Outline

The appendix is organized as follows. We first provide the proofs for the concentration bound of RIPS (Theorem 1), the computation of the inverse of the bilinear form (Lemma 1), the guarantees of the PTR procedure (Theorem 2), the regret bound of the RIPS regret minimization algorithm (Theorem 3), the sample complexity of the RIPS pure exploration algorithm (Theorem 4). We also establish the regret bound and the sample complexity guarantees of the PTR procedure. Then, we provide the proofs of the comparison of our variance term $f(\mathcal{X}, 1/T)$ with the information gain of (Srinivas et al., 2009) (Lemma 2) and with the effective dimension of (Alaoui & Mahoney, 2015) (Lemma 3) and prove a corollary of Theorem 1 of (Degenne et al., 2020). Last, we complete the details of the experiments.

A. Concentration of RIPS, Proof of Theorem 1

Proof. First note that

$$\begin{aligned}
\max_{v\in\mathcal{V}} \frac{|\langle\widehat{\theta}, v\rangle - \langle\theta_*, v\rangle|}{\|v\|_{A^{(\gamma)}(\lambda)^{-1}}} &= \max_{v\in\mathcal{V}} \frac{|\langle\widehat{\theta}, v\rangle - W^{(v)} + W^{(v)} - \langle\theta_*, v\rangle|}{\|v\|_{A^{(\gamma)}(\lambda)^{-1}}} \\
&\le \max_{v\in\mathcal{V}} \frac{|\langle\widehat{\theta}, v\rangle - W^{(v)}|}{\|v\|_{A^{(\gamma)}(\lambda)^{-1}}} + \max_{v\in\mathcal{V}} \frac{|W^{(v)} - \langle\theta_*, v\rangle|}{\|v\|_{A^{(\gamma)}(\lambda)^{-1}}} \\
&= \min_{\theta}\max_{v\in\mathcal{V}} \frac{|\langle\theta, \phi(v)\rangle - W^{(v)}|}{\|v\|_{A^{(\gamma)}(\lambda)^{-1}}} + \max_{v\in\mathcal{V}} \frac{|W^{(v)} - \langle\theta_*, v\rangle|}{\|v\|_{A^{(\gamma)}(\lambda)^{-1}}} \\
&\le 2\max_{v\in\mathcal{V}} \frac{|\langle\theta_*, v\rangle - W^{(v)}|}{\|v\|_{A^{(\gamma)}(\lambda)^{-1}}}
\end{aligned}$$

which completes the second part of the lemma, so it suffices to show that each $|\langle\theta_*, v\rangle - W^{(v)}|$ is small.

We begin by bounding the variance of $v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_t)y_t$ for any $t \in \mathbb{N}$ which is necessary to use the robust estimator. Note that

$$\mathbb{V}\text{ar}(v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_t)y_t) = \mathbb{E}[(v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_t)y_t)^2] - \mathbb{E}[v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_t)y_t]^2$$

which means we can drop the second term to bound the variance by

$$\begin{aligned}
\mathbb{E}[\Big(v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_t)y_t\Big)^2] &= \mathbb{E}[\Big(v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_t)(\phi(x_t)^\top\theta_* + \eta_t + \xi_t)\Big)^2] \\
&= \mathbb{E}[\Big(v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_t)(\phi(x_t)^\top\theta_* + \eta_t)\Big)^2] + \mathbb{E}[\Big(v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_t)\Big)^2\xi_t^2] \\
&\le B^2\mathbb{E}[\Big(v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_t)\Big)^2] + \sigma^2\mathbb{E}[\Big(v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_t)\Big)^2] \\
&= (B^2+\sigma^2)\mathbb{E}[v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_t)\phi(x_t)^\top A^{(\gamma)}(\lambda)^{-1}v] \\
&\le (B^2+\sigma^2)\|v\|^2_{A^{(\gamma)}(\lambda)^{-1}}.
\end{aligned}$$

Recalling that

$$|\widehat{\mu}(\{v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_t)y_t\}_{t=1}^T) - \mathbb{E}[v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_1)y_1]| \le c_0\sqrt{\mathbb{V}\text{ar}(v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_1)y_1)\frac{\log(\frac{2}{\delta})}{T}}$$

we have

$$\begin{aligned}
|\langle\theta_*, v\rangle - W^{(v)}| &= |\langle\theta_*, v\rangle - \mathbb{E}[v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_1)y_1] + \mathbb{E}[v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_1)y_1] - W^{(v)}| \\
&\le |\langle\theta_*, v\rangle - \mathbb{E}[v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_1)y_1]| + |\widehat{\mu}(\{v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_t)y_t\}_{t=1}^T) - \mathbb{E}[v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_1)y_1]|.
\end{aligned}$$

We now recall that $y_t = \langle\phi(x_t), \theta_*\rangle + \xi_t + \eta_{x_t}$ where ξ_t is a mean-zero, independent random variable with variance σ^2, and $|\eta_{x_t}| \le h$. Thus,

$$\begin{aligned}
|\langle\theta_*, v\rangle - \mathbb{E}[v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_1)y_1]| &= |\langle\theta_*, v\rangle - \mathbb{E}[v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_1)\phi(x_1)^\top\theta_*] - \mathbb{E}[v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_1)\eta_{x_1}]| \\
&\le |\langle\theta_*, v\rangle - \mathbb{E}[v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_1)\phi(x_1)^\top\theta_*]| + |\mathbb{E}[v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_1)\eta_{x_1}]|
\end{aligned}$$

which we bound separately. Firstly,

$$\begin{aligned}|\langle\theta_*, v\rangle - \mathbb{E}[v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_1)\phi(x_1)^\top\theta_*]| &= |\langle\theta_*, v\rangle - v^\top A^{(\gamma)}(\lambda)^{-1}A(\lambda)\theta_*| \\ &= \gamma|v^\top A^{(\gamma)}(\lambda)^{-1}\theta_*| \\ &= \gamma^{1/2}|v^\top(A(\lambda)+\gamma I)^{-1/2}(A(\lambda)/\gamma + I)^{-1/2}\theta_*| \\ &\leq \gamma^{1/2}|v^\top(A(\lambda)+\gamma I)^{-1/2}\theta_*| \\ &\leq \gamma^{1/2}\|v\|_{A^{(\gamma)}(\lambda)^{-1}}\|\theta_*\|\end{aligned}$$

and secondly,

$$\begin{aligned}|\mathbb{E}[v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_1)\eta_{x_1}]| &\leq \mathbb{E}[|v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_1)\eta_{x_1}|] \\ &\leq h\sqrt{\mathbb{E}[|v^\top A^{(\gamma)}(\lambda)^{-1}\phi(x_1)|^2]} \\ &= h\sqrt{v^\top A^{(\gamma)}(\lambda)^{-1}A(\lambda)A^{(\gamma)}(\lambda)^{-1}v} \\ &\leq h\|v\|_{A^{(\gamma)}(\lambda)^{-1}}.\end{aligned}$$

Thus, putting it all together we have

$$|\langle\theta_*, v\rangle - W^{(v)}| \leq (\sqrt{\gamma}\|\theta_*\|_2 + h + c_0\sqrt{(B^2+\sigma^2)\frac{\log(2/\delta)}{T}})\|v\|_{A^{(\gamma)}(\lambda)^{-1}}.$$

Union bounding over all $v \in \mathcal{V}$ completes the proof. □

B. Inverses and bilinear forms, Proof of Lemma 1

Proof of Proposition 1. The following manipulations are well-known, but we include them from completeness. Define

$$\Phi := [\phi(x_1)^\top, \ldots, \phi(x_\tau)^\top]^\top$$

Holds

$$\Phi^\top(\Phi\Phi^\top + \gamma I) = (\Phi^\top\Phi + \gamma I)\Phi^\top\ .$$

And thus

$$(\Phi^\top\Phi + \gamma I)^{-1}\Phi^\top = \Phi^\top(\Phi\Phi^\top + \gamma I)^{-1}\ .$$

Now we use the expansion

$$\left(\Phi^\top\Phi + \gamma I\right)a = \Phi^\top\Phi a + \gamma a$$

to write

$$\begin{aligned}a &= \left(\Phi^\top\Phi + \gamma I\right)^{-1}\left(\Phi^\top\Phi a + \gamma a\right) \\ &= \left(\Phi^\top\Phi + \gamma I\right)^{-1}\Phi^\top\Phi a + \left(\Phi^\top\Phi + \gamma I\right)^{-1}\gamma a \\ &= \Phi^\top\left(\Phi\Phi^\top + \gamma I\right)^{-1}\Phi a + \gamma\left(\Phi^\top\Phi + \gamma I\right)^{-1}a\ .\end{aligned}$$

Then multiplying on the left side by $b^\top$ leads to

$$b^\top a = b^\top\Phi^\top\left(\Phi\Phi^\top + \gamma I\right)^{-1}\Phi a + \gamma b^\top\left(\Phi^\top\Phi + \gamma I\right)^{-1}a\ .$$

So

$$\begin{aligned}b^\top\left(\sum_{x'\in\mathcal{X}}\phi(x')\phi(x')^\top + \gamma I\right)^{-1}a &= \frac{1}{\gamma}b^\top a - \frac{1}{\gamma}b^\top\Phi^\top\left(\Phi\Phi^\top + \gamma I\right)^{-1}\Phi a \\ &= \frac{1}{\gamma}a^\top b - \frac{1}{\gamma}k(a)^\top\left(K + \gamma I\right)^{-1}k(b)\ .\end{aligned}$$

We now simply repeat with the same calculations with

$$\Phi_\lambda := [\sqrt{\lambda_1}\phi(x_1)^\top, \ldots, \sqrt{\lambda_\tau}\phi(x_\tau)^\top]^\top \ ,$$

$$K_\lambda = \Phi_\lambda \Phi_\lambda^\top = \left[\sqrt{\lambda_i}\sqrt{\lambda_j}\phi(x_i)^\top \phi(x_j)\right]_{1\le i,j\le \tau} \ ,$$

and

$$k_\lambda(x) := \Phi_\lambda \phi(x) \in \mathbb{R}^\tau \ .$$

□

C. Guarantees of the PTR procedure, Proof of Theorem 2

We establish the proof in a finite dimension case where ϕ is the identity map and then extend it to any feature map ϕ. In both cases, we fix $\mathcal{X} \subset \mathbb{R}^d$ and consider $\lambda \in \triangle_{\mathcal{X}}$ to be the design we wish to round.

C.1. Finite dimension

Lemma 4. *Let $VDV^\top$ be the eigenvalue decomposition of the matrix $\sum_{x\in\mathcal{X}} \lambda_x x x^\top$, and denote $D = diag(d_1, \ldots, d_d)$. For any $z \in \mathcal{V}$, as long as $\tau = \Omega(k/\epsilon)$, we can find an allocation $\{\widetilde{x}_i\}_{i=1}^\tau \subset \mathcal{X}$ such that*

$$z^\top \left(\sum_{i=1}^\tau \widetilde{x}_i \widetilde{x}_i^\top + \tau\gamma I_d\right)^{-1} z \le \max\{1+\epsilon, 2\} z^\top \left(\tau \sum_{x\in\mathcal{X}} \lambda_x x x^\top + \tau\gamma I_d\right)^{-1} z \ ,$$

where we defined $k = \max\{i : d_i \ge \gamma\}$.

Proof. Start by also denoting $V = [v_1, \ldots, v_d]$. Then

$$\begin{aligned}
z^\top(\tau \sum_{x\in\mathcal{X}} \lambda_x x x^\top + \tau\gamma I)^{-1} z &= z^\top(\tau V D V^\top + \tau\gamma I)^{-1} z \\
&= z^\top(\tau V D V^\top + \tau\gamma V V^\top)^{-1} z \\
&= z^\top V(\tau D + \tau\gamma I)^{-1} V^\top z \\
&= z^\top \left(\sum_{i=1}^d v_i v_i^\top \frac{1}{\tau d_i + \tau\gamma}\right) z
\end{aligned}$$

Now, for any $k = \max\{i : d_i \ge \gamma\}$ we have

$$\begin{aligned}
z^\top \left(\sum_{i=1}^d v_i v_i^\top \frac{1}{\tau d_i + \tau\gamma}\right) z &= z^\top \left(\sum_{i=1}^k v_i v_i^\top \frac{1}{\tau d_i + \tau\gamma}\right) z + z^\top \left(\sum_{i=k+1}^d v_i v_i^\top \frac{1}{\tau d_i + \tau\gamma}\right) z \\
&\ge z^\top \left(\sum_{i=1}^k v_i v_i^\top \frac{1}{\tau d_i + \tau\gamma}\right) z + \frac{1}{2} z^\top \left(\sum_{i=k+1}^d v_i v_i^\top \frac{1}{\tau\gamma}\right) z \\
&= (V_k^\top z)^\top V_k^\top \left(\sum_{i=1}^k v_i v_i^\top \frac{1}{\tau d_i + \tau\gamma}\right) V_k (V_k^\top z) + \frac{1}{2} z^\top \left(\sum_{i=k+1}^d v_i v_i^\top \frac{1}{\tau\gamma}\right) z \\
&= (V_k^\top z)^\top V_k^\top \left(\tau \sum_{x\in\mathcal{X}} \lambda_x x x^\top + \tau\gamma I_d\right)^{-1} V_k (V_k^\top z) + \frac{1}{2} z^\top \left(\sum_{i=k+1}^d v_i v_i^\top \frac{1}{\tau\gamma}\right) z \\
&= (V_k^\top z)^\top \left(\tau \sum_{x\in\mathcal{X}} \lambda_x (V_k^\top x)(V_k^\top x)^\top + \tau\gamma I_k\right)^{-1} (V_k^\top z) + \frac{1}{2} z^\top \left(\sum_{i=k+1}^d v_i v_i^\top \frac{1}{\tau\gamma}\right) z.
\end{aligned}$$

where we denote V_k and V_{-k} as the top k and bottom $d-k$ eigenvectors, respectively. But now we notice that for this first term, we have $\max\{\text{dim}(\text{span}(\{V_k^\top z\}_{z\in\mathcal{V}})), \text{dim}(\text{span}(\{V_k^\top x\}_{x\in\mathcal{X}}))\} \leq k$ which now means that thanks to (Allen-Zhu et al., 2017) we can find an allocation $\{\widetilde{x}_i\}_{i=1}^{\tau} \subset \mathcal{X}$ such that

$$(V_k^\top z)^\top \left(\tau \sum_{x\in\mathcal{X}} \lambda_x (V_k^\top x)(V_k^\top x)^\top + \tau\gamma I_k\right)^{-1} (V_k^\top z) \geq \frac{1}{1+\epsilon} (V_k^\top z)^\top \left(\sum_{i=1}^{\tau} (V_k^\top \widetilde{x})(V_k^\top \widetilde{x})^\top + \tau\gamma I_k\right)^{-1} (V_k^\top z)$$

as long as $\tau = \Omega(k/\epsilon)$. Putting it altogether we have

$$\begin{aligned}
&z^\top \left(\sum_{i=1}^{\tau} \widetilde{x}_i \widetilde{x}_i^\top + \tau\gamma I_d\right)^{-1} z \\
&= (V_k^\top z)^\top \left(\sum_{i=1}^{\tau} (V_k^\top \widetilde{x}_i)(V_k^\top \widetilde{x}_i)^\top + \tau\gamma V_k^\top V_k\right)^{-1} (V_k^\top z) + (V_{-k}^\top z)^\top \left(\sum_{i=1}^{\tau} (V_{-k}^\top \widetilde{x}_i)(V_{-k}^\top \widetilde{x}_i)^\top + \tau\gamma V_{-k}^\top V_{-k}\right)^{-1} (V_{-k}^\top z) \\
&\leq (V_k^\top z)^\top \left(\sum_{i=1}^{\tau} (V_k^\top \widetilde{x}_i)(V_k^\top \widetilde{x}_i)^\top + \tau\gamma I_k\right)^{-1} (V_k^\top z) + (V_{-k}^\top z)^\top \left(\tau\gamma V_{-k}^\top V_{-k}\right)^{-1} (V_{-k}^\top z) \\
&= (V_k^\top z)^\top \left(\sum_{i=1}^{\tau} (V_k^\top \widetilde{x}_i)(V_k^\top \widetilde{x}_i)^\top + \tau\gamma I_k\right)^{-1} (V_k^\top z) + z^\top \left(\sum_{i=k+1}^{d} v_i v_i^\top \frac{1}{\tau\gamma}\right) z \\
&\leq (V_k^\top z)^\top \left(\sum_{i=1}^{\tau} (V_k^\top \widetilde{x}_i)(V_k^\top \widetilde{x}_i)^\top + \tau\gamma I_k\right)^{-1} (V_k^\top z) + 2z^\top \left(\sum_{i=k+1}^{d} v_i v_i^\top \frac{1}{\tau\gamma + \tau d_i}\right) z \\
&\leq (1+\epsilon)(V_k^\top z)^\top \left(\tau \sum_{x\in\mathcal{X}} \lambda_x (V_k^\top x)(V_k^\top x)^\top + \tau\gamma I_k\right)^{-1} (V_k^\top z) + 2z^\top \left(\sum_{i=k+1}^{d} v_i v_i^\top \frac{1}{\tau\gamma + \tau d_i}\right) z \\
&\leq \max\{1+\epsilon, 2\} z^\top \left(\tau \sum_{x\in\mathcal{X}} \lambda_x x x^\top + \tau\gamma I_d\right)^{-1} z.
\end{aligned}$$

□

Oftentimes k can be much smaller than $\min\{d, |\mathcal{X}|\}$, especially for large γ. For instance, for $\mathcal{X} = \mathcal{Z} = \{a\mathbf{e}_1\} \cup \{\mathbf{e}_i : i \in [d]\}$ with $a \gg \gamma = 1$, even as $d \to \infty$ we have that $k = 1$ since λ^* will be the majority of its mass on $\mathbf{e}_1$.

C.2. Connection to kernels

We now get back to our initial setting. Consider K the kernel matrix of $\mathcal{X} = \{x_1, \ldots, x_n\}$. Take $\widetilde{\Phi} \in \mathbb{R}^{n\times n}$ such that $K = \widetilde{\Phi}\widetilde{\Phi}^\top$ (can easily done by diagonalizing K). Consider the rows of $\widetilde{\Phi}$ and name these $\widetilde{\phi}(x_i)$. Then, we have by definition $\phi(x_i)^\top \phi(x_j) = [K]_{ij}$ and we have by construction $\widetilde{\phi}(x_i)^\top \widetilde{\phi}(x_j) = [K]_{ij}$, which importantly leads to $\phi(x_i)^\top \phi(x_j) = \widetilde{\phi}(x_i)^\top \widetilde{\phi}(x_j)$.
Fix $v \in \mathcal{V} \subset \mathcal{X}$. We have from Lemma 1

$$\phi(v)^\top \left(\sum_{i=1}^{\tau} \phi(x_i)\phi(x_i)^\top + \tau\gamma I\right)^{-1} \phi(v) = \phi(v)^\top \phi(v)/(\tau\gamma) - \phi(v)^\top \Phi^\top (\Phi\Phi^\top + \tau\gamma I_n)^{-1} \Phi\phi(v)/(\tau\gamma)\ .$$

This only involves scalar products of the form $\phi(x_i)^\top\phi(x_j)$, such that the property $\phi(x_i)^\top\phi(x_j)=\widetilde{\phi}(x_i)^\top\widetilde{\phi}(x_j)$ allows us to write the variance as

$$\begin{aligned}\phi(v)^\top\left(\sum_{i=1}^{\tau}\phi(x_i)\phi(x_i)^\top+\tau\gamma I\right)^{-1}\phi(v) &= \phi(v)^\top\phi(v)/(\tau\gamma)-\phi(v)^\top\Phi^\top(\Phi\Phi^\top+\tau\gamma I_n)^{-1}\Phi\phi(v)/(\tau\gamma)\\ &= \widetilde{\phi}(v)^\top\widetilde{\phi}(v)/(\tau\gamma)-\widetilde{\phi}^\top\widetilde{\Phi}^\top(\widetilde{\Phi}\widetilde{\Phi}^\top+\tau\gamma I_n)^{-1}\widetilde{\Phi}\widetilde{\phi}(v)/(\tau\gamma)\\ &= \widetilde{\phi}(v)^\top\left(\sum_{i=1}^{\tau}\widetilde{\phi}(x_i)\widetilde{\phi}(x_i)^\top+\tau\gamma I_n\right)^{-1}\widetilde{\phi}(v)\ .\end{aligned}$$

The same trick allows us to write

$$\phi(v)^\top\left(\tau\sum_{x\in\mathcal{X}}\lambda_x\phi(x)\phi(x)^\top+\tau\gamma I\right)^{-1}\phi(v)=\widetilde{\phi}(v)^\top\left(\tau\sum_{x\in\mathcal{X}}\lambda_x\widetilde{\phi}(x)\widetilde{\phi}(x)^\top+\tau\gamma I_n\right)^{-1}\widetilde{\phi}(v)\ .$$

Let $V\Delta V^\top$ be the eigenvalue decomposition of the matrix $\sum_{x\in\mathcal{X}}\lambda_x\widetilde{\phi}(x)\widetilde{\phi}(x)^\top$, and denote $\Delta=\mathrm{diag}(d_1,\ldots,d_n)$. We know from lemma 4 that with $\tau=\Omega(\widetilde{d}(\lambda,\gamma)/\epsilon)$ and $\widetilde{d}(\lambda,\gamma)=\max\{i: d_i\geq\gamma\}$ we can find an allocation $\{\widetilde{x}_i\}_{i=1}^{\tau}\subset\mathcal{X}$ such that

$$\widetilde{\phi}(v)^\top\left(\sum_{i=1}^{\tau}\widetilde{\phi}(\widetilde{x}_i)\widetilde{\phi}(\widetilde{x}_i)+\tau\gamma I_n\right)^{-1}\widetilde{\phi}(v)\leq\max\{1+\epsilon,2\}\widetilde{\phi}(v)^\top\left(\tau\sum_{x\in\mathcal{X}}\lambda_x\widetilde{\phi}(x)\widetilde{\phi}(x)^\top+\tau\gamma I_n\right)^{-1}\widetilde{\phi}(v)\ ,$$

which yields to the following result.
For any $v\in\mathcal{V}\subset\mathcal{X}$, as long as $\tau=\Omega(\widetilde{d}(\lambda,\gamma)/\epsilon)$, we can find an allocation $\{\widetilde{x}_i\}_{i=1}^{\tau}\subset\mathcal{X}$ such that

$$\phi(v)^\top\left(\sum_{i=1}^{\tau}\phi(\widetilde{x}_i)\phi(\widetilde{x}_i)^\top+\tau\gamma I_d\right)^{-1}\phi(v)\leq\max\{1+\epsilon,2\}\phi(v)^\top\left(\tau\sum_{x\in\mathcal{X}}\lambda_x\phi(x)\phi(x)^\top+\tau\gamma I_d\right)^{-1}\phi(v)$$

and $\widetilde{d}(\lambda,\gamma)=\max\{i: d_i\geq\gamma\}$.

And we can take the supremum over $v\in\mathcal{V}$ to get to the result of Theorem 2.

D. Main regret argument, Proof of Theorem 3

In this section, we can consider without loss of generality that ϕ is the identity map. Indeed, the features of the actions - thus denoted x here and $\phi(x)$ in the rest of the paper - appear in this proof only through scalar products.

Define $f(\mathcal{V};\gamma)=\inf_{\lambda\in\triangle_\mathcal{V}}\max_{v\in\mathcal{V}}\|v\|^2_{(\sum_{y\in\mathcal{V}}\lambda_y yy^\top+\gamma I)^{-1}}$ and $\bar{f}(\mathcal{X};\gamma):=\max_{\mathcal{V}\subseteq\mathcal{X}}f(\mathcal{V};\gamma)$.

Define the event

$$\mathcal{E}:=\bigcap_{\ell=1}^{\infty}\bigcap_{x\in\mathcal{X}_\ell}\left\{|\langle x,\widehat{\theta}_\ell-\theta_*\rangle|\leq\epsilon_\ell+(\sqrt{\gamma}\|\theta_*\|_2+h)\sqrt{\bar{f}(\mathcal{X};\gamma)}\right\}$$

Lemma 5. *We have* $\mathbb{P}(\mathcal{E})\geq 1-\delta$.

Proof. For any $\mathcal{V}\subseteq\mathcal{X}$ and $x\in\mathcal{V}$ define

$$\mathcal{E}_{x,\ell}(\mathcal{V})=\{|\langle x,\widehat{\theta}_\ell(\mathcal{V})-\theta_*\rangle|\leq\epsilon_\ell+(\sqrt{\gamma}\|\theta_*\|_2+h)\sqrt{\bar{f}(\mathcal{X};\gamma)}\}$$

where $\widehat{\theta}_\ell(\mathcal{V})$ is the estimator that would be constructed by the algorithm at stage ℓ with $\mathcal{X}_\ell = \mathcal{V}$. For fixed $\mathcal{V} \subset \mathcal{X}$ and $\ell \in \mathbb{N}$ we apply Theorem 1 with $\tau = \tau_\ell$ so that with probability at least $1 - \frac{\delta}{2\ell^2|\mathcal{X}|}$ we have that for any $x \in \mathcal{V}$

$$\begin{aligned}
|\langle x, \widehat{\theta}_\ell(\mathcal{V}) - \theta_* \rangle| &\leq \|x\|_{A^{(\gamma)}(\lambda_\ell)^{-1}} \left(\sqrt{\gamma}\|\theta_*\|_2 + h + c_0\sqrt{(B^2+\sigma^2)\frac{\log(4\ell^2|\mathcal{X}|/\delta)}{\tau_\ell}} \right) \\
&\leq \sqrt{f(\mathcal{V};\gamma)} \left(\sqrt{\gamma}\|\theta_*\|_2 + h + \epsilon_\ell/\sqrt{f(\mathcal{V};\gamma)} \right) \\
&\leq \epsilon_\ell + (\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{\bar{f}(\mathcal{X};\gamma)}
\end{aligned}$$

Noting that $\mathcal{E} := \bigcap_{\ell=1}^\infty \bigcap_{x\in\mathcal{X}_\ell} \mathcal{E}_{x,\ell}(\mathcal{X}_\ell)$ we have

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{\ell=1}^\infty \bigcup_{x\in\mathcal{X}_\ell} \{\mathcal{E}^c_{x,\ell}(\mathcal{X}_\ell)\} \right) &\leq \sum_{\ell=1}^\infty \mathbb{P}\left(\bigcup_{x\in\mathcal{X}_\ell} \{\mathcal{E}^c_{x,\ell}(\mathcal{X}_\ell)\} \right) \\
&= \sum_{\ell=1}^\infty \sum_{\mathcal{V}\subseteq\mathcal{X}} \mathbb{P}\left(\bigcup_{x\in\mathcal{V}} \{\mathcal{E}^c_{x,\ell}(\mathcal{V})\}, \mathcal{X}_\ell = \mathcal{V} \right) \\
&= \sum_{\ell=1}^\infty \sum_{\mathcal{V}\subseteq\mathcal{X}} \mathbb{P}\left(\bigcup_{x\in\mathcal{V}} \{\mathcal{E}^c_{x,\ell}(\mathcal{V})\} \right) \mathbb{P}(\mathcal{X}_\ell = \mathcal{V}) \\
&\leq \sum_{\ell=1}^\infty \sum_{\mathcal{V}\subseteq\mathcal{X}} \tfrac{\delta}{2\ell^2|\mathcal{X}|} |\mathcal{V}| \mathbb{P}(\mathcal{X}_\ell = \mathcal{V}) \\
&\leq \sum_{\ell=1}^\infty \sum_{\mathcal{V}\subseteq\mathcal{X}} \tfrac{\delta}{2\ell^2} \mathbb{P}(\widehat{\mathcal{X}}_\ell = \mathcal{V}) \leq \delta
\end{aligned}$$

□

Lemma 6. *For all* $\ell \in \mathbb{N}$ *we have* $\max_{x\in\mathcal{X}_\ell} \mu_* - \mu_x \leq \max\{16\epsilon_\ell, 32(\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{\bar{f}(\mathcal{X};\gamma)}\}$.

Proof. An arm $x \in \mathcal{X}_\ell$ is discarded (i.e., not in $\mathcal{X}_{\ell+1}$) if $\max_{x'\in\mathcal{X}_\ell}\langle x', \widehat{\theta}\rangle - \langle x, \widehat{\theta}\rangle > 4\epsilon_\ell$. Let $\bar{\ell} := \max\{\ell : \epsilon_\ell > 2(\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{\bar{f}(\mathcal{X};\gamma)}\}$. If $x_* = \arg\max_{x\in\mathcal{X}} \mu_x$ then $x_* \in \mathcal{X}_1$. Now if $x_* \in \mathcal{X}_\ell$ for some $\ell \leq \bar{\ell}$, then for any $x' \in \mathcal{X}_\ell$ we have

$$\begin{aligned}
\langle x', \widehat{\theta}\rangle - \langle x_*, \widehat{\theta}\rangle &\leq \langle x' - x_*, \theta_*\rangle + 2\epsilon_\ell + 2(\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{\bar{f}(\mathcal{X};\gamma)} \\
&\leq \mu_x - \mu_{x_*} + 2h + 2\epsilon_\ell + 2(\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{\bar{f}(\mathcal{X};\gamma)} \\
&\leq 2\epsilon_\ell + 4(\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{\bar{f}(\mathcal{X};\gamma)} \\
&\leq 4\epsilon_\ell
\end{aligned}$$

which implies $x_* \in \mathcal{X}_{\ell+1}$. Moreover, suppose that $\ell \leq \bar{\ell}$ and there exists some $x \in \mathcal{X}_\ell$ such that $\mu_* - \mu_x > 8\epsilon_\ell$, then

$$\begin{aligned}
\max_{x'\in\mathcal{X}_\ell}\langle x', \widehat{\theta}\rangle - \langle x, \widehat{\theta}\rangle &\geq \langle x_*, \widehat{\theta}\rangle - \langle x, \widehat{\theta}\rangle \\
&\geq \langle x_* - x, \theta_*\rangle - 2\epsilon_\ell - 2(\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{\bar{f}(\mathcal{X};\gamma)} \\
&\geq \mu_* - \mu_x - 2h - 2\epsilon_\ell - 2(\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{\bar{f}(\mathcal{X};\gamma)} \\
&\geq \mu_* - \mu_x - 2\epsilon_\ell - 4(\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{\bar{f}(\mathcal{X};\gamma)} \\
&\geq \mu_* - \mu_x - 4\epsilon_\ell \\
&> 4\epsilon_\ell
\end{aligned}$$

which implies $\max_{x\in\mathcal{X}_{\ell+1}} \mu_* - \mu_x \leq 8\epsilon_\ell = 16\epsilon_{\ell+1}$. Because $\mathcal{X}_{\ell+1} \subseteq \mathcal{X}_\ell$ we have for $\ell > \bar{\ell}$ that

$$\begin{aligned}\max_{x\in\mathcal{X}_\ell} \mu_* - \mu_x &\leq \max_{x\in\mathcal{X}_{\bar{\ell}+1}} \mu_* - \mu_x \\ &\leq 16\epsilon_{\bar{\ell}+1} \\ &\leq 32(\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{\bar{f}(\mathcal{X};\gamma)}.\end{aligned}$$

Thus, $\max_{x\in\mathcal{X}_\ell} \mu_* - \mu_x \leq \max\{16\epsilon_\ell, 32(\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{\bar{f}(\mathcal{X};\gamma)}\}$. □

We now compute the final regret bound. After T steps of the algorithm, let T_x denote the number of times arm x is played. Let $\Gamma = (\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{\bar{f}(\mathcal{X};\gamma)}$. If L is the final round reached after T steps, we have

$$\begin{aligned}\sum_{x\in\mathcal{X}} (\mu_* - \mu_x)T_x &\leq \sum_{\ell=1}^{L} \max_{x\in\mathcal{X}_\ell} (\mu_* - \mu_x)\tau_\ell \\ &\leq \sum_{\ell=1}^{L} \tau_\ell \max\{16\epsilon_\ell, 32(\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{\bar{f}(\mathcal{X};\gamma)}\} \\ &\leq \sum_{\ell=1}^{L} \tau_\ell \max\{16\epsilon_\ell, 32\Gamma\} \\ &\leq \sum_{\ell:\epsilon_\ell<2\Gamma} 32\Gamma\tau_\ell + \sum_{\ell:\epsilon_\ell\geq 2\Gamma} \epsilon_\ell\tau_\ell \\ &\leq \sum_{\ell:\epsilon_\ell<2\Gamma} 32\Gamma\tau_\ell + 16\nu T + \sum_{\ell:\epsilon_\ell\geq 2\Gamma\vee\nu} 16\epsilon_\ell\tau_\ell \\ &\leq \sum_{\ell:\epsilon_\ell<2\Gamma} 32\Gamma\tau_\ell + 16\nu T + \sum_{\ell:\epsilon_\ell\geq \nu} 16\epsilon_\ell\tau_\ell \\ &\leq c\left(\Gamma T + \nu T + \sum_{\ell:\epsilon_\ell\geq\nu} \epsilon_\ell \cdot \left(c_0(B^2+\sigma^2)\epsilon_\ell^{-2} f(\mathcal{X}_\ell;\gamma)\log(4\ell^2|\mathcal{X}|/\delta) + c_1\log(|\mathcal{X}|/\delta)\right)\right) \\ &\leq c\left(\Gamma T + \nu T + \cdot \left(c_0(B^2+\sigma^2)\epsilon_\ell^{-2} f(\mathcal{X}_\ell;\gamma)\log(4\ell^2|\mathcal{X}|/\delta) + c_1\log(|\mathcal{X}|/\delta)\right) \sum_{\ell:\epsilon_\ell\geq\nu} \epsilon_\ell^{-1}\right) \\ &\leq c\left(\Gamma T + \nu T + \nu^{-1} c_0(B^2+\sigma^2)\bar{f}(\mathcal{X};\gamma)\log(4\lceil\log_2(1/\nu)\rceil^2|\mathcal{X}|/\delta) + c_1\log(|\mathcal{X}|/\delta)\right).\end{aligned}$$

Choosing $\nu = \sqrt{c_0(B^2+\sigma^2)\bar{f}(\mathcal{X};\gamma)\log(|\mathcal{X}|/\delta)/T}$ and plugging Γ back in yields

$$\sum_{x\in\mathcal{X}} (\mu_* - \mu_x)T_x \leq c'\sqrt{\bar{f}(\mathcal{X};\gamma)}\left(T(\sqrt{\gamma}\|\theta_*\|_2 + h) + \sqrt{(B^2+\sigma^2)\log(|\mathcal{X}|\log(T)/\delta)T}\right) + c_1\log(|\mathcal{X}|/\delta).$$

Choosing $\gamma = 1/T$ yields

$$\sum_{x\in\mathcal{X}} (\mu_* - \mu_x)T_x \leq c'\sqrt{\bar{f}(\mathcal{X};1/T)}\left(hT + \sqrt{(\|\theta_*\|^2 + \max_{x\in\mathcal{X}}\langle x,\theta_*\rangle + \sigma^2)\log(|\mathcal{X}|\log(T)/\delta)T}\right) + c_1\log(|\mathcal{X}|/\delta).$$

E. Main robust pure exploration result, Proof of Theorem 4

For any $\mathcal{V} \subset \mathcal{Z}$ define $f(\mathcal{X},\mathcal{V};\gamma) = \min_{\lambda\in\triangle_\mathcal{X}} \max_{v,v'\in\mathcal{V}} \|v - v'\|^2_{(\sum_{x\in\mathcal{X}}\lambda_x\phi(x)\phi(x)^\top + \gamma I)^{-1}}$

Lemma 7. *Define*

$$\bar{\epsilon} = 8\min\{\epsilon \geq 0 : 4(\sqrt{\gamma}\|\theta_*\|_2 + h)(2 + \sqrt{f(\mathcal{X},\{z\in\mathcal{Z} : \langle\phi(z_*) - \phi(z),\theta_*\rangle \leq \epsilon\};\gamma)}) \leq \epsilon\}.$$

Then $\max_{z\in\mathcal{Z}_\ell} \mu_* - \mu_z \leq 8\max\{\epsilon_\ell, \bar{\epsilon}\}$ *for all* $\ell \geq 0$ *with probability at least* $1-\delta$.

We use Theorem 1 again, with here $\mathcal{V} \subset \mathcal{Z} \subset \mathbb{R}^d$:

$$\max_{v \in \mathcal{V}} \frac{|\langle \theta_*, \phi(v)\rangle - W^{(\phi(v))}|}{\|\phi(v)\|_{(\sum_{x\in\mathcal{X}} \lambda_x \phi(x)\phi(x)^\top + \gamma I)^{-1}}} \leq \sqrt{\gamma}\|\theta_*\|_2 + h + c_0\sqrt{\tfrac{(B^2+\sigma^2)}{T}\log(2|\mathcal{V}|/\delta)}$$

which motivates the choice

$$\tau_\ell = c_0^2 \epsilon_\ell^{-2} f(\mathcal{X}, \mathcal{Z}_\ell; \gamma)(B^2+\sigma^2)\log(2\ell^2|\mathcal{Z}|^2/\delta)$$

Define the event

$$\mathcal{E} := \bigcap_{\ell=1}^{\infty} \bigcap_{z,z' \in \mathcal{Z}_\ell} \left\{ |\langle \phi(z) - \phi(z'), \widehat{\theta}_\ell - \theta_* \rangle| \leq \epsilon_\ell + (\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{f(\mathcal{X}, \mathcal{Z}_\ell; \gamma)} \right\}$$

Lemma 8. *We have* $\mathbb{P}(\mathcal{E}) \geq 1 - \delta$.

Proof. This proof follows the analogous result for regret almost identically. We include it for completeness. For any $\mathcal{V} \subseteq \mathcal{Z}$ and $x \in \mathcal{X}$ define

$$\mathcal{E}_{z,z',\ell}(\mathcal{V}) = \{ |\langle \phi(z) - \phi(z'), \widehat{\theta}_\ell(\mathcal{V}) - \theta_* \rangle| \leq \epsilon_\ell + (\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{f(\mathcal{X}, \mathcal{Z}_\ell; \gamma)} \}$$

where $\widehat{\theta}_\ell(\mathcal{V})$ is the estimator that would be constructed by the algorithm at stage ℓ with $\mathcal{Z}_\ell = \mathcal{V}$. For fixed $\mathcal{V} \subset \mathcal{Z}$ and $\ell \in \mathbb{N}$ we apply Theorem 1 with $T = \tau_\ell$ so that with probability at least $1 - \frac{\delta}{\ell^2|\mathcal{Z}|^2}$ we have that for any $z, z' \in \mathcal{V}$

$$\begin{aligned}
|\langle \phi(z) - \phi(z'), \widehat{\theta}_\ell(\mathcal{V}) - \theta_* \rangle| &\leq \|\phi(z) - \phi(z')\|_{A^{(\gamma)}(\lambda_\ell)^{-1}} (\sqrt{\gamma}\|\theta_*\|_2 + h + c_0\sqrt{(B^2+\sigma^2)\frac{\log(2\ell^2|\mathcal{Z}|^2/\delta)}{\tau_\ell}}) \\
&\leq \sqrt{f(\mathcal{X}, \mathcal{V}; \gamma)} \left(\sqrt{\gamma}\|\theta_*\|_2 + h + \epsilon_\ell / \sqrt{f(\mathcal{X}, \mathcal{V}; \gamma)} \right) \\
&\leq \epsilon_\ell + (\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{f(\mathcal{X}, \mathcal{V}; \gamma)}
\end{aligned}$$

Noting that $\mathcal{E} := \bigcap_{\ell=1}^{\infty} \bigcap_{z,z' \in \mathcal{Z}_\ell} \mathcal{E}_{z,z',\ell}(\mathcal{Z}_\ell)$ we have

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{\ell=1}^{\infty} \bigcup_{z,z' \in \mathcal{Z}_\ell} \{\mathcal{E}^c_{z,z',\ell}(\mathcal{Z}_\ell)\} \right) &\leq \sum_{\ell=1}^{\infty} \mathbb{P}\left(\bigcup_{z,z' \in \mathcal{Z}_\ell} \{\mathcal{E}^c_{z,z',\ell}(\mathcal{Z}_\ell)\} \right) \\
&= \sum_{\ell=1}^{\infty} \sum_{\mathcal{V} \subseteq \mathcal{Z}} \mathbb{P}\left(\bigcup_{z,z' \in \mathcal{V}} \{\mathcal{E}^c_{z,z',\ell}(\mathcal{V})\}, \mathcal{Z}_\ell = \mathcal{V} \right) \\
&= \sum_{\ell=1}^{\infty} \sum_{\mathcal{V} \subseteq \mathcal{Z}} \mathbb{P}\left(\bigcup_{z,z' \in \mathcal{V}} \{\mathcal{E}^c_{z,z',\ell}(\mathcal{V})\} \right) \mathbb{P}(\mathcal{Z}_\ell = \mathcal{V}) \\
&\leq \sum_{\ell=1}^{\infty} \sum_{\mathcal{V} \subseteq \mathcal{Z}} \tfrac{\delta}{\ell^2|\mathcal{Z}|^2} \binom{|\mathcal{V}|}{2} \mathbb{P}(\mathcal{Z}_\ell = \mathcal{V}) \\
&\leq \sum_{\ell=1}^{\infty} \sum_{\mathcal{V} \subseteq \mathcal{Z}} \tfrac{\delta}{2\ell^2} \mathbb{P}(\mathcal{Z}_\ell = \mathcal{V}) \leq \delta
\end{aligned}$$

□

Lemma 9. *Define* $S_1 = \mathcal{Z}$ *and* $S_{\ell+1} = \{z \in S_\ell : \langle \phi(z_*) - \phi(z), \theta_* \rangle \leq 3\epsilon_\ell + (\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{f(\mathcal{X}, S_\ell; \gamma)}\}$. *Define*

$$\bar{\ell} = \max\{\ell : (\sqrt{\gamma}\|\theta_*\|_2 + h)(2 + \sqrt{f(\mathcal{X}, S_\ell; \gamma)}) \leq \epsilon_\ell\}.$$

For all $\ell \in \mathbb{N}$ *we have* $\max_{z \in \mathcal{Z}_\ell} \mu_* - \mu_z \leq 8\max\{\epsilon_\ell, \epsilon_{\bar{\ell}}\}$.

Proof. An arm $z \in \mathcal{Z}_\ell$ is discarded (i.e., not in $\mathcal{Z}_{\ell+1}$) if $\max_{z' \in \mathcal{Z}_\ell} \langle \phi(z') - \phi(z), \widehat{\theta}_\ell \rangle > 2\epsilon_\ell$.

We will show $\{z_* \in \mathcal{Z}_\ell\} \cap \{\mathcal{Z}_\ell \subset S_\ell\} \cap \{\ell \leq \bar{\ell}\} \implies \{z_* \in \mathcal{Z}_{\ell+1}\} \cap \{\mathcal{Z}_{\ell+1} \subset S_{\ell+1}\}$. Noting that $\{z_* \in \mathcal{Z}_\ell\} \cap \{\mathcal{Z}_\ell \subset S_\ell\}$ holds for $\ell = 1$, we will assume an inductive hypothesis of this condition for some $\ell \leq \bar{\ell}$.

First we will show $\{z_* \in \mathcal{Z}_\ell\} \cap \{\mathcal{Z}_\ell \subset S_\ell\} \cap \{\ell \leq \bar{\ell}\} \implies \{z_* \in \mathcal{Z}_{\ell+1}\}$. On $\{z_* \in \mathcal{Z}_\ell\} \cap \{\mathcal{Z}_\ell \subset S_\ell\} \cap \{\ell \leq \bar{\ell}\}$, we have for any $z' \in \mathcal{Z}_\ell$ that

$$\begin{aligned}
\langle \phi(z') - \phi(z_*), \widehat{\theta}_\ell \rangle &\leq \langle \phi(z') - \phi(z_*), \theta_* \rangle + \epsilon_\ell + (\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{f(\mathcal{X}, \mathcal{Z}_\ell; \gamma)} \\
&\leq \mu_{z'} - \mu_{z_*} + 2h + \epsilon_\ell + (\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{f(\mathcal{X}, \mathcal{Z}_\ell; \gamma)} \\
&\leq \epsilon_\ell + (\sqrt{\gamma}\|\theta_*\|_2 + h)(2 + \sqrt{f(\mathcal{X}, \mathcal{Z}_\ell; \gamma)}) \\
&\leq \epsilon_\ell + (\sqrt{\gamma}\|\theta_*\|_2 + h)(2 + \sqrt{f(\mathcal{X}, S_\ell; \gamma)}) \\
&\leq 2\epsilon_\ell
\end{aligned}$$

which implies z_* is not eliminated, that is, $z_* \in \mathcal{Z}_{\ell+1}$. The second-to-last inequality follows from

$$\begin{aligned}
f(\mathcal{X}, \mathcal{Z}_\ell; \gamma) &= \inf_{\lambda \in \triangle_\mathcal{X}} \max_{z, z' \in \mathcal{Z}_\ell} \|\phi(z) - \phi(z')\|^2_{(\sum_{x \in \mathcal{X}} \lambda_x \phi(x)\phi(x)^\top + \gamma I)^{-1}} \\
&\leq \inf_{\lambda \in \triangle_\mathcal{X}} \max_{z, z' \in S_\ell} \|\phi(z) - \phi(z')\|^2_{(\sum_{x \in \mathcal{X}} \lambda_x \phi(x)\phi(x)^\top + \gamma I)^{-1}} \\
&= f(\mathcal{X}, S_\ell; \gamma).
\end{aligned}$$

Now we will show $\{z_* \in \mathcal{Z}_\ell\} \cap \{\mathcal{Z}_\ell \subset S_\ell\} \cap \{\ell \leq \bar{\ell}\} \implies \{\mathcal{Z}_{\ell+1} \subset S_{\ell+1}\}$. For any $z \in \mathcal{Z}_\ell \cap S^c_{\ell+1}$ we have

$$\begin{aligned}
\max_{z' \in \mathcal{Z}_\ell} \langle \phi(z') - \phi(z), \widehat{\theta}_\ell \rangle &\geq \langle \phi(z_*) - \phi(z), \widehat{\theta}_\ell \rangle \\
&\geq \langle \phi(z_*) - \phi(z), \theta_* \rangle - \epsilon_\ell - (\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{f(\mathcal{X}, \mathcal{Z}_\ell; \gamma)} \\
&> 3\epsilon_\ell + (\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{f(\mathcal{X}, S_\ell; \gamma)} - \epsilon_\ell - (\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{f(\mathcal{X}, \mathcal{Z}_\ell; \gamma)} \\
&\geq 2\epsilon_\ell
\end{aligned}$$

which implies $z \notin \mathcal{Z}_{\ell+1}$, and $\mathcal{Z}_{\ell+1} \subset S_{\ell+1}$.

Thus, for $\ell \leq \bar{\ell}$ we have

$$\begin{aligned}
\max_{z \in \mathcal{Z}_\ell} \mu_* - \mu_z &\leq \max_{z \in \mathcal{Z}_\ell} \langle \phi(z_*) - \phi(z), \theta_* \rangle + 2h \\
&\leq \max_{z \in \mathcal{S}_\ell} \langle \phi(z_*) - \phi(z), \theta_* \rangle + 2h \\
&\leq 3\epsilon_{\ell-1} + 2h + (\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{f(\mathcal{X}, S_{\ell-1}; \gamma)} \\
&\leq 3\epsilon_{\ell-1} + (\sqrt{\gamma}\|\theta_*\|_2 + h)(2 + \sqrt{f(\mathcal{X}, S_{\ell-1}; \gamma)}) \\
&\leq 4\epsilon_{\ell-1} = 8\epsilon_\ell.
\end{aligned}$$

And because $\mathcal{Z}_{\ell+1} \subseteq \mathcal{Z}_\ell$ we always have that $\max_{z \in \mathcal{Z}_\ell} \mu_* - \mu_z \leq 8 \max\{\epsilon_\ell, \epsilon_{\bar{\ell}}\}$. Note that

$$\begin{aligned}
\bar{\ell} &= \max\{\ell : (\sqrt{\gamma}\|\theta_*\|_2 + h)(2 + \sqrt{f(\mathcal{X}, S_\ell; \gamma)}) \leq \epsilon_\ell\} \\
&\geq \max\{\ell : (\sqrt{\gamma}\|\theta_*\|_2 + h)(2 + \sqrt{f(\mathcal{X}, \{z \in \mathcal{Z} : \langle \phi(z_*) - \phi(z), \theta_* \rangle \leq 4\epsilon_\ell\}; \gamma)}) \leq \epsilon_\ell\} \\
&= \max\{\ell : 4(\sqrt{\gamma}\|\theta_*\|_2 + h)(2 + \sqrt{f(\mathcal{X}, \{z \in \mathcal{Z} : \langle \phi(z_*) - \phi(z), \theta_* \rangle \leq 4\epsilon_\ell\}; \gamma)}) \leq 4\epsilon_\ell\} \\
&= -2 + \max\{\ell : 4(\sqrt{\gamma}\|\theta_*\|_2 + h)(2 + \sqrt{f(\mathcal{X}, \{z \in \mathcal{Z} : \langle \phi(z_*) - \phi(z), \theta_* \rangle \leq \epsilon_\ell\}; \gamma)}) \leq \epsilon_\ell\} \\
&\geq -3 - \log_2(\min\{\epsilon > 0 : 4(\sqrt{\gamma}\|\theta_*\|_2 + h)(2 + \sqrt{f(\mathcal{X}, \{z \in \mathcal{Z} : \langle \phi(z_*) - \phi(z), \theta_* \rangle \leq \epsilon\}; \gamma)}) \leq \epsilon\})
\end{aligned}$$

which defines $\bar{\epsilon}$. □

The sample complexity to return an $8(\Delta \vee \bar{\epsilon})$-good arm is equal to

$$\sum_{\ell=1}^{\lceil \log_2(8(\Delta\vee\bar{\epsilon})^{-1})\rceil} \tau_\ell = \sum_{\ell=1}^{\lceil \log_2(8(\Delta\vee\bar{\epsilon})^{-1})\rceil} \left\lceil \max\left\{c_1 \log(|\mathcal{Z}|/\delta), c_0 \epsilon_\ell^{-2} f(\mathcal{X}, \mathcal{Z}_\ell; \gamma)(B^2+\sigma^2)\log(2\ell^2|\mathcal{Z}|^2/\delta)\right\}\right\rceil$$

$$\leq c\left(c_1 \log(|\mathcal{Z}|/\delta)\log_2(8(\Delta\vee\bar{\epsilon})^{-1}) + c_0(B^2+\sigma^2)\log(2\lceil\log_2(8(\Delta\vee\bar{\epsilon})^{-1})\rceil^2|\mathcal{Z}|^2/\delta) \sum_{\ell=1}^{\lceil \log_2(8(\Delta\vee\bar{\epsilon})^{-1})\rceil} \epsilon_\ell^{-2} f(\mathcal{X}, \mathcal{Z}_\ell;\gamma)\right)$$

$$\leq c\left(c_1 \log(|\mathcal{Z}|/\delta)\log_2(8(\Delta\vee\bar{\epsilon})^{-1}) + c_0(B^2+\sigma^2)\log(2\lceil\log_2(8(\Delta\vee\bar{\epsilon})^{-1})\rceil^2|\mathcal{Z}|^2/\delta) \sum_{\ell=1}^{\lceil \log_2(8(\Delta\vee\bar{\epsilon})^{-1})\rceil} \epsilon_\ell^{-2} f(\mathcal{X}, S_\ell;\gamma)\right)$$

$$\leq c\left(c_1 \log(|\mathcal{Z}|/\delta)\log_2(8(\Delta\vee\bar{\epsilon})^{-1}) + 16\lceil\log_2(8(\Delta\vee\bar{\epsilon})^{-1})\rceil c_0(B^2+\sigma^2)\log(2\lceil\log_2(8(\Delta\vee\bar{\epsilon})^{-1})\rceil^2|\mathcal{Z}|^2/\delta)\rho^*(\gamma,\bar{\epsilon})\right)$$

where the last line follows from

$$\begin{aligned}
\rho^*(\gamma,\bar{\epsilon}) &= \inf_{\lambda\in\Delta_\mathcal{X}} \sup_{z\in\mathcal{Z}} \frac{\|\phi(z_*)-\phi(z)\|^2_{\left(\sum_{x\in\mathcal{X}}\lambda_x\phi(x)\phi(x)^\top+\gamma I\right)^{-1}}}{\max\{\bar{\epsilon}^2, \langle\phi(z_*)-\phi(z),\theta_*\rangle^2)\}} \\
&= \inf_{\lambda\in\Delta_\mathcal{X}} \sup_{\ell\leq\lceil\log_2(8(\Delta\vee\bar{\epsilon})^{-1})\rceil} \sup_{z\in S_\ell} \frac{\|\phi(z_*)-\phi(z)\|^2_{\left(\sum_{x\in\mathcal{X}}\lambda_x\phi(x)\phi(x)^\top+\gamma I\right)^{-1}}}{\max\{\bar{\epsilon}^2, ((\phi(z_*)-\phi(z))^\top\theta_*)^2\}} \\
&\geq \inf_{\lambda\in\Delta_\mathcal{X}} \frac{1}{\lceil\log_2(8(\Delta\vee\bar{\epsilon})^{-1})\rceil} \sum_{\ell=1}^{\lceil \log_2(8(\Delta\vee\bar{\epsilon})^{-1})\rceil} \sup_{z\in S_\ell} \frac{\|\phi(z_*)-\phi(z)\|^2_{\left(\sum_{x\in\mathcal{X}}\lambda_x\phi(x)\phi(x)^\top+\gamma I\right)^{-1}}}{\max\{\bar{\epsilon}^2, ((\phi(z_*)-\phi(z))^\top\theta_*)^2\}} \\
&\geq \frac{1}{\lceil\log_2(8(\Delta\vee\bar{\epsilon})^{-1})\rceil} \sum_{\ell=1}^{\lceil \log_2(8(\Delta\vee\bar{\epsilon})^{-1})\rceil} \inf_{\lambda\in\Delta_\mathcal{X}} \sup_{z\in S_\ell} \frac{\|\phi(z_*)-\phi(z)\|^2_{\left(\sum_{x\in\mathcal{X}}\lambda_x\phi(x)\phi(x)^\top+\gamma I\right)^{-1}}}{\max\{\bar{\epsilon}^2, ((\phi(z_*)-\phi(z))^\top\theta_*)^2\}} \\
&\geq \frac{1}{4\lceil\log_2(8(\Delta\vee\bar{\epsilon})^{-1})\rceil} \sum_{\ell=1}^{\lceil \log_2(8(\Delta\vee\bar{\epsilon})^{-1})\rceil} 2^{2\ell} \inf_{\lambda\in\Delta_\mathcal{X}} \sup_{z\in S_\ell} \|\phi(z_*)-\phi(z)\|^2_{\left(\sum_{x\in\mathcal{X}}\lambda_x\phi(x)\phi(x)^\top+\gamma I\right)^{-1}} \\
&\geq \frac{1}{16\lceil\log_2(8(\Delta\vee\bar{\epsilon})^{-1})\rceil} \sum_{\ell=1}^{\lceil \log_2(8(\Delta\vee\bar{\epsilon})^{-1})\rceil} 2^{2\ell} \inf_{\lambda\in\Delta_\mathcal{X}} \sup_{z,z'\in S_\ell} \|\phi(z)-\phi(z')\|^2_{\left(\sum_{x\in\mathcal{X}}\lambda_x\phi(x)\phi(x)^\top+\gamma I\right)^{-1}} \\
&= \frac{1}{16\lceil\log_2(8(\Delta\vee\bar{\epsilon})^{-1})\rceil} \sum_{\ell=1}^{\lceil \log_2(8(\Delta\vee\bar{\epsilon})^{-1})\rceil} 2^{2\ell} f(\mathcal{X}, S_\ell;\gamma)\ .
\end{aligned}$$

F. Proofs for the regret bound and the sample complexity of the alternative baseline

Note importantly that in this section the stochastic noise is sub-gaussian.

F.1. Concentration of the sparse estimator

Lemma 10. *Fix a finite set $\mathcal{V}\subset\mathcal{H}$, finite set $\mathcal{X}$, and let $\phi:\mathcal{X}\to\mathcal{H}$. Fix any $x_1,\ldots,x_T$ and assume $y_t = \langle\phi(x_t),\theta_*\rangle + \xi_t+\eta_t$ where each ξ_t is independent, mean-zero, sub-gaussian with parameter σ^2, and each η_t satisfies $|\eta_t|\leq h$. If $\widehat{\theta}$ is the regularized least squares estimator with regularization $\gamma>0$ then*

$$\max_{v\in\mathcal{V}} \frac{|\langle\widehat{\theta},v\rangle-\langle\theta_*,v\rangle|}{\|v\|_{(\sum_{i=1}^T\phi(x_i)\phi(x_i)^\top+\gamma I)^{-1}}} \leq (\sqrt{\gamma}\|\theta_*\|_2 + h\sqrt{T} + \sqrt{2\sigma^2\log(2|\mathcal{V}|/\delta)})$$

with probability at least $1-\delta$.

Proof. Recall first the definition of regularized least squares estimator $\widehat{\theta}$:

$$\widehat{\theta} = G_\gamma^{-1}\Phi^\top Y$$

where $G_\gamma := \sum_{i=1}^T \phi(x_i)\phi(x_i)^\top + \gamma I$. Defining $\xi := (\xi_1, \dots, \xi_T)$ and $\eta := (\eta_1, \dots, \eta_T)$, holds

$$\begin{aligned}
v^\top(\widehat{\theta} - \theta_*) &= v^\top G_\gamma^{-1} \Phi^\top Y - v^\top \theta_* \\
&= v^\top G_\gamma^{-1} \Phi^\top (\Phi\theta_* + \xi + \eta) - v^\top \theta_* \\
&= v^\top G_\gamma^{-1} \Phi^\top \Phi \theta_* - v^\top \theta_* + v^\top G_\gamma^{-1} \Phi^\top \xi + v^\top G_\gamma^{-1} \Phi^\top \eta \; .
\end{aligned}$$

We study each term separately:

$$\begin{aligned}
|v^\top G_\gamma^{-1} \Phi^\top \eta| &\leq \|\eta\|_\infty \|\Phi G_\gamma^{-1} v\|_1 \\
&\leq h\sqrt{T} \|\Phi G_\gamma^{-1} v\|_2 \\
&\leq h\sqrt{T} \|v\|_{G_\gamma^{-1}} \; ,
\end{aligned}$$

with probability at least $1 - \delta$

$$|v^\top G_\gamma^{-1} \Phi^\top \xi| \leq \|v\|_{G_\gamma^{-1}} \sqrt{2\sigma^2 \log\left(\frac{2}{\delta}\right)} \; ,$$

and

$$v^\top G_\gamma^{-1} \Phi^\top \Phi \theta_* - v^\top \theta_* = -\gamma v^\top (\Phi^\top \Phi + \gamma I)^{-1} \theta_*$$

is bounded using Cauchy-Schwarz inequality:

$$\begin{aligned}
&|\gamma v (\Phi^\top \Phi + \gamma I)^{-1} \theta_*| \\
&\leq \gamma \|\theta_*\| \sqrt{v^\top (\Phi^\top \Phi + \gamma I)^{-1} I (\Phi^\top \Phi + \gamma I)^{-1} v} \\
&= \gamma \|\theta_*\| \sqrt{v^\top (\Phi^\top \Phi + \gamma I)^{-1} \gamma^{-1} \gamma I (\Phi^\top \Phi + \gamma I)^{-1} v} \\
&\leq \gamma^{1/2} \|\theta_*\| \sqrt{v^\top (\Phi^\top \Phi + \gamma I)^{-1} (\gamma I + \Phi^\top \Phi)(\Phi^\top \Phi + \gamma I)^{-1} v} \\
&= \gamma^{1/2} \|\theta_*\| \|v\|_{(\Phi^\top \Phi + \gamma I)^{-1}} \\
&= \gamma^{1/2} \|\theta_*\| \|v\|_{G_\gamma^{-1}} \; .
\end{aligned}$$

So

$$\begin{aligned}
|v^\top(\widehat{\theta} - \theta_*)| &\leq \sqrt{\gamma} \|\theta_*\|_2 \|v\|_{G_\gamma^{-1}} + \|v\|_{G_\gamma^{-1}} \sqrt{2\sigma^2 \log\left(\frac{2}{\delta}\right)} + h\sqrt{n} \|v\|_{G_\gamma^{-1}} \\
&= \|v\|_{G_\gamma^{-1}} \left(\sqrt{\gamma} \|\theta_*\|_2 + h\sqrt{T} + \sqrt{2\sigma^2 \log\left(\frac{2}{\delta}\right)} \right) .
\end{aligned}$$

Union bounding over all $v \in \mathcal{V}$ completes the proof. □

F.2. Regret bound

For the same reason as in section D, we can consider without loss of generality that ϕ is the identity map in this section. Indeed, the features of the actions - thus denoted x here and $\phi(x)$ in the rest of the paper - appear in this proof only through scalar products.

Theorem 5. *With probability at least $1 - \delta$, the regret of Algorithm 5 satisfies*

$$\sum_{t=1}^T \mu_x - \mu_{x_t} \leq c \left(\widetilde{d}(\gamma) + \sqrt{\max_{\mathcal{V} \subset \mathcal{X}} f(\mathcal{V}, \gamma)} \left(T(\sqrt{\gamma} \|\theta_*\|_2 + h) + \sqrt{(\sigma^2 \log(|\mathcal{X}| \log(T)/\delta))\, T} \right) \right)$$

where $f(\mathcal{V}, \gamma) = \inf_{\lambda \in \triangle_{\mathcal{V}}} \sup_{y \in \mathcal{V}} \|y\|^2_{(\sum_{x \in \mathcal{X}} \lambda_x x x^\top + \gamma I)^{-1}}$ *and* $\widetilde{d}(\gamma) = \max_{\ell \leq L} \widetilde{d}(\gamma, \lambda_\ell) \leq \max_{\mathcal{V} \subset \mathcal{X}} \sup_{\lambda \in \triangle_{\mathcal{V}}} \widetilde{d}(\gamma, \lambda)$.

Algorithm 5 PTR for Regret minimization

Input: Finite sets $\mathcal{X} \subset \mathbb{R}^d$ ($|\mathcal{X}| = n$), feature map ϕ, confidence level $\delta \in (0,1)$, regularization γ, sub-gaussian parameter σ.
Set $\mathcal{X}_1 \leftarrow \mathcal{X}, \ell \leftarrow 1$
while $|\mathcal{X}_\ell| > 1$ **do**
Let $\lambda_\ell \in \triangle_{\mathcal{X}_\ell}$ be a minimizer of $f(\lambda, \mathcal{X}_\ell, \gamma)$ where

$$f(\mathcal{V};\gamma) = \inf_{\lambda\in\triangle_{\mathcal{V}}} f(\lambda,\mathcal{V},\gamma) = \inf_{\lambda\in\triangle_{\mathcal{V}}} \max_{v\in\mathcal{V}} \|\phi(v)\|^2_{(\sum_{y\in\mathcal{V}}\lambda_y\phi(y)\phi(y)^\top+\gamma I)^{-1}}$$

Set $\epsilon_\ell = 2^{-\ell}$ and $\tau_\ell := \lceil \max\{2\sigma^2\epsilon_\ell^{-2} f(\mathcal{X}_\ell;\gamma)\log(4\ell^2|\mathcal{X}|/\delta), \widetilde{d}(\gamma,\lambda_\ell)\}\rceil$
Use the PTR procedure of section 2.4 to find sparse allocation $\{\widetilde{x}_i\}_{i=1}^{\tau_\ell} \subset \mathcal{X}_\ell$ from λ_ℓ.
Take each action $x \in \{\widetilde{x}_i\}_{i=1}^{\tau_\ell}$ with corresponding features Φ and rewards Y
Compute $\widehat{\theta}_\ell = (\Phi^\top\Phi + \tau_\ell\gamma I)^{-1}\Phi^\top Y$
Update active set:

$$\mathcal{X}_{\ell+1} = \left\{x \in \mathcal{X}_\ell, \max_{x'\in\mathcal{X}_\ell} \langle \phi(x') - \phi(x), \widehat{\theta}_\ell\rangle < 8\epsilon_\ell\right\}$$

$\ell \leftarrow \ell + 1$
end while
Play unique element of $\mathcal{X}_\ell$ indefinitely.

Recall the definition of $f(\mathcal{V};\gamma) = \inf_{\lambda\in\triangle_{\mathcal{V}}}\max_{v\in\mathcal{V}}\|v\|^2_{(\sum_{y\in\mathcal{V}}\lambda_y yy^\top+\gamma I)^{-1}}$ and $\bar{f}(\mathcal{X};\gamma) := \max_{\mathcal{V}\subseteq\mathcal{X}} f(\mathcal{V};\gamma)$. Define the event

$$\mathcal{E} := \bigcap_{\ell=1}^{\infty}\bigcap_{x\in\mathcal{X}_\ell}\left\{|\langle x, \widehat{\theta}_\ell - \theta_*\rangle| \le 2\epsilon_\ell + 2\left(\sqrt{\gamma}\|\theta_*\|_2 + h\right)\sqrt{\bar{f}(\mathcal{X};\gamma)}\right\}$$

Lemma 11. *We have* $\mathbb{P}(\mathcal{E}) \geq 1 - \delta$.

Proof. For any $\mathcal{V} \subseteq \mathcal{X}$ and $x \in \mathcal{V}$ define

$$\mathcal{E}_{\ell,x} = \left\{|x^\top(\widehat{\theta}_\ell(\mathcal{V}) - \theta_*)| \le 2\epsilon_\ell + 2\left(\sqrt{\gamma}\|\theta_*\|_2 + h\right)\sqrt{\bar{f}(\mathcal{X};\gamma)}\right\}$$

where $\widehat{\theta}_\ell(\mathcal{V})$ is the estimator that would be constructed by the algorithm at stage ℓ with $\mathcal{X}_\ell = \mathcal{V}$. For fixed $\mathcal{V} \subset \mathcal{X}$, $\ell \in \mathbb{N}$, τ_ℓ actions are taken. Thus we apply Lemma 10 with $\tau = \tau_\ell$ and with regularization factor $\tau_\ell\gamma$, so that with probability at least $1 - \frac{\delta}{2\ell^2|\mathcal{X}|}$ we have for any $x \in \mathcal{V}$

$$\begin{aligned}
|x^\top(\widehat{\theta}_\ell - \theta_*)| &\le \|x\|_{\left(\sum_{i=1}^{\tau_\ell}\widetilde{x}_i\widetilde{x}_i^\top+\tau_\ell\gamma I\right)^{-1}}\left(\sqrt{\tau_\ell\gamma}\|\theta_*\|_2 + h\sqrt{\tau_\ell} + \sqrt{2\sigma^2\log(4\ell^2|\mathcal{X}|/\delta)}\right)\\
&\le 2\|x\|_{(\tau_\ell A(\lambda_\ell)+\tau_\ell\gamma I)^{-1}}\left(\sqrt{\tau_\ell\gamma}\|\theta_*\|_2 + h\sqrt{\tau_\ell} + \sqrt{2\sigma^2\log(4\ell^2|\mathcal{X}|/\delta)}\right)\\
&= 2\|x\|_{(A(\lambda_\ell)+\gamma I)^{-1}}\left(\sqrt{\gamma}\|\theta_*\|_2 + h + \sqrt{\frac{2\sigma^2\log(4\ell^2|\mathcal{X}|/\delta)}{\tau_\ell}}\right)\\
&\le 2\sqrt{f(\mathcal{V};\gamma)}\left(\sqrt{\gamma}\|\theta_*\|_2 + h + \epsilon_\ell/\sqrt{f(\mathcal{V};\gamma)}\right)\\
&\le 2\epsilon_\ell + 2\sqrt{\bar{f}(\mathcal{X};\gamma)}\,(\sqrt{\gamma}\|\theta_*\|_2 + h)
\end{aligned}$$

Noting that $\mathcal{E} := \bigcap_{\ell=1}^{\infty}\bigcap_{x\in\mathcal{X}_\ell}\mathcal{E}_{x,\ell}(\mathcal{X}_\ell)$, the rest of the proof with the robust estimator applies here. □

The next lemma is similar to the one for the robust estimator, and the proof will follow the same argument as for the robust estimator.

Lemma 12. *For all* $\ell \in \mathbb{N}$ *we have* $\max_{x\in\mathcal{X}_\ell} \mu_* - \mu_x \leq \max\{32\epsilon_\ell, 32(4\sqrt{\gamma}\|\theta_*\|_2 + 6h)\sqrt{\bar{f}(\mathcal{X};\gamma)}\}$.

Proof. An arm $x \in \mathcal{X}_\ell$ is discarded (i.e., not in $\mathcal{X}_{\ell+1}$) if $\max_{x'\in\mathcal{X}_\ell}\langle x', \widehat{\theta}\rangle - \langle x, \widehat{\theta}\rangle > 8\epsilon_\ell$. Let $\bar{\ell} := \max\{\ell : \epsilon_\ell > (4\sqrt{\gamma}\|\theta_*\|_2 + 6h)\sqrt{\bar{f}(\mathcal{X};\gamma)}\}$. If $x_* = \arg\max_{x\in\mathcal{X}} \mu_x$ then $x_* \in \mathcal{X}_1$. Now if $x_* \in \mathcal{X}_\ell$ for some $\ell \leq \bar{\ell}$, then for any $x' \in \mathcal{X}_\ell$ we have

$$\begin{aligned}
\langle x', \widehat{\theta}\rangle - \langle x_*, \widehat{\theta}\rangle &\leq \langle x' - x_*, \theta_*\rangle + 4\epsilon_\ell + 4(\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{\bar{f}(\mathcal{X};\gamma)} \\
&\leq \mu_x - \mu_{x_*} + 2h + 4\epsilon_\ell + 4(\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{\bar{f}(\mathcal{X};\gamma)} \\
&\leq 4\epsilon_\ell + (4\sqrt{\gamma}\|\theta_*\|_2 + 6h)\sqrt{\bar{f}(\mathcal{X};\gamma)} \\
&\leq 8\epsilon_\ell
\end{aligned}$$

which implies $x_* \in \mathcal{X}_{\ell+1}$. Moreover, suppose that $\ell \leq \bar{\ell}$ and there exists some $x \in \mathcal{X}_\ell$ such that $\mu_* - \mu_x > 16\epsilon_\ell$, then

$$\begin{aligned}
\max_{x'\in\mathcal{X}_\ell}\langle x', \widehat{\theta}\rangle - \langle x, \widehat{\theta}\rangle &\geq \langle x_*, \widehat{\theta}\rangle - \langle x, \widehat{\theta}\rangle \\
&\geq \langle x_* - x, \theta_*\rangle - 4\epsilon_\ell - 4(\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{\bar{f}(\mathcal{X};\gamma)} \\
&\geq \mu_* - \mu_x - 2h - 4\epsilon_\ell - 4(\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{\bar{f}(\mathcal{X};\gamma)} \\
&\geq \mu_* - \mu_x - 4\epsilon_\ell - (4\sqrt{\gamma}\|\theta_*\|_2 + 6h)\sqrt{\bar{f}(\mathcal{X};\gamma)} \\
&\geq \mu_* - \mu_x - 8\epsilon_\ell \\
&> 8\epsilon_\ell
\end{aligned}$$

which implies $\max_{x\in\mathcal{X}_{\ell+1}} \mu_* - \mu_x \leq 16\epsilon_\ell = 32\epsilon_{\ell+1}$. Because $\mathcal{X}_{\ell+1} \subseteq \mathcal{X}_\ell$ we have for $\ell > \bar{\ell}$ that

$$\begin{aligned}
\max_{x\in\mathcal{X}_\ell} \mu_* - \mu_x &\leq \max_{x\in\mathcal{X}_{\bar{\ell}+1}} \mu_* - \mu_x \\
&\leq 32\epsilon_{\bar{\ell}+1} \\
&\leq 32(4\sqrt{\gamma}\|\theta_*\|_2 + 6h)\sqrt{\bar{f}(\mathcal{X};\gamma)}.
\end{aligned}$$

Thus, $\max_{x\in\mathcal{X}_\ell} \mu_* - \mu_x \leq \max\{32\epsilon_\ell, 32(4\sqrt{\gamma}\|\theta_*\|_2 + 6h)\sqrt{\bar{f}(\mathcal{X};\gamma)}\}$. □

We now compute the final regret bound. After T steps of the algorithm, let T_x denote the number of times arm x is played.

Let $\Gamma = (4\sqrt{\gamma}\|\theta_*\|_2 + 6h)\sqrt{\bar{f}(\mathcal{X};\gamma)}$. If L is the final round reached after T steps, we have

$$\begin{aligned}
\sum_{x\in\mathcal{X}}(\mu_* - \mu_x)T_x &\le \sum_{\ell=1}^{L}\max_{x\in\mathcal{X}_\ell}(\mu_* - \mu_x)\tau_\ell \\
&\le \sum_{\ell=1}^{L}\tau_\ell \max\{32\epsilon_\ell, 32(4\sqrt{\gamma}\|\theta_*\|_2 + 6h)\sqrt{\bar{f}(\mathcal{X};\gamma)}\} \\
&\le \sum_{\ell=1}^{L}\tau_\ell \max\{32\epsilon_\ell, 32\Gamma\} \\
&\le \sum_{\ell:\epsilon_\ell<\Gamma} 32\Gamma\tau_\ell + 32\sum_{\ell:\epsilon_\ell\ge\Gamma}\epsilon_\ell\tau_\ell \\
&\le \sum_{\ell:\epsilon_\ell<\Gamma} 32\Gamma\tau_\ell + 32\nu T + \sum_{\ell:\epsilon_\ell\ge\Gamma\vee\nu}32\epsilon_\ell\tau_\ell \\
&\le \sum_{\ell:\epsilon_\ell<\Gamma} 32\Gamma\tau_\ell + 32\nu T + \sum_{\ell:\epsilon_\ell\ge\nu}32\epsilon_\ell\tau_\ell \\
&\le c\left(\Gamma T + \nu T + \sum_{\ell:\epsilon_\ell\ge\nu}\epsilon_\ell\cdot\left(2\sigma^2\epsilon_\ell^{-2}f(\mathcal{X}_\ell;\gamma)\log(4\ell^2|\mathcal{X}|/\delta) + \widetilde{d}(\gamma,\lambda_\ell)\right)\right) \\
&\le c\left(\Gamma T + \nu T + \left(2\sigma^2\bar{f}(\mathcal{X};\gamma)\log(4\lceil\log_2(1/\nu)\rceil^2|\mathcal{X}|/\delta) + \widetilde{d}(\gamma)\right)\sum_{\ell:\epsilon_\ell\ge\nu}\epsilon_\ell^{-1}\right) \\
&\le c\left(\Gamma T + \nu T + \nu^{-1}\left(2\sigma^2\bar{f}(\mathcal{X};\gamma)\log(4\lceil\log_2(1/\nu)\rceil^2|\mathcal{X}|/\delta)\right) + 32\widetilde{d}(\gamma)\right).
\end{aligned}$$

Where we denote $\widetilde{d}(\gamma) = \max_{\ell\le L}\widetilde{d}(\gamma,\lambda_\ell)$. Choosing $\nu = \sqrt{\left(2\sigma^2\bar{f}(\mathcal{X};\gamma)\log(|\mathcal{X}|/\delta)\right)/T}$ and plugging Γ back in yields

$$\sum_{x\in\mathcal{X}}(\mu_* - \mu_x)T_x \le c\left(\widetilde{d}(\gamma) + \sqrt{\bar{f}(\mathcal{X};\gamma)}\left(T(\sqrt{\gamma}\|\theta_*\|_2 + h) + \sqrt{(\sigma^2\log(|\mathcal{X}|\log(T)/\delta))\,T}\right)\right).$$

Choosing $\gamma = 1/T$ yields

$$\sum_{x\in\mathcal{X}}(\mu_* - \mu_x)T_x \le c\left(\widetilde{d}(\gamma) + \sqrt{\bar{f}(\mathcal{X};1/T)}\left(hT + \sqrt{((\|\theta_*\|^2 + \sigma^2)\log(|\mathcal{X}|\log(T)/\delta))\,T}\right)\right).$$

F.3. Sample complexity bound

For any $\mathcal{V}\subset\mathcal{Z}$ define $f(\mathcal{X},\mathcal{V};\gamma) = \min_{\lambda\in\triangle_\mathcal{X}}\max_{z,z'\in\mathcal{V}}\|\phi(z)-\phi(z')\|^2_{(\sum_{x\in\mathcal{X}}\lambda_x\phi(x)\phi(x)^\top+\gamma I)^{-1}}$

Theorem 6. *With $z_*\in\arg\max_{z\in\mathcal{Z}}\langle z,\theta_*\rangle$, fix any $\epsilon\ge\bar{\epsilon}$ where*

$$\bar{\epsilon} = 8\min\{\epsilon\ge 0 : 4(\sqrt{\gamma}\|\theta_*\|_2 + h)(2 + \sqrt{f(\mathcal{X},\{z\in\mathcal{Z} : \langle\phi(z_*)-\phi(z),\theta_*\rangle\le\epsilon\};\gamma)})\le\epsilon\}.$$

Then with probability at least $1-\delta$, once the algorithm has taken at least τ samples where

$$\tau \le c'\left(\widetilde{d}(\gamma)\log_2(8(\Delta\vee\bar{\epsilon})^{-1}) + \lceil\log_2(8(\Delta\vee\bar{\epsilon})^{-1})\rceil\sigma^2\log(2\lceil\log_2(8(\Delta\vee\bar{\epsilon})^{-1})\rceil^2|\mathcal{Z}|^2/\delta)\rho^*(\gamma,\bar{\epsilon})\right)$$

we have that $\mu_{\widehat{z}}\ge\max_{z'\in\mathcal{Z}} - \epsilon$ where $\widehat{z}$ is any arm in the set $\mathcal{Z}_\ell$ under consideration after τ pulls and

$$\rho^*(\gamma,\epsilon) = \inf_{\lambda\in\Delta_\mathcal{X}}\sup_{z\in\mathcal{Z}}\frac{\|\phi(z_*)-\phi(z)\|^2_{A^{(\gamma)}(\lambda)^{-1}}}{\max\{\epsilon^2,\langle\theta_*,\phi(z_*)-\phi(z)\rangle^2\}} \tag{10}$$

and $\widetilde{d}(\gamma) = \max_{\ell\le\lceil\log_2(8(\Delta\vee\epsilon)^{-1})\rceil}\widetilde{d}(\gamma,\lambda_\ell)\le\max_{\mathcal{V}\subset\mathcal{X}}\sup_{\lambda\in\triangle_\mathcal{V}}\widetilde{d}(\gamma,\lambda)$.

Algorithm 6 PTR for Pure exploration

Input: Finite sets $\mathcal{X} \subset \mathbb{R}^d$, $\mathcal{Z} \subset \mathbb{R}^d$, feature map ϕ, confidence level $\delta \in (0,1)$, regularization γ, sub-gaussian parameter σ, norm of model parameter B, bound on the misspecification noise h.
Let $\mathcal{Z}_1 \leftarrow \mathcal{Z}, \ell \leftarrow 1$
while $|\mathcal{Z}_\ell| > 1$ **do**
Let $\lambda_\ell \in \triangle_{\mathcal{X}}$ be a minimizer of $f(\lambda, \mathcal{Z}_\ell, \gamma)$ where

$$f(\mathcal{V};\gamma) = \inf_{\lambda \in \triangle_{\mathcal{X}}} f(\lambda, \mathcal{V}, \gamma) = \inf_{\lambda \in \triangle_{\mathcal{X}}} \max_{v,v' \in \mathcal{V}} \|\phi(v) - \phi(v')\|^2_{(\sum_{x\in\mathcal{X}} \lambda_y \phi(x)\phi(x)^\top + \gamma I)^{-1}}$$

Set $\epsilon_\ell = 2^{-\ell}$ and $\tau_\ell := \lceil \max\{2\sigma^2 \epsilon_\ell^{-2} f(\mathcal{Z}_\ell;\gamma)\log(4\ell^2|\mathcal{Z}|/\delta), \widetilde{d}(\gamma, \lambda_\ell)\}\rceil$
Use the PTR procedure of section 2.4 to find sparse allocation $\{\widetilde{x}_i\}_{i=1}^{\tau_\ell} \subset \mathcal{X}_\ell$ from λ_ℓ.
Take each action $x \in \{\widetilde{x}_i\}_{i=1}^{\tau_\ell}$ with corresponding features Φ and rewards Y
Compute $\widehat{\theta}_\ell = (\Phi^\top\Phi + \tau_\ell \gamma I)^{-1}\Phi^\top Y$

$$\mathcal{Z}_{\ell+1} \leftarrow \mathcal{Z}_\ell \setminus \left\{ z \in \mathcal{Z}_\ell : \max_{z' \in \mathcal{Z}_\ell} \langle \phi(z') - \phi(z), \widehat{\theta}_\ell \rangle > \epsilon_\ell \right\}$$

$\ell \leftarrow \ell + 1$
end while
Output: $\mathcal{Z}_\ell$

We first prove the following intermediate result.

Theorem 7. *Recall that we defined*

$$\bar{\epsilon} = 8\min\{\epsilon \geq 0 : 4(\sqrt{\gamma}\|\theta_*\|_2 + h)(2 + \sqrt{f(\mathcal{X}, \{z \in \mathcal{Z} : \langle \phi(z_*) - \phi(z), \theta_* \rangle \leq \epsilon\};\gamma)}) \leq \epsilon\}.$$

Then $\max_{z\in\mathcal{Z}_\ell} \mu_* - \mu_z \leq 16\max\{\epsilon_\ell, \bar{\epsilon}\}$ *for all* $\ell \geq 0$ *with probability at least* $1-\delta$.

Define the event

$$\mathcal{E} := \bigcap_{\ell=1}^{\infty} \bigcap_{z,z' \in \mathcal{Z}_\ell} \left\{ |\langle \phi(z) - \phi(z'), \widehat{\theta}_\ell - \theta_* \rangle| \leq 2\epsilon_\ell + 2\left(\sqrt{\gamma}\|\theta_*\|_2 + h\right)\sqrt{f(\mathcal{X}, \mathcal{Z}_\ell;\gamma)} \right\}$$

Lemma 13. *We have* $\mathbb{P}(\mathcal{E}) \geq 1 - \delta$.

Proof. For any $\mathcal{V} \subseteq \mathcal{Z}$ and $x \in$ define

$$\mathcal{E}_{z,z',\ell}(\mathcal{V}) = \left\{ |\langle \phi(z) - \phi(z'), \widehat{\theta}_\ell(\mathcal{V}) - \theta_* \rangle| \leq 2\epsilon_\ell + 2\left(\sqrt{\gamma}\|\theta_*\|_2 + h\right)\sqrt{f(\mathcal{X}, \mathcal{V};\gamma)} \right\}$$

where $\widehat{\theta}_\ell(\mathcal{V})$ is the estimator that would be constructed by the algorithm at stage ℓ with $\mathcal{Z}_\ell = \mathcal{V}$. For fixed $\mathcal{V} \subset \mathcal{Z}$, $\ell \in \mathbb{N}$ and $x \in \mathcal{V}$, τ_ℓ actions are taken. Thus we apply Lemma 10 with $\tau = \tau_\ell$ and with regularization factor $\tau_\ell\gamma$, so that with probability at least $1 - \frac{\delta}{2\ell^2|\mathcal{Z}|}$ we have for any $z, z' \in \mathcal{V}$

$$\begin{aligned}
&|(\phi(z) - \phi(z'))^\top(\widehat{\theta}_\ell - \theta_*)| \\
&\leq 2\|\phi(z) - \phi(z')\|_{\left(\sum_{i=1}^{\tau_\ell} \phi(\widetilde{x}_i)\phi(\widetilde{x}_i)^\top + \tau_\ell\gamma I\right)^{-1}} \left(\sqrt{\tau_\ell\gamma}\|\theta_*\|_2 + h\sqrt{\tau_\ell} + \sqrt{2\sigma^2\log(4\ell^2|\mathcal{Z}|/\delta)}\right) \\
&\leq 2\|\phi(z) - \phi(z')\|_{(\tau_\ell A(\lambda) + \tau_\ell\gamma I)^{-1}} \left(\sqrt{\tau_\ell\gamma}\|\theta_*\|_2 + h\sqrt{\tau_\ell} + \sqrt{2\sigma^2\log(4\ell^2|\mathcal{Z}|/\delta)}\right) \\
&= 2\|\phi(z) - \phi(z')\|_{(A(\lambda) + \gamma I)^{-1}} \left(\sqrt{\gamma}\|\theta_*\|_2 + h + \sqrt{\frac{2\sigma^2\log(4\ell^2|\mathcal{Z}|/\delta)}{\tau_\ell}}\right) \\
&\leq 2\sqrt{f(\mathcal{X},\mathcal{V};\gamma)}\left(\sqrt{\gamma}\|\theta_*\|_2 + h + \epsilon_\ell/\sqrt{f(\mathcal{X},\mathcal{V};\gamma)}\right) \\
&= 2\epsilon_\ell + 2\sqrt{f(\mathcal{X},\mathcal{V};\gamma)}\left(\sqrt{\gamma}\|\theta_*\|_2 + h\right)
\end{aligned}$$

Noting that $\mathcal{E} := \bigcap_{\ell=1}^{\infty} \bigcap_{z,z' \in \mathcal{Z}_\ell} \mathcal{E}_{z,z',\ell}(\mathcal{Z}_\ell)$, the rest of the proof with the robust estimator applies here. $\square$

Lemma 14. *For all $\ell \in \mathbb{N}$ we have* $\max_{z \in \mathcal{Z}_\ell} \mu_* - \mu_z \leq 16 \max\{\epsilon_\ell, \epsilon_{\bar{\ell}}\}$.

Proof. An arm $z \in \mathcal{Z}_\ell$ is discarded (i.e., not in $\mathcal{Z}_{\ell+1}$) if $\max_{z' \in \mathcal{Z}_\ell} \langle \phi(z') - \phi(z), \widehat{\theta} \rangle > 4\epsilon_\ell$.

Define $S_1 = \mathcal{Z}$ and $S_{\ell+1} = \{z \in S_\ell : \langle \phi(z_*) - \phi(z), \theta_* \rangle \leq 6\epsilon_\ell + 2(\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{f(\mathcal{X}, S_\ell; \gamma)}\}$. Define

$$\bar{\ell} = \max\{\ell : (\sqrt{\gamma}\|\theta_*\|_2 + h)(2 + \sqrt{f(\mathcal{X}, S_\ell; \gamma)}) \leq \epsilon_\ell\}.$$

We will show $\{z_* \in \mathcal{Z}_\ell\} \cap \{\mathcal{Z}_\ell \subset S_\ell\} \cap \{\ell \leq \bar{\ell}\} \implies \{z_* \in \mathcal{Z}_{\ell+1}\} \cap \{\mathcal{Z}_{\ell+1} \subset S_{\ell+1}\}$. Noting that $\{z_* \in \mathcal{Z}_\ell\} \cap \{\mathcal{Z}_\ell \subset S_\ell\}$ holds for $\ell = 1$, we will assume an inductive hypothesis of this condition for some $\ell \leq \bar{\ell}$.

First we will show $\{z_* \in \mathcal{Z}_\ell\} \cap \{\mathcal{Z}_\ell \subset S_\ell\} \cap \{\ell \leq \bar{\ell}\} \implies \{z_* \in \mathcal{Z}_{\ell+1}\}$. On $\{z_* \in \mathcal{Z}_\ell\} \cap \{\mathcal{Z}_\ell \subset S_\ell\} \cap \{\ell \leq \bar{\ell}\}$, we have for any $z' \in \mathcal{Z}_\ell$ that

$$\begin{aligned}
\langle \phi(z') - \phi(z_*), \widehat{\theta} \rangle &\leq \langle \phi(z') - \phi(z_*), \theta_* \rangle + 2\epsilon_\ell + 2(\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{f(\mathcal{X}, \mathcal{Z}_\ell; \gamma)} \\
&\leq \mu_{z'} - \mu_{z_*} + 2h + 2\epsilon_\ell + 2(\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{f(\mathcal{X}, \mathcal{Z}_\ell; \gamma)} \\
&\leq 2\epsilon_\ell + 2(\sqrt{\gamma}\|\theta_*\|_2 + h)(2 + \sqrt{f(\mathcal{X}, \mathcal{Z}_\ell; \gamma)}) \\
&\leq 2\epsilon_\ell + 2(\sqrt{\gamma}\|\theta_*\|_2 + h)(2 + \sqrt{f(\mathcal{X}, S_\ell; \gamma)}) \\
&\leq 4\epsilon_\ell
\end{aligned}$$

which implies z_* is not eliminated, that is, $z_* \in \mathcal{Z}_{\ell+1}$. The second-to-last inequality follows from

$$\begin{aligned}
f(\mathcal{X}, \mathcal{Z}_\ell; \gamma) &= \inf_\lambda \max_{z,z' \in \mathcal{Z}_\ell} \|\phi(z) - \phi(z')\|^2_{(\sum_{x \in \mathcal{X}} \lambda_x \phi(x)\phi(x)^\top + \gamma I)^{-1}} \\
&\leq \inf_\lambda \max_{z,z' \in S_\ell} \|\phi(z) - \phi(z')\|^2_{(\sum_{x \in \mathcal{X}} \lambda_x \phi(x)\phi(x)^\top + \gamma I)^{-1}} \\
&= f(\mathcal{X}, S_\ell; \gamma).
\end{aligned}$$

Now we will show $\{z_* \in \mathcal{Z}_\ell\} \cap \{\mathcal{Z}_\ell \subset S_\ell\} \cap \{\ell \leq \bar{\ell}\} \implies \{\mathcal{Z}_{\ell+1} \subset S_{\ell+1}\}$. For any $z \in \mathcal{Z}_\ell \cap S_{\ell+1}^c$ we have

$$\begin{aligned}
\max_{z' \in \mathcal{Z}_\ell} \langle \phi(z') - \phi(z), \widehat{\theta} \rangle &\geq \langle \phi(z_*) - \phi(z), \widehat{\theta} \rangle \\
&\geq \langle \phi(z_*) - \phi(z), \theta_* \rangle - 2\epsilon_\ell - 2(\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{f(\mathcal{X}, \mathcal{Z}_\ell; \gamma)} \\
&> 6\epsilon_\ell + 2(\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{f(\mathcal{X}, S_\ell; \gamma)} - 2\epsilon_\ell - 2(\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{f(\mathcal{X}, \mathcal{Z}_\ell; \gamma)} \\
&\geq 4\epsilon_\ell
\end{aligned}$$

which implies $z \notin \mathcal{Z}_{\ell+1}$, and $\mathcal{Z}_{\ell+1} \subset S_{\ell+1}$.

Thus, for $\ell \leq \bar{\ell}$ we have

$$\begin{aligned}
\max_{z \in \mathcal{Z}_\ell} \mu_* - \mu_z &\leq \max_{z \in \mathcal{Z}_\ell} \langle \phi(z_*) - \phi(z), \theta_* \rangle + 2h \\
&\leq \max_{z \in \mathcal{S}_\ell} \langle \phi(z_*) - \phi(z), \theta_* \rangle + 2h \\
&\leq 6\epsilon_{\ell-1} + 2h + 2(\sqrt{\gamma}\|\theta_*\|_2 + h)\sqrt{f(\mathcal{X}, S_{\ell-1}; \gamma)} \\
&\leq 6\epsilon_{\ell-1} + 2(\sqrt{\gamma}\|\theta_*\|_2 + h)(2 + \sqrt{f(\mathcal{X}, S_{\ell-1}; \gamma)}) \\
&\leq 8\epsilon_{\ell-1} = 16\epsilon_\ell.
\end{aligned}$$

And because $\mathcal{Z}_{\ell+1} \subseteq \mathcal{Z}_\ell$ we always have that $\max_{z\in\mathcal{Z}_\ell} \mu_* - \mu_z \leq 16\max\{\epsilon_\ell, \epsilon_{\bar\ell}\}$. Note that

$$\begin{aligned}
\bar\ell &= \max\{\ell : (\sqrt{\gamma}\|\theta_*\|_2 + h)(2 + \sqrt{f(\mathcal{X}, S_\ell; \gamma)}) \leq \epsilon_\ell\} \\
&\geq \max\{\ell : (\sqrt{\gamma}\|\theta_*\|_2 + h)(2 + \sqrt{f(\mathcal{X}, \{z \in \mathcal{Z} : \langle \phi(z_*) - \phi(z), \theta_*\rangle \leq 4\epsilon_\ell\}; \gamma)}) \leq \epsilon_\ell\} \\
&= \max\{\ell : 4(\sqrt{\gamma}\|\theta_*\|_2 + h)(2 + \sqrt{f(\mathcal{X}, \{z \in \mathcal{Z} : \langle \phi(z_*) - \phi(z), \theta_*\rangle \leq 4\epsilon_\ell\}; \gamma)}) \leq 4\epsilon_\ell\} \\
&= -2 + \max\{\ell : 4(\sqrt{\gamma}\|\theta_*\|_2 + h)(2 + \sqrt{f(\mathcal{X}, \{z \in \mathcal{Z} : \langle \phi(z_*) - \phi(z), \theta_*\rangle \leq \epsilon_\ell\}; \gamma)}) \leq \epsilon_\ell\} \\
&\geq -3 - \log_2(\min\{\epsilon > 0 : 4(\sqrt{\gamma}\|\theta_*\|_2 + h)(2 + \sqrt{f(\mathcal{X}, \{z \in \mathcal{Z} : \langle \phi(z_*) - \phi(z), \theta_*\rangle \leq \epsilon\}; \gamma)}) \leq \epsilon\})
\end{aligned}$$

which defines $\bar\epsilon$. □

Denoting $\widetilde{d}(\gamma) = \max_{\ell \leq \lceil \log_2(8(\Delta\vee\bar\epsilon)^{-1})\rceil} \widetilde{d}(\gamma, \lambda_\ell)$, the sample complexity to return an $\bar\epsilon$-good arm is equal to

$$\begin{aligned}
&\sum_{\ell=1}^{\lceil \log_2(8(\Delta\vee\bar\epsilon)^{-1})\rceil} \tau_\ell \\
&\leq \sum_{\ell=1}^{\lceil \log_2(8(\Delta\vee\bar\epsilon)^{-1})\rceil} (2\epsilon_\ell^{-2}(\gamma) f(\mathcal{X}, \mathcal{Z}_\ell; \gamma)\sigma^2 \log(2\ell^2|\mathcal{Z}|^2/\delta) + \widetilde{d}(\gamma, \lambda_\ell)) \\
&\leq c\left(\widetilde{d}(\gamma)\log_2(8(\Delta\vee\bar\epsilon)^{-1}) + \sigma^2\log(2\lceil\log_2(8(\Delta\vee\bar\epsilon)^{-1})\rceil^2|\mathcal{Z}|^2/\delta)\sum_{\ell=1}^{\lceil \log_2(8(\Delta\vee\bar\epsilon)^{-1})\rceil} \epsilon_\ell^{-2} f(\mathcal{X}, \mathcal{Z}_\ell; \gamma)\right) \\
&\leq c\left(\widetilde{d}(\gamma)\log_2(8(\Delta\vee\bar\epsilon)^{-1}) + \sigma^2\log(2\lceil\log_2(8(\Delta\vee\bar\epsilon)^{-1})\rceil^2|\mathcal{Z}|^2/\delta)\sum_{\ell=1}^{\lceil \log_2(8(\Delta\vee\bar\epsilon)^{-1})\rceil} \epsilon_\ell^{-2} f(\mathcal{X}, S_\ell; \gamma)\right) \\
&\leq c'\left(\widetilde{d}(\gamma)\log_2(8(\Delta\vee\bar\epsilon)^{-1}) + \lceil\log_2(8(\Delta\vee\bar\epsilon)^{-1})\rceil\sigma^2\log(2\lceil\log_2(8(\Delta\vee\bar\epsilon)^{-1})\rceil^2|\mathcal{Z}|^2/\delta)\rho^*(\gamma, \bar\epsilon)\right)
\end{aligned}$$

where the last line follows from

$$\begin{aligned}
\rho^*(\gamma, \bar\epsilon) &= \inf_{\lambda\in\Delta_\mathcal{X}} \sup_{z\in\mathcal{Z}} \frac{\|\phi(z_*) - \phi(z)\|^2_{\left(\sum_{x\in\mathcal{X}} \lambda_x \phi(x)\phi(x)^\top + \gamma I\right)^{-1}}}{\max\{\bar\epsilon^2, \langle \phi(z_*) - \phi(z), \theta_*\rangle^2)\}} \\
&= \inf_{\lambda\in\Delta_\mathcal{X}} \sup_{\ell \leq \lceil \log_2(8(\Delta\vee\bar\epsilon)^{-1})\rceil} \sup_{z\in S_\ell} \frac{\|\phi(z_*) - \phi(z)\|^2_{\left(\sum_{x\in\mathcal{X}} \lambda_x \phi(x)\phi(x)^\top + \gamma I\right)^{-1}}}{\max\{\bar\epsilon^2, ((\phi(z_*) - \phi(z))^\top\theta_*)^2\}} \\
&\geq \inf_{\lambda\in\Delta_\mathcal{X}} \frac{1}{\lceil\log_2(8(\Delta\vee\bar\epsilon)^{-1})\rceil} \sum_{\ell=1}^{\lceil \log_2(8(\Delta\vee\bar\epsilon)^{-1})\rceil} \sup_{z\in S_\ell} \frac{\|\phi(z_*) - \phi(z)\|^2_{\left(\sum_{x\in\mathcal{X}} \lambda_x \phi(x)\phi(x)^\top + \gamma I\right)^{-1}}}{\max\{\bar\epsilon^2, ((\phi(z_*) - \phi(z))^\top\theta_*)^2\}} \\
&\geq \frac{1}{\lceil\log_2(8(\Delta\vee\bar\epsilon)^{-1})\rceil} \sum_{\ell=1}^{\lceil \log_2(8(\Delta\vee\bar\epsilon)^{-1})\rceil} \inf_{\lambda\in\Delta_\mathcal{X}} \sup_{z\in S_\ell} \frac{\|\phi(z_*) - \phi(z)\|^2_{\left(\sum_{x\in\mathcal{X}} \lambda_x \phi(x)\phi(x)^\top + \gamma I\right)^{-1}}}{\max\{\bar\epsilon^2, ((\phi(z_*) - \phi(z))^\top\theta_*)^2\}} \\
&\geq \frac{1}{4\lceil\log_2(8(\Delta\vee\bar\epsilon)^{-1})\rceil} \sum_{\ell=1}^{\lceil \log_2(8(\Delta\vee\bar\epsilon)^{-1})\rceil} 2^{2\ell} \inf_{\lambda\in\Delta_\mathcal{X}} \sup_{z\in S_\ell} \|\phi(z_*) - \phi(z)\|^2_{\left(\sum_{x\in\mathcal{X}} \lambda_x \phi(x)\phi(x)^\top + \gamma I\right)^{-1}} \\
&\geq \frac{1}{16\lceil\log_2(8(\Delta\vee\bar\epsilon)^{-1})\rceil} \sum_{\ell=1}^{\lceil \log_2(8(\Delta\vee\bar\epsilon)^{-1})\rceil} 2^{2\ell} \inf_{\lambda\in\Delta_\mathcal{X}} \sup_{z,z'\in S_\ell} \|\phi(z) - \phi(z')\|^2_{\left(\sum_{x\in\mathcal{X}} \lambda_x \phi(x)\phi(x)^\top + \gamma I\right)^{-1}} \\
&= \frac{1}{16\lceil\log_2(8(\Delta\vee\bar\epsilon)^{-1})\rceil} \sum_{\ell=1}^{\lceil \log_2(8(\Delta\vee\bar\epsilon)^{-1})\rceil} 2^{2\ell} f(\mathcal{X}, S_\ell; \gamma)\ .
\end{aligned}$$

G. Related work results

Lemma 15. *If* $\lambda^* \in \arg\max_{\lambda \in \triangle_{\mathcal{V}}} f(\lambda)$ *where* $f(\lambda) = \log\left(\det\left(\sum_{x \in \mathcal{X}} \lambda_x \phi(x)\phi(x)^\top + \gamma I\right)\right)$, *then*

$$\begin{aligned}\sup_{x \in \mathcal{X}} \|\phi(x)\|^2_{A^{(\gamma)}(\lambda^*)^{-1}} &= \sum_{x \in \mathcal{X}} \lambda^*_x \|\phi(x)\|^2_{A^{\gamma}(\lambda^*)^{-1}} \\ &= \textit{Trace}\left(A(\lambda^*)(A(\lambda^*) + \gamma I)^{-1}\right) \\ &= \textit{Trace}\left(K_{\lambda^*}(K_{\lambda^*} + \gamma I)^{-1}\right)\end{aligned}$$

Proof. We first state that

$$\begin{aligned}\sum_{x \in \mathcal{X}} \lambda^*_x \|\phi(x)\|^2_{A^{\gamma}(\lambda^*)^{-1}} &= \sum_{x \in \mathcal{X}} \lambda^*_x \phi(x)^\top A^{\gamma}(\lambda^*)^{-1}\phi(x) \\ &= \text{Trace}\left(\sum_{x \in \mathcal{X}} \lambda^*_x \phi(x)^\top A^{\gamma}(\lambda^*)^{-1}\phi(x)\right) \\ &= \text{Trace}\left(\sum_{x \in \mathcal{X}} \lambda^*_x \phi(x)\phi(x)^\top A^{\gamma}(\lambda^*)^{-1}\right) \\ &= \text{Trace}\left(A(\lambda^*)(A(\lambda^*) + \gamma I)^{-1}\right)\end{aligned}$$

This implies

$$\sup_{x \in \mathcal{X}} \|\phi(x)\|^2_{A^{(\gamma)}(\lambda^*)^{-1}} \geq \sum_{x \in \mathcal{X}} \lambda^*_x \|\phi(x)\|^2_{A^{\gamma}(\lambda^*)^{-1}} = \text{Trace}\left(A(\lambda^*)(A(\lambda^*) + \gamma I)^{-1}\right)$$

Further, one can compute that

$$[\nabla_\lambda f(\lambda^*)]_x = \text{Trace}\left(A^{(\gamma)}(\lambda^*)^{-1} x x^\top\right) = \|\phi(x)\|^2_{A^{(\gamma)}(\lambda^*)^{-1}}$$

And last, λ^* satisfies the first order conditions on f: for any $\lambda \in \triangle_{\mathcal{X}}$

$$\begin{aligned}0 &\geq \langle \nabla_\lambda f(\lambda^*), \lambda - \lambda^* \rangle \\ &= \sum_{x \in \mathcal{X}} \lambda_x \|\phi(x)\|^2_{A^{\gamma}(\lambda^*)^{-1}} - \sum_{x \in \mathcal{X}} \lambda^*_x \|\phi(x)\|^2_{A^{\gamma}(\lambda^*)^{-1}} \\ &= \sum_{x \in \mathcal{X}} \lambda_x \|\phi(x)\|^2_{A^{\gamma}(\lambda^*)^{-1}} - \text{Trace}\left(A(\lambda^*)(A(\lambda^*) + \gamma I)^{-1}\right)\end{aligned}$$

Choosing λ to be a Dirac at $\arg\max_{x \in \mathcal{X}} \|\phi(x)\|^2_{A^{\gamma}(\lambda^*)^{-1}}$, we get to

$$\max_{x \in \mathcal{X}} \|\phi(x)\|^2_{A^{\gamma}(\lambda^*)^{-1}} \leq \text{Trace}\left(A(\lambda^*)(A(\lambda^*) + \gamma I)^{-1}\right).$$

Hence the result of the lemma. □

Lemma 16. *We can lower bound* γ_T *the notion of information gain from (Srinivas et al., 2009) as*

$$\gamma_T \geq \frac{2}{3} \max_{\mathcal{V} \subset \mathcal{X}} \inf_{\lambda \in \triangle_{\mathcal{V}}} \sup_{x \in \mathcal{V}} \|\phi(x)\|^2_{\left(\sum_{x \in \mathcal{V}} \lambda_x \phi(x)\phi(x)^\top + \gamma/T I\right)^{-1}} + |\mathcal{X}| \log(\gamma).$$

Proof. Recall the definition of (Srinivas et al., 2009) notion of information gain:

$$\gamma_T := \sup_{\lambda \in \triangle_{\mathcal{X}}} \log(\det(T K_\lambda + \gamma I))$$

where K_λ is defined in Section 2.3. Note that the case where we have an infinite dimensional RKHS and ϕ is any feature map reduces to the finite one with ϕ being the identity map by computing $\Phi_\lambda \in \mathbb{R}^{|\mathcal{X}|\times|\mathcal{X}|}$ such that $K_\lambda = \Phi_\lambda \Phi_\lambda^\top$ and then looking at the (finite dimension) columns of Φ_λ. So we can write without loss of generality

$$\gamma_T = \sup_{\lambda \in \triangle_{\mathcal{X}}} \log\left(\det\left(T\sum_{x\in\mathcal{X}} \lambda_x x x^\top + \gamma I\right)\right)$$

Thus

$$\gamma_T = \sup_{\lambda \in \triangle_{\mathcal{X}}} \log\left(\det\left(T\sum_{x\in\mathcal{X}} \lambda_x x x^\top + \gamma I\right)\right) \geq \sup_{\mathcal{V}\subset\mathcal{X}} \sup_{\lambda \in \triangle_{\mathcal{V}}} \log\left(\det\left(T\sum_{x\in\mathcal{V}} \lambda_x x x^\top + \gamma I\right)\right)$$

Fix for now $\mathcal{V}\subset\mathcal{X}$ and let $\lambda^* \in \triangle_{\mathcal{V}}$ be such that

$$\lambda^* \in \arg\max_{\lambda \in \triangle_{\mathcal{V}}} \log\left(\det\left(T\sum_{x\in\mathcal{V}} \lambda_x x x^\top + \gamma I\right)\right) = \arg\max_{\lambda \in \triangle_{\mathcal{V}}} \log\left(\det\left(\sum_{x\in\mathcal{V}} \lambda_x x x^\top + \gamma/TI\right)\right)$$

Inspired from equation 19.9 of (Lattimore & Szepesvári, 2020), we write for some $x_0 \in \mathcal{V}$

$$\begin{aligned}
&\det\left(T\sum_{x\in\mathcal{V}} \lambda_x^* x x^\top + \gamma I\right)\\
&= \det\left(T\sum_{x\in\mathcal{V}\setminus\{x_0\}} \lambda_x^* x x^\top + \gamma I + T\lambda_{x_0}^* x_0 x_0^\top\right)\\
&= \det\left(T\sum_{x\in\mathcal{V}\setminus\{x_0\}} \lambda_x^* x x^\top + \gamma I\right) \det\left(I + \left(T\sum_{x\in\mathcal{V}\setminus\{x_0\}} \lambda_x^* x x^\top + \gamma I\right)^{-1/2} T\lambda_{x_0}^* x_0 x_0^\top \left(T\sum_{x\in\mathcal{V}\setminus\{x_0\}} \lambda_x^* x x^\top + \gamma I\right)^{-1/2}\right)\\
&= \det\left(T\sum_{x\in\mathcal{V}\setminus\{x_0\}} \lambda_x^* x x^\top + \gamma I\right)\left(1 + T\lambda_{x_0}^* \|x_0\|^2_{\left(T\sum_{x\in\mathcal{V}\setminus\{x_0\}} \lambda_x^* x x^\top + \gamma I\right)^{-1}}\right)\\
&\geq \det\left(T\sum_{x\in\mathcal{V}\setminus\{x_0\}} \lambda_x^* x x^\top + \gamma I\right)\left(1 + T\lambda_{x_0}^* \|x_0\|^2_{\left(T\sum_{x\in\mathcal{V}} \lambda_x^* x x^\top + \gamma I\right)^{-1}}\right)
\end{aligned}$$

We can now iterate with all the remaining $x \in \mathcal{V}\setminus\{x_0\}$, to get

$$\det\left(T\sum_{x\in\mathcal{V}} \lambda_x^* x x^\top + \gamma I\right) \geq \det(\gamma I) \prod_{x_0\in\mathcal{V}} \left(1 + T\lambda_{x_0}^* \|x_0\|^2_{\left(T\sum_{x\in\mathcal{V}} \lambda_x^* x x^\top + \gamma I\right)^{-1}}\right)$$

equivalent to

$$\log\det\left(T\sum_{x\in\mathcal{V}} \lambda_x^* x x^\top + \gamma I\right) \geq \log\det(\gamma I) + \sum_{x\in\mathcal{V}} \log\left(1 + T\lambda_x^* \|x\|^2_{\left(T\sum_{x\in\mathcal{V}} \lambda_x^* x x^\top + \gamma I\right)^{-1}}\right)$$

We know that if $x \geq 0$ holds $\log(1+x) \geq 2x/(2+x)$. Note that $\|x\|^2_{\left(\sum_{x'\in\mathcal{V}} \lambda_{x'} x' x'^\top + \gamma/TI\right)^{-1}} \leq 1$ always holds:

$$\|x\|^2_{\left(\sum_{x'\in\mathcal{V}} \lambda_{x'} x' x'^\top + \gamma/TI\right)^{-1}} = \|x\|^2_{\left(\sum_{x'\in\mathcal{V}\setminus\{x\}} \lambda_{x'} x' x'^\top + \gamma/TI + xx^\top\right)^{-1}} = \alpha - \alpha^2/(1+\alpha) = \alpha/(1+\alpha) \leq 1\ .$$

where we used Sherman–Morrison formula and defined $\alpha = \|x\|^2_{\left(\sum_{x'\in\mathcal{V}\setminus\{x\}} \lambda_{x'} x' x'^\top + \gamma/TI\right)^{-1}}$. Thus holds $0 \leq T\lambda_x^* \|x\|^2_{\left(T\sum_{x'\in\mathcal{V}} \lambda_{x'}^* x' x'^\top + \gamma I\right)^{-1}} \leq 1$. So

$$\sum_{x\in\mathcal{V}} \log\left(1 + T\lambda_x^* \|x\|^2_{\left(T\sum_{x\in\mathcal{V}} \lambda_x^* x x^\top + \gamma I\right)^{-1}}\right) \geq \frac{2}{2+1} \sum_{x\in\mathcal{V}} T\lambda_x^* \|x\|^2_{\left(T\sum_{x\in\mathcal{V}} \lambda_x^* x x^\top + \gamma I\right)^{-1}}$$

And thus

$$\frac{3}{2}\log\left(\frac{\det\left(T\sum_{x\in\mathcal{V}}\lambda_x^* xx^\top+\gamma I\right)}{\det(\gamma I)}\right)\geq\sum_{x\in\mathcal{V}}T\lambda_x^*\|x\|^2_{\left(T\sum_{x\in\mathcal{V}}\lambda_x^* xx^\top+\gamma I\right)^{-1}}$$
$$=\sum_{x\in\mathcal{V}}\lambda_x^*\|x\|^2_{(\sum_{x\in\mathcal{V}}\lambda_x^* xx^\top+\gamma/TI)^{-1}}$$
$$=\sup_{x\in\mathcal{V}}\|x\|^2_{(\sum_{x\in\mathcal{V}}\lambda_x^* xx^\top+\gamma/TI)^{-1}}\ .$$

Where the last equality comes from lemma 15 with $\lambda^*\in\arg\max_{\lambda\in\triangle_{\mathcal{V}}}\log\left(\det\left(\sum_{x\in\mathcal{V}}\lambda_x xx^\top+\gamma/TI\right)\right)$.
So to summarize

$$\begin{aligned}
\gamma_T &:= \sup_{\lambda\in\triangle_{\mathcal{X}}}\log\left(\det\left(T\sum_{x\in\mathcal{X}}\lambda_x\phi(x)\phi(x)^\top+\gamma I\right)\right)\\
&\geq \sup_{\mathcal{V}\subset\mathcal{X}}\sup_{\lambda\in\triangle_{\mathcal{V}}}\log\left(\det\left(T\sum_{x\in\mathcal{V}}\lambda_x\phi(x)\phi(x)^\top+\gamma I\right)\right)\\
&= \sup_{\mathcal{V}\subset\mathcal{X}}\log\left(\frac{\det\left(T\sum_{x\in\mathcal{V}}\lambda_x^*\phi(x)\phi(x)^\top+\gamma I\right)}{\det(\gamma I)}\right)+\log(\det(\gamma I))\\
&\geq \sup_{\mathcal{V}\subset\mathcal{X}}\frac{2}{3}\sup_{x\in\mathcal{V}}\|x\|^2_{(A(\lambda^*)+\gamma/TI)^{-1}}+\log(\det(\gamma I))\\
&\geq \frac{2}{3}\sup_{\mathcal{V}\subset\mathcal{X}}\inf_{\lambda\in\triangle_{\mathcal{V}}}\sup_{x\in\mathcal{X}}\|x\|^2_{(A(\lambda)+\gamma/TI)^{-1}}+d\log(\gamma)
\end{aligned}$$

□

Corollary 1 (Consequence of Theorem 1 of (Degenne et al., 2020))**.** *Let τ_δ be the expected number of sample needed to find the best arm with probability at least $1-\delta$. For any $\theta_*\in\mathcal{E}$ we have*

$$\liminf_{\delta\to 0}\frac{\mathbb{E}_{\theta_*}[\tau_\delta]}{\log(1/\delta)}\geq T^*(\theta_*)$$

where

$$T^{*-1}(\theta_*)=\max_{\lambda\in\Delta_{\mathcal{X}}}\inf_{x'\neq x_*}\sup_{\gamma\geq 0}F(\lambda,x',\gamma,\theta_*)$$

$$F(\lambda,x',\gamma,\theta_*)=\frac{\max\{(x'-x_*)^\top(A(\lambda)+\gamma I)^{-1}A(\lambda)\theta_*,0\}^2}{2\|x'-x_*\|^2_{(A(\lambda)+\gamma I)^{-1}}}+\frac{\gamma}{2}\left(\|\theta_*\|^2_{(A(\lambda)+\gamma I)^{-1}A(\lambda)}-R^2\right)$$

Proof. Recall theorem 1 of (Degenne et al., 2020). For any $\theta_*\in\mathcal{E}$ we have

$$\liminf_{\delta\to 0}\frac{\mathbb{E}_{\theta_*}[\tau_\delta]}{\log(1/\delta)}\geq T^*(\theta_*)$$

where we define the characteristic time through

$$T^{*-1}(\theta_*):=\max_{\lambda\in\Delta_{\mathcal{X}}}\inf_{\theta\in\overline{S}_{x_*}}\|\theta-\theta_*\|^2_{\left(\sum_{x\in\mathcal{X}}\lambda_x xx^\top\right)}$$

with $\overline{S}_{x_*}=\{\theta\in\mathcal{E}\text{ s.t. }\exists x'\neq x_*,\theta^\top(x'-x_*)>0\}$ and with here $\mathcal{E}=\{\theta\in\mathbb{R}^d:\|\theta\|_2^2\leq R^2\}$.

We can now start the proof by writing $T^{*-1}(\theta)$ as

$$T^{*-1}(\theta)=\max_{\lambda\in\Delta_{\mathcal{X}}}\inf_{x'\neq x_*}\inf_{\theta\in\mathcal{E},\theta^\top(x'-x_*)>0}\|\theta-\theta_*\|^2_{\left(\sum_{x\in\mathcal{X}}\lambda_x xx^\top\right)}\ .$$

Then, instead of

$$\inf_{\theta \in \mathcal{E}, \theta^\top (x' - x_*) > 0} \|\theta - \theta_*\|^2_{\left(\sum_{x \in \mathcal{X}} \lambda_x x x^\top\right)}$$

we use $y = x' - x_*$ to write

$$\inf_{\theta \in \mathbb{R}^d, \theta^\top y \geq 0, \|\theta\|_2^2 \leq R^2} \frac{1}{2}\|\theta - \theta_*\|^2_{A(\lambda)} .$$

We introduce the Lagrangian of this convex program

$$L(\theta, \gamma, \nu) = \frac{1}{2}\|\theta - \theta_*\|^2_{A(\lambda)} - \nu(\theta^\top y) + \frac{\gamma}{2}(\|\theta\|_2^2 - R^2) .$$

and solve

$$\inf_{\theta \in \mathbb{R}^d} L(\theta, \gamma, \nu) = \inf_{\theta \in \mathbb{R}^d} \frac{1}{2}\|\theta - \theta_*\|^2_{A(\lambda)} - \nu(\theta^\top y) + \frac{\gamma}{2}(\|\theta\|_2^2 - R^2)$$

$\theta \mapsto L(\theta, \gamma, \nu)$ is differentiable and convex so we compute the gradient

$$\nabla_\theta L(\theta, \gamma, \nu) = (A(\lambda) + \gamma I)\theta - A(\lambda)\theta_* - \nu y$$

and set it to zero to get

$$\widehat{\theta} = \arg\min_{\theta \in \mathbb{R}^d} L(\theta, \gamma, \nu) = (A(\lambda) + \gamma I)^{-1}(A(\lambda)\theta_* + \nu y) = \theta_* + (A(\lambda) + \gamma I)^{-1}(\nu y - \gamma \theta_*)$$

The cross term of both norms has absolute value $\gamma\nu\theta_*(A(\lambda) + \gamma I)^{-1}A(\lambda)y$, and they cancel. So we get

$$\begin{aligned}
L(\widehat{\theta}, \gamma, \nu) &= \frac{1}{2}\|(A(\lambda) + \gamma I)^{-1}(\nu y - \gamma\theta_*)\|^2_{A(\lambda)} - \nu y^\top \left((A(\lambda) + \gamma I)^{-1}(A(\lambda)\theta_* + \nu y)\right) + \frac{\gamma}{2}(\|(A(\lambda) + \gamma I)^{-1}(A(\lambda)\theta_* + \nu y)\|_2^2 - R^2) \\
&= \frac{\gamma^2}{2}\|(A(\lambda) + \gamma I)^{-1}\theta_*\|^2_{A(\lambda)} + \frac{\gamma}{2}\|(A(\lambda) + \gamma I)^{-1}A(\lambda)\theta_*\|^2 - \frac{\gamma}{2}R^2 - \nu y^\top (A(\lambda) + \gamma I)^{-1}A(\lambda)\theta_* \\
&\quad + \frac{\nu^2}{2}\|(A(\lambda) + \gamma I)^{-1}y\|^2_{A(\lambda)} - \nu^2 y^\top (A(\lambda) + \gamma I)^{-1}y + \frac{\nu^2\gamma}{2}\|(A(\lambda) + \gamma I)^{-1}y\|^2 \\
&= \frac{\gamma}{2}\left(\|\theta_*\|^2_{(A(\lambda)+\gamma I)^{-1}A(\lambda)} - R^2\right) - \nu y^\top (A(\lambda) + \gamma I)^{-1}A(\lambda)\theta_* - \frac{\nu^2}{2}\|y\|^2_{(A(\lambda)+\gamma I)^{-1}}
\end{aligned}$$

so

$$\sup_{\nu \geq 0} L(\widehat{\theta}, \gamma, \nu) = \frac{\gamma}{2}\left(\|\theta_*\|^2_{(A(\lambda)+\gamma I)^{-1}A(\lambda)} - R^2\right) + \frac{(\max\{y^\top (A(\lambda) + \gamma I)^{-1}A(\lambda)\theta_*, 0\})^2}{2\|y\|^2_{(A(\lambda)+\gamma I)^{-1}}}$$

Conclusion:

$$\inf_{\theta \in \mathbb{R}^d, \theta^\top y \geq 0, \|\theta\|_2^2 \leq R^2} \frac{1}{2}\|\theta - \theta_*\|^2_{A(\lambda)} = \sup_{\gamma \geq 0}\left\{\frac{(\max\{y^\top (A(\lambda) + \gamma I)^{-1}A(\lambda)\theta_*, 0\})^2}{2\|y\|^2_{(A(\lambda)+\gamma I)^{-1}}} + \frac{\gamma}{2}\left(\|\theta_*\|^2_{(A(\lambda)+\gamma I)^{-1}A(\lambda)} - R^2\right)\right\}$$

Then for any $\theta_* \in \mathcal{E}$ we have

$$\begin{aligned}
\liminf_{\delta \to 0} \frac{\mathbb{E}_{\theta_*}[\tau_\delta]}{\log(1/\delta)} \geq T^*(\theta_*) &= \frac{1}{\max_{\lambda \in \Delta_{\mathcal{X}}} \inf_{x' \neq x_*} \sup_{\gamma \geq 0}\left\{\frac{(\max\{(x'-x_*)^\top (A(\lambda)+\gamma I)^{-1}A(\lambda)\theta_*, 0\})^2}{2\|x'-x_*\|^2_{(A(\lambda)+\gamma I)^{-1}}} + \frac{\gamma}{2}\left(\|\theta_*\|^2_{(A(\lambda)+\gamma I)^{-1}A(\lambda)} - R^2\right)\right\}} \\
&= \inf_{\lambda \in \Delta_{\mathcal{X}}} \sup_{x' \neq x_*} \inf_{\gamma \geq 0} \frac{1}{F(\lambda, x', \gamma, \theta_*)}
\end{aligned}$$

With

$$F(\lambda, x', \gamma, \theta_*) := \frac{\max\{(x' - x_*)^\top (A(\lambda) + \gamma I)^{-1}A(\lambda)\theta_*, 0\}^2}{2\|x' - x_*\|^2_{(A(\lambda)+\gamma I)^{-1}}} + \frac{\gamma}{2}\left(\|\theta_*\|^2_{(A(\lambda)+\gamma I)^{-1}A(\lambda)} - R^2\right)$$

□

Note that this result and its proof can be written in the case where ϕ is any feature map without any changes.

H. Experiments details

We briefly provide some additional details on the experiments. We used Python 3 and parallelized the simulations on a 2.9 GHz Intel Core i7. We computed the designs in each of the three experiments using mirror descent. We repeated the G-optimal design experiment 16 times, the kernels experiment 40 times, and the IPS vs. RIPS experiment 16 times. The G-optimal design experiment and the IPS vs. RIPS experiment used noise $\eta \sim N(0, 1)$ while in the kernels experiment used noise $\eta \sim N(0, 0.05)$. The confidence bounds in our plots are based on standard errors.