : x(1-\tilde{F}(x)) \ge R\}$, we can define $\tilde{x}_1 = \sup\{x : x(1-\tilde{F}(x)) \ge R\}$, then define $\tilde{\omega}$ equal to ω below $\tilde{x}_1$ and equal to the identity above. Then we can use again Lemma 8 to show the utility is improved as the reserve price didn't change. ☐

Lemma 10. *Given two real-values functions f and g quasi-concave. we denote $R = \sup_x (f \wedge g)(x)$. We have*

$$\{x : f(x) \ge R\} \cup \{x : g(x) \ge R\} \text{ is convex.}$$

Proof. First, both $\{x : f(x) \ge R\}$ and $\{x : g(x) \ge R\}$ are convex (f, g are quasi-concave). By contradiction, we assume there exists $x < z < y$ such that $f(x) \ge R$, $g(y) \ge R$, $f(z) < R$ and $g(z) < R$. Because g is increasing before z, $\sup_{x<z} (f \wedge g)(x) \le g(z) < R$. With the same argument over f, we have $\sup_{x>z} (f \wedge g)(x) \le f(z) < R$. ☐

B.3 Welfare Sharing between Seller and Bidder

Theorem 2. *Given a two-step process (G, H, r^*_β, F) with $r^*_\beta = \inf \operatorname{argmax}_r R_B(r)$, the best response is of the form of $\tilde{\beta}_{x_0^*, x_1^*}$ and the utility $U_\alpha(\tilde{\beta}_{x_0,x_1})$ has the following derivatives:*

$$\frac{\partial U_\alpha(\tilde{\beta}_{x_0,x_1})}{\partial x_0} = \alpha(1 - F(x_0))G(x_0)H(x_0) - x_0 f(x_0) G_\alpha\left(\frac{x_1(1-F(x_1))}{(1-F(x_0))}\right) \tag{9}$$

$$\frac{\partial U_\alpha(\tilde{\beta}_{x_0,x_1})}{\partial x_1} = f(x_1)\psi(x_1)\left(G_\alpha(x_1) - \mathbb{E}_X\left(\mathbf{1}_{[x_0 \le X \le x_1]} \frac{X}{(1-F(X))} g_\alpha\left(\frac{x_1(1-F(x_1))}{(1-F(X))}\right)\right)\right) \tag{10}$$

where $G_\alpha(x) = \alpha G(x)H(x) + (1-\alpha)G(x)$.

[2] We say a distribution is quasi-regular if the associated revenue curve R_{X_i} is quasi-concave

Proof. The push-forward of F through $\tilde{\beta}_{x_0,x_1}$ has the following virtual value:

$$\psi_B(\beta(X)) = \mathbf{1}_{[X\le x_0]}\psi(X) + \mathbf{1}_{[x_0<X\le x_1]}0 + \mathbf{1}_{[X>x_1]}\psi(X)$$

Then, we can use Myerson's Lemma Myerson (1981) to write the utility:

$$\begin{aligned} U_\alpha(\tilde{\beta}_{x_0,x_1}) = &\ \mathbb{E}_X\left(\mathbf{1}_{[X<x_0]}(X-\psi(X))\alpha G(x)H(x)\right) \\ &+ \mathbb{E}_X\left(\mathbf{1}_{[x_0<X\le x_1]}XG_\alpha\left(\frac{x_1(1-\tilde{F}(x_1))}{1-\tilde{F}(x)}\right)\right) \\ &+ \mathbb{E}_X\left(\mathbf{1}_{[X<x_1]}(X-\psi(X))G_\alpha(X)\right) \end{aligned} \tag{11}$$

Straigtforward calculations gives the derivatives. □

Theorem 1. *Given a two-step process (G, H, r^*_β, F) with $r^*_\beta = \inf \operatorname{argmax}_r R_B(r)$. We assume here $\forall x \in (0, x^*], f(x) > 0$ with $x^* = \operatorname{argmax}_x R_X(x)$. If $\exists y \in (0, x^*], \forall x < y, \frac{G(x)H(x)}{f(x)} < x$ and $\frac{G(x)H(x)}{f(x)}$ is bounded on $(0, x^*]$, then $\exists \alpha_0 > 0$ such that $\forall \alpha < \alpha_0$,*

$$(x^*_0, x^*_1) \in \operatorname*{argmax}_{x_0,x_1} U_\alpha(\tilde{\beta}_{x_0,x_1}) \quad \Rightarrow \quad x^*_0 = 0$$

Proof. Denoting $\overline{x}_1 = \sup_\alpha x^*_1$,

$$\begin{aligned} \frac{\partial U_\alpha(\tilde{\beta}_{x_0,x_1})}{\partial x_0} &= \alpha(1-F(x_0))G(x_0)H(x_0) - x_0 f(x_0)G_\alpha\left(\frac{x_1(1-F(x_1))}{(1-F(x_0))}\right) && (12) \\ &\le \alpha G(x_0)H(x_0) - x_0 f(x_0)G_\alpha\left(\overline{x}_1(1-F(\overline{x}_1))\right) && (13) \\ &\le \alpha G(x_0)H(x_0) - x_0 f(x_0)\inf_{\gamma\in[0,1]} G_\gamma\left(\overline{x}_1(1-F(\overline{x}_1))\right) && (14) \\ & && (15) \end{aligned}$$

So $\frac{\partial U_\alpha(\tilde{\beta}_{x_0,x_1})}{\partial x_0} \le 0 \Leftrightarrow \alpha\frac{G(x_0)H(x_0)}{f(x_0)} \le x_0 \inf_{\gamma\in[0,1]} G_\gamma\left(\overline{x}_1(1-F(\overline{x}_1))\right)$. Finally, as $\inf_{\gamma\in[0,1]} G_\gamma\left(\overline{x}_1(1-F(\overline{x}_1))\right) \in (0, b)$ (from Lemma 11), I can choose $\alpha_0 = 1/\inf_{\gamma\in[0,1]} G_\gamma\left(\overline{x}_1(1-F(\overline{x}_1))\right)$.

Case $x_0 < y$: from the assumptions, we can directly get $\frac{\partial U(x_0,x_1)}{\partial x_0} \le 0$.

Case $y \le x_0$: Denoting $C = \sup_{x\in[y,x^*]}\frac{G(x)H(x)}{f(x)}$, we can denote $\alpha_1 = \frac{y}{x\alpha_0}$.

Finally, for $\alpha < \alpha_2 = \min(\alpha_0, \alpha_1)$, we have $\frac{\partial U_\alpha(\tilde{\beta}_{x_0,x_1})}{\partial x_0} \le 0$ for any x_0. □

Lemma 11. *Given a two-step process (G, H, r^*_β, F) with $r^*_\beta = \inf \operatorname{argmax}_r R_B(r)$, we have $\overline{x}_1 = \sup_\alpha x^*_1 < b$.*

Proof. For this we study the different terms of U_α when x_1 goes to b, to prove that in such case, the utility goes to 0.

$$\begin{aligned} \mathbb{E}_X\left(\mathbf{1}_{[X\le x^*_0]}(X-\psi_X(X))\alpha G(X)H(X)\right) &\le \int \mathbf{1}_{[X\le x^*_0]}(1-F(X))G(X)H(X)\alpha \mathrm{d}X \\ &\le \int \mathbf{1}_{[X\le x^*_0]}\mathrm{d}X \\ &\le x^*_0 = o_{x^*_1\to b}(1) \end{aligned}$$

$$\begin{aligned} \mathbb{E}_X\left(\mathbf{1}_{[X>x^*_1]}(X-\psi_X(X))G_\alpha(X)\right) &\le \int \mathbf{1}_{[X>x^*_1]}(1-F(X))G_\alpha(X)H(X)\mathrm{d}X \\ &\le \int \mathbf{1}_{[X>x^*_1]}(1-F(X))\mathrm{d}X \\ &= o_{x^*_1\to b}(1) \quad \text{as either } 1-F(x) = o(x^{-1}) \text{ or } b \text{ is finite.} \end{aligned}$$

Fix $y \in (x_0^*, x_1^*)$,

$$\mathbb{E}_X\left(\mathbf{1}_{[x_0^*<X\leq x_1^*]} X G_\alpha\left(\frac{x_1^*(1-F(x_1^*))}{1-F(x)}\right)\right) = \mathbb{E}_X\left(\mathbf{1}_{[x_0^*<X\leq y]} X G_\alpha\left(\frac{x_1^*(1-F(x_1^*))}{1-F(x)}\right)\right) + \mathbb{E}_X\left(\mathbf{1}_{[y<X\leq x_1^*]} X G_\alpha\left(\frac{x_1^*(1-F(x_1^*))}{1-F(x)}\right)\right)$$

We have $x_1^*(1-F(x_1^*)) \to 0$ when $x_1^* \to b$ as either b is finite or $1-F(x) = o(x^{-1})$. Hence, we have $\mathbb{E}_X\left(\mathbf{1}_{[x_0^*<X\leq y]} X G_\alpha\left(\frac{x_1^*(1-F(x_1^*))}{1-F(x)}\right)\right) = o_{x_1^*\to b}(1)$. On the other side,

$$\begin{aligned}\mathbb{E}_X\left(\mathbf{1}_{[y<X\leq x_1^*]} X G_\alpha\left(\frac{x_1^*(1-F(x_1^*))}{1-F(x)}\right)\right) &\leq \int_y^b X f(X)\mathrm{d}X \\ &= y(1-F(y)) + \int_y^b 1-F(X)\mathrm{d}X \\ &= o_{y\to b}(1)\end{aligned}$$

By contradiction, assume $\forall y < b, \exists \alpha \in [0,1]$ such that $x_1^* > y$. Then for any $\varepsilon > 0$, we can set y such that $\exists \alpha, U_\alpha(\tilde{\beta}_{x_0^*,x_1^*}) \leq \varepsilon$, which contradict the optimality of (x_0^*, x_1^*) (as bidding truthfully brings a strictly positive utility). $\square$

C Proof of the existence of a Nash equilibrium ($\alpha = 0$)

More formally, the game we are considering is the following: all bidders have the same value distribution F_X. The mechanism is a lazy second price auction with reserve price equal to the monopoly price of bidders' bid distributions. We denote by $\beta_1, ..., \beta_n$ the bidding strategies used by each player and by $U_i(\beta_1, ..., \beta_n)$ the utility of bidder i when using β_i. We say that strategy β is a symmetric Nash equilibrium for this game if for all bidding strategies $\hat{\beta}$ and for all bidders i, if all players except bidder i are using β,

$$U_i(\beta, ..., \hat{\beta}, ..., \beta) \leq U_i(\beta, ..., \beta, ..., \beta)$$

We can reuse the same type of formulas we introduced in the previous subsection to compute the expected utility of each bidder. We focus on bidder i. We note G_β the distribution of the maximum bid of the competitors of bidder i. The distribution of the highest bid of the competition depends now on the strategy of bidder i. With this notation,

$$U_i(\beta, ..., \hat{\beta}, ..., \beta) = \mathbb{E}\left((X - h_{\hat{\beta}}(X))G_\beta(\hat{\beta}(X))\mathbf{1}(X \geq h_{\hat{\beta}}^{-1}(0))\right).$$

As in the previous setting, we can compute directional derivatives when $\hat{\beta} = \beta + t\rho$. We can find different sorts of Bayes-Nash equilibria depending on the class of function bidders are optimizing in. We call them *restricted Nash equilibrium* since they are equilibrium in a restricted class of bidding strategies. In practice, bidders may not be able or willing to implement all possible bidding strategies. They often limit themselves to strategies that are easy to practically implement.

We have obtained results on *restricted Nash equilibria* when the bidders restrict themselves to the class of linear shading or affine shading, though we do not present them here as they are a bit tangential to the main thrust of this paper. We detail the results on a larger and more interesting class of functions: the class of thresholded bidding strategies that we introduced in Definition 1. We assume that bidders can choose any strategies in this large class of functions.

Our first theorem on a Nash equilibrium between bidders is the following:

Theorem 7. *We consider the symmetric setting where all the bidders have the same value distributions. We assume for simplicity that the distribution is supported on [0,1]. Suppose this distribution has density that is continuous at 0 and 1 with $f(1) \neq 0$ and X has a distribution for which ψ_X crosses 0 exactly once and is positive beyond that crossing point.*

There is a unique symmetric Nash equilibrium in the class of thresholded bidding strategies defined in Definition 1. It is found by solving Equation (16) to determine r^* and bidding truthfully beyond r^*

Moreover, if all the bidders are playing this strategy corresponding to the unique Nash equilibrium, the revenue of the seller is the same as in a second price auction with no reserve price. The same is true of the utility of the buyers.

Sketch of proof:

- State a directional derivative result in this class of strategies by directionally-differentiating the expression of $U_i(\beta, ..., \beta_t, ..., \beta)$.
- Prove uniqueness of the Nash equilibrium.
- Prove existence of the Nash equilibrium.
- Show equivalence of revenue with a second price auction without reserve.

C.1 A directional derivative result

We can state a directional derivative result in this class of strategies by directionally-differentiating the expression of $U_i(\beta, ..., \beta_t, ..., \beta)$.It implies the only strategy with 0 “gradient" in this class is truthful beyond a value r (r is different from the one in strategic case), where r can be determined through a non-linear equation.

Lemma 12. *Consider the symmetric setting. Suppose that all the bidders are strategic and that they are all except one using the strategy β, G_β denotes the CDF of the maximum bid of the competition when they use β and g_β the corresponding pdf, consider r and γ such that $\beta = \beta_r^\gamma$. Assuming that the seller is welfare-benevolent and that $\beta_i^t = \beta_r^{\gamma+t\rho}$. Then*

$$\left.\frac{\partial U(\beta, ..., \beta_i^t, ..., \beta)}{\partial t}\right|_{t=0} = \mathbb{E}\Big((X - \gamma(X))g_\beta(\gamma(X))\rho(X)\mathbf{1}_{[X\geq r]}\Big) + \rho(r)$$
$$(1 - F(r))\left(\mathbb{E}\left(\frac{X}{1 - F(X)}g_\beta\left(\frac{\gamma(r)(1 - F(r))}{1 - F(X)}\right)\mathbf{1}_{[X\leq r]}\right) - G_\beta(\gamma(r))\right)$$

Also,

$$\frac{\partial U(\beta, ..., \beta_i, ..., \beta)}{\partial r} = -h_\beta(r)f(r)\left(\mathbb{E}\left(\frac{X}{1 - F(X)}g_\beta\left(\frac{\gamma(r)(1 - F(r))}{1 - F(X)}\right)\mathbf{1}_{[X\leq r]}\right) - G_\beta(\gamma(r))\right)$$

The only strategy γ and threshold r for which we can cancel the derivatives in all directions ρ consists in bidding truthfully beyond r^*_{all}, where r^*_{all} satisfies the equation

$$\frac{K - 1}{r^*_{all}(1 - F(r^*_{all}))}\mathbb{E}\Big(XF^{K-2}(X)(1 - F(X))\mathbf{1}_{[X\leq r^*_{all}]}\Big) = F^{K-1}(r^*_{all})\,. \tag{16}$$

Proof. In the case of K symmetric bidders with the same value distributions F, we have

$$G_\beta(x) = F^{K-1}(\beta^{-1}(x)) \qquad \text{and} \qquad g_\beta(\beta(x)) = \frac{(K - 1)F^{K-2}(x)}{\beta'(x)}f(x)\,.$$

The last result follows by plugging-in these expressions in the corresponding equations. □

C.2 Uniqueness of the Nash equilibrium

We show that this strategy represents the unique symmetric Bayes-Nash equilibrium in the class of shading functions defined in Definition 1. At this equilibrium, the bidders recover the utility they would get in a second price auction without reserve price.

Lemma 13. *Suppose X has a distribution for which ψ_X crosses 0 exactly once and is positive beyond that crossing point. Then Equation (16) has a unique non-zero solution.*

Proof. We have by integration by parts

$$\begin{aligned}&(K - 1)\mathbb{E}\left(\left(XF^{K-2}(X)(1 - F(X))\mathbf{1}_{[X\leq r]}\right)\right)\\ &= r(1 - F(r))F^{K-1}(r) + \mathbb{E}\left(\left(\psi_X(X)F^{K-1}(X)\mathbf{1}_{[X\leq r]}\right)\right)\,.\end{aligned}$$

Hence finding the root of

$$\frac{K - 1}{r(1 - F(r))}\mathbb{E}\left(\left(XF^{K-2}(X)(1 - F(X))\mathbf{1}_{[X\leq r]}\right)\right) = F^{K-1}(r)$$

amounts to finding the root(s) of

$$\mathfrak{R}(r) \triangleq \mathbb{E}\left(\left(\psi_X(X)F^{K-1}(X)\mathbf{1}_{[X\leq r]}\right)\right) = 0\,.$$

0 is an obvious root of the above equation but does not work for the penultimate one. Note that for the class of distributions we consider (which is much larger than regular distributions but contains it), this function $\mathfrak{R}$ is decreasing and then increasing after $\psi_X^{-1}(0)$, since the virtual value is negative and then positive. Since $\mathfrak{R}(0) = 0$, it will have at most one non-zero root for the distributions we consider. The fact that this function is positive at infinity (or at the end of the support of X) comes from the fact that its value then is the revenue-per-buyer of a seller performing a second price auction with K symmetric buyers bidding truthfully with a reserve price of 0. And this is by definition positive. So we have shown that for regular distributions and the much broader class of distributions we consider the function $\mathfrak{R}$ has exactly one non-zero root. □

C.3 Existence of the Nash equilibrium

C.3.1 Best response strategy: one strategic case

Definition 1 (Thresholded bidding strategies)**.** *A bidding strategy β is called a thresholded bidding strategy if and only if there exists $r > 0$ such that for all $x < r, h_\beta(x) = \psi_B(\beta(x)) = 0$. This family of functions can be parametrized with*

$$\beta_r^\gamma(x) = \frac{\gamma(r)(1-F(r))}{1-F(x)}\mathbf{1}_{[x<r]} + \gamma(x)\mathbf{1}_{[x\geq r]} \ ,$$

with $r \in \mathbb{R}$ and $\gamma : \mathbb{R} \to \mathbb{R}$ strictly increasing.

Lemma 14. *Suppose that only one bidder is strategic, let G denote the CDF of the maximum value of the competition and g the corresponding pdf. Denote by $U(\beta_r^\gamma)$ the utility of the bidder using the strategy β_r^γ according to the parametrization of Definition 1. Assume that the seller is welfare-benevolent. Then if $\beta_t = \beta_r^{\gamma+t\rho}$,*

$$\begin{aligned}\left.\frac{\partial U(\beta_t)}{\partial t}\right|_{t=0} &= \mathbb{E}\Big((X-\gamma(X))g(\gamma(X))\rho(X)\mathbf{1}_{[X\geq r]}\Big) + \rho(r) \\ &(1-F(r))\left(\mathbb{E}\left(\frac{X}{1-F(X)}g\left(\frac{\gamma(r)(1-F(r))}{1-F(X)}\right)\mathbf{1}_{[X\leq r]}\right) - G(\gamma(r))\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial U(\beta_r^\gamma)}{\partial r} &= -\, h_\beta(r)f(r) \\ &\left(\mathbb{E}\left(\frac{X}{1-F(X)}g\left(\frac{\gamma(r)(1-F(r))}{1-F(X)}\right)\mathbf{1}_{[X\leq r]}\right) - G(\gamma(r))\right)\end{aligned}$$

The only strategy γ and threshold r for which we can cancel the derivatives in all directions ρ consists in bidding truthfully beyond r^*, where r^* satisfies the equation

$$G(r^*) = \mathbb{E}\left(\frac{X}{1-F(X)}g\left(\frac{r^*(1-F(r^*))}{1-F(X)}\right)\mathbf{1}_{[X\leq r^*]}\right) . \tag{17}$$

Proof. Recall that our revenue in such a strategy (we take $\epsilon = 0$) is just, when the seller is welfare-benevolent (and hence s/he will push the reserve value to 0 as long as $\psi_B(\beta(x)) \geq 0$ for all x)

$$U(\beta_r^\gamma) = \mathbb{E}\left((X-\psi_B(\beta(X)))G(\beta(X))\mathbf{1}_{[X\geq r]}\right) + \mathbb{E}\left(XG(\frac{\beta(r)(1-F(r))}{1-F(x)})\mathbf{1}_{[X\leq r]}\right) .$$

Now if $\beta_t = \beta + t\rho$, as usual, we have $\psi_{B_t}(\beta_t) = h_\beta + th_\rho$. Because we assumed that $\psi_B(\beta(r)) > 0$, changing β to β_t won't drastically change that; in particular if $\psi_{B_t}(\beta_t(r)) > 0$ the seller is still going to take all bids after $\beta_t(r)$. In particular, we don't have to deal with the fact that the optimal reserve value for the seller may be completely different for β and β_t. So the assumption $\psi_B(\beta(r)) > 0$ is here for convenience and to avoid technical nuisances. In any case, we have

$$\begin{aligned}\frac{\partial U(\beta_t)}{\partial t} &= \mathbb{E}\left(\left[-h_\rho(X)G(\beta(X)) + (X-h_\beta(X))\rho(X)g(\gamma(X))\right]\mathbf{1}_{[X\geq r]}\right) \\ &+\mathbb{E}\left(X\rho(r)\frac{1-F(r)}{1-F(X)}g(\frac{\gamma(r)(1-F(r))}{1-F(x)})\mathbf{1}_{[X\leq r]}\right) .\end{aligned}$$

Now using integration by parts on $\int_r^\infty \rho'(x)(1-F(x))G(\gamma(x))dx$, we have

$$\begin{aligned}\mathbb{E}\left(-h_\rho(X)G(\beta(X))\mathbf{1}_{[X\geq r]}\right) &= \rho(x)(1-F(x))G(\gamma(x))|_r^\infty \\ &+ \int_r^\infty [\rho(x)f(x)G(\gamma(x)) - \rho(x)\beta'(x)g(\gamma(x))(1-F(x))]dx \\ &- \int_r^\infty \rho(x)f(x)G(\gamma(x))dx \\ &= -\rho(r)(1-F(r))G(\beta(r)) - \mathbb{E}\left(\rho(X)\beta'(X)g(\gamma(X))\frac{1-F(X)}{f(X)}\right) .\end{aligned}$$

Hence,

$$\frac{\partial U(\beta_t)}{\partial t} = \mathbb{E}\left((X-\gamma(X))\rho(X)g(\gamma(X))\mathbf{1}_{[X\geq r]}\right) - \rho(r)(1-F(r))G(\gamma(r))$$
$$+\rho(r)(1-F(r))\mathbb{E}\left(\left(\frac{X}{1-F(X)}\right)g(\frac{\gamma(r)(1-F(r))}{1-F(x)})\mathbf{1}_{[X\leq r]}\right) .$$

This gives the first equation of Lemma 14.

On the other hand, we have

$$\frac{\partial U(\beta_r^\gamma)}{\partial r} = -(r-\psi_B(\gamma(r)))G(\gamma(r))f(r) + rG(\gamma(r))f(r)$$
$$+\mathbb{E}\left(\frac{X}{1-F(X)}g(\frac{\gamma(r)(1-F(r))}{1-F(x)})\mathbf{1}_{[X\leq r]}\right)[\gamma'(r)(1-F(r))-\beta(r)f(r)] .$$

Since

$$[\gamma'(r)(1-F(r))-\gamma(r)f(r)] = -f(r)\psi_B(\beta(r)) ,$$

we have established that

$$\frac{\partial U(\beta_r^\gamma)}{\partial r} = \psi_B(\gamma(r))f(r)\left[G(\gamma(r)) - \mathbb{E}\left(\frac{X}{1-F(X)}g(\frac{\gamma(r)(1-F(r))}{1-F(x)})\mathbf{1}_{[X\leq r]}\right)\right] .$$

We see that the only strategy β and threshold r for which we can cancel the derivatives in all directions ρ consists in bidding truthfully beyond r, where r satisfies the equation

$$G(r) = \mathbb{E}\left(\frac{X}{1-F(X)}g\left(\frac{r(1-F(r))}{1-F(X)}\right)\mathbf{1}_{[X\leq r]}\right) .$$

Lemma 14 is shown. □

C.3.2 Proof of existence of the Nash equilibrium

• On Equation (17) and consequences

Recall the statement of Equation (17).

$$\mathbb{E}\left(\frac{X}{1-F(X)}g\left(\frac{r(1-F(r))}{1-F(X)}\right)\mathbf{1}_{[X\leq r]}\right) = G(r) . \tag{17}$$

Lemma 15. *Suppose X has a regular distribution that is compactly supported (on [0,1] for convenience). Equation (17) can be re-written as*

$$\mathbb{E}\left(\frac{X}{1-F(X)}g\left(\frac{r(1-F(r))}{1-F(X)}\right)\mathbf{1}_{[X\leq r]}\right) = G(r) + \frac{1}{r(1-F(r))}\mathbb{E}\left(\psi_X(X)G\left(\frac{r(1-F(r))}{1-F(X)}\right)\mathbf{1}_{[X\leq r]}\right) . \tag{18}$$

Equation (17) has at most one solution on (0,1). This possible root is greater than $\psi^{-1}(0)$. 0 is also a (trivial) solution of Equation (17).

The assumption that X is supported on [0,1] can easily be replaced by the assumption that it is compactly supported, but it made notations more convenient.

Proof of Lemma 15. We use again integration by parts :

$$\mathbb{E}\left(\frac{X}{1-F(X)}g\left(\frac{r(1-F(r))}{1-F(X)}\right)\mathbf{1}_{[X\leq r]}\right) = \int_0^r \frac{x(1-F(x))}{(1-F(x))^2}g\left(\frac{r(1-F(r))}{1-F(x)}\right)f(x)dx$$
$$= x(1-F(x))\frac{G\left(\frac{r(1-F(r))}{1-F(x)}\right)}{r(1-F(r))}\Bigg|_0^r - \frac{1}{r(1-F(r))}\int_0^r (x(1-F(x)))'G\left(\frac{r(1-F(r))}{1-F(x)}\right)dx$$
$$= G(r) + \frac{1}{r(1-F(r))}\mathbb{E}\left(\psi_X(X)G\left(\frac{r(1-F(r))}{1-F(X)}\right)\mathbf{1}_{[X\leq r]}\right) .$$

So we are really looking at the properties of the solution of

$$\frac{1}{r(1-F(r))}\mathbb{E}\left(\psi_X(X)G\left(\frac{r(1-F(r))}{1-F(X)}\right)\mathbf{1}_{[X\leq r]}\right)=0\ .$$

We call

$$I(r)=\mathbb{E}\left(\psi_X(X)G\left(\frac{r(1-F(r))}{1-F(X)}\right)\mathbf{1}_{[X\leq r]}\right)\ .$$

For regular distributions, it is clear that if $r=\psi^{-1}(0)$, $I(r)<0$.

Now we note that

$$\frac{\partial I(r)}{\partial r}=\psi_X(r)G\left(\frac{r(1-F(r))}{1-F(r)}\right)-\psi_X(r)\mathbb{E}\left(\frac{\psi_X(X)}{1-F(X)}g\left(\frac{r(1-F(r))}{1-F(X)}\right)\mathbf{1}_{[X\leq r]}\right)\ .$$

Using $\psi_X(x)=x-(1-F(x))/f(x)<x$, we see that

$$\mathbb{E}\left(\frac{\psi_X(X)}{1-F(X)}g\left(\frac{r(1-F(r))}{1-F(X)}\right)\mathbf{1}_{[X\leq r]}\right)<\mathbb{E}\left(\frac{X}{1-F(X)}g\left(\frac{r(1-F(r))}{1-F(X)}\right)\mathbf{1}_{[X\leq r]}\right)$$

So if r is a solution of

$$\mathbb{E}\left(\frac{X}{1-F(X)}g\left(\frac{r(1-F(r))}{1-F(X)}\right)\mathbf{1}_{[X\leq r]}\right)=G(r)$$

we have, if $r>\psi_X^{-1}(0)$ and $r\neq 1$,

$$\frac{\partial I(r)}{\partial r}>0\ .$$

So r needs to be a solution of $I(r)=0$ (which is equivalent to the initial equation for non-trivial solutions) and must have $\frac{\partial I(r)}{\partial r}>0$.

So we have shown that I is a function such that its (non-trivial) zeros are such that I is strictly increasing at those roots. Because I is differentiable and hence continuous, this implies that I can have at most one non-trivial root. (0 is a trivial root of $I(r)=0$, though it is not an acceptable solution of our initial problem.)

We note that the end point of the support of X is also a trivial solution of $I(r)=0$, by the dominated convergence theorem, though not an acceptable solution of our initial problem, as shown by a simple inspection.

□

Lemma 16. *Suppose that $G(0)=0$, G is continuous and either $G(x)=g^k(0)x^k+o(x^k)$ near 0 for some k or G is constant near 0. Assume f has a continuous density near 1 with $f(1)\neq 0$ and the Assumptions of Lemma 15 are satisfied. Then Equation*

$$\mathfrak{G}(r)=\frac{1}{r(1-F(r))}\mathbb{E}\left(\psi_X(X)G\left(\frac{r(1-F(r))}{1-F(x)}\right)\mathbf{1}_{[X\leq r]}\right)=0$$

has a unique root in $(0,1)$.

In particular in this situation there exists an optimal strategy in the class of shading functions defined in Definition 1 and it is unique. It is defined by being truthful beyond $r^*:\mathfrak{G}(r^*)=0$ and shading in such a way that our virtual value is 0 below r^*.

Proof. We have already seen that this equation has at most one zero on (0,1) so we now just need to show that the function $\mathfrak{G}$ is positive somewhere to have established that it has a zero. Of course, for $r=\psi^{-1}(0)$, the function is negative.

• **G not locally constant near 0** Since G is a cdf, and hence a non-decreasing function, the first k such that $g^{(k)}(0)\neq 0$ has $g^{(k)}(0)>0$. Otherwise, G would be decreasing around 0. We treat the case where G is constant near 0 later so we now assume that k exists and is finite.

Let r be such that $1 - F(r) = \epsilon$ very small (e.g. 10^{-6}). Let $c < r$ be such that $(1 - F(r))/(1 - F(c)) < \eta$ very small (e.g. 10^{-3}) and $\psi_X(c) > 0$. Hence we can use a Taylor approximation to get that

$$G\left(\frac{r(1-F(r))}{1-F(x)}\right)\mathbf{1}_{[x\leq c]} \simeq g^{(k)}(0)\frac{(r(1-F(r)))^k}{(1-F(x))^k}\mathbf{1}_{[x\leq c]} \ .$$

Integrating this out (ignoring at this point possible integration questions), we get

$$\mathbb{E}\left(\psi_X(X)G\left(\frac{r(1-F(r))}{1-F(x)}\right)\mathbf{1}_{[X\leq c]}\right) \simeq g^{(k)}(0)(r(1-F(r)))^k\mathbb{E}\left(\frac{\psi_X(X)}{[1-F(X)]^k}\mathbf{1}_{[X\leq c]}\right) \ .$$

Now integration by parts shows that, if $k > 1$

$$\begin{aligned}\mathbb{E}\left(\frac{\psi_X(X)}{[1-F(X)]^k}\mathbf{1}_{[X\leq c]}\right) &= \int_0^c x\frac{f(x)}{(1-F(x))^k} - \frac{1}{(1-F(x))^{k-1}}dx\\ &= \left.\frac{xf(x)}{(1-F(x))^k}\right|_0^c - \left(1+\frac{1}{k-1}\right)\int_0^c \frac{dx}{[1-F(x)]^{k-1}} \ .\end{aligned}$$

If $k = 1$, using the fact that $(\ln(1 - F(x))' = -f(x)/(1 - F(x))$, we have

$$\mathbb{E}\left(\frac{\psi_X(X)}{1-F(X)}\mathbf{1}_{[X\leq c]}\right) = \int_0^c x\frac{f(x)}{1-F(x)} - 1dx$$

$$= -x\ln(1-F(x))|_0^c - c + \int_0^c \ln(1-F(x))dx = -c\ln(1-F(c)) - c + \int_0^c \ln(1-F(x))dx$$

We now assume that $k > 1$; the adjustments for $k = 1$ are trivial and are left to the reader. Clearly, when $F(c)$ is close to 1, we have, since we assume that $f(c) \neq 0$,

$$\int_0^c \frac{dx}{[1-F(x)]^{k-1}} \leq \frac{c}{(1-F(c))^{k-1}} = o\left(cf(c)/(1-F(x))^k\right) \ .$$

So we have, as c increases so that $F(c) \simeq 1$ (while of course having $(1 - F(r))/(1 - F(c)) < \eta$),

$$\frac{1}{r(1-F(r))}\mathbb{E}\left(\psi_X(X)G\left(\frac{r(1-F(r))}{1-F(x)}\right)\mathbf{1}_{[X\leq c]}\right) \sim [r(1-F(r))]^{k-1}\, g^{(k)}(0)\, \frac{cf(c)}{(1-F(c))^k} > 0 \ .$$

So we have, if $\psi_X(x)f(x)$ is continuous near r, i.e. $f(x)$ is continuous near r,

$$\begin{aligned}&\mathbb{E}\left(\psi_X(X)G\left(\frac{r(1-F(r))}{1-F(X)}\right)\mathbf{1}_{[c\leq X\leq r]}\right) \geq G\left(\frac{r(1-F(r))}{1-F(c)}\right)\mathbb{E}\left(\psi_X(X)\mathbf{1}_{[c\leq X\leq r]}\right)\\ &\simeq G\left(\frac{r(1-F(r))}{1-F(c)}\right)(r-c)(rf(r)-(1-F(r)) \simeq G\left(\frac{r(1-F(r))}{1-F(c)}\right)(r-c)rf(r) \ .\end{aligned}$$

Now we note that using a Taylor expansion of $1 - F(x)$ around r, we have

$$x \simeq r + \frac{F(x)-F(r)}{f(r)} \ .$$

So we see that

$$\begin{aligned}&\frac{1}{r(1-F(r))}\mathbb{E}\left(\psi_X(X)G\left(\frac{r(1-F(r))}{1-F(X)}\right)\mathbf{1}_{[c\leq X\leq r]}\right)\\ &\simeq rG\left(\frac{r(1-F(r))}{1-F(c)}\right)\frac{F(r)-F(c)}{1-F(r)} \simeq rG\left(\frac{r(1-F(r))}{1-F(c)}\right)\left(\frac{1-F(c)}{1-F(r)}-1\right) \ .\end{aligned}$$

If now we take c_2 such that $1 - F(r)/(1 - F(c_2)) = 1/3$, the reasoning above applies and we have

$$\frac{1}{r(1-F(r))}\mathbb{E}\left(\psi_X(X)G\left(\frac{r(1-F(r))}{1-F(X)}\right)\mathbf{1}_{[c_2\leq X\leq r]}\right) \geq rG\left(r/3\right) > 0 \ .$$

Because $\pi(r) = (1 - F(r))/(1 - F(x))$ is increasing, we have $c \leq c_2$, since $\eta = \pi(c) \leq \pi(c_2) = 1/3$. So we have

$$\frac{1}{r(1-F(r))}\mathbb{E}\left(\psi_X(X)G\left(\frac{r(1-F(r))}{1-F(X)}\right)\mathbf{1}_{[c\leq X\leq c_2]}\right) \geq 0 \; .$$

Of course the choice of $1/3$ above is arbitrary and it could be replaced by any fixed number $s < 1$ such that $G(sr) > 0$. We conclude that $\mathfrak{G}$ is positive in a neighborhood of 1.

• **G locally constant near 0** In this case we can pick c such that

$$\mathbb{E}\left(\psi_X(X)G\left(\frac{r(1-F(r))}{1-F(x)}\right)\mathbf{1}_{[X\leq c]}\right) = 0 \; .$$

If c is such that $\psi(c) > 0$ our arguments above immediately carry through. In fact we can ensure that this is always true by picking such a c and picking a corresponding r as function of the ratio $(1 - F(r))/(1 - F(c))$ we would like.

So we conclude that even in this case, $\mathfrak{G}$ is positive in a neighborhood of 1 □

Theorem 8. *We consider the symmetric case and assume that bidders have a compactly supported and regular distribution. We assume for simplicity that the distribution is supported on [0,1].*

Suppose this distribution has density that is continuous at 0 and 1 with $f(1) \neq 0$.

Then there is a unique symmetric equilibrium in the class of shading functions defined in Definition 1.

It is found by solving Equation (16) to determine r^* *and bidding truthfully beyond* r^*.

Proof. We already know that there is at most one solution since Equation (16) has exactly one solution.

If all the bidders but one put themselves at this strategy, we know from Lemma 16, which applies because of our assumptions on f, that the optimal strategy for bidder one is unique in the class we consider and consists in using a shading that is truthful beyond r. This r is uniquely determined by Equation (17) but given the shading used by the other players we know that the r determined by Equation (16) is a solution. Hence we have an equilibrium. □

Uniform[0,1] example When $K = 2$, the solution of Equation (16) and hence the equilibrium is obtained at $r = 3/4$. For $K = 3$, $r = 2/3$; $K = 4$ gives $r = 15/24 = 0.625$; $K = 5$ gives $r = .6$.

With appropriate shading functions, the bidders can recover the utility they would get when the seller was not optimizing her mechanism to maximize her revenue. Nevertheless, this equilibrium can be weakly collusive since we restrict ourselves to the class of functions introduced in Definition 1. It is not obvious that the strategy exhibited in Theorem 9 is an equilibrium in a larger class of functions. However, as mentioned previously, from a practical standpoint, as of now there exists no other clear way to increase drastically bidders utility that is independent from a precise estimation of the competition. The fact that at symmetric equilibrium bidders recover the same utility as in a second price auction with no reserves arguably makes it an even more natural class of bidding strategies to consider from the bidder standpoint.

C.4 Equivalence of revenue

Theorem 9. *Suppose we are in a symmetric situation and all buyers use the symmetric optimal strategy described above.*

Then the revenue of the seller is the same as in a second price auction with no reserve price. The same is true of the revenue of the buyers.

Interestingly, the theorem shows that this shading strategy completely cancels the effect of the reserve price. This is a result akin to our result on the Myerson auction.

Proof. The revenue of the seller per buyer is

$$\mathbb{E}\left(\psi_X(X)F^{K-1}(X)\mathbf{1}_{[X\geq r^*]}\right), \text{ with } \mathfrak{R}(r^*) = 0 \; .$$

Hence it is also

$$\mathbb{E}\left(\psi_X(X)F^{K-1}(X)\mathbf{1}_{[X\geq r^*]}\right) + \mathfrak{R}(r^*) = \mathbb{E}\left(\psi_X(X)F^{K-1}(X)\right) .$$

This is exactly the revenue of the seller in a second price auction with no reserve price.

From the buyer standoint, his/her revenue is, since all buyers are using the same increasing strategy and the reserve value has been sent to 0,

$$\mathbb{E}\left((X-\psi_B(\beta(X;r^*)))F^{K-1}(X)\right) = \mathbb{E}\left(XF^{K-1}(X)\mathbf{1}_{[X\leq r^*]}\right) + \mathbb{E}\left((X-\psi_X(X))F^{K-1}(X)\mathbf{1}_{[X>r^*]}\right) .$$

We know however that $\mathfrak{R}(r^*) = 0$ and therefore

$$\mathbb{E}\left(XF^{K-1}(X)\mathbf{1}_{[X\leq r^*]}\right) = \mathbb{E}\left((X-\psi_X(X))F^{K-1}(X)\mathbf{1}_{[X\leq r^*]}\right) .$$

Summing things up we get that his/her payoff

$$\mathbb{E}\left((X-\psi_X(X))F^{K-1}(X)\right)$$

as in a second price auction with no reserve. □

D Proofs for Subsections 3.2 and 3.3

Proof of Theorem 3 We recall the statement of the theorem.

Theorem 10. *We call $\mathcal{G}_\alpha = \alpha G_1 + (1-\alpha)G_2$. If the buyer uses the thresholding strategy $\tilde{\beta}_{x_1}$ of Definition 1 and commits to it, we have for their utility, if the seller is welfare benevolent : if $x_1 \geq \psi^{-1}(0)$,*

$$U(x_1) = \mathbb{E}\left(X\mathcal{G}_\alpha\left(\frac{x_1(1-F(x_1))}{1-F(X)}\right)\mathbf{1}_{[X\leq x_1]}\right) + \mathbb{E}\left((X-\psi(X))\mathcal{G}_\alpha(X)\mathbf{1}_{[X\geq x_1]}\right) . \tag{19}$$

if $x_1 < \psi^{-1}(0)$,

$$U(x_1) = \alpha\mathbb{E}\left(XG_1\left(\frac{x_1(1-F(x_1))}{1-F(X)}\right)\mathbf{1}_{[X\leq x_1]}\right) + \mathbb{E}\left((X-\psi(X))\left[\alpha G_1(X)\mathbf{1}_{[X\geq x_1]} + (1-\alpha)G_2(X)\mathbf{1}_{[X\geq\psi^{-1}(0)]}\right]\right) . \tag{20}$$

This utility has (in general) a discontinuity at $x_1 = \psi^{-1}(0)$. For $x_1 < \psi^{-1}(0)$, we have $U(0) \geq U(x_1)$. For $x_1 > \psi^{-1}(0)$, the first order condition are the same as in Nedelec et al. (2018); Tang and Zeng (2018), i.e.

$$\mathcal{G}_\alpha(r) = \mathbb{E}\left(\frac{X}{1-F(X)}g_\alpha\left(\frac{r(1-F(r))}{1-F(X)}\right)\mathbf{1}_{[X\leq r]}\right) .$$

where the distribution of the competition is now $\mathcal{G}_\alpha = \alpha G_1 + (1-\alpha)G_2$ and g_α is its density. Call $x_1^*(\alpha)$ the unique solution of this problem.

Hence, the optimal threshold is $\operatorname{argmax}_{x_1\in\{0,x_1^*(\alpha)\}} U(x_1)$. An optimal threshold at $x_1 = 0$ corresponds to bidding truthfully.

Proof of Theorem 3. We first note that for the thresholding strategy at level x_1 we have

$$\psi_B(\beta(x)) = x\mathbf{1}_{[x\geq x_1]} \text{ and } \beta(x) = \frac{x_1(1-F(x_1))}{1-F(x)}\mathbf{1}_{[x\leq x_1]} + x\mathbf{1}_{[x>x_1]} .$$

Two situations are possible: if the bidder thresholds at $x_1 > \psi^{-1}(0)$, then the welfare benevolent seller has an incentive to take all the bids in the second phase as their revenue curve is non-increasing in the value (because $\psi_B(\beta(x)) \geq 0$ for all x). Hence the reserve price in the second phase is $x_1(1-F(x_1))$ and we have for the utility of the bidder in this case

$$U(x_1) = \mathbb{E}\left(X\mathcal{G}_\alpha\left(\frac{x_1(1-F(x_1))}{1-F(X)}\right)\mathbf{1}_{[X\leq x_1]}\right) + \mathbb{E}\left((X-\psi(X))\mathcal{G}_\alpha(X)\mathbf{1}_{[X\geq x_1]}\right) ,$$

as announced in the text.

On the other hand if $x_1 < \psi^{-1}(0)$, the revenue of the seller is constant on $[0, x_1]$, increasing on $[x_1, \psi^{-1}(0)]$ and decreasing beyond $\psi^{-1}(0)$. Hence the optimal reserve price is $\psi^{-1}(0)$ regardless of $x_1 < \psi^{-1}(0s)$. In this case we get

$$U(x_1) = \alpha \mathbb{E}\left(X G_1\left(\frac{x_1(1-F(x_1))}{1-F(X)}\right) \mathbf{1}_{[X \le x_1]}\right) + \mathbb{E}\left((X-\psi(X))\left[\alpha G_1(X)\mathbf{1}_{[X\ge x_1]} + (1-\alpha)G_2(X)\mathbf{1}_{[X\ge \psi^{-1}(0)]}\right]\right) .$$

The function $U(x_1)$ is clearly discontinuous at $\psi^{-1}(0)$.

The fact that the optimal x_1 in $[0, \psi^{-1}(0))$ is 0 comes from the fact that the buyer derives no benefit from overbidding below x_1 in terms of reserve price in the second stage, so the auctions are effectively not optimized on past bids. Hence bidding truthfully is preferable in both stages. This corresponds to picking $x_1 = 0$. The claim about first order conditions follows from Appendix C. Existence and uniqueness of $x_1^*(\alpha)$ follows from Lemma 15. □

Proof of Lemma 2

Proof of Lemma 2. When $G_1 = G_2 = G$, we have $\mathcal{G}_\alpha = G$. Hence $U(r)$ does not depend on α, when $r > \psi^{-1}(0)$. Hence the optimal utility for the seller is independent of α; let us call it $U^*(G)$.

On the other hand, we have

$$U(0) = \alpha \mathbb{E}\left((X-\psi(X))G(X)\right) + (1-\alpha)\mathbb{E}\left((X-\psi(X))G(X)\mathbf{1}_{[X\ge\psi^{-1}(0)]}\right) .$$

Obviously, $\mathbb{E}\left((X-\psi(X))G(X)\right) > \mathbb{E}\left((X-\psi(X))G(X)\mathbf{1}_{[X\ge\psi^{-1}(0)]}\right)$. Hence $U(0)$ is increasing in α. In the rest of the proof we denote $U(0)$ by $U(0;\alpha)$ to stress its dependence on α.

The results for $\alpha = 0$ correspond to Nedelec et al. (2018). So we know that as $\alpha \to 0$ $U(0;\alpha) < U^*(G)$. Indeed, thresholding at the monopoly price $r = \psi^{-1}(0)$ beats bidding truthful in that case. And optimal thresholding beats thresholding at the monopoly price. On the other hand, $U(0; \alpha = 1) > U^*(G)$ since truthful bidding is optimal when there is no exploitation phase. Since $U(0;\alpha)$ is an increasing function of α, and is obviously continuous, we see that there exists a unique α_c such that if $\alpha < \alpha_c$, it is preferable for the buyer to shade her bid and above it, it is preferable to bid truthful.

□

Proof of Theorem 4

Proof of Theorem 4. It is clear that each buyer's best response is either to bid truthful or to threshold optimally in the class of strategies we consider.

The theorem follows from Lemma 2. **1) Part 1.** For the first part, we suppose that all players except 1 plays the optimal thresholded strategy characterized by r^* the unique solution of the equation

$$\frac{K-1}{r(1-F(r))}\mathbb{E}\left(XF^{K-2}(X)(1-F(X))\mathbf{1}_{[X\le r]}\right) = F^{K-1}(r) .$$

This gives rise to the competition distribution

$$G_\beta(x; r^*) = F^{K-1}(\beta^{-1}(x)) = \left(1 - \frac{r^*(1-F(r^*))}{x}\right)^{K-1} \mathbf{1}_{[r^*(1-F(r^*))\le x\le r^*]} + F^{K-1}(x)\mathbf{1}_{[x\ge r^*]} .$$

According to Lemma 2, for this distribution $G_\beta(x;r^*)$, there exists $\alpha_{c,\text{thresh}}$ such that if $\alpha < \alpha_{c,\text{thresh}}$, the best response of the buyer in the class we consider is to threshold at $r > \psi^{-1}(0)$. Our results on Nash equilibria in the case $\alpha = 0$ apply. By construction the optimal r for this buyer is r^* and we have the Nash equilibrium.

Of course, if $\alpha > \alpha_{c,\text{thresh}}$, the best response of the buyer is to bid truthfully and hence there cannot be a Nash equilibrium in thresholding above $\psi^{-1}(0)$.

2) Part 2. If all the players except one bid truthfully, Lemma 2 applies with $G = F^{K-1}$ and says that there exists $\alpha_{c,\text{truthful}}$, such that if $\alpha > \alpha_{c,\text{truthful}}$ the best response of the last player is to bid truthful. Hence there is a Nash equilibrium in the class we consider. On the other hand, if $\alpha < \alpha_{c,\text{truthful}}$, the best response of the last player is to threshold above $\psi^{-1}(0)$ and hence there is no Nash equilibrium.

□

E Proof of results in Section 4.1

E.1 Proof of Lemma 17 and Corollary 2

Lemma 17. *Suppose that the buyer uses a strategy β under her value distribution F. Suppose the seller thinks that the value distribution of the buyer is G. Call λ_F and λ_G the hazard rate functions of the two distributions Then the seller computes the virtual value function of the buyer under G, denoted $\psi_{B,G}$, as*

$$\psi_{B,G}(\beta(x)) = \psi_{B,F}(\beta(x)) - \beta'(x)\left(\frac{1}{\lambda_G(x)} - \frac{1}{\lambda_F(x)}\right) .$$

Corollary 2. *If the buyer uses the strategy $\tilde{\beta}_r^{(\epsilon)}(x)$ defined as*

$$\tilde{\beta}_r^{(\epsilon)}(x) = \left(\frac{(r-\epsilon)(1-F(r))}{1-F(x)} + \epsilon\right) \boldsymbol{1}_{[x\leq r]} + x\boldsymbol{1}_{[x>r]} ,$$

we have, for $x \neq r$, $\psi_{B,F}(\tilde{\beta}_r^{(\epsilon)}(x)) = \epsilon\boldsymbol{1}_{[x\leq r]} + \psi_F(x)\boldsymbol{1}_{[x>r]}$. In particular, we have for $x \neq r$

$$\left|\psi_{B,F}(\tilde{\beta}_r^{(\epsilon)}(x)) - \psi_{B,G}(\tilde{\beta}_r^{(\epsilon)}(x))\right| \leq |\psi_F(x) - \psi_G(x)| \left[(r-\epsilon)\boldsymbol{1}_{[x\leq r]} + \boldsymbol{1}_{[x>r]}\right] .$$

If for all x, $\psi_{B,F}(\tilde{\beta}_r^{(\epsilon)}) \geq \epsilon$ and $|\psi_F(x) - \psi_G(x)| \leq \delta$, we have $\psi_{B,G}(\tilde{\beta}_r^{(\epsilon)}(x)) \geq \epsilon - \delta\max((r-\epsilon), 1)$.

Proof. As we have seen before we have

$$\psi_{B,F}(\beta(x)) = \beta(x) - \beta'(x)\frac{1-F(x)}{f(x)} .$$

By construction, we have

$$\beta(x) - \beta'(x)\frac{1-F(x)}{f(x)} = 0 \text{ for } x \leq r .$$

If the seller perceives the behavior of the buyer under the distribution G, we have

$$\psi_{B,G}(\beta(x)) = \beta(x) - \beta'(x)\frac{1-G(x)}{g(x)} .$$

Hence, we have

$$|\psi_{B,G}(\beta(x)) - \psi_{B,F}(\beta(x))| = |\beta'(x)|\left|\frac{1-F(x)}{f(x)} - \frac{1-G(x)}{g(x)}\right| .$$

Recall the hazard function $\lambda_F(x) = f(x)/(1-F(x))$. With this notation, we simply have

$$|\psi_{B,G}(\beta(x)) - \psi_{B,F}(\beta(x))| = |\beta'(x)|\left|\frac{1}{\lambda_F(x)} - \frac{1}{\lambda_G(x)}\right| .$$

The corollary follows by noting that when $x \leq r$,

$$\left|\beta'_{r,\epsilon}(x)\frac{1-F(x)}{f(x)}\right| \leq (r-\epsilon)\frac{1-F(r)}{1-F(x)}$$

□

Hence, a natural way to quantify the proximity of distributions in this context is of course in terms of their virtual value functions. Furthermore, if the buyer uses a shading function such that, under her strategy and with her value distribution, the perceived virtual value is positive, as long as the seller computes the virtual value using a nearby distribution, she will also perceive a positive virtual value and hence have no incentive to put a reserve price above the lowest bid. In particular, if δ comes from an approximation error that the buyer can predict or measure, she can also adjust her ϵ so as to make sure that the seller perceives a positive virtual value for all x.

E.2 Proof of Theorem 5

Theorem 5. *Suppose the buyer has a continuous and increasing value distribution F, supported on $[0, b]$, $b \leq \infty$, with the property that if $r \geq y \geq x$, $F(y) - F(x) \geq \gamma_F(y - x)$, where $\gamma_F > 0$. Suppose finally that $\sup_{t \geq r} t(1 - F(t)) = r(1 - F(r))$.*

Suppose the buyer uses the strategy $\tilde{\beta}_r^{(\epsilon)}$ described above and samples n values $\{x_i\}_{i=1}^n$ i.i.d according to the distribution F and bids accordingly in second price auctions. Call $x_{(n)} = \max_{1 \leq i \leq n} x_i$. In this case the (population) reserve value x^* is equal to 0.

Assume that the seller uses empirical risk minimization to determine the monopoly price in a (lazy) second price auction, using these n samples.

Call $\hat{x}_n^*$ the reserve value determined by the seller using ERM.

We have, if $C_n(\delta) = n^{-1/2}\sqrt{\log(2/\delta)/2}$ and $\epsilon > x_{(n)} C_n(\delta)/F(r)$ with probability at least $1 - \delta_1$,

$$\hat{x}_n^* < \frac{2rC_n(\delta)}{\epsilon \gamma_F} \text{ with probability at least } 1 - (\delta + \delta_1) \ .$$

In particular, if ϵ is replaced by a sequence ϵ_n such that
$n^{1/2}\epsilon_n min(1, 1/x_{(n)}) \to \infty$ in probability, $\hat{x}_n^*$ goes to 0 in probability like $n^{-1/2}\max(1, x_{(n)})/\epsilon_n$.

Examples : Our theorem applies for value distributions that are bounded, with ϵ_n of order $n^{-1/2+\eta}$, $\eta > 0$ and fixed. If the value distribution is log-normal(μ, σ) truncated away from 0 so all values are greater than a very small threshold t, standard results on the maximum of i.i.d $\mathcal{N}(\mu, \sigma)$ random variables guarantee that $x_{(n)} \leq \exp(\mu + \sigma\sqrt{2\log(n)})$ with probability going to 1. In that case too, picking ϵ_n of order $n^{-1/2+\eta}$, $\eta > 0$ and fixed, guarantees that the reserve value computed by the seller by ERM will converge to the population reserve value, which is of course 0.

Comment : The requirement on γ_F, which essentially means that the density f is bounded away from 0 could also be weakened with more technical work to make this requirement hold only around 0, at least for the convergence in probability result. Similarly one could handle situations, like the log-normal case, where γ_F is close to 0 at 0 by refining slightly the first part of the argument given in the proof. The formal proof is in Appendix E.

Proof. • **Preliminaries**
Notations : We use the standard notation for order statistics $b_{(1)} \leq b_{(2)} \leq \ldots \leq b_{(n)}$ to denote our n increasingly ordered bids. We denote as usual by $\hat{F}_n$ the empirical cumulative distribution function obtained from a sample of n i.i.d observations drawn from a population distribution F.

Setting the monopoly price by ERM amounts to finding, if $\hat{B}_n$ is the empirical cdf of the bids,

$$b_n^* = \operatorname*{argmax}_t t(1 - \hat{B}_n(t))$$

We note in particular than

$$b_n^* \leq \max_{1 \leq i \leq n} b_i = b_{(n)} \ ,$$

since $(1 - \hat{B}_n(t)) = 0$ for $t > b_{(n)}$.

Because $(1 - \hat{B}_n(t))$ is piecewise constant and the function $t \mapsto t$ is increasing, on $[b_{(i)}, b_{(i+1)})$ the function $t(1 - \hat{B}_n(t))$ reaches its supremum at $b_{(i+1)}^-$.

$$b_n^* = \operatorname*{argmax}_t t(1 - \hat{B}_n(t)) = \operatorname*{argmax}_{1 \leq i \leq n-1} b_{(i+1)}^- \left(1 - \frac{i}{n}\right) \ .$$

Since our shading function $\beta_{r,\epsilon}$ is increasing and if $x_{(i)}$ are our ordered values, we have, if $\hat{F}_n$ is the empirical cdf of our value distribution,

$$\operatorname*{argmax}_{1 \leq i \leq n-1} b_{(i+1)}^- \left(1 - \frac{i}{n}\right) = \operatorname*{argmax}_{1 \leq i \leq n-1} \beta_{r,\epsilon}(u_{(i+1)}^-) \left(1 - \frac{i}{n}\right) = \operatorname*{argmax}_u \beta_{r,\epsilon}(u)(1 - \hat{F}_n(u)) \ .$$

The last equality comes again from the fact that $(1-\hat{F}_n(u))$ is piecewise constant and $\beta_{r,\epsilon}(u)$ is increasing. So in our analysis we can act as if the seller had perfect information of our shading function $\beta_{r,\epsilon}$.

In what follows we focus on reserve values and denote

$$\hat{x}^*_{r,n} = \underset{0\leq x\leq r}{\operatorname{argmax}}\, \beta_{r,\epsilon}(x)(1-\hat{F}_n(x))\ ,\ \hat{x}^*_n = \underset{0\leq x}{\operatorname{argmax}}\, \beta_{r,\epsilon}(x)(1-\hat{F}_n(x))\ ,$$
$$x^* = \operatorname{argmax} \beta_{r,\epsilon}(x)(1-F(x))$$

The arguments we gave above imply that $\hat{x}^*_{r,n} \leq x_{(n)}$. We will otherwise study the continuous version of the problem. We also note that by construction, $x^*=0$, though we keep it in the proof as it makes it clearer.

We recall one main result of Massart (1990) on the Dvoretzky-Kiefer-Wolfowitz (DKW) inequality: if $C_n(\delta) = n^{-1/2}\sqrt{\log(2/\delta)/2}$,

$$P(\sup_x |\hat{F}_n(x) - F(x)| > C_n(\delta)) \leq \delta\ .$$

In what follows, we therefore assume that we have a uniform approximation

$$\forall x\ ,\ |\hat{F}_n(x) - F(x)| \leq C_n(\delta)\ ,$$

since it holds with probability $1-\delta$. In what follows we write C_n instead of $C_n(\delta)$ for the sake of clarity. Using the fact that $\beta_{r,\epsilon}$ is increasing, this immediately implies that with probability at least $1-\delta$, for any $c>0$

$$\forall x \in [0,c]\ ,\ |\beta_{r,\epsilon}(x)(1-\hat{F}_n(x)) - \beta_{r,\epsilon}(x)(1-F(x))| \leq \beta_{r,\epsilon}(c)C_n\ .$$

• $\hat{x}^*_{r,n} = \operatorname{argmax}_{y\leq r} \beta_{r,\epsilon}(y)(1-\hat{F}_n(y))$ cannot be too far from x*
Now for our construction of $\beta_{r,\epsilon}(x)$, we have by construction that

$$\frac{\partial}{\partial u}\left[\beta(u)(1-F(u))\right] = -\epsilon f(u) \text{ when } x \leq r\ .$$

In particular, it means that when $x, y \leq r$

$$\beta_{r,\epsilon}(x)(1-F(x)) - \beta_{r,\epsilon}(y)(1-F(y)) = -\epsilon(F(x)-F(y))\ .$$

Also $x^*=0$ since $\beta_{r,\epsilon}(1-F)$ is decreasing on $[0,r]$, as we have just seen that its derivative is negative. Here we used the fact that F is increasing.

If $r > y \geq x^* + tC_n/\epsilon$, we have, using the previous inequality and the fact that $\beta_{r,\epsilon}(1-F)$ is decreasing on $[0,r]$,

$$\beta_{r,\epsilon}(y)(1-F(y)) \leq \beta_{r,\epsilon}(x^*)(1-F(x^*)) - \epsilon(F(x^*+tC_n/\epsilon) - F(x^*))\ .$$

Since we assumed that $F(y)-F(x) \geq \gamma_F(y-x)$, we have

$$-\epsilon(F(x^*+tC_n/\epsilon) - F(x^*)) \leq -tC_n\gamma_F\ .$$

Since $\beta_{r,\epsilon}$ is increasing, we have $\sup_{0\leq x\leq r}\beta_{r,\epsilon}(x) \leq \beta_{r,\epsilon}(r) = r$. Picking $t > 2r/\gamma_F$, it is clear that for $r > y \geq x + tC_n/\epsilon$,

$$\begin{aligned}\max_{r\geq u\geq x+tC_n/\epsilon} \beta_{r,\epsilon}(u)(1-\hat{F}_n(u)) &\leq \max_{r\geq u\geq x+tC_n/\epsilon} \beta_{r,\epsilon}(u)(1-F(u)) + \max_{r\geq u\geq x+tC_n/\epsilon} \beta_{r,\epsilon}(u)C_n \\ &\leq \beta_{r,\epsilon}(x^*)(1-F(x^*)) + (r-t\gamma_F)C_n < \beta_{r,\epsilon}(x^*)(1-F(x^*)) - rC_n \leq \beta_{r,\epsilon}(x^*)(1-\hat{F}_n(x^*))\ .\end{aligned}$$

We conclude that $\hat{x}^*_{r,n}$ cannot be greater than $x + 2rC_n/(\epsilon\gamma_F)$ and therefore

$$\hat{x}^*_{r,n} - \hat{x} < \frac{2rC_n}{\epsilon\gamma_F}.$$

• **Dealing with** $\max_{y>r}\beta_{r,\epsilon}(y)(1-\hat{F}_n(y))$
Recall that $\max_x \beta_{r,\epsilon}(x)(1-F(x)) = \beta_{r,\epsilon}(0)(1-F(0)) = r(1-F(r)) + \epsilon F(r)$. We now assume that $\max_{y\geq r} y(1-$

$F(y)) = r(1 - F(r))$. This is in particular the case for regular distributions, which are commonly assumed in auction theory.

To show that the argmax cannot be in $[r, b]$ with pre-specified probability we simply show that the estimated value of the seller revenue at reserve value 0 is higher than $\max_{y>r} \beta_{r,\epsilon}(y)(1 - \hat{F}_n(y))$. Of course,

$$\max_{y>r} \beta_{r,\epsilon}(y)(1 - \hat{F}_n(y)) = \max_{x_{(n)} \geq y > r} \beta_{r,\epsilon}(y)(1 - \hat{F}_n(y)) \ .$$

Recall that $\beta_{r,\epsilon}(0) = r(1 - F(r)) + \epsilon F(r)$. Under our assumptions, we have

$$\beta_{r,\epsilon}(0)(1 - \hat{F}_n(0)) = \beta_{r,\epsilon}(0) = r(1 - F(r)) + \epsilon F(r) \text{ and}$$

$$\max_{x_{(n)} \geq y \geq r} \beta_{r,\epsilon}(y)(1 - \hat{F}_n(y))$$

$$\leq \max_{x_{(n)} \geq y \geq r} \beta_{r,\epsilon}(y)(1 - F(y)) + C_n \max_{x_{(n)} \geq y \geq r} \beta_{r,\epsilon}(y) \leq r(1 - F(r)) + C_n x_{(n)}$$

So as long as $\epsilon > x_{(n)} C_n / F(r)$, the result we seek holds. By assumption this property holds with probability $1 - \delta_1$.

The statement of the theorem holds when both parts of the proof hold. Since they hold with probability at least $1 - \delta$ and $1 - \delta_1$ the intersection event holds with probability at least $1 - \delta - \delta_1$, as announced. $\square$

• **Asymptotic statement/Convergence in probability issue**
This is a straightforward application of the previous result and we give no further details.

F Proof of results of Section 4.2

This theorem works for non-regular value distributions and in the asymmetric case when the bidders have different value distributions.

Theorem 6 (Thresholding at the monopoly price)**.** *Consider the one-strategic setting in a lazy second price auction with F_{X_i} the value distribution of the strategic bidder i with a seller computing the reserve prices to maximize her revenue. Suppose β_r is an increasing strategy with associated reserve value $r > 0$. Suppose $\alpha = 0$, i.e. there is only an exploitation phase. Then there exists $\tilde{\beta}_r$ which does not depend on G such that :* ***2)*** *$U_i(\tilde{\beta}_r) \geq U_i(\beta_r)$, U_i being the utility of the strategic bidder.* ***3)*** *$R_i(\tilde{\beta}_r) \geq R_i(\beta_r)$, R_i being the payment of bidder i to the seller. The following continuous functions fulfill these conditions for $\epsilon \geq 0$ small enough:*

$$\tilde{\beta}_r^{(\epsilon)}(x) = \left(\frac{[\beta_r(r) - \epsilon](1 - F_{X_i}(r))}{1 - F_{X_i}(x)} + \epsilon \right) \mathbf{1}_{x<r} + \beta_r(x) \mathbf{1}_{x \geq r}$$

In the two-stage game, there is a critical value α_c for which, if $\alpha < \alpha_c$, it is preferable for the bidder to threshold at the monopoly price and if $\alpha > \alpha_c$, it is preferable for the bidder to bid truthfully.

Proof: Important special case of $\beta_r(x) = x$; We assumed that the seller computes the reserve price to maximize her revenue. Hence $r = \operatorname{argmax} x(1 - F_{X_i}(x))$. In this case, $\int_0^r \psi_i(x_i) G(x_i) f_i(x_i) dx_i \leq 0$ (otherwise the reserve price would be lower and since the payment of bidder i is equal to $\int_r^{+\infty} \psi_i(x_i) G(x_i) f_i(x_i) dx_i$.

$\tilde{\beta}_r^{(\epsilon)}$ defined in Theorem 6 verifies the ODE defined in Lemma 7 such that $h_{\tilde{\beta}_r^{(\epsilon)}}(x) = \psi_{B_{\tilde{\beta}_r^{(\epsilon)}}}(\tilde{\beta}_r^{(\epsilon)}(x)) = \epsilon$ for $x \in [0, r]$ and $h_{\tilde{\beta}^{(\epsilon)}}(x) = \psi_{X_i}(x)$ for $x \in [r, +\infty]$. $\tilde{\beta}_r^{(\epsilon)}$ is trivially increasing.

Hence, the virtual value of the distribution induced by $\tilde{\beta}_r^{(\epsilon)}$ is non-negative on $[0, r]$ and the new reserve value is equal to zero. The new reserve price is therefore equal to the minimum bid of bidder i and

$$R_i(\tilde{\beta}_r^{(\epsilon)}) \ = R_i(\beta_r) + \mathbb{E}_{X_i \sim F_i} \left(\epsilon G_i(\tilde{\beta}_r^{(\epsilon)}(X_i)) \mathbf{1}(X_i \leq r) \right) \geq R_i(\beta_r) \ .$$

The new bidder's utility is

$$U_i(\tilde{\beta}_r^{(\epsilon)}) = U_i(\beta_r) + \mathbb{E}_{X_i \sim F_i} \left((X_i - \epsilon) G_i(\tilde{\beta}_r^{(\epsilon)}(X_i)) \mathbf{1}(X_i \leq r) \right)$$

For $\epsilon = 0$ we have clearly $U(\tilde{\beta}_r^{(\epsilon)}) \geq U(\beta_r)$ and $R_i(\tilde{\beta}_r^{(\epsilon)}) = R_i(\beta_r)$. Outside pathological cases, it is a strict inequality and by continuity with respect to ϵ, i.e. assuming G continuous, it is true in a neighborhood of zero so for some $\epsilon > 0$. □

We now handle rigorously the general case:

Proof. The reserve value $r > 0$ is given. Consider

$$\tilde{\beta}_r(x) = \begin{cases} t_r(x) & \text{if } x \leq r \\ \beta_r(x) & \text{if } x > r \end{cases}$$

To make things simple we require $t_r(r) = \beta_r(r^+)$, so we have continuity. Note that beyond r the seller revenue is unaffected. If the seller sets the reserve value at r_0 the extra benefit compared to setting it at r is

$$\mathbb{E}\left(\psi_{t_r}(t_r(X)G(t_r(X))\mathbf{1}_{[r_0 \leq X < r]}\right) .$$

Hence, as long as $\psi_{t_r}(x) \geq 0$, the seller has an incentive to lower the reserve value. The extra gain to the buyer is

$$\mathbb{E}\left((X - \psi_{t_r}(t_r(X)))G(t_r(X))\mathbf{1}_{[r_0 \leq X < r]}\right) .$$

Now, if we take

$$t_r(x) = \frac{t_r(0)}{1 - F(x)} ,$$

it is easy to verify that

$$\psi_{t_r}(t_r(x)) = t_r(x) - t_r'(x)\frac{1 - F(x)}{f(x)} = 0 .$$

So in this limit case, there is no change in buyer's payment and when the reserve price is moved by the seller to any value on $[0, r]$. If we assume that the seller is welfare benevolent, she will set the reserve value to 0. To have continuity of the bid function, we just require that

$$\frac{t_r(0)}{1 - F(r)} = \beta_r(r^+) .$$

Since there is no extra cost for the buyer, it is clear that his/her payoff is increased with this strategy. Taking $t_r^{(\epsilon)}$ such that

$$\psi_{t_r^{(\epsilon)}}(t_r^{(\epsilon)}(x)) = \epsilon ,$$

gives a strict incentive to the seller to move the reserve value to 0, (so the assumption that s/he is welfare benevolent is not required) even if it is slightly suboptimal for the buyer. Note that we explained in Lemma 7 how to construct such a ψ. In particular,

$$t_r^\epsilon = \frac{C_\epsilon}{1 - F(x)} + \epsilon, \text{ with } \frac{C_\epsilon}{1 - F(r)} + \epsilon = \beta_r(r^+)$$

works. Taking limits proves the result, i.e. for ϵ small enough the Lemma holds, since everything is continuous in ϵ.

To extend to the two stage process, we can use exactly the same proof as in Lemma 2. □

Comment We note that the flexibility afforded by ϵ is two-fold: when $\epsilon > 0$, the extra seller revenue is a strictly decreasing function of the reserve price; hence even if for some reason reserve price movements are required to be small, the seller will have an incentive to make such move. The other reason is more related to estimation issues: if the reserve price is determined by empirical risk minimization, and hence affected by even small sampling noise, having ϵ big enough will guarantee that the mean extra gain of the seller will be above this sampling noise. Of course, the average cost for the bidder can be interpreted to just be ϵ at each value under the current reserve price and hence may not be a too hefty price to bear.

For the sake of clarity, we first consider the case where the strategic bidder is only optimizing her utility in the second stage. We consider the robust-optimization framework Aghassi and Bertsimas (2006). We show that thresholding at the monopoly price is the best-response in the worst case of the competition when the number of players is not known to the strategic bidder.

We assume that the bidders know that they all have the same value distribution but they do not know the number of bidders K. The problem is now to find the optimal thresholding parameter in the worst case :

$$r^* = \underset{r}{\operatorname{argmax}}\, min_{K\in\mathbb{N}} U(\beta_r, K)$$

We call this strategy the best robust-optimization strategy.

Theorem 11. *Consider the one-strategic setting in a lazy second price auction with F_{X_i} regular. Assume the symmetric setting where all the other bidders have the same value distributions and assume the non-strategic bidders are bidding truthfully. Consider the setting where $\alpha = 1$. In the class of thresholded strategies, the best robust-optimization strategy is to threshold at the monopoly price and bid truthfully after.*

Proof. Based on Lemma 14, we know that given a competition distribution G, the best response in the class of thresholded strategies is to bid truthfully above a certain threshold r.

Given Lemma 6, we know that for any given competition distribution G, the optimal threshold r verifies $r \geq \psi^{-1}(0)$.

For any $r > \psi^{-1}(0)$, we show that there exists K_{lim} such that $\forall K \geq K_{lim}$, it is better to bid truthfully on $[a, r]$ with $\psi^{-1}(0) < a < r$.

$$U(\beta_{tr}) - U(\beta_r) = \int_{\psi^{-1}(0)}^{r} (x - \psi(x))F(x)^{K-1} f(x)dx - \int_{\psi^{-1}(0)}^{r} x\left(F\left(\frac{(1-F(r))r}{1-F(x)}\right)\right)^{K-1} f(x)dx$$

Since F is regular and by definition of the monopoly price,

$$\forall x \in [\psi^{-1}(0), r], \frac{(1-F(r))r}{1-F(x)} \leq x$$

(In the case of equality $\psi(x) = 0$). By definition of the virtual value, $\frac{x-\psi(x)}{x} \in]0, 1[$. Hence, if F is increasing (defined on all its support), for all $r > \psi^{-1}(0)$ there exists $a \in]\psi^{-1}(0), r[$ and K_{lim} such that $\forall K \geq K_{lim}$, and

$$\forall x \in [a, r], \frac{x - \psi(x)}{x} \geq \frac{\left(F\left(\frac{(1-F(r))r}{1-F(x)}\right)\right)^{K-1}}{F(x)^{K-1}}.$$

Hence thresholding at the monopoly price is optimal in the worst case of the competition distribution. $\square$

G Proof of results of Section 4.3

G.1 Thresholding at the monopoly price in the Myerson auction

Here we ask what happens when one player is strategic, the others are truthful and she thresholds her virtual value at her monopoly price in the Myerson auction.

Lemma 3. *Consider the one-strategic setting. Assume all bidders have the same value distribution, F_X and that F_X is regular. Assume that bidder i is strategic that the $K-1$ other bidders bid truthfully. Let us denote by β_{truth} the truthful strategy and β_{thresh} the thresholded strategy at the monopoly price. The utility of the truthful bidder in the Myerson auction $U_i^{Myerson}$ and in the lazy second price auction U_i^{Lazy} satisfy*

$$U_i^{Myerson}(\beta_{thresh}) - U_i^{Myerson}(\beta_{truth}) \geq U_i^{Lazy}(\beta_{thresh}) - U_i^{Lazy}(\beta_{truth})$$

Proof. By definition the thresholded strategy at monopoly price β_{thresh} verifies

$$\psi_{B_i}(b_i) = \begin{cases} \psi_{X_i}(x_i) \text{ if } x_i \geq \psi_{X_i}^{-1}(0) \\ \epsilon = 0^+ \text{ if } x_i < \psi_{X_i}^{-1}(0) \;. \end{cases}$$

The revenue/utility of the buyer in the Myerson auction is

$$U_i^{\text{Myerson}}(\beta) = \mathbb{E}\left((X_i - \psi_{B_i}(B_i))\mathbf{1}_{\left[\psi_{B_i}(B_i)\geq \max_{j\neq i}(0,\psi_{B_j}(B_j))\right]}\right) .$$

If we assume that all players except the i-th are truthful and the i-th player employs the strategy described above, we get that

$$\begin{aligned}U_i^{\text{Myerson}}(\beta_{\text{thresh}}) &= \mathbb{E}\left((X_i - \psi_{B_i}(B_i))\mathbf{1}_{\left[\psi_{B_i}(B_i)\geq \max_{j\neq i}(0,\psi_{B_j}(B_j))\right]}\right)\\ &= \mathbb{E}\left((X_i - \psi_{X_i}(X_i))\mathbf{1}_{\left[\psi_{X_i}(X_i)\geq \max_{j\neq i}(0,\psi_{X_j}(X_j))\right]}\mathbf{1}_{\left[X_i\geq \psi_{X_i}^{-1}(0)\right]}\right)\\ &\quad +\mathbb{E}\left((X_i-\epsilon)\mathbf{1}_{\left[\epsilon\geq \max_{j\neq i}\psi_{X_j}(X_j)\right]}\mathbf{1}_{\left[X_i< \psi_{X_i}^{-1}(0)\right]}\right)\end{aligned}$$

Note that here we've just split the integral into two according to whether X_i was greater of less than the monopoly price and adjusted the definitions of $\psi_{B_i}(b_i)$ accordingly. We conclude immediately that

$$U_i^{\text{Myerson}}(\beta_{\text{thresh}}) = U_i^{\text{Myerson}}(\beta_{\text{truth}}) + \mathbb{E}\left((X_i-\epsilon)\mathbf{1}_{\left[\epsilon\geq \max_{j\neq i}\psi_{X_j}(X_j)\right]}\mathbf{1}_{\left[X_i< \psi_{X_i}^{-1}(0)\right]}\right) .$$

Therefore the extra utility derived from our shading is just

$$U_i^{\text{Myerson}}(\beta_{\text{thresh}}) - U_i^{\text{Myerson}}(\beta_{\text{truth}}) = \mathbb{E}\left((X_i-\epsilon)\mathbf{1}_{\left[\epsilon\geq \max_{j\neq i}\psi_{X_j}(X_j)\right]}\mathbf{1}_{\left[X_i< \psi_{X_i}^{-1}(0)\right]}\right)$$

If X_j's are independent of X_i, we have

$$U_i^{\text{Myerson}}(\beta_{\text{thresh}})(\epsilon) - U_i^{\text{Myerson}}(\beta_{\text{truth}})(\epsilon) = \mathbb{E}\left((X_i-\epsilon)P(\epsilon\geq \max_{j\neq i}\psi_{X_j}(X_j))\mathbf{1}_{\left[X_i< \psi_{X_i}^{-1}(0)\right]}\right)$$

When the X_j's are independent of each other and we pick $\epsilon = 0$ we get

$$\lim_{\epsilon\to 0} U_i^{\text{Myerson}}(\beta_{\text{thresh}})(\epsilon) - U_i^{\text{Myerson}}(\beta_{\text{truth}})(\epsilon) = \left[\prod_{j\neq i} P(X_j \leq \psi_j^{-1}(0))\right]\mathbb{E}\left(X_i\mathbf{1}_{\left[X_i< \psi_{X_i}^{-1}(0)\right]}\right)$$

Going back to our work on second price auctions, the gain from thresholding in a continuous manner at 0 was

$$U_i^{\text{Lazy}}(\beta_{\text{thresh}}) - U_i^{\text{Lazy}}(\beta_{\text{truth}}) = \mathbb{E}\left(XG(t_0(X))\mathbf{1}_{[X\leq \psi^{-1}(0)]}\right) .$$

In this case, $t_0(x) = \psi^{-1}(0)(1-F(\psi^{-1}(0)))/(1-F(x))$, $G(x) = F^{K-1}(x)$ using symmetry and independence. As we've seen above,

$$U_i^{\text{Myerson}}(\beta_{\text{thresh}}) - U_i^{\text{Myerson}}(\beta_{\text{truth}}) = \mathbb{E}\left(X\mathbf{1}_{[X\leq\psi^{-1}(0)]}\right)F^{K-1}(\psi^{-1}(0)) = \mathbb{E}\left(X\mathbf{1}_{[X\leq\psi^{-1}(0)]}\right)G(\psi^{-1}(0)) .$$

Now since $t_0(x)\leq \psi^{-1}(0)$, we see that in the symmetric case, for the 1-strategic player we have

$$\text{extra gain}_{\text{Myerson}} \geq \text{extra gain}_{\text{second price}} ,$$

since $G(\psi^{-1}(0)) \geq G(t_0(x))$ when $x\leq \psi^{-1}(0)$.

Of course, in the symmetric case, the truthful revenue is the same in the Myerson and 2nd price lazy auction. As such the relative gain is also higher in the Myerson auction than in the second price auction. □

Example: K symmetric bidders, unif[0,1] distribution, 1 strategic In this case, $\psi_i^{-1}(0) = 1/2$, $\int_0^{1/2} xf(x)dx = \int_0^{1/2} xdx = 1/8$. And $\prod_{j\neq i} P(X_j \leq \psi_j^{-1}(0)) = 2^{-(K-1)}$. The truthful revenue/utility is the same as in a 2nd price auction with reserve at 1/2. Elementary computations show that this utility is $\int_{1/2}^1 x^{K-1} - x^K dx = (1-(K+2)/2^{K+1})/(K(K+1))$.

Numerics for $K = 2$ When $K = 2$, the utility is $1/12$ in the truthful case. And the extra utility is $1/16$, i.e. 75% of the utility in the truthful case. Hence the utility of the shaded strategy is $7/48$. We note that the gain is even larger than for a 2nd price auction with monopoly reserve where the extra utility was 57%

Remarks about the asymmetric case In the asymmetric case, it becomes harder to make a relative comparison of gains. That is because the truthful revenue in Myerson and 2nd price auctions are different. Furthemore, for the extra gain, even if X_j's are independent, we have to compare

$$\mathbb{E}\left(X_i \mathbf{1}_{[X_i \leq \psi_i^{-1}(0)]} \prod_{j \neq i} F_j(\psi_j^{-1}(0))\right) \text{ and } \mathbb{E}\left(X_i \mathbf{1}_{[X_i \leq \psi_i^{-1}(0)]} \prod_{j \neq i} F_j\left(\frac{\psi_i^{-1}(0)(1 - F_i(\psi_i^{-1}(0)))}{1 - F_i(X_i)}\right)\right)$$

In general, the comparison seem like it could go either way. An exception is the case $\psi_j^{-1}(0) \geq \psi_i^{-1}(0)$ for all $j \neq i$: then the extra revenue in the Myerson auction is greater than the extra revenue in the second price auction.

G.2 Thresholding at the monopoly price in the eager second price auction with monopoly price

Lemma 4. *Consider the one-strategic setting with F_{X_i} the value distribution of the strategic bidder i that we assume regular. The K-1 other bidders have the same value distribution than bidder i and bid truthfully. Lets not β_{truth} the truthful strategy and β_{thresh} the thresholded strategy at the monopoly price. The utility of the truthful bidder in the Myerson auction $U_i^{Myerson}$ and in the lazy second price auction U_i^{Lazy} verifies*

$$U_i^{Eager}(\beta_{thresh}) - U_i^{Eager}(\beta_{truth}) \geq U_i^{Lazy}(\beta_{thresh}) - U_i^{Lazy}(\beta_{truth})$$

Proof. The proof is very similar to the previous one. We recall that in the eager version of the second price auction, the winner of the auction is the bidder with the highest bid among bidders who clear their reserve price and she pays the maximum between the second highest bid and her reserve price.

To complete the proof, we need to remark that in the symmetric case, the utility of the strategic bidder in the eager second price auction is

$$U_i^{\text{Eager}}(\beta) = \mathbb{E}\left((X_i - \psi_{B_i}(B_i))G_i(\beta(X_i))\mathbf{1}_{\left[B_i \geq \max_{j \neq i}(psi_{B_j}^{-1}(0))\right]}\right) .$$

with G_i the distribution of the highest bid of the competition above their reserve price. As we are in the symmetric case and all the bidders different than i are bidding truthfully:

$$\forall x \in [0, \psi_X^{-1}(0)], G_i(x) = F_X^{K-1}(\psi_X^{-1}(0)) .$$

Hence,

$$U_i^{\text{Eager}}(\beta_{\text{thresh}}) - U_i^{\text{Lazy}}(\beta_{\text{lazy}}) = \mathbb{E}\left(X\left(F_X^{K-1}(\psi_{F_X}^{-1}(0)) - F_X^{K-1}(x)\right)\mathbf{1}_{[X \leq \psi_{F_X}^{-1}(0)]}\right) \geq 0$$

□