Appendix

A Proof of Lemma 2

Assume that the samples of D_i datasets are $x_1^i, ..., x_m^i$. Without loss of generality, we assume that $x_i^0 = x_i^1$ for $1 \leq i \leq m-1$. Then we have

$$\begin{aligned}
d_{\mathcal{F}}\left(\hat{\mu}_m^0, \hat{\mu}_m^1\right) &= \sup_{f \in \mathcal{F}} \left\{\mathbb{E}_{x \sim \hat{\mu}_m^0}\left[f(x)\right] - \mathbb{E}_{x \sim \hat{\mu}_m^1}\left[f(x)\right]\right\} \\
&= \sup_{f \in \mathcal{F}} \left\{\frac{1}{m}\sum_{i=1}^{m} f(x_i^0) - \frac{1}{m}\sum_{i=1}^{m} f(x_i^1)\right\} \\
&= \frac{1}{m}\sup_{f \in \mathcal{F}} \left\{f(x_m^0) - f(x_m^1)\right\} \\
&\leq \frac{2\Delta}{m}
\end{aligned}$$

B Proof of Lemma 3

From Lemma 1, we know that

$$\mathbb{P}\left[d_{\mathcal{F}}\left(\hat{\mu}_m^0, g_0\right) > \frac{1}{2}\tau_{k,\xi}\right] \leq \xi$$
$$\mathbb{P}\left[d_{\mathcal{F}}\left(\hat{\mu}_m^1, g_1\right) > \frac{1}{2}\tau_{k,\xi}\right] \leq \xi$$

Therefore,

$$\mathbb{P}\left[d_{\mathcal{F}}\left(\hat{\mu}_m^0, g_0\right) \leq \frac{1}{2}\tau_{k,\xi} \wedge d_{\mathcal{F}}\left(\hat{\mu}_m^1, g_1\right) \leq \frac{1}{2}\tau_{k,\xi}\right] \geq 1 - 2\xi.$$

With probability at least $1-2\xi$, we have

$$\begin{aligned}
d_{\mathcal{F}}\left(g_0, g_1\right) =& \sup_{f \in \mathcal{F}} \left\{\mathbb{E}_{x \sim g_0}\left[f(x)\right] - \mathbb{E}_{x \sim g_1}\left[f(x)\right]\right\} \\
=& \sup_{f \in \mathcal{F}} \left\{\mathbb{E}_{x \sim g_0}\left[f(x)\right] - \mathbb{E}_{x \sim \hat{\mu}_m^0}\left[f(x)\right] + \mathbb{E}_{x \sim \hat{\mu}_m^0}\left[f(x)\right] - \mathbb{E}_{x \sim \hat{\mu}_m^1}\left[f(x)\right] + \mathbb{E}_{x \sim \hat{\mu}_m^1}\left[f(x)\right] - \mathbb{E}_{x \sim g_1}\left[f(x)\right]\right\} \\
\leq& \sup_{f \in \mathcal{F}} \left\{\mathbb{E}_{x \sim g_0}\left[f(x)\right] - \mathbb{E}_{x \sim \hat{\mu}_m^0}\left[f(x)\right]\right\} + \sup_{f \in \mathcal{F}} \left\{\mathbb{E}_{x \sim \hat{\mu}_m^0}\left[f(x)\right] - \mathbb{E}_{x \sim \hat{\mu}_m^1}\left[f(x)\right]\right\} \\
& + \sup_{f \in \mathcal{F}} \left\{\mathbb{E}_{x \sim \hat{\mu}_m^1}\left[f(x)\right] - \mathbb{E}_{x \sim g_1}\left[f(x)\right]\right\} \\
=& \sup_{f \in \mathcal{F}} \left\{\mathbb{E}_{x \sim \hat{\mu}_m^0}\left[f(x)\right] - \mathbb{E}_{x \sim g_0}\left[f(x)\right]\right\} + \sup_{f \in \mathcal{F}} \left\{\mathbb{E}_{x \sim \hat{\mu}_m^0}\left[f(x)\right] - \mathbb{E}_{x \sim \hat{\mu}_m^1}\left[f(x)\right]\right\} \\
& + \sup_{f \in \mathcal{F}} \left\{\mathbb{E}_{x \sim \hat{\mu}_m^1}\left[f(x)\right] - \mathbb{E}_{x \sim g_1}\left[f(x)\right]\right\} \\
& (\mathcal{F} \text{ is even}) \\
=& d_{\mathcal{F}}\left(\hat{\mu}_m^0, g_0\right) + d_{\mathcal{F}}\left(\hat{\mu}_m^0, \hat{\mu}_m^1\right) + d_{\mathcal{F}}\left(\hat{\mu}_m^1, g_1\right) \\
\leq& \tau_{k,\xi} + \frac{2\Delta}{m}
\end{aligned}$$

C Proof of Lemma 5

Define ρ_p, ρ_q as the probability (density) functions of p and q respectively. Assume set $S_0 = \{x : \log \rho_p(x) - \log \rho_q(x) \geq \epsilon\}$, then $\forall x \in S_0$, we have $\rho_p(x) \geq \rho_q(x)e^\epsilon$, and

$$\begin{aligned}
s \geq & d_{\mathrm{KL}}\left(p, q\right) + d_{\mathrm{KL}}\left(q, p\right) \\
= & \int_x \left(\rho_p(x) - \rho_q(x)\right)\left(\log \rho_p(x) - \log \rho_q(x)\right) \\
\geq & \int_S \left(\rho_p(x) - \rho_q(x)\right)\left(\log \rho_p(x) - \log \rho_q(x)\right) \\
& \text{(because } \left(\rho_p(x) - \rho_q(x)\right)\left(\log \rho_p(x) - \log \rho_q(x)\right) \geq 0 \quad \forall x) \\
\geq & \int_S \rho_p(x)(1 - e^{-\epsilon})\epsilon
\end{aligned}$$

i.e. $\mathbb{P}\left[M(D_0) \in S_0\right] \leq \frac{s}{\epsilon(1-e^{-\epsilon})}$. For any set S, we have

$$\begin{aligned}
\mathbb{P}\left[M(D_0) \in S \setminus S_0\right] = & \int_{S \setminus S_0} \rho_p(x) dx \\
\leq & \int_{S \setminus S_0} \rho_q(x) e^\epsilon dx \\
= & e^\epsilon \mathbb{P}\left[M(D_1) \in S \setminus S_0\right]
\end{aligned}$$

D Proof of Lemma 6

Recall that in Appendix D we get

$$\mathbb{P}\left[d_{\mathcal{F}}\left(\hat{\mu}_m^0, g_0\right) \leq \frac{1}{2}\tau_{k,\xi} \wedge d_{\mathcal{F}}\left(\hat{\mu}_m^1, g_1\right) \leq \frac{1}{2}\tau_{k,\xi}\right] \geq 1 - 2\xi.$$

With probability at least $1 - 2\xi$, we have

$$\begin{aligned}
d_{\mathcal{F}}\left(g_0, g_1\right) = & \sup_{f \in \mathcal{F}} \left\{\mathbb{E}_{x \sim g_0}\left[f(x)\right] - \mathbb{E}_{x \sim g_1}\left[f(x)\right]\right\} \\
= & \sup_{f \in \mathcal{F}} \left\{\mathbb{E}_{x \sim g_0}\left[f(x)\right] - \mathbb{E}_{x \sim \hat{\mu}_m^0}\left[f(x)\right] + \mathbb{E}_{x \sim \hat{\mu}_m^0}\left[f(x)\right] - \mathbb{E}_{x \sim \hat{\mu}_m^1}\left[f(x)\right] + \mathbb{E}_{x \sim \hat{\mu}_m^1}\left[f(x)\right] - \mathbb{E}_{x \sim g_1}\left[f(x)\right]\right\} \\
\geq & - \sup_{f \in \mathcal{F}} \left\{\mathbb{E}_{x \sim g_0}\left[f(x)\right] - \mathbb{E}_{x \sim \hat{\mu}_m^0}\left[f(x)\right]\right\} + \sup_{f \in \mathcal{F}} \left\{\mathbb{E}_{x \sim \hat{\mu}_m^0}\left[f(x)\right] - \mathbb{E}_{x \sim \hat{\mu}_m^1}\left[f(x)\right]\right\} \\
& - \sup_{f \in \mathcal{F}} \left\{\mathbb{E}_{x \sim \hat{\mu}_m^1}\left[f(x)\right] - \mathbb{E}_{x \sim g_1}\left[f(x)\right]\right\} \\
= & - d_{\mathcal{F}}\left(\hat{\mu}_m^0, g_0\right) + d_{\mathcal{F}}\left(\hat{\mu}_m^0, \hat{\mu}_m^1\right) - d_{\mathcal{F}}\left(\hat{\mu}_m^1, g_1\right) \\
\geq & \frac{\Delta'}{2m} - \tau_{k,\xi}
\end{aligned}$$

E Proof of Lemma 7

Assume that $S = \{x \in X | p(x) > q(x)\}$ and $T = X \setminus S = \{x \in X | p(x) <= q(x)\}$. Let $a_1 = \int_{x \in S} p(x)dx$, $b_1 = \int_{x \in S} q(x)dx$, $a_2 = \int_{x \in T} p(x)dx$, and $b_2 = \int_{x \in T} q(x)dx$. Because of the the differential privacy guarantee, we have

$$\begin{aligned}
a_1 - \delta \leq e^\epsilon b_1 \\
b_2 - \delta \leq e^\epsilon a_2
\end{aligned}$$

Note that $a_1 + a_2 = 1$, $b_1 + b_2 = 1$. Therefore, we have

$$b_1 + a_2 \geq \frac{2 - 2\delta}{1 + e^\epsilon}$$

and

$$d_{\mathrm{TV}}\left(p, q\right) = \frac{a_1 + b_2 - b_1 - a_2}{2} \leq \frac{e^{\epsilon} + 2\delta - 1}{e^{\epsilon} + 1}.$$