Supplementary Materials for "Finite-Time Error Bounds for Biased Stochastic Approximation with Applications to Q-learning" by G. Wang and G. B. Giannakis

Remark. *The equations* (1)–(39) *and Assumptions 1–4 are referenced with respect to the indexing used in the paper.*

A Proof of Proposition 1

We start off the proof by introducing the following auxiliary function

$$g(k,T,\Theta_k) := \Theta_{k+T} - \Theta_k - \epsilon \sum_{j=k}^{k+T-1} f(\Theta_k, X_j), \quad \forall T \geq 1 \tag{40}$$

which is evidently well defined under our working Assumptions 1 and 3. Regarding the function $g(k,T,\Theta_k)$ above, we present the following useful bound, whose proof details are, however, postponed to Appendix E for readability.

Lemma 2. *For any $\Theta_k \in \mathbb{R}^d$, the function $g(k,T,\Theta_k)$ satisfies for all $k \geq 0$*

$$\|g(k,T,\Theta_k)\| \leq \epsilon^2 L^2 T^2 (1+\epsilon L)^{T-2}, \quad \forall T \geq 1. \tag{41}$$

On the other hand, note from (8) that

$$g'(k,T,\Theta_k) = \Theta_{k+T} - \Theta_k - \epsilon T \bar{f}(\Theta_k) \tag{42}$$

which, in conjunction with (40), suggests that we can write

$$\begin{aligned} g'(k,T,\Theta_k) &= g(k,T,\Theta_k) + \epsilon \sum_{j=k}^{k+T-1} f(\Theta_k, X_j) - \epsilon T \bar{f}(\Theta_k) \\ &= g(k,T,\Theta_k) + \epsilon \sum_{j=k}^{k+T-1} \left(f(\Theta_k, X_j) - \bar{f}(\Theta_k) \right). \end{aligned} \tag{43}$$

By taking expectation of both sides of (43) conditioned on the σ-field $\mathcal{F}_k$, along with the fact that Θ_k is $\mathcal{F}_k$-measurable, we obtain

$$\begin{aligned} \mathbb{E}\big[g'(k,T,\Theta_k)\big|\mathcal{F}_k\big] &= \mathbb{E}\big[g(k,T,\Theta_k)\big|\mathcal{F}_k\big] + \epsilon\, \mathbb{E}\left[\sum_{j=k}^{k+T-1} \left(f(\Theta_k, X_j) - \bar{f}(\Theta_k)\right) \Big| \mathcal{F}_k\right] \\ &= \mathbb{E}\big[g(k,T,\Theta_k)\big|\mathcal{F}_k\big] + \epsilon T \left(\frac{1}{T} \sum_{j=k}^{k+T-1} \mathbb{E}\big[f(\Theta_k, X_j)\big|\mathcal{F}_k\big] - \bar{f}(\Theta_k) \right) \\ &\leq \epsilon L T \Big[\epsilon L T (1+\epsilon L)^{T-2} + \sigma(T;k)\Big] (\|\Theta_k\| + 1) \end{aligned} \tag{44}$$

where the last inequality follows from Lemma 2 as well as the property of the averaged operator $\bar{f}$ in (7) under our working Assumption 3. This concludes the proof.

B Proof of Theorem 1

We prove this theorem by carefully constructing function for $W'(k,\Theta_k)$ from $W(\Theta_k)$ (recall under our working assumption 2 that $W(\Theta_k)$ exists and satisfies properties (52)—(6c)). Toward this objective, let us start with the following candidate

$$W'(k,\Theta_k) = \sum_{j=k}^{k+T-1} W(\Theta_j(k,\Theta_k)) \tag{45}$$

where, to make the dependence of $\Theta_{j\geq k}$ on Θ_k explicit, we maintain the notation $\Theta_j = \Theta_j(k, \Theta_k)$, which is understood as the state of the recursion (1) at time instant $j \geq k$, with an initial condition Θ_k at time instant k.

In the following, we will show that there exists and also determine a value for the parameter $T \in \mathbb{N}^+$ such that the inequalities (11) and (12) are satisfied.

For ease of exposition, we start by proving the second inequality (12). To this end, observe from the definition of $W'(k, \Theta_k)$ in (45) that

$$\begin{aligned} W'(k+1, \Theta_k + \epsilon f(\Theta_k, X_k)) - W'(k, \Theta_k) &= \sum_{j=k+1}^{k+T} W(\Theta_j(k, \Theta_k)) - \sum_{j=k}^{k+T-1} W(\Theta_j(k, \Theta_k)) \\ &= W(\Theta_{k+T}(k, \Theta_k)) - W(\Theta_k(k, \Theta_k)) \\ &= W(\Theta_{k+T}(k, \Theta_k)) - W(\Theta_k) \end{aligned} \tag{46}$$

where the last equality is due to the fact that $\Theta_k(k, \Theta_k) = \Theta_k$.

To upper bound the term in (46), we will focus on bound the first term $W(\Theta_{k+T}(k, \Theta_k))$. Recall from (8) that

$$\Theta_{k+T}(k, \Theta_k) = \Theta_k + \epsilon T \bar{f}(\Theta_k) + g'(k, T, \Theta_k)$$

based on which we can find the second-order Taylor expansion of $W(\Theta_{k+T}(k, \Theta_k))$ (which is twice differentiable under Assumption 2) around Θ_k, as follows

$$\begin{aligned} W(\Theta_{k+T}(k, \Theta_k)) = W(\Theta_k) &+ \left(\frac{\partial W}{\partial \theta}\bigg|_{\Theta_k} \right)^\top \big[\epsilon T \bar{f}(\Theta_k) + g'(k, T, \Theta_k)\big] \\ &+ \big[\epsilon T \bar{f}(\Theta_k) + g'(k, T, \Theta_k)\big]^\top \nabla^2 W(\Theta_k') \big[\epsilon T \bar{f}(\Theta_k) + g'(k, T, \Theta_k)\big] \end{aligned} \tag{47}$$

where we have employed the so-called mean-value theorem, suggesting that (47) holds with $\Theta_k' := \Theta_k + \eta\big[\epsilon T \bar{f}(\Theta_k) + g'(k, T, \Theta_k)\big]$ for some constant $\eta \in [0, 1]$.

Next, we will pursue an upper bound for each individual term on the right hand side of (47) by conditioning on the σ-field $\mathcal{F}_k$. Again, using the fact that Θ_k is $\mathcal{F}_k$-measurable and invoking (6b), we have that

$$\mathbb{E}\left[\epsilon T \Big(\frac{\partial W}{\partial \theta}\bigg|_{\Theta_k} \Big)^\top \bar{f}(\Theta_k) \Big| \mathcal{F}_k \right] \leq -c_3 \epsilon L T \|\Theta_k\|^2. \tag{48}$$

One can further verify the following bounds

$$\begin{aligned} \mathbb{E}\left[\left(\frac{\partial W}{\partial \theta}\bigg|_{\Theta_k} \right)^\top g'(k, T, \Theta_k) \Big| \mathcal{F}_k \right] &= \left(\frac{\partial W}{\partial \theta}\bigg|_{\Theta_k} \right)^\top \mathbb{E}\big[g'(k, T, \Theta_k)\big|\mathcal{F}_k\big] \\ &\leq \left\| \frac{\partial W}{\partial \theta}\bigg|_{\Theta_k} \right\| \cdot \big\|\mathbb{E}\big[g'(k, T, \Theta_k)\big|\mathcal{F}_k\big]\big\| \qquad (49) \\ &\leq c_4 \|\Theta_k\| \cdot \epsilon L T \beta_k(T, \epsilon)(\|\Theta_k\| + 1) \qquad (50) \\ &\leq 2 c_4 \epsilon L T \beta_k(T, \epsilon)(\|\Theta_k\|^2 + 1). \qquad (51) \end{aligned}$$

In particular, (49) uses the Cauchy-Schwartz inequality, (50) calls for Proposition 1, and the last one follows from the inequality $\|\theta\|(\|\theta\| + 1) \leq 2(\|\theta\|^2 + 1)$.

As far as the last term of (46) is concerned, it is clear that

$$\begin{aligned} &\mathbb{E}\Big\{ \big[\epsilon T \bar{f}(\Theta_k) + g'(k, T, \Theta_k)\big]^\top \nabla^2 W(\Theta_k') \big[\epsilon T \bar{f}(\Theta_k) + g'(k, T, \Theta_k)\big] \Big| \mathcal{F}_k \Big\} \\ &\leq c_4\, \mathbb{E}\Big[\big\|\epsilon T \bar{f}(\Theta_k) + g'(k, T, \Theta_k)\big\|^2 \Big| \mathcal{F}_k \Big] \qquad (52) \\ &\leq 2 c_4 \epsilon^2 T^2 \big\|\bar{f}(\Theta_k)\big\|^2 + 2 c_4 \mathbb{E}\Big[\big\|g'(k, T, \Theta_k)\,\big\|^2 \Big| \mathcal{F}_k \Big] \qquad (53) \end{aligned}$$

$$\leq 2c_4\epsilon^2T^2L^2\|\Theta_k\|^2 + 2c_4\mathbb{E}\Big[\big\|g'(k,T,\Theta_k)\big\|^2\Big|\mathcal{F}_k\Big] \tag{54}$$

where (52) leverages the upper bound on the Hessian matrix of $W(\theta)$ arising from the property (6c), (53) follows from the inequality $\|a+b\|^2 \leq 2(\|a\|^2+\|b\|^2)$ for any real-valued vectors $a, b \in \mathbb{R}^d$, and (54) uses the Lipschitz property of function $\bar{f}(\theta)$ that can be easily verified since $f(\theta, x)$ is Lipschitz in θ.

To further upper bound the last term of (54), we establish the following helpful result whose proof is also postponed to Appendix F for readability.

Lemma 3. *The following bound holds for any fixed $\theta_k \in \mathbb{R}^d$*

$$\mathbb{E}\Big[\big\|g'(k,T,\theta_k)\big\|^2\big|\mathcal{F}_k\Big] \leq \epsilon^2L^2T^2\Big[\epsilon^2L^2T^2(1+\epsilon L)^{2T-4}+12\Big]\|\theta_k\|^2 + 8\epsilon^2L^2T^2. \tag{55}$$

Coming back to inequality (54), as $\mathbb{E}[\Theta_k|\mathcal{F}_k] = \Theta_k$, Lemma 3 now applies. Plugging (55) into (54), we establish an upper bound on the last term of (46) as follows

$$\begin{aligned}
&\mathbb{E}\Big\{\big[\epsilon T\bar{f}(\Theta_k) + g'(k,T,\Theta_k)\big]^\top \nabla^2 W(\Theta_k')\,\big[\epsilon T\bar{f}(\Theta_k) + g'(k,T,\Theta_k)\big]\Big|\mathcal{F}_k\Big\} \\
&\leq 2c_4\epsilon^2T^2L^2\Big[\epsilon^2L^2T^2(1+\epsilon L)^{2T-4}+13\Big]\|\Theta_k\|^2 + 16c_4\epsilon^2L^2T^2.
\end{aligned} \tag{56}$$

Putting together the bounds in (48), (51), and (56), it follows from (47) that

$$\begin{aligned}
&\mathbb{E}\big[W(\Theta_{k+T}(k,\Theta_k)) - W(\Theta_k)\big|\mathcal{F}_k\big] \\
&= \mathbb{E}\left[\epsilon T\left(\frac{\partial W}{\partial\theta}\Big|_{\Theta_k}\right)^\top \bar{f}(\Theta_k) + \left(\frac{\partial W}{\partial\theta}\Big|_{\Theta_k}\right)^\top g'(k,T,\Theta_k)\Big|\mathcal{F}_k\right] \\
&\quad + \mathbb{E}\Big\{\big[\epsilon T\bar{f}(\Theta_k) + g'(k,T,\Theta_k)\big]^\top \nabla^2 W(\Theta_k')\big[\epsilon T\bar{f}(\Theta_k) + g'(k,T,\Theta_k)\big]\Big|\mathcal{F}_k\Big\} \\
&\leq -\epsilon LT\Big\{c_3 - 2c_4\beta_k(T,\epsilon) - 2c_4\epsilon LT\Big[\epsilon^2L^2T^2(1+\epsilon L)^{2T-4}+13\Big]\Big\}\|\Theta_k\|^2 \\
&\quad + 2c_4\epsilon LT\beta_k(T,\epsilon) + 16c_4\epsilon^2L^2T^2 \\
&= -\epsilon LT[c_3 - c_4\rho_k(T,\epsilon)]\|\Theta_k\|^2 + c_4\epsilon LT\kappa_k(T,\epsilon)
\end{aligned} \tag{57}$$

where in the last equality, we have defined for notational brevity the following two functions

$$\rho_k(T,\epsilon) := 2\beta_k(T,\epsilon) + 2\epsilon LT\Big[\epsilon^2L^2T^2(1+\epsilon L)^{2T-4}+13\Big] \tag{58}$$

$$\kappa_k(T,\epsilon) := 2\beta_k(T,\epsilon) + 16\epsilon LT \tag{59}$$

both of which depend on parameters $T \in \mathbb{N}^+$ and $\epsilon > 0$.

In the sequel, we will show that there exist parameters $\epsilon > 0$ and $T \geq 1$ such that the coefficient of (57) obeys $c_3 - c_4\rho_k(T,\epsilon) > 0$ for all $k \in \mathbb{N}^+$. Formally, such a result is summarized in Proposition 2 below, whose proof is relegated to Appendix G.

Proposition 2. *Consider functions $\beta_k(T,\epsilon)$ and $\rho_k(T,\epsilon)$ defined in* (10) *and* (58)*, respectively. Then for any $\delta > 0$, there exist constants $\epsilon_\delta > 0$ and $T_\delta \geq 1$, such that the following inequality holds for each $\epsilon \in (0, \epsilon_\delta)$*

$$\sigma(T_\delta, k) < \rho_k(T_\delta,\epsilon) < \rho_0(T_\delta,\epsilon) < \rho_0(T_\delta,\epsilon_\delta) \leq \delta, \quad \forall k \geq 1. \tag{60}$$

As such, by taking any $\delta < c_3/c_4$, feasible parameter values T^* and ϵ_c can be obtained according to (114) and (116), respectively. Now by choosing

$$T^* = T_\delta \tag{61}$$

$$\epsilon_c = \epsilon_\delta \tag{62}$$

it follows that

$$c_3' := LT^*\left[c_3 - c_4\rho_0(T^*,\epsilon_\delta)\right] = LT^*\left(c_3 - c_4\delta\right) > 0. \tag{63}$$

It follows from (57) that

$$\mathbb{E}\big[W(\Theta_{k+T}(k,\Theta_k)) - W(\Theta_k)\big|\mathcal{F}_k\big] \le -c_3'\epsilon\|\Theta_k\|^2 + c_4\epsilon LT^*\kappa_k(T^*,\epsilon)$$
$$= -c_3'\epsilon\|\Theta_k\|^2 + c_4'\epsilon^2 + c_5'\sigma(T^*;k)\epsilon \tag{64}$$

where we have defined constants $c_4' := c_4LT^*\big[2L(1+\epsilon_\delta L)^{T^*-2} + 16LT^*\big]$, and $c_5' := 2c_4LT^*$.

Finally, recalling (46), we deduce that

$$\mathbb{E}\big[W'(k+1,\Theta_k + \epsilon f(\Theta_k, X_k)) - W'(k,\Theta_k)\big|\mathcal{F}_k\big] \le -c_3'\epsilon\|\Theta_k\|^2 + c_4'\epsilon^2 + c_5'\sigma(T^*;k)\epsilon \tag{65}$$

concluding the proof of (12).

Now, we turn to show the first inequality. It is evident from the properties of $W(\Theta_k)$ in Assumption 2 that

$$W'(k,\Theta_k) = \sum_{j=k}^{k+T-1} W(\Theta_j(k,\Theta_k)) \ge W(\Theta_k(k,\Theta_k))$$
$$\ge c_1\|\Theta_k(k,\Theta_k)\|^2$$
$$= c_1\|\Theta_k\|^2 \tag{66}$$

where the second inequality follows from (6a), and the last equality from the fact that $\Theta_k(k,\Theta_k) = \Theta_k$. Therefore, by taking $c_1' = c_1$, we have shown that the first part of inequality (11) holds true. For the second part, it follows that

$$\|\Theta_{j+1}\| = \|\Theta_j + \epsilon f(\Theta_j, X_j)\| \le (1+\epsilon L)\|\Theta_j\| + \epsilon L, \quad \forall j \ge k \tag{67}$$

yielding by means of telescoping series

$$\|\Theta_j(k,\Theta_k)\| \le (1+\epsilon L)^{j-k}\|\Theta_k\| + \sum_{j=1}^{j-k}(1+\epsilon L)^{j-1}\epsilon L$$
$$\le (1+\epsilon L)^{j-k}\|\Theta_k\| + (1+\epsilon L)^{j-k} - 1, \quad \forall j \ge k.$$

Using further the inequality $(a+b)^2 \le 2(a^2+b^2)$, we deduce that

$$\|\Theta_j(k,\Theta_k)\|^2 \le 2(1+\epsilon L)^{2(j-k)}\|\Theta_k\|^2 + 2\Big[(1+\epsilon L)^{j-k} - 1\Big]^2. \tag{68}$$

Taking advantage of the properties of $W(\Theta_k)$ in Assumption 2 and (68), it follows that

$$W'(k,\Theta_k) = \sum_{j=k}^{k+T-1} W(\Theta_j(k,\Theta_k))$$
$$\le \sum_{j=k}^{k+T-1} c_2\|\Theta_j(k,\Theta_k)\|^2$$
$$\le 2c_2\sum_{j=k}^{k+T-1}(1+\epsilon L)^{2(j-k)}\|\Theta_k\|^2 + 2c_2\sum_{j=k}^{k+T-1}\Big[(1+\epsilon L)^{j-k} - 1\Big]^2. \tag{69}$$

Let us now examine the two coefficients of (69) more carefully. Note that

$$\sum_{j=k}^{k+T-1}(1+\epsilon L)^{2(j-k)} = \frac{(1+\epsilon L)^{2T}-1}{(1+\epsilon L)^2 - 1} = T\,\frac{2+(2T-1)(1+\epsilon' L)^{2T-2}\epsilon L}{2+\epsilon L} \tag{70}$$

$$\sum_{j=k}^{k+T-1}\big[(1+\epsilon L)^{j-k} - 1\big]^2 = \sum_{j=k+1}^{k+T-1}\left[(j-k)\epsilon L\left(1 + \frac{1}{2}(j-k-1)\big(1+\epsilon_{j-k}' L\big)^{j-k-2}\epsilon L\right)\right]^2 \tag{71}$$

$$= (\epsilon L)^2 \sum_{j=1}^{T-1} j^2 \left[1 + \frac{1}{2}(j-1)\big(1+\epsilon'_j L\big)^{j-2}\right]^2 \tag{72}$$

where both (70) and (71) follow from the mean-value theorem $(1+\epsilon L)^{j-k} = 1 + (j-k)\epsilon L + \frac{1}{2}(j-k-1)(1+\epsilon'_{j-k}L)^{j-k-2}(\epsilon L)^2$ for any $j-k \geq 1$ and some constants $\epsilon'_j \in [0,\epsilon]$.

According to Proposition 2, or more specifically, the inequalities (61) and (62), we see that $\epsilon'_j \leq \epsilon \leq \epsilon_\delta$ for all $1 \leq j \leq T-1$.

On the other hand, it is easy to check that both terms [(70) and (72)] are monotonically increasing functions of $\epsilon > 0$. Therefore, if we define constants

$$c'_2 := 2c_2 T^* \frac{2 + (2T^*-1)(1+\epsilon_\delta L)^{2T^*-2}\epsilon_\delta L}{2+\epsilon_\delta L} \tag{73}$$

$$c''_2 := 2c_2 \sum_{j=1}^{T^*-1} j^2 \left[1 + \frac{1}{2}(j-1)(1+\epsilon_\delta L)^{j-2}\right]^2 \tag{74}$$

which are independent of ϵ, then we draw from (69), (70), and (72) that

$$W'(k,\Theta_k) \leq c'_2\|\Theta_k\|^2 + c''_2(\epsilon L)^2. \tag{75}$$

concluding the proof of the second part of (11).

C Proof of Lemma 1

Taking expectation of both sides of (11) conditioned on $\mathcal{F}_k$ gives rise to

$$\mathbb{E}\big[W'(k,\Theta_k)|\mathcal{F}_k\big] \leq c'_2\|\Theta_k\|^2 + c''_2(\epsilon L)^2. \tag{76}$$

On the other hand, it is evident from (12) that

$$\begin{aligned}
&\mathbb{E}\big[W'(k+1,\Theta_{k+1})|\mathcal{F}_k\big] \\
&\leq \mathbb{E}\big[W'(k,\Theta_k)|\mathcal{F}_k\big] - c'_3\epsilon\|\Theta_k\|^2 + c'_4\epsilon^2 + c'_5\sigma(T^*;k)\epsilon \\
&= \mathbb{E}\big[W'(k,\Theta_k)|\mathcal{F}_k\big] - \frac{c'_3\epsilon}{c'_2}\left[c'_2\|\Theta_k\|^2 + c''_2(\epsilon L)^2\right] + \frac{c'_3}{c'_2}c''_2\epsilon(\epsilon L)^2 + c'_4\epsilon^2 + c'_5\sigma(T^*;k)\epsilon \\
&\leq \mathbb{E}\big[W'(k,\Theta_k)|\mathcal{F}_k\big] - \frac{c'_3\epsilon}{c'_2}\mathbb{E}\big[W'(k,\Theta_k)|\mathcal{F}_k\big] + \frac{c'_3}{c'_2}c''_2\epsilon_\delta(\epsilon L)^2 + c'_4\epsilon^2 + c'_5\sigma(T^*;k)\epsilon && (77) \\
&= \left(1 - \frac{c'_3\epsilon}{c'_2}\right)\mathbb{E}\big[W'(k,\Theta_k)|\mathcal{F}_k\big] + c''_4\epsilon^2 + c'_5\sigma(T^*;k)\epsilon && (78)
\end{aligned}$$

where, in order to obtain (77), we have employed the inequality in (76), and used the fact that $\epsilon < \epsilon_\delta$ to derive (78); and the last equality follows from $c''_4 := c'_4 + c'_3 c''_2 \epsilon_\delta L^2/c'_2$.

Finally, taking expectation of both sides of (78) with respect to the σ-field $\mathcal{F}_k$, concludes the proof.

D Proof of Theorem 2

Let us start with a basic Lemma, whose proof is elementary and is hence omitted here.

Lemma 4. *Consider the recursion $z_{t+1} = az_t + b$, where $a \neq 1$ and b are given constants. Then the following holds for all $t \geq t_0 \geq 0$*

$$z_t = a^{t-t_0}z_{t_0} + \frac{b(a^{t-t_0}-1)}{a-1}. \tag{79}$$

Proof of Theorem 2 is established in two phases depending on the k values. Specifically, let us define $k_\epsilon := \min\{k \in \mathbb{N}^+ | \sigma(T^*;k) \leq \epsilon\}$; then the first phase is from $k=0$ to k_ϵ, while the second phase consists of all $k > k_\epsilon$.

Phase I ($k \le k_\epsilon$). We have from 2 that $\sigma(T^*;k) \le \delta$ for all $0 \le k (\le k_\epsilon)$. Then, fixing $t_0 = 0$, and substituting $a := 1 - c_3'\epsilon/c_2' > 0$ and $b := c_4''\epsilon^2 + c_5'\delta\epsilon$ in (79), the recursion $\{\mathbb{E}[W'(k,\Theta_k)]\}$ in (14) can be recursively expressed as follows

$$\begin{aligned}
\mathbb{E}\big[W'(k,\Theta_k)\big] &\le \left(1 - \frac{c_3'\epsilon}{c_2'}\right)\mathbb{E}\big[W'(k-1,\Theta_{k-1})\big] + c_4''\epsilon^2 + c_5'\delta\epsilon \\
&\le \left(1 - \frac{c_3'\epsilon}{c_2'}\right)^k \mathbb{E}\big[W'(0,\Theta_0)\big] + \left[1 - \left(1 - \frac{c_3'\epsilon}{c_2'}\right)^k\right]\frac{c_2'}{c_3'}\big(c_4''\epsilon + c_5'\delta\big) \qquad (80)\\
&\le \left(1 - \frac{c_3'\epsilon}{c_2'}\right)^k \mathbb{E}\big[W'(0,\Theta_0)\big] + \frac{c_2'}{c_3'}\big(c_4''\epsilon + c_5'\delta\big) \\
&\le \left(1 - \frac{c_3'\epsilon}{c_2'}\right)^k \mathbb{E}\big[W'(0,\Theta_0)\big] + \frac{c_2'}{c_3'}\big(c_4'' + c_5'\big)\delta \qquad (81)\\
&\le c_2'\left(1 - \frac{c_3'\epsilon}{c_2'}\right)^k \|\Theta_0\|^2 + c_2''L^2\epsilon^2 + c_6\delta \qquad (82)
\end{aligned}$$

where the last inequality follows from $\epsilon \le \delta$ and the fact [cf. (11)] that

$$\mathbb{E}[W'(0,\Theta_0)] \le c_2'\mathbb{E}[\|\Theta_0\|^2] + c_2''\epsilon^2 L^2 \le c_2'\|\Theta_0\|^2 + c_2''\epsilon^2 L^2 \qquad (83)$$

where the initial guess $\Theta_0 \in \mathbb{R}^d$ is assumed given for simplicity; and $c_6 := c_2(c_4'' + c_5')/c_3'$.

On the other hand, using (11), the term $\mathbb{E}\big[W'(k,\Theta_k)\big]$ can be lowered bounded as follows

$$\mathbb{E}\big[W'(k,\Theta_k)\big] \ge c_1'\|\Theta_k\|^2 \qquad (84)$$

which, combined with (82), yields the finite-time error bound for iterations $k \le k_\epsilon$

$$\mathbb{E}[\|\Theta_k\|^2] \le \frac{c_2'}{c_1'}\left(1 - \frac{c_3'\epsilon}{c_2'}\right)^k \|\Theta_0\|^2 + \frac{c_2''L^2}{c_1'}\epsilon^2 + \frac{c_6}{c_1'}\delta. \qquad (85)$$

Phase II ($k > k_\epsilon$). Using now the fact that $\sigma(T^*;k) \le \epsilon$ due to the definition of k_ϵ, the recursion $\{\mathbb{E}[W'(k,\Theta_k)]\}$ for all $k > k_\epsilon$ becomes

$$\begin{aligned}
\mathbb{E}\big[W'(k+1,\Theta_{k+1})\big] &\le \left(1 - \frac{c_3'\epsilon}{c_2'}\right)\mathbb{E}\big[W'(k,\Theta_k)\big] + c_4''\epsilon^2 + c_5'\sigma(T^*;k)\epsilon \qquad (86)\\
&\le \left(1 - \frac{c_3'\epsilon}{c_2'}\right)\mathbb{E}\big[W'(k,\Theta_k)\big] + (c_4'' + c_5')\epsilon^2. \qquad (87)
\end{aligned}$$

Letting $t_0 = k_\epsilon$, and replacing a and b in (79) by constants $(1 - c_3'\epsilon/c_2')$ and $(c_4'' + c_5')\epsilon^2$ accordingly, we arrive at

$$\begin{aligned}
\mathbb{E}\big[W'(k,\Theta_k)\big] &\le \left(1 - \frac{c_3'\epsilon}{c_2'}\right)^{k-k_\epsilon}\mathbb{E}\big[W'(k_\epsilon,\Theta_{k_\epsilon})\big] + \left[1 - \left(1 - \frac{c_3'\epsilon}{c_2'}\right)^{k-k_\epsilon}\right]\frac{c_2'}{c_3'}\big(c_4'' + c_5'\big)\epsilon \\
&\le \left(1 - \frac{c_3'\epsilon}{c_2'}\right)^{k-k_\epsilon}\left[\left(1 - \frac{c_3'\epsilon}{c_2'}\right)^{k_\epsilon}\mathbb{E}\big[W'(0,\Theta_0)\big] + \frac{c_2'}{c_3'}\big(c_4''\epsilon + c_5'\delta\big)\right] + \frac{c_2'(c_4'' + c_5')}{c_3'}\epsilon \\
&\le \left(1 - \frac{c_3'\epsilon}{c_2'}\right)^k \mathbb{E}\big[W'(0,\Theta_0)\big] + \left(1 - \frac{c_3'\epsilon}{c_2'}\right)^{k-k_\epsilon}\frac{c_2'\big(c_4'' + c_5'\big)}{c_3'}\delta + \frac{c_2'(c_4'' + c_5')}{c_3'}\epsilon \qquad (88)\\
&\le c_2'\left(1 - \frac{c_3'\epsilon}{c_2'}\right)^k \|\Theta_0\|^2 + c_2''\epsilon^2 L^2 + \left(1 - \frac{c_3'\epsilon}{c_2'}\right)^{k-k_\epsilon} c_6\delta + c_6\epsilon \qquad (89)
\end{aligned}$$

where we have used the following bound at $k = k_\epsilon$ from Phase I in (80) along with (83)

$$\mathbb{E}[W'(k_\epsilon,\Theta_{k_\epsilon})] \le \left(1 - \frac{c_3'\epsilon}{c_2'}\right)^{k_\epsilon}\mathbb{E}\big[W'(0,\Theta_0)\big] + \left[1 - \left(1 - \frac{c_3'\epsilon}{c_2'}\right)^{k_\epsilon}\right]\frac{c_2'}{c_3'}\big(c_4''\epsilon + c_5'\delta\big). \qquad (90)$$

Plugging (84) into (89), yields the finite-time error bound for $k \geq k_\epsilon$

$$\mathbb{E}\big[\|\Theta_k\|^2\big] \leq \frac{c_2'}{c_1'}\left(1-\frac{c_3'\epsilon}{c_2'}\right)^k\|\Theta_0\|^2 + \frac{c_2''L^2}{c_1'}\epsilon^2 + \left(1-\frac{c_3'\epsilon}{c_2'}\right)^{k-k_\epsilon}\frac{c_6}{c_1'}\delta + \frac{c_6}{c_1'}\epsilon \tag{91}$$

which converges to a small (size-ϵ) neighborhood of the optimal solution $\Theta^* = 0$ at a linear rate.

Combining the results in the two phases, we deduce the following error bound that holds at any $k \in \mathbb{N}^+$

$$\mathbb{E}\big[\|\Theta_k\|^2\big] \leq \frac{c_2'}{c_1'}\left(1-\frac{c_3'\epsilon}{c_2'}\right)^k\|\Theta_0\|^2 + \frac{c_2''L^2}{c_1'}\epsilon^2 + \left(1-\frac{c_3'\epsilon}{c_2'}\right)^{\max\{k-k_\epsilon,0\}}\frac{c_6}{c_1'}\delta + \frac{c_6}{c_1'}\epsilon \tag{92}$$

concluding the proof of Theorem 2.

E Proof of Lemma 2

When $T = 1$ and for any $\Theta_k \in \mathbb{R}^d$, one can easily check that

$$g(k,1,\Theta_k) = \Theta_{k+1} - \Theta_k - \epsilon f(\Theta_k, X_k) = 0$$

implying $G_1 := \|g(k,1,\Theta_k)\| = 0$. To proceed, let us start by introducing the function

$$h(k,T,\Theta_k) := \sum_{j=k}^{k+T-1} f(\Theta_k, X_j)$$

which can be bounded as follows

$$\begin{aligned}
\big\|h(k,T,\Theta_k)\big\| = \left\|\sum_{j=k}^{k+T-1} f(\Theta_k, X_j)\right\| &\leq \sum_{j=k}^{k+T-1}\big\|f(\Theta_k, X_j)\big\| \\
&\leq L\sum_{j=k}^{k+T-1}(\|\Theta_k\|+1) \\
&= TL(\|\Theta_k\|+1)
\end{aligned} \tag{93}$$

where the second inequality follows from (5) in Assumption 1.

It is evident that

$$\begin{aligned}
g(k,T+1,\Theta_k) &= \Theta_{k+T+1} - \Theta_k - \epsilon\sum_{j=k}^{k+T} f(\Theta_k, X_j) \\
&= \Theta_{k+T} + \epsilon f(\Theta_{k+T}, X_{k+T}) - \Theta_k - \epsilon\left[f(\Theta_k, X_{k+T_0}) + \sum_{j=k}^{k+T-1} f(\Theta_k, X_j)\right] \\
&= g(k,T,\Theta_k) + \epsilon\big[f(\Theta_{k+T}, X_{k+T}) - f(\Theta_k, X_{k+T})\big].
\end{aligned} \tag{94}$$

By means of triangle inequality, it follows that

$$\begin{aligned}
G_{T+1} = \|g(k,T+1,\Theta_k)\| &\leq \big\|g(k,T,\Theta_k)\big\| + \epsilon\big\|f(\Theta_{k+T}, X_{k+T}) - f(\Theta_k, X_{k+T})\big\| \\
&\leq G_T + \epsilon L\big\|\Theta_{k+T} - \Theta_k\big\| && (95)\\
&\leq G_T + \epsilon L\left[\epsilon\big\|h(k,T,\Theta_k)\big\| + \big\|g(k,T,\Theta_k)\big\|\right] && (96)\\
&\leq (1+\epsilon L)G_T + \epsilon^2 L^2 T(\|\Theta_k\|+1) && (97)\\
&\leq \epsilon^2 L^2(\|\Theta_k\|+1)\sum_{k=0}^{T}(1+\epsilon L)^{T-k}k && (98)
\end{aligned}$$

where the inequality (95) follows from the Lipschitz continuity of $f(\theta, x)$ in θ, (96) from the fact that $\Theta_{k+T} = \Theta_k + \epsilon h(k,T,\Theta_k) + g(k,T,\Theta_k)$, (97) from (93) as well as the definition $G_T := \|g(k,T,\Theta_k)\|$, and the last inequality is obtained by telescoping series and uses $G_1 = 0$.

Lemma 5. *Given any positive constant $d \neq 1$, the following holds for all $T \geq 1$*

$$S_{T+1} = \sum_{k=0}^{T} k d^k = \frac{d(1-d^T)}{(1-d)^2} - \frac{T d^{T+1}}{1-d}. \tag{99}$$

Taking $d = (1+\epsilon L)^{-1}$ in (99), then (98) can be simplified as follows

$$\begin{aligned} G_T &\leq \epsilon^2 L^2 (1+\epsilon L)^{T-1} (\|\Theta_k\| + 1) \sum_{k=0}^{T-1} (1+\epsilon L)^{-k} k \\ &= \left[(1+\epsilon L)^T - \epsilon L T - 1\right] (\|\Theta_k\| + 1). \end{aligned} \tag{100}$$

To further simplify this bound, the Taylor expansion along with the mean-value theorem confirms that the following holds for some $\epsilon' \in (0,1)$

$$(1+\epsilon L)^T = 1 + \epsilon L T + \frac{1}{2} T(T-1)(1+\epsilon' L)^{T-2} (\epsilon L)^2, \quad \forall T \geq 1 \tag{101}$$

or equivalently,

$$(1+\epsilon L)^T - 1 - \epsilon L T = \frac{1}{2} T(T-1)(1+\epsilon' L)^{T-2} (\epsilon L)^2 \tag{102}$$

$$\leq \epsilon^2 L^2 T^2 (1+\epsilon L)^{T-2}. \tag{103}$$

F Proof of Lemma 3

Recalling that $g'(k,T,\Theta_k) = g(k,T,\Theta_k) + \epsilon \sum_{j=k}^{k+T-1} [f(\Theta_k, X_j) - \bar{f}(\Theta_k)]$, we have

$$\begin{aligned} \left\|g'(k,T,\Theta_k)\right\|^2 &= \left\| g(k,T,\Theta_k) + \epsilon \sum_{j=k}^{k+T-1} \left(f(\Theta_k, X_j) - \bar{f}(\Theta_k)\right) \right\|^2 \\ &\leq 2 \left\|g(k,T,\Theta_k)\right\|^2 + 2\epsilon^2 T^2 \left\| \frac{1}{T} \sum_{j=k}^{k+T-1} f(\Theta_k, X_j) - \bar{f}(\Theta_k) \right\|^2 \end{aligned} \tag{104}$$

$$\begin{aligned} &\leq 4 \left[(1+\epsilon L)^T - \epsilon L T - 1\right]^2 (\|\Theta_k\|^2 + 1) \\ &\quad + 4\epsilon^2 T^2 \left\| \frac{1}{T} \sum_{j=k}^{k+T-1} f(\Theta_k, X_j) \right\|^2 + 4\epsilon^2 T^2 \left\|\bar{f}(\Theta_k)\right\|^2 \end{aligned} \tag{105}$$

where we have used the property $\|a+b\|^2 \leq 2(\|a\|^2 + \|b\|^2)$ for any real-valued vectors a, b in deriving (104) and (105), as well as Proposition 1.

Squaring both sides of (102) yields

$$\left[(1+\epsilon L)^T - 1 - \epsilon T L\right]^2 = \frac{1}{4} T^2 (T-1)^2 (\epsilon L)^4 (1+\epsilon' L)^{2T-4} \leq \frac{1}{4} \epsilon^4 L^4 T^4 (1+\epsilon L)^{2T-4}. \tag{106}$$

Thus, the first term of (105) can be upper bounded by

$$4 \left[(1+\epsilon L)^T - \epsilon L T - 1\right]^2 (\|\Theta_k\|^2 + 1) \leq \epsilon^4 L^4 T^4 (1+\epsilon L)^{2T-4} (\|\Theta_k\|^2 + 1). \tag{107}$$

Regarding the second term of (105), we have that

$$\left\| \frac{1}{T} \sum_{j=k}^{k+T-1} f(\Theta_k, X_j) \right\|^2 \leq \frac{1}{T} \sum_{j=k}^{k+T-1} \left\|f(\Theta_k, X_j)\right\|^2 \tag{108}$$

$$\leq \frac{1}{T}\sum_{j=k}^{k+T-1} L^2(\|\Theta_k\|+1)^2 \tag{109}$$

$$\leq 2L^2\|\Theta_k\|^2 + 2L^2 \tag{110}$$

where (108) and (110) follow from the inequality $\|\sum_{i=1}^n z_i\|^2 \leq n\sum_{i=1}^n \|z_i\|^2$ for all real-valued vectors $\{z_i\}_{i=1}^n$, and (109) from our working assumption on function $f(\theta, x)$.

Withe regards to the last term of (105), it follows directly from the Lipschitz property of the average operator $\bar{f}(\theta)$ that

$$\left\|\bar{f}(\Theta_k)\right\|^2 \leq L^2\|\Theta_k\|^2. \tag{111}$$

Substituting the bounds in (107), (110), and (111) into (105), we arrive at

$$\|g'(k,T,\Theta_k)\|^2 \leq \epsilon^2 L^2 T^2\Big[\epsilon^2 L^2 T^2 (1+\epsilon L)^{2T-4} + 12\Big]\|\Theta_k\|^2 + 8\epsilon^2 L^2 T^2 \tag{112}$$

concluding the proof.

G Proof of Proposition 2

We prove this claim by construction. By definition, it follows that for all $k \in \mathbb{N}^+$

$$\rho_k(T,\epsilon) \leq \rho_0(T,\epsilon) = 2\epsilon LT\Big[(1+\epsilon L)^{T-2} + 13\Big] + 2(\epsilon LT)^3(1+\epsilon L)^{2T-4} + 2\sigma(T;0). \tag{113}$$

Under the assumption that $\lim_{T\to+\infty}\sigma(T;0) = 0$, the function value $\sigma(T;0) \geq 0$ can be made arbitrarily small by taking a sufficiently large integer $T \in \mathbb{N}^+$ in constructing the function $W'(k,\Theta_k)$. Without loss of generality, let us work with T such that

$$T_\delta := \min\left\{T \in \mathbb{N}^+ \,\middle|\, \sigma(T;0) \leq \frac{\delta}{4}\right\}. \tag{114}$$

It is clear that $T_\delta \geq 1$. Define function

$$\nu(\epsilon) := \epsilon LT_\delta\Big[(1+\epsilon L)^{T_\delta - 2} + 13\Big] + (\epsilon LT_\delta)^3(1+\epsilon L)^{2T_\delta - 4} \tag{115}$$

which can be easily shown to be a monotonically decreasing function of $\epsilon > 0$, and which attains its minimum $\nu = 0$ at $\epsilon = 0$. Let ϵ_δ be the unique solution to the equation

$$\nu(\epsilon) = \frac{\delta}{4}, \quad \epsilon > 0. \tag{116}$$

As a result, for all $\epsilon \in (0, \epsilon_\delta]$, it holds that

$$\nu(\epsilon) \leq \frac{\delta}{4}. \tag{117}$$

Combining (114) and (117) concludes the proof of Proposition 2.