Supplementary Material
A Stein Goodness-of-fit Test for Directional Distributions

A Uniformity test

We present Rayleigh test and Kuiper test for uniformity.

A.1 Rayleigh Test

The test statistic of Rayleigh test is

$$R_n := \frac{2}{n}\left[\left(\sum_{i=1}^{n}\cos\theta_i\right)^2 + \left(\sum_{i=1}^{n}\sin\theta_i\right)^2\right].$$

Under the null, we have $R_n \sim \chi_2^2$. Therefore, the critical value is given by the quantile of chi-square distribution. For example, if the significance level is set to $\alpha = 0.01$, then the critical value is 9.210.

A.2 Kuiper Test

Kuiper test for uniformity is based on the cumulative distribution function (cdf). The cdf of the uniform distribution is

$$F(\theta) = \frac{\theta}{2\pi}.$$

We sort the samples to $0 \leq \theta_1 \leq \cdots \leq \theta_n \leq 2\pi$ and compute

$$D_n^+ := \sqrt{n}\sup_{\theta\in[0,2\pi)}\{F_n(\theta) - F(\theta)\} = \sqrt{n}\max_{1\leq i\leq n}\left(\frac{i}{n} - U_i\right),$$

$$D_n^- := \sqrt{n}\sup_{\theta\in[0,2\pi)}\{F(\theta) - F_n(\theta)\} = \sqrt{n}\max_{1\leq i\leq n}\left(U_i - \frac{i-1}{n}\right),$$

where $U_i = \theta_i/(2\pi)$. Then, the test statistic is defined as

$$V_n := D_n^+ + D_n^-.$$

The critical value is found in the statistical table. For example, for the significance level $\alpha = 0.01$, the critical value is 2.001.