A PROOFS

A.1 PROOF OF PROPOSITION 3.4

Proof. **(Proof of Proposition 3.4)**

Consider the most basic case when there is only one Boolean variable b in theory θ. Let θ' be an SMT($\mathcal{LRA}$) theory defined as follow

$$\theta' = \theta\{b : \lambda_b\} \wedge (-1 \leq \lambda_b \leq 1)$$

where $\theta\{b : \lambda_b\}$ is obtained by replacing all atom b by $0 < \lambda_b$ and replacing all its negation $\neg b$ by $\lambda_b < 0$ in theory θ.

Recall that weight functions are defined by a set of literals $\mathcal{L}$ and a set of per-literal weight functions $\mathcal{P} = \{p_\ell(\boldsymbol{x})\}_{\ell \in \mathcal{L}}$. When a literal ℓ is satisfied in a world, denoted by $\boldsymbol{x} \wedge \boldsymbol{b} \models \ell$, weights are defined as follows

$$w(\boldsymbol{x}, \boldsymbol{b}) = \prod_{\substack{\ell \in \mathcal{L} \\ \boldsymbol{x} \wedge \boldsymbol{b} \models \ell}} p_\ell(\boldsymbol{x}).$$

Let $\mathcal{L}'$ be a set of literals obtained by replacing Boolean literal b by $0 < \lambda_b$ and replacing its negation $\neg b$ by $\lambda_b < 0$ in theory θ as we do for theory. For the set of per-literal weight functions $\mathcal{P}'$, we define it for introduced real variable λ_b by $p_{(\lambda_b > 0)} = p_b$ and $p_{(\lambda_b < 0)} = p_{\neg b}$.

Then we have that for any $\boldsymbol{x}^*$,

$$w'(\boldsymbol{x}^*, \lambda_b) = \begin{cases} w(\boldsymbol{x}^*, b), & 1 > \lambda_b > 0 \\ w(\boldsymbol{x}^*, \neg b), & -1 < \lambda_b < 0 \end{cases}$$

By definition of WMI, we write $\mathrm{WMI}(\theta, w \mid \boldsymbol{x}, \boldsymbol{b})$ in its integration form as follows.

$$\begin{aligned} &\mathrm{WMI}(\theta, w \mid \boldsymbol{x}, \boldsymbol{b}) \\ &= \int_{\theta(\boldsymbol{x}, b)} w(\boldsymbol{x}, b) d\boldsymbol{x} + \int_{\theta(\boldsymbol{x}, \neg b)} w(\boldsymbol{x}, \neg b) d\boldsymbol{x} \end{aligned}$$

For the first term in the above equation, we can rewrite it s.t. Boolean variable b is replaced by real variable λ_b in the following way.

$$\begin{aligned} \int_{\theta(\boldsymbol{x}, b)} w(\boldsymbol{x}, b) d\boldsymbol{x} &= \int_0^1 \int_{\theta(\boldsymbol{x}, b)} w(\boldsymbol{x}, b) d\boldsymbol{x} d\lambda_b \\ &= \int_{\theta'(\boldsymbol{x}, \lambda_b)} w'(\boldsymbol{x}, \lambda_b) d\boldsymbol{x} d\lambda_b \end{aligned}$$

By doing this to the other integration term of $\mathrm{WMI}(\theta, w \mid \boldsymbol{x}, \boldsymbol{b})$, and also by the definition of WMI, we finally obtain that

$$\mathrm{WMI}(\theta, w \mid \boldsymbol{x}, \boldsymbol{b}) = \mathrm{WMI}(\theta', w' \mid \boldsymbol{x}')$$

where $\boldsymbol{x}' = \boldsymbol{x} \cup \{\lambda_b\}$ is a set of real variables. The proof above can be easily adapted to multiple Boolean variable cases, which proves our proposition. □

A.2 PROOF OF PROPOSITION 3.5

Proof. **(Proof of Proposition 3.5)** To start with, we consider SMT($\mathcal{LRA}$) theory θ with no Boolean variables with a simple weight function w where the set of literal $\mathcal{L} = \{\ell\}$ has only one literal and literal weight function $p_\ell(\boldsymbol{x}) = \prod_{i=0}^n x_i^{p_i}$.

Claim A.1. *For a monomial function $f(\boldsymbol{x}) = \prod_{i=0}^n x_i^{p_i}$, let $\theta_f = \bigwedge_{i=0}^n \bigwedge_{j=1}^{p_i} (0 \leq z_j^i \leq x_i)$. Then we have the monomial $f(\boldsymbol{x}) = MI(\theta_f \mid \boldsymbol{z}; \boldsymbol{x})$, where $\boldsymbol{z}$ is the set of real variables z_j^i in theory θ_f, and $\boldsymbol{x}$ is parameters of theory θ_f.*

Let $\theta' = \theta \wedge (\ell \Rightarrow \theta_p) \wedge (\neg \ell \Rightarrow \hat{\theta}_p)$ where $p = p_\ell$ for brevity, θ_p is as defined in Claim A.1 and $\hat{\theta}_p := \bigwedge_{i=0}^n \bigwedge_{j=1}^{p_i} (0 \leq z_j^i \leq 1)$. Then we can rewrite $\mathrm{WMI}(\theta, w \mid \boldsymbol{x})$ as MI problem by Claim A.1 as follows.

$$\begin{aligned} &\mathrm{WMI}(\theta, w \mid \boldsymbol{x}) = \int_{\theta(\boldsymbol{x})} w(\boldsymbol{x}) d\boldsymbol{x} \\ &= \int_{\theta(\boldsymbol{x}) \wedge \ell(\boldsymbol{x})} p(\boldsymbol{x}) d\boldsymbol{x} + \int_{\theta(\boldsymbol{x}) \wedge \neg \ell(\boldsymbol{x})} 1 d\boldsymbol{x} \\ &= \int_{\theta(\boldsymbol{x}) \wedge \ell(\boldsymbol{x})} MI(\theta_p \mid \boldsymbol{z}; \boldsymbol{x}) d\boldsymbol{x} + \int_{\theta(\boldsymbol{x}) \wedge \neg \ell(\boldsymbol{x})} 1 d\boldsymbol{x} \\ &= \int_{\theta(\boldsymbol{x}) \wedge \ell(\boldsymbol{x})} \int_{\theta_p(\boldsymbol{z})} 1 d\boldsymbol{z} d\boldsymbol{x} + \int_{\theta(\boldsymbol{x}) \wedge \neg \ell(\boldsymbol{x}) \wedge \hat{\theta}_p} 1 d\boldsymbol{x} d\boldsymbol{z} \\ &= \mathrm{MI}(\theta \wedge (\ell \Rightarrow \theta_p) \wedge (\neg \ell \Rightarrow \hat{\theta}_p) \mid \boldsymbol{x}, \boldsymbol{z}) \end{aligned}$$

Take $\boldsymbol{x}' = \boldsymbol{x} \cup \boldsymbol{z}$ then the proposition holds. The proof can be easily adapted for monomials with non-trivial coefficient by inducing more real variables z. It also holds for more general weight functions with literal set $\mathcal{L} = \{\ell_i\}_{i=1}^k$ and set of monomial per-literal weight functions $\mathcal{P} = \{p_{\ell_i}\}_{i=1}^k$, by taking theory θ' as follows which completes the proof of proposition.

$$\theta' = \theta \wedge \bigwedge_{i=1}^k (\ell_i \Rightarrow \theta_{p_{\ell_i}}) \wedge \bigwedge_{i=1}^k (\neg \ell_i \Rightarrow \hat{\theta}_{p_{\ell_i}}).$$

□

Proof. **(Proof of Claim A.1)** By definition of theory θ_f,

$$
\begin{aligned}
MI(\theta_f \mid \boldsymbol{z}; \boldsymbol{x}) &= \int_{\theta_f(\boldsymbol{z})} 1 d\boldsymbol{z} \\
&= \prod_{i=1}^{n} \prod_{j=1}^{p_i} \int_0^{x_i} 1 dz_j^i \\
&= \prod_{i=1}^{n} \prod_{j=1}^{p_i} x_i = \prod_{i=1}^{n} x_i^{p_i} = f(\boldsymbol{x}).
\end{aligned}
$$

□

A.3 REDUCTION TO MI WITH POLYNOMIAL WEIGHTS

The reduction from WMI problems to MI problems in Proposition 3.5 can also be done for arbitrary polynomial weight functions but can increase treewidth of primal graphs. We give a formal description on this reduction as follows.

Let θ be an SMT($\mathcal{LRA}$) theory with no Boolean variables with weight functions where the set of literal $\mathcal{L} = \{\ell\}$ has only one literal and literal weight function is a polynomial, denoted by $p(\boldsymbol{x}) = \sum_{i=1}^{k} \alpha_i f_i(\boldsymbol{x})$ with each f_i a monomial function.

It has been shown in the proof of Proposition 3.5 in Section A.2 that for each monomial function f_i, there exist two SMT($\mathcal{LRA}$) theories θ_i and $\hat{\theta}_i$ such that MI$(\theta_i \mid \boldsymbol{z}_i; \boldsymbol{x}) = f_i(\boldsymbol{x})$ and MI$(\hat{\theta}_i \mid \boldsymbol{z}_i; \boldsymbol{x}) = 1$.

Let's define theories $\theta'_i = \theta_i \wedge (0 < v_i < \alpha_i)$ and $\hat{\theta}'_i = \hat{\theta}_i \wedge (0 < v_i < 1)$ with parameter variables v_i. Also define an indicator variable λ with real domain $[0, k]$ and literals $\ell_i = i - 1 < \lambda < i$ with $i \in \{1, 2, \cdots, k\}$. Then we have that for an SMT($\mathcal{LRA}$) theory θ' defined as follows, it holds that $\mathrm{WMI}(\theta, w \mid \boldsymbol{x}) = \mathrm{MI}(\theta' \mid \boldsymbol{x}, \boldsymbol{z})$ with $\boldsymbol{z}$ denoting all auxiliary variables.

$$
\theta' = \theta \wedge (\ell \iff \vee_{i=1}^{k} \ell_i) \bigwedge_{i=1}^{k} (\ell_i \Rightarrow \theta'_i) \bigwedge_{i=1}^{k} (\neg \ell_i \Rightarrow \hat{\theta}'_i)
$$

Why the WMI problem and the MI problem are equal can be proved by the following observations.

$$
\begin{aligned}
\mathrm{WMI}(\theta, w \mid \boldsymbol{x}) &= \int_{\theta(\boldsymbol{x})} w(\boldsymbol{x}) d\boldsymbol{x} && (4) \\
&= \int_{\theta(\boldsymbol{x}) \wedge \ell(\boldsymbol{x})} p(\boldsymbol{x}) d\boldsymbol{x} + \int_{\theta(\boldsymbol{x}) \wedge \neg \ell(\boldsymbol{x})} 1 d\boldsymbol{x} && (5)
\end{aligned}
$$

For the first term in Equation 5, we have that

$$
\begin{aligned}
\int_{\theta(\boldsymbol{x}) \wedge \ell(\boldsymbol{x})} p(\boldsymbol{x}) d\boldsymbol{x} &= \sum_{i=1}^{k} \int_{\theta(\boldsymbol{x}) \wedge \ell(\boldsymbol{x})} \alpha_i f_i(\boldsymbol{x}) d\boldsymbol{x} \\
&= \sum_{i=1}^{k} \int_{\theta(\boldsymbol{x}) \wedge \ell(\boldsymbol{x}) \wedge \ell_i} \alpha_i f_i(\boldsymbol{x}) d\boldsymbol{x} d\lambda \\
&= \sum_{i=1}^{k} \int_{\theta(\boldsymbol{x}) \wedge \ell(\boldsymbol{x}) \wedge \ell_i \wedge \theta_i} 1 d\boldsymbol{x} d\boldsymbol{z} \\
&= \mathrm{MI}(\theta' \wedge \ell \mid \boldsymbol{x}, \boldsymbol{z})
\end{aligned}
$$

Also for the second term in Equation 5, it equates to MI$(\theta' \wedge \neg \ell \mid \boldsymbol{x}, \boldsymbol{z})$. Therefore, reduction from the WMI problem to the MI problem holds. Although the reduction process we show here is for theories with one polynomial weight function, this process can be generalized to theories with multiple polynomial weight functions with little modification.

A.4 PROOF OF PROPOSITION 4.1

Proof. **(Proof of Proposition 4.1)** It follows from definition of WMI. Denote the set of real variables $\boldsymbol{x} \backslash \{y\}$ by $\hat{\boldsymbol{x}}$. From the definition of WMI in Equation 2.2, we can obtain the following partial derivative of WMI of theory θ w.r.t. variable y.

$$
\begin{aligned}
&\frac{\partial}{\partial x} \mathrm{WMI}(\theta, w \mid \boldsymbol{x}, b) \mid_{y=y^*} \\
&= \sum_{\mu \in \mathbb{B}^m} \int_{\theta(y^*, \hat{\boldsymbol{x}}, \mu)} w(y^*, \hat{\boldsymbol{x}}, \mu) d\hat{\boldsymbol{x}}
\end{aligned}
$$

where the variable y is fixed to value y^* in weight function, μ are total truth assignments to Boolean variables as defined before. The weight function is integrated over set $\{\hat{\boldsymbol{x}}^* \mid \theta(y^*, \hat{\boldsymbol{x}}^*, \mu) \text{ is } \texttt{true}\}$. We define $p(y)$ as follows

$$
p(y) := \sum_{\mu \in \mathbb{B}^m} \int_{y, \theta(\hat{\boldsymbol{x}}, \mu)} w(y, \hat{\boldsymbol{x}}, \mu) d\hat{\boldsymbol{x}}
$$

Since weight functions w are piecewise polynomial, function $p(y)$ is a univariate piecewise polynomial $p(y)$, and $\mathrm{WMI}(\theta, w \mid \boldsymbol{x}, b)$ is an integration over $p(y)$, which finishes our proof. □

A.5 PROOF OF THEOREM 4.4

Claim A.2. *For each path in the primal graph that starts with the root and ends with a leaf, and each real variable in path with height i, its number of polynomial pieces is $O(n \cdot c^{i+1})$.*

Algorithm 2 Polynomial pieces and degree enumeration algorithms

a) PE_EDGE – For Two Variable Theory
Input: θ: SMT($\mathcal{LRA}$) theory with two real variables
I : interval and degree tuples of variable x
Output: I_y: pieces and degrees for variable y

$B \leftarrow$ collect integration bounds on variable x
$Y \leftarrow y$ values where two bounds in B meet
for all interval $[l, u]$ resulting from Y **do**
  $\theta' \leftarrow \theta \wedge (l \leq y \leq u)$
  if θ' is SAT **then**
    $\{l(y), u(y), d\} \leftarrow$ get_bound_degree(x,θ',I)
    $d' \leftarrow$ argmax$_d$ get_degree($l(y), u(y), d)\}$)
    $I_y \leftarrow I_y \cup ([l, u], d')$
Return I_y

b) PE_NODE – For Tree Primal Graph
Input: θ: SMT theory with tree primal graph
G: primal graph for theory θ
Output: I_y: interval and degree tuples of root variable y

if root y has no child **then**
  $I_y \leftarrow$ get_bound_degree(θ)
return I_y
$\theta_{y,c}$'s, θ_{G_c}'s $\leftarrow$ partition SMT($\mathcal{LRA}$) theory θ
for all child c **do**
  $I_c \leftarrow$ PE_NODE(θ_c, G_c)
  $I_y^c \leftarrow$ PE_EDGE($\theta_{y,c}, I_c$)
Return $I_y = \mathit{shatter}\big(\{I_y^c\}_c\big)$

Proof. The proof can be done by mathematical induction. Denote the real variable with height i in the path by x_i For $i = 0$, since the number of $\mathcal{LRA}$ literals is c, then there are at most c critical points for real variable x_0 and therefore there are at most $c + 1$ polynomial pieces for x_0.

Suppose that the claim holds for i, that is, the number of polynomial pieces for x_i is $O(n \cdot c^{i+1})$. To obtain critical points for variable x_{i+1}, we collect integration bounds on variable x_i whose size is $O(n \cdot c^{i+1})$ by assumption. Since the critical points of variable x_{i+1} are obtained by solving $b_1 = b_2$ w.r.t. variable x_{i+1} for b_1, b_2 in bounds on variable x_i, where there are at most c bounds containing x_{i+1} and the rest bounds are numerical ones, there are at most $O(n \cdot c^{i+2})$ solutions. Therefore, the number of polynomial pieces for x_{i+1} is $O(n \cdot c^{i+2})$, which finishes our proof. □

Proof. **(Proof of Theorem 4.4)** Let p be an arbitrary path in the pseudo tree T that starts with the root and ends with a leaf. Denote the maximum polynomial degree in weight functions by d. By Claim A.2 for each variable, it has at most $O(n \cdot c^{h_p})$ polynomial pieces. Moreover from Prop. 4.1, polynomials defined over each pieces have at most $n(d + h)$ polynomial degree. Therefore the set of values chosen to do instantiation on a certain real variable has size $O(n^3 \cdot c^{h_p})$ and each path p induces a search space with size $O((n^3 \cdot c^{h_p})^{h_t})$ since length of each path is bounded by h_t.

The pseudo tree T is covered by l such directed paths. The union of their individual search spaces covers the whole search space, where every distinct full path in the search space appears exactly once. Therefore, the size of the search space is bounded by $O(l \cdot (n^3 \cdot c^{h_p})^{h_t})$. □

B CACHING

Our algorithm allows caching in two sense. The first is the caching of pieces, i.e. intervals and polynomial degrees obtained from child nodes, which can be considered as constraints from child nodes. The pieces of a certain nodes is decided both by instantiation values from its father node as well as pieces from child nodes. Although we instantiate root nodes with distinct values, the constraints from child nodes for a certain node remains unchanged as long as they have the same father-child relation in subtree.

Another case where caching is possible is values of $p(y)$ as defined in Prop. 4.1 at instantiations of variable x. This is possible because for a certain node, its pieces resulting from different instantiation values of its grandfather node might intersects. This is especially helpful when there is a long path in primal graphs and caching can save a lot computational effort.

C PIECE ENUMERATION ALGORITHM

We summarize piece enumeration algorithms for two variable theory and for theory with tree primal graphs as described in Section 4.2 in Algorithm 2. Both get_bound_degree and get_degree are trivial operations for specifying integration bounds and polynomial degree. They are applied when the magnitude order of integration bounds are fixed and thus they can be done by scanning through related theories.