A. Missing Proofs from Section 2

Proof of Theorem 2.4. Let $\gamma = a/b$ where $a, b \in \mathbb{Z}^+$. Set n to be any integer multiple of bk such that $n \geqslant n_0$. Let $\hat{l} = \mathrm{argmin}_{l\in[\ell]} \min\{\alpha_l, 1 - \sum_{j\neq l}\beta_j\}$. Consider a true ranking where all the n items from group $\hat{l}$ are placed in top n ranks followed by the items from all the other groups. Observe that by our choice of parameters, γn is an integer. Let $\gamma n = ck$ for some $c \in \mathbb{Z}^+$. Now, consider any ranking of these items satisfying γ underranking in the top γn ranks and $(\boldsymbol{\alpha}, \boldsymbol{\beta}, k)$ group fairness in the top $\frac{\gamma n}{k}$ blocks of size k. By the definition of underranking, we get that the top γn ranks must contain all the n items from group $\hat{l}$. Since the ranking satisfies all the upper bound constraints, any of the top c blocks must have at most $\alpha_{\hat{l}}k$ items from group $\hat{l}$. Similarly, since the ranking also satisfies all the lower bounds, any of the top c blocks must have at least $\beta_l k$ items from group l, $\forall l \in [\ell]$. Hence, there could be at most $\left(1 - \sum_{l\neq\hat{l}}\beta_l\right)\cdot k$ items from group $\hat{l}$. This implies that, the top ck ranks have at most $\left(\min\left\{\alpha_{\hat{l}}, 1 - \sum_{l\neq\hat{l}}\beta_l\right\}\right)\cdot ck$ items from group $\hat{l}$. Therefore, the top $ck = \gamma n$ ranks contain at most $\left(\min\left\{\alpha_{\hat{l}}, 1 - \sum_{l\neq\hat{l}}\beta_l\right\}\right)\cdot\gamma n$ items from group $\hat{l}$. If $\gamma < \frac{1}{\min\left\{\alpha_{\hat{l}}, 1-\sum_{l\neq\hat{l}}\beta_l\right\}}$, then top γn ranks contain strictly less than n elements from group $\hat{l}$, which is a contradiction. Therefore, we must have $\gamma \geqslant \frac{1}{\min_{l\in[\ell]}\min\{\alpha_l, 1-\sum_{j\neq l}\beta_j\}} = \frac{1}{\min\{\alpha_{\min}, 1-\sum_{l\neq l_*}\beta_l\}}$, where $\alpha_{\min} = \min_l \alpha_l$ and $l_* = \mathrm{argmin}_l \beta_l$. □

Before proving the underranking and group fairness guarantess of Algorithm 1, we prove the following useful lemma,

Lemma A.1. *In a block of size $\lfloor \epsilon k/2\rfloor$, the following always hold,*

1. *If the block contains any empty ranks, then there is at least one group that can be assinged to this rank without violating the upper bound constraint.*
2. *If $\forall l \in [\ell]$ there are $\lceil \beta_l \lfloor \epsilon k/2\rfloor\rceil$ items available from the group l, then we can always assign $\lceil \beta_l \lfloor \epsilon k/2\rfloor\rceil$ ranks to group l, $\forall l \in [\ell]$, such that all the lower bound constraints in the block are satisfied.*
3. *For each group $l \in [\ell]$, the upper bound on the number of ranks to be assigned to the group is always greater than the lower bound.*

Proof. By our choice of parameters, we have,

$$\epsilon \geqslant \frac{2}{k}\cdot\max\left\{\left\lfloor 1 + \frac{\ell}{\sum_{l\in[\ell]}\alpha_l - 1}\right\rfloor, \left\lfloor 1 + \frac{\ell}{1 - \sum_{l\in[\ell]}\beta_l}\right\rfloor, \max_{l\in[\ell]}\left\lfloor 1 + \frac{2}{\alpha_l - \beta_l}\right\rfloor\right\} \geqslant \frac{2}{k}.$$

Therefore, the size of each block $\lfloor \epsilon k/2\rfloor$ is at least 1. Moreover,

$$\begin{aligned}
&\epsilon \geqslant \frac{2}{k}\left(1 + \frac{\ell}{\sum_{l\in[\ell]}\alpha_l - 1}\right)\\
\implies &(\epsilon k/2 - 1) \geqslant \frac{\ell}{\sum_{l\in[\ell]}\alpha_l - 1}\\
\implies &\lfloor \epsilon k/2\rfloor > \frac{\ell}{\sum_{l\in[\ell]}\alpha_l - 1}\\
\implies &\sum_{l\in[\ell]}\left(\alpha_l\lfloor \epsilon k/2\rfloor - 1\right) > \lfloor \epsilon k/2\rfloor\\
\implies &\sum_{l\in[\ell]}\lfloor \alpha_l\lfloor \epsilon k/2\rfloor\rfloor > \lfloor \epsilon k/2\rfloor.
\end{aligned}$$

Therefore, if the block has an empty rank, there is at least one group that can be assinged to this empty rank without violating the upper bound constraints. Hence, the statement 1 is true. We also have,

$$\begin{aligned}
&\epsilon \geqslant \frac{2}{k}\left(1 + \frac{\ell}{1 - \sum_{l\in[\ell]}\beta_l}\right)\\
\implies &\epsilon k/2 - 1 \geqslant \frac{\ell}{1 - \sum_{l\in[\ell]}\beta_l}\\
\implies &\lfloor \epsilon k/2\rfloor > \frac{\ell}{1 - \sum_{l\in[\ell]}\beta_l}\\
\implies &\lfloor \epsilon k/2\rfloor\left(1 - \sum_{l\in[\ell]}\beta_l\right) > \ell \qquad (1)\\
\implies &\sum_{l\in[\ell]}\left(\beta_l\lfloor \epsilon k/2\rfloor + 1\right) < \lfloor \epsilon k/2\rfloor\\
\implies &\sum_{l\in[\ell]}\lceil \beta_l\lfloor \epsilon k/2\rfloor\rceil < \lfloor \epsilon k/2\rfloor.
\end{aligned}$$

Hence, if $\forall l \in [\ell]$, there are $\lceil \beta_l \lfloor \epsilon k/2\rfloor\rceil$ items available from the group l, we can assign $\lceil \beta_l \lfloor \epsilon k/2\rfloor\rceil$ ranks in the block to group l since the sum of the minimum number of ranks needed to be assigned to satisfy the lower bound constraints is strictly less than the number of empty ranks in the block. Therefore statement 2 is also true. Finally, for all $l \in [\ell]$,

$$\begin{aligned}
&\epsilon \geqslant \frac{2}{k}\left(1+\frac{2}{\alpha_l-\beta_l}\right)\\
\implies &\epsilon k/2 - 1 \geqslant \frac{2}{\alpha_l-\beta_l}\\
\implies &\lfloor \epsilon k/2 \rfloor > \frac{2}{\alpha_l-\beta_l}\\
\implies &\beta_l \lfloor \epsilon k/2 \rfloor + 1 < \alpha_l \lfloor \epsilon k/2 \rfloor - 1 \qquad (2)\\
\implies &\beta_l \lfloor \epsilon k/2 \rfloor + 1 < \lfloor \alpha_l \lfloor \epsilon k/2 \rfloor \rfloor\\
\implies &\lceil \beta_l \lfloor \epsilon k/2 \rfloor \rceil < \lfloor \alpha_l \lfloor \epsilon k/2 \rfloor \rfloor .
\end{aligned}$$

Hence, the statement 3 is true. □

Throughout the results shown below, let $\alpha_{\text{max}} = \max_l \alpha_l$, $\alpha_{\text{min}} = \min_l \alpha_l$, $l_* = \text{argmin}_l \beta_l$,

Lemma A.2. *The underranking of the ranking output by Algorithm 1 is*

$$\gamma = \frac{1}{\min\left\{\alpha_{min} - \frac{1}{\lfloor \epsilon k/2 \rfloor}, \left(1 - \sum_{l \neq l_*} \beta_l\right) - \frac{\ell - 1}{\lfloor \epsilon k/2 \rfloor}\right\}}.$$

Proof. Fix an item having true rank $j \in [N]$.

Case 1: $\lfloor \alpha_{\text{min}} \lfloor \epsilon k/2 \rfloor \rfloor \leqslant \lfloor \epsilon k/2 \rfloor - \sum_{l \neq l_*} \lceil \beta_l \lfloor \epsilon k/2 \rfloor \rceil$.

Let $\eta = \lfloor \alpha_{\text{min}} \lfloor \epsilon k/2 \rfloor \rfloor$. At the end of step 8, its rank is

$$\begin{aligned}
&\left(\left\lceil \frac{j}{\eta} \right\rceil - 1\right)\left\lfloor \frac{\epsilon k}{2} \right\rfloor + \left(j - \left(\left\lceil \frac{j}{\eta} \right\rceil - 1\right)\eta\right)\\
&\leqslant \frac{j}{\eta}\left(\left\lfloor \frac{\epsilon k}{2} \right\rfloor - \eta\right) + j = j\frac{\left\lfloor \frac{\epsilon k}{2} \right\rfloor}{\eta}\\
&= j\frac{\lfloor \epsilon k/2 \rfloor}{\lfloor \alpha_{\text{min}} \lfloor \epsilon k/2 \rfloor \rfloor} \qquad (3)\\
&< \frac{j}{\frac{\alpha_{\text{min}} \lfloor \epsilon k/2 \rfloor - 1}{\lfloor \epsilon k/2 \rfloor}} = \frac{j}{\alpha_{\text{min}} - \frac{1}{\lfloor \epsilon k/2 \rfloor}}.
\end{aligned}$$

From Equation (2) we have,

$$\begin{aligned}
&\alpha_l \lfloor \epsilon k/2 \rfloor - 1 > \beta_l \lfloor \epsilon k/2 \rfloor + 1\\
\implies &\alpha_l - \frac{1}{\lfloor \epsilon k/2 \rfloor} > \beta_l + \frac{1}{\lfloor \epsilon k/2 \rfloor} > 0.
\end{aligned}$$

Since this holds true even for the group corresponding to α_{min}, the underranking is positive.

Case 2: $\lfloor \alpha_{\text{min}} \lfloor \epsilon k/2 \rfloor \rfloor > \lfloor \epsilon k/2 \rfloor - \sum_{l \neq l_*} \lceil \beta_l \lfloor \epsilon k/2 \rfloor \rceil$.

Let $\eta = \lfloor \epsilon k/2 \rfloor - \sum_{l \neq l_*} \lceil \beta_l \lfloor \epsilon k/2 \rfloor \rceil$. At the end of step 8, similar to case 1, its rank is

$$\begin{aligned}
&\left(\left\lceil \frac{j}{\eta} \right\rceil - 1\right)\left\lfloor \frac{\epsilon k}{2} \right\rfloor + \left(j - \left(\left\lceil \frac{j}{\eta} \right\rceil - 1\right)\eta\right)\\
&\leqslant \frac{j}{\eta}\left(\left\lfloor \frac{\epsilon k}{2} \right\rfloor - \eta\right) + j = j\frac{\left\lfloor \frac{\epsilon k}{2} \right\rfloor}{\eta}\\
&= j\frac{\left\lfloor \frac{\epsilon k}{2} \right\rfloor}{\left\lfloor \frac{\epsilon k}{2} \right\rfloor - \sum_{l \neq l_*} \left\lceil \beta_l \left\lfloor \frac{\epsilon k}{2} \right\rfloor \right\rceil} \qquad (4)\\
&\leqslant j\frac{\left\lfloor \frac{\epsilon k}{2} \right\rfloor}{\left\lfloor \frac{\epsilon k}{2} \right\rfloor - \sum_{l \neq l_*} \left(\beta_l \left\lfloor \frac{\epsilon k}{2} \right\rfloor + 1\right)}\\
&= \frac{j}{\left(1 - \sum_{l \neq l_*} \beta_l\right) - \frac{\ell - 1}{\lfloor \epsilon k/2 \rfloor}}.
\end{aligned}$$

From Equation (1) we have,

$$\begin{aligned}
&\lfloor \epsilon k/2 \rfloor \left(1 - \sum_{l \in [\ell]} \beta_l\right) > \ell\\
\implies &\left(1 - \sum_{l \in [\ell]} \beta_l\right) - \frac{\ell - 1}{\lfloor \epsilon k/2 \rfloor} > \frac{1}{\lfloor \epsilon k/2 \rfloor}\\
\implies &\left(1 - \sum_{l \neq l_*} \beta_l\right) - \frac{\ell - 1}{\lfloor \epsilon k/2 \rfloor} > \beta_{l_*} + \frac{1}{\lfloor \epsilon k/2 \rfloor} > 0.
\end{aligned}$$

Therefore, the underranking in this case is also positive. □

Lemma A.3. *At the end of step 20, none of the top* $\left\lfloor \frac{n}{\alpha_{max}} \right\rfloor - \lfloor \epsilon k/2 \rfloor$ *ranks will be empty.*

Proof. Consider step 20 of the algorithm. A rank j will be left unassigned if either (i) fairness constraints for each group are satisfied and assigning it to any item will only violate the fairness constraints, or (ii) there is no item ranked higher than j that can be assigned the rank j and still satisfies the fairness constraints.

Let i be the block the rank j belongs to, $i = \lceil j / \lfloor \epsilon k/2 \rfloor \rceil$. From statement 2 in Lemma A.1 we know that, if the ranks in block i are assigned such that $\forall l \in [\ell]$, $\lceil \beta_l \lfloor \epsilon k/2 \rfloor \rceil$ ranks are assigned to group l, there could still be empty ranks in the block i. Also from statement 3 in Lemma A.1 we know that for each group l, the number of ranks to be assigned to in order to satisfy the upper bound constraints is always greater than the number of ranks to be assigned to in order to satisfy the lower bound constraints. Therefore we can still add items to the block i until the upper bound constraints are not violated. Now, from statement 1 of Lemma A.1 we also know that if there are empty ranks in the block, then there is at least one group which can be assinged to the empty rank and not violate the upper bound constraints. Hence, case (i) can not happen.

We know that $\lfloor \alpha_{\max} \lfloor \epsilon k/2 \rfloor \rfloor \geqslant \lfloor \alpha_l \lfloor \epsilon k/2 \rfloor \rfloor$ by definition and $\lfloor \alpha_{\max} \lfloor \epsilon k/2 \rfloor \rfloor \geqslant \lceil \beta_l \lfloor \epsilon k/2 \rfloor \rceil$ by Lemma A.1 for any l. Since there are at least n items from each group, we have that as long as i satisfies $i \lfloor \alpha_{\max} \lfloor \epsilon k/2 \rfloor \rfloor \leqslant n$, there will be at least one item available from each group to move into an empty rank in the top i blocks without violating the fairness constraints for any of the top i blocks. Thus, the top i blocks will be filled at the end of step 20. Therefore, the number of ranks filled is at least

$$
\begin{aligned}
i \lfloor \epsilon k/2 \rfloor &= \left\lfloor \frac{n}{\lfloor \alpha_{\max} \lfloor \epsilon k/2 \rfloor \rfloor} \right\rfloor \lfloor \epsilon k/2 \rfloor \\
&> \left(\frac{n}{\lfloor \alpha_{\max} \lfloor \epsilon k/2 \rfloor \rfloor} - 1 \right) \lfloor \epsilon k/2 \rfloor \\
&\geqslant \left(\frac{n}{\alpha_{\max} \lfloor \epsilon k/2 \rfloor} - 1 \right) \lfloor \epsilon k/2 \rfloor \\
&= \frac{n}{\alpha_{\max}} - \lfloor \epsilon k/2 \rfloor \geqslant \left\lfloor \frac{n}{\alpha_{\max}} \right\rfloor - \lfloor \epsilon k/2 \rfloor .
\end{aligned}
$$

Therefore, case (ii) will not happen for the top $\left\lfloor \frac{n}{\alpha_{\max}} \right\rfloor - \lfloor \epsilon k/2 \rfloor$ ranks. Thus, at the end of step 20, none of the top $\left\lfloor \frac{n}{\alpha_{\max}} \right\rfloor - \lfloor \epsilon k/2 \rfloor$ ranks will be empty. □

Lemma A.4. *At the end of step 20, in each of the top $M/\lfloor \epsilon k/2 \rfloor$ blocks of size $\lfloor \epsilon k/2 \rfloor$, for each group $l \in [\ell]$ we have that the number of items from that group is at most $\lfloor \alpha_l \lfloor \epsilon k/2 \rfloor \rfloor$, where M is the length of the intermediate ranking as in Algorithm 1.*

Proof. For any block i, we observe that at the end of step 8, block i of size $\lfloor \epsilon k/2 \rfloor$ has at most $\min\{\lfloor \alpha_{\min} \lfloor \epsilon k/2 \rfloor \rfloor, \lfloor \epsilon k/2 \rfloor - \sum_{l \neq l_*} \lceil \beta_l \lfloor \epsilon k/2 \rfloor \rceil\}$ non-empty ranks and therefore has at most $\lfloor \alpha_{\min} \lfloor \epsilon k/2 \rfloor \rfloor$ items from any particular group. Step 14 ensures that when the algorithm terminates, each block has at most $\lfloor \alpha_l \lfloor \epsilon k/2 \rfloor \rfloor$ items from group l for all the groups. Since the length of the ranking is M, and each block is of size $\lfloor \epsilon k/2 \rfloor$, the statement follows. □

Lemma A.5. *At the end of step 20, each of the top $\left\lfloor \frac{n}{\lfloor \alpha_{max} \lfloor \epsilon k/2 \rfloor \rfloor} \right\rfloor$ blocks have at least $\lceil \beta_l \lfloor \epsilon k/2 \rfloor \rceil$ items from group l, forall $l \in [\ell]$.*

Proof. For any block i, we observe that at the end of step 8, block i of size $\lfloor \epsilon k/2 \rfloor$ has exactly $\min\{\lfloor \alpha_{\min} \lfloor \epsilon k/2 \rfloor \rfloor, \lfloor \epsilon k/2 \rfloor - \sum_{l \neq l_*} \lceil \beta_l \lfloor \epsilon k/2 \rfloor \rceil\}$ ranks non empty. Let n_l be the number of items from group l assigned ranks in block i after step 8. We know that $\lfloor \alpha_{\min} \lfloor \epsilon k/2 \rfloor \rfloor \geqslant \lceil \beta_{l_*} \lfloor \epsilon k/2 \rfloor \rceil$ and from Lemma A.1 we also have that $\lfloor \epsilon k/2 \rfloor - \sum_{l \neq l_*} \lceil \beta_l \lfloor \epsilon k/2 \rfloor \rceil \geqslant \lceil \beta_{l_*} \lfloor \epsilon k/2 \rfloor \rceil$. Therefore,

$$
\begin{aligned}
\sum_l n_l &= \min\{\lfloor \alpha_{\min} \lfloor \epsilon k/2 \rfloor \rfloor, \lfloor \epsilon k/2 \rfloor - \sum_{l \neq l_*} \lceil \beta_l \lfloor \epsilon k/2 \rfloor \rceil\} \\
&\geqslant \lceil \beta_{l_*} \lfloor \epsilon k/2 \rfloor \rceil .
\end{aligned}
$$

To satisfy all the lower bounds, there have to be at least $\sum_l (\lceil \beta_l \lfloor \epsilon k/2 \rfloor \rceil - n_l)$ empty ranks in block i.

The number of empty ranks in block i after step 8 are,

$$
\begin{aligned}
&\lfloor \epsilon k/2 \rfloor - \min\{\lfloor \alpha_{\min} \lfloor \epsilon k/2 \rfloor \rfloor, \lfloor \epsilon k/2 \rfloor - \sum_{l \neq l_*} \lceil \beta_l \lfloor \epsilon k/2 \rfloor \rceil\} \\
&\geqslant \lfloor \epsilon k/2 \rfloor - \left(\lfloor \epsilon k/2 \rfloor - \sum_{l \neq l_*} \lceil \beta_l \lfloor \epsilon k/2 \rfloor \rceil \right) \\
&= \sum_{l \neq l_*} \lceil \beta_l \lfloor \epsilon k/2 \rfloor \rceil \\
&= \sum_l \lceil \beta_l \lfloor \epsilon k/2 \rfloor \rceil - \lceil \beta_{l_*} \lfloor \epsilon k/2 \rfloor \rceil \\
&\geqslant \sum_l \lceil \beta_l \lfloor \epsilon k/2 \rfloor \rceil - \sum_l n_l \\
&= \sum_l (\lceil \beta_l \lfloor \epsilon k/2 \rfloor \rceil - n_l) .
\end{aligned}
$$

Therefore, after Step 8, there are enough empty ranks left in the block to satisfy the lower bounds of all the groups.

Step 14 ensures that as long as the items are available, we always satisfy these lower bounds. Since from Lemma A.3 the top $\left\lfloor \frac{n}{\lfloor \alpha_{\max} \lfloor \epsilon k/2 \rfloor \rfloor} \right\rfloor$ blocks have no empty ranks, there are at least $\lceil \beta_l \lfloor \epsilon k/2 \rfloor \rceil$ items from group l in each of these blocks. □

Lemma A.6. *The ranking output by Algorithm 1 satisfies $((1+\epsilon)\boldsymbol{\alpha}, (1-\epsilon)\boldsymbol{\beta}, k)$ group fairness in every k consecutive ranks in the top $\left\lfloor \frac{n}{\alpha_{max}} \right\rfloor - \lfloor \epsilon k/2 \rfloor$ ranks.*

Proof. Lemma A.3 shows that none of the top $\left\lfloor \frac{n}{\alpha_{\max}} \right\rfloor - \lfloor \epsilon k/2 \rfloor$ ranks will be empty at the end of step 20; therefore, these ranks will remain unchanged in the steps after step 25.

Consider any k consecutive ranks $j, \ldots, j+k-1$. Let $i_1 \stackrel{\text{def}}{=} \lceil j/\lfloor \epsilon k/2 \rfloor \rceil$ and $i_2 \stackrel{\text{def}}{=} \lceil (j+k-1)/\lfloor \epsilon k/2 \rfloor \rceil$. By construction, the blocks $i_1+1, \ldots, i_2-1$ are fully contained in the ranks $\{j, j+1, \ldots, j+k-1\}$. For any $l \in [\ell]$, the number of items from group l in ranks j to $j+k-1$ is at most the number of items from group l in blocks i_1 to i_2. Using Lemma A.4 we get that this is at most

$$
\lfloor \alpha_l k \rfloor + 2 \lfloor \alpha_l \lfloor \epsilon k/2 \rfloor \rfloor \leqslant \alpha_l (1+\epsilon) k.
$$

We note that this bound also holds for cases when $i_2 = i_1+1$ or $i_2 = i_1$.

Let m be the number of blocks fully contained in these k ranks. Then $k = m\lfloor \epsilon k/2\rfloor + (x+y)$ where $0 \leqslant x, y < \lfloor \epsilon k/2\rfloor$. For any $l \in [\ell]$, the number of items from group l in k consecutive ranks is at least the number of items from group l in m blocks. Using Lemma A.5 we get that this is at least

$$m\lceil \beta_l \lfloor \epsilon k/2\rfloor\rceil = \left(\frac{k-(x+y)}{\lfloor \epsilon k/2\rfloor}\right)\lceil \beta_l \lfloor \epsilon k/2\rfloor\rceil$$
$$\geqslant \left(\frac{k-(x+y)}{\lfloor \epsilon k/2\rfloor}\right)\beta_l \lfloor \epsilon k/2\rfloor = \beta_l k - (x+y)\beta_l$$
$$> \beta_l k - 2\beta_l \lfloor \epsilon k/2\rfloor > \beta_l(1-\epsilon)k.$$

□

Proof of Theorem 2.5. Follows from the choice of ϵ and from Lemma A.2, Lemma A.3, Lemma A.6. □

Proof of Theorem 2.6. We use Algorithm 1 with $\epsilon := 2$. Now, the ith "block" is of size $\left\lfloor \frac{\epsilon k}{2} \right\rfloor = k$.

Fix an item $j \in [N]$ in the true ranking. If $\alpha_{\text{min}} k \leqslant k - \sum_{l \neq l_*} \beta_l k$, from Equation (3) we have that its final rank will be at most

$$\frac{j\lfloor \epsilon k/2\rfloor}{\lfloor \alpha_{\text{min}}\lfloor \epsilon k/2\rfloor\rfloor} = \frac{j}{\alpha_{\text{min}}}.$$

Here, the equality follows from our choice of $\epsilon = 2$.

Otherwise, from Equation (4) we have that its final rank will be at most

$$j\frac{\left\lfloor \frac{\epsilon k}{2}\right\rfloor}{\left\lfloor \frac{\epsilon k}{2}\right\rfloor - \sum_{l\neq l_*}\left\lceil \beta_l \left\lfloor \frac{\epsilon k}{2}\right\rfloor\right\rceil}$$
$$= \frac{jk}{k-\sum_{l\neq l_*}\lceil \beta_l k\rceil} \qquad \text{(substituting } \epsilon = 2\text{)}$$
$$= \frac{j}{1-\sum_{l\neq l_*}\frac{\lceil \beta_l k\rceil}{k}} = \frac{j}{1-\sum_{l\neq l_*}\beta_l}.$$

Hence, the ranking output by Algorithm 1 with $\epsilon = 2$ satisfies $\frac{1}{\min\left\{\alpha_{\text{min}}, 1-\sum_{l\neq l_*}\beta_l\right\}}$ underranking.

Lemma A.4 shows that at the end of step 25, each block has at most $\lfloor \alpha_l k\rfloor$ items from group l. Then we have,

$$\sum_{l\in[\ell]}\lfloor \alpha_l k\rfloor = k\left(\sum_{l\in[\ell]}\alpha_l\right) \geqslant k.$$

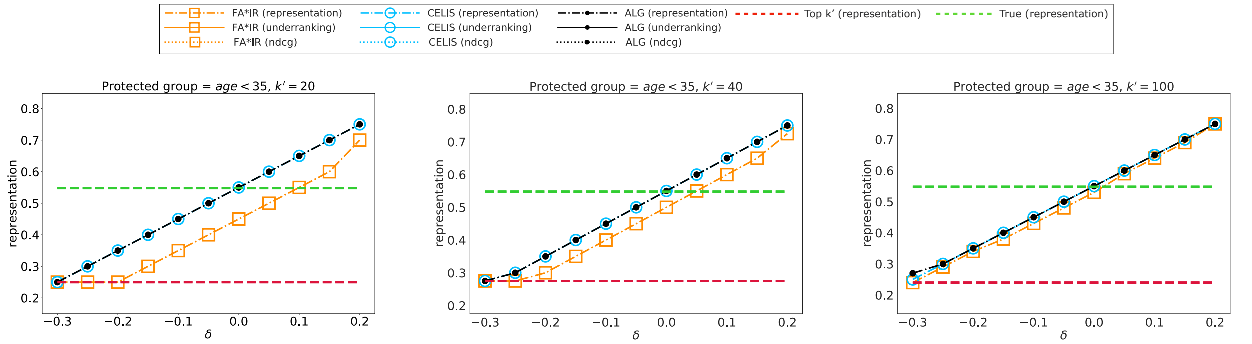
Therefore, if a block contains $\lfloor \alpha_l k\rfloor$ items from group $l \in [\ell]$, it can not have any empty ranks. Consequently, as long as $i\lfloor \alpha_{\text{max}} k\rfloor$ items fom each group are available, blocks 1 to i will not contain empty ranks and also have required number of items to satisfy the lower bounds since $\alpha_l k \geqslant \beta_l k$. Since there are at least n items from each group, no rank in the top i blocks will be empty, where i satisfies $i\lfloor \alpha_{\text{max}} k\rfloor \leqslant n$. Hence, for $i \in \mathbb{Z}^+$ such that $i \leqslant \frac{n}{\lfloor \alpha_{\text{max}} k\rfloor} = \frac{n}{\alpha_{\text{max}} k}$, blocks 1 to i each of size k contain at most $\alpha_{\text{max}} k$ items from each group. That is, the top $\left\lfloor \frac{n}{\alpha_{\text{max}} k}\right\rfloor$ blocks each of size k satisfy $(\boldsymbol{\alpha}, \boldsymbol{\beta}, k)$ group fairness. □

B. Additional Details of Experiments

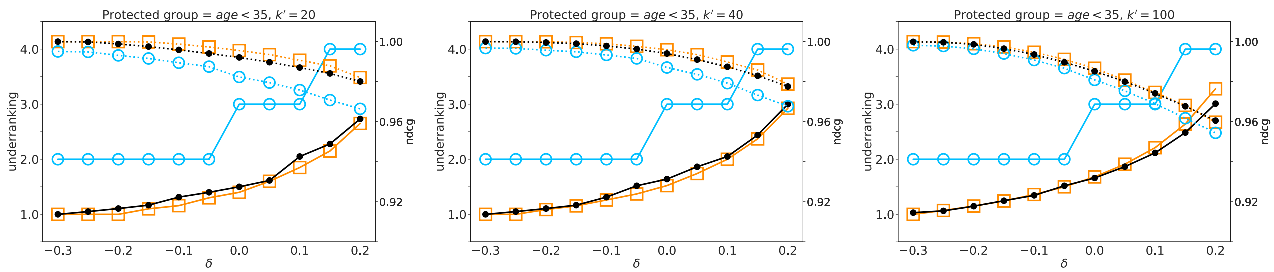
In this section, we describe additional experimental resutls on the German credit risk and the COMPAS recidivism datasets. Figure 6 shows the evaluation of ALG and the baselines on the German credit risk dataset with *age* < 35 as the protected group. The candidates with *age* < 35 are underrepresented in the top k' ranks (dashed red lines), even though they have very high represention (dashed green lines) in the whole dataset. Hence, we observe a trade-off between representation of the protected group and underranking with all the algorithms. However, ALG again achieves the best of both representation and underranking even in this case.

Figures 7 to 10 show evaluation of the algorithms in consecutive ranks. Plot (a) shows the evaluation at top 1 to 20 ranks, plot (b) shows the evaluation at top 21 to 60 ranks, and plot (c) shows the evaluation at top 61 to 100 ranks. The plots (d), (e), (f) however still show underranking and nDCG in the top $20, 60$, and 100 ranks respectively. ALG achieves good trade-offs between underranking and representation in all different consecutive ranks compared to the baselines.

In Figure 11, we show results of ALG with the group fairness constraints for the groups formed by the intersection of two groups *age* and *gender*. We partition the candidates into six groups and their true representations are as show in Table 2. ALG is run with the constraints $\alpha_l = p_l^* + \delta$ and $\beta_l = p_l^* - \delta$, where l represents the group. ALG achieves proportional representation while the underranking decreases as δ increases, hence confirming the trade-off.

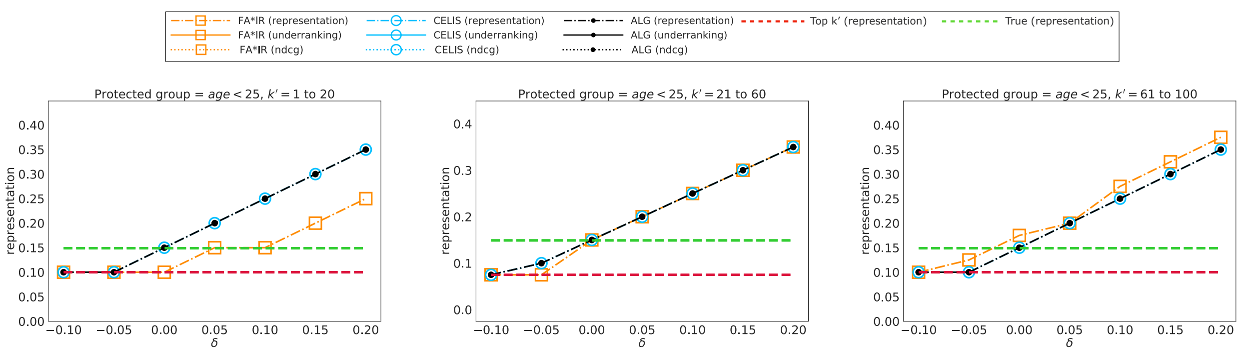


(a) Representation at top 20 ranks. (b) Representation at top 40 ranks. (c) Representation at top 100 ranks.

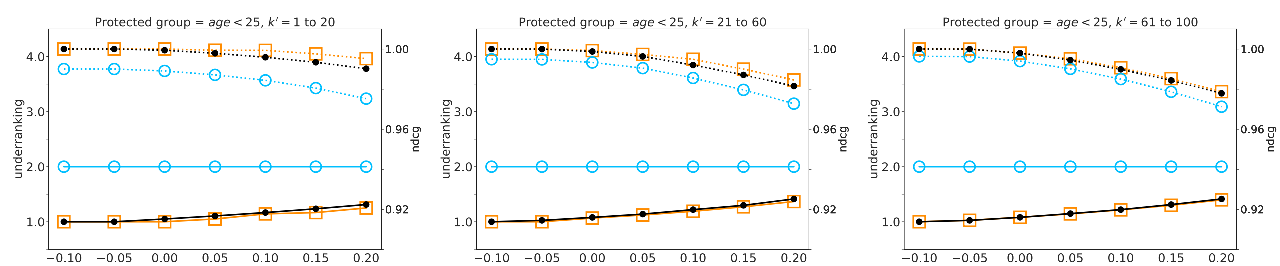


(d) Underranking, nDCG at top 20 ranks. (e) Underranking, nDCG at top 40 ranks. (f) Underranking, nDCG at top 100 ranks.

Figure 6: Results on the German Credit Risk dataset with *age*$<$ 35 as the protected group.

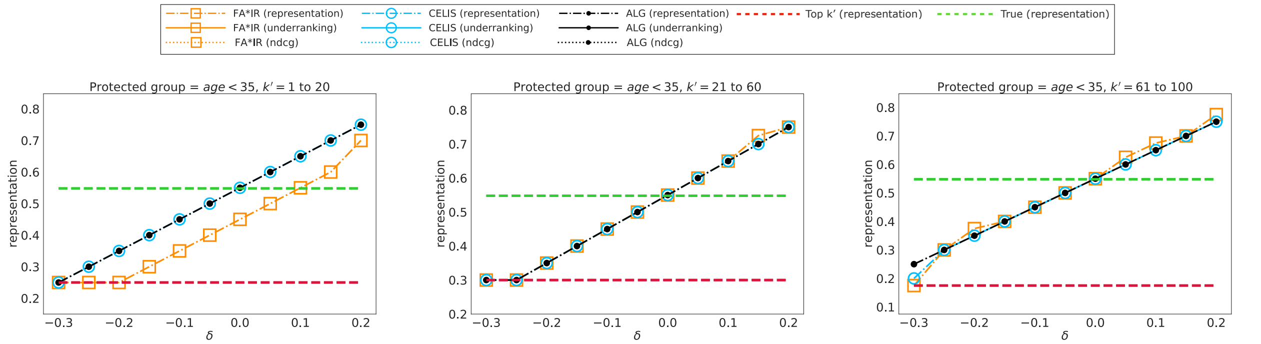


(a) Representation at top 20 ranks. (b) Representation at top 40 ranks. (c) Representation at top 100 ranks.

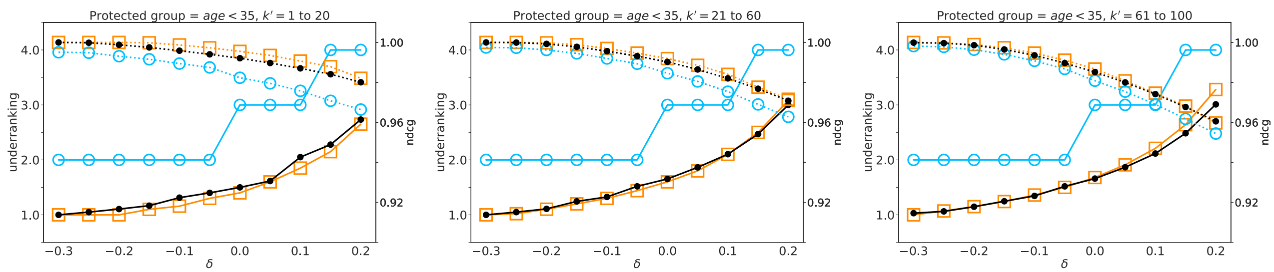


(d) Underranking, nDCG at top 20 ranks. (e) Underranking, nDCG at top 40 ranks. (f) Underranking, nDCG at top 100 ranks.

Figure 7: Results on the German Credit Risk dataset with *age*$<$ 25 as the protected group.

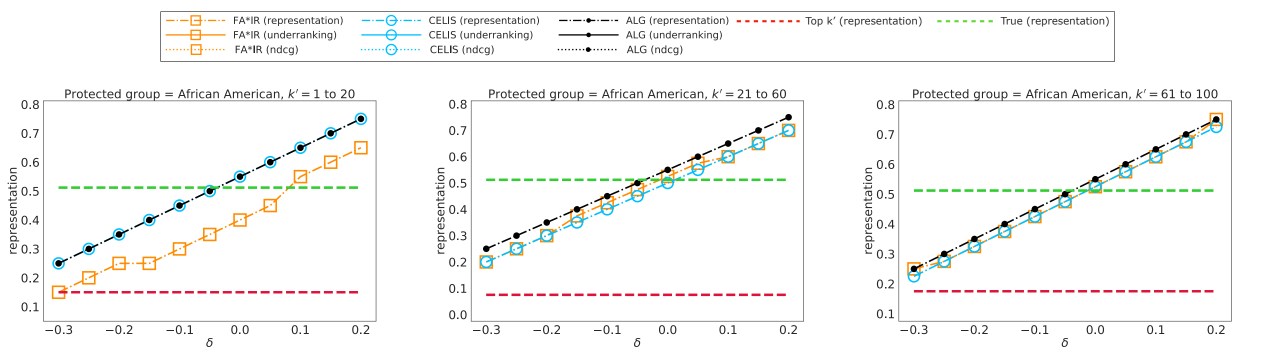


(a) Representation at ranks 1 to 20. (b) Representation at ranks 21 to 60. (c) Representation at ranks 61 to 100.

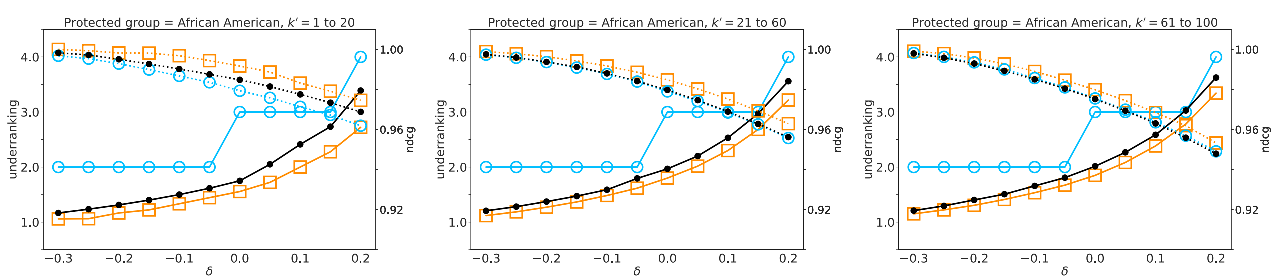


(d) Underranking, nDCG at top 20 ranks. (e) Underranking, nDCG at top 60 ranks. (f) Underranking, nDCG at top 100 ranks.

Figure 8: Results on the German Credit Risk dataset with *age* < 35 as the protected group.

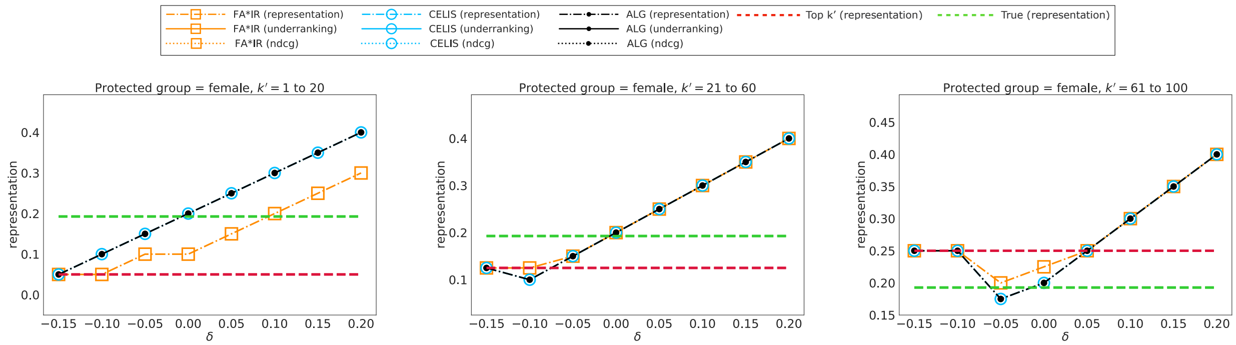


(a) Representation at ranks 1 to 20. (b) Representation at ranks 21 to 60. (c) Representation at ranks 61 to 100.

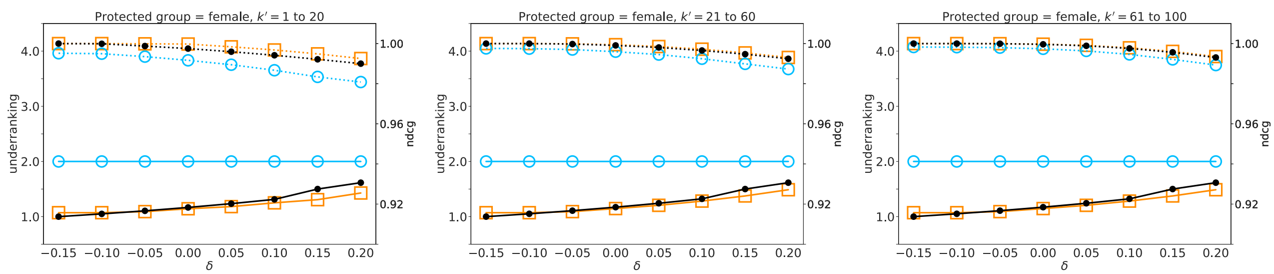


(d) Underranking, nDCG at top 20 ranks. (e) Underranking, nDCG at top 60 ranks. (f) Underranking, nDCG at top 100 ranks.

Figure 9: Results on the COMPAS Recidivism dataset with *African American* as the protected group.

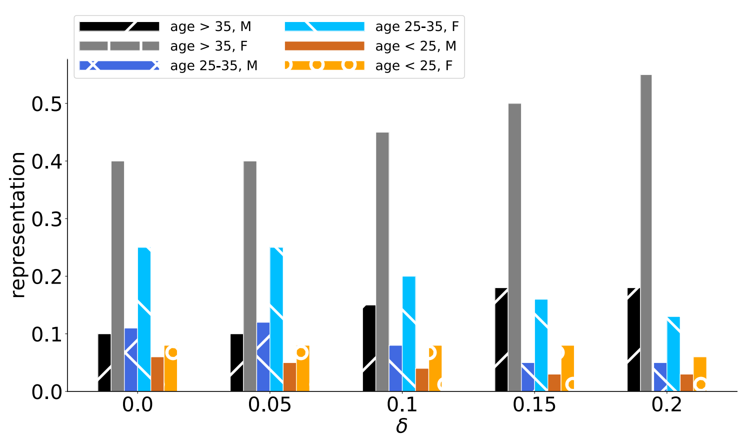


(a) Representation at ranks 1 to 20. (b) Representation at ranks 21 to 60. (c) Representation at ranks 61 to 100.

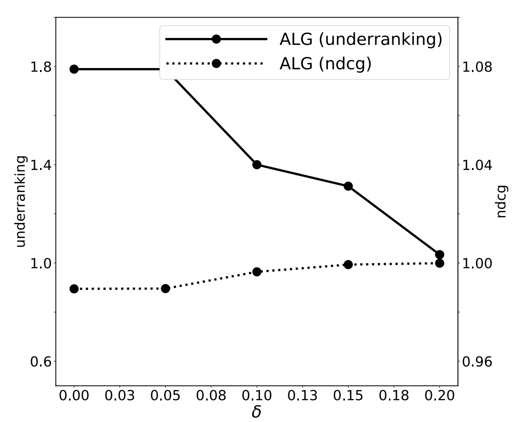


(d) Underranking, nDCG at top 20 ranks. (e) Underranking, nDCG at top 60 ranks. (f) Underranking, nDCG at top 100 ranks.

Figure 10: Results on the COMPAS Recidivism dataset with *Female* as the protected group.



(a)



(b)

Figure 11: Results on the German Credit Risk dataset with six groups based on *age, gender*.

Group	age >35, M	age >35, F	age 25 to 35, M	age 25 to 35, F	age <25, M	age <25, F
p^*	0.10	0.36	0.13	0.27	0.08	0.06

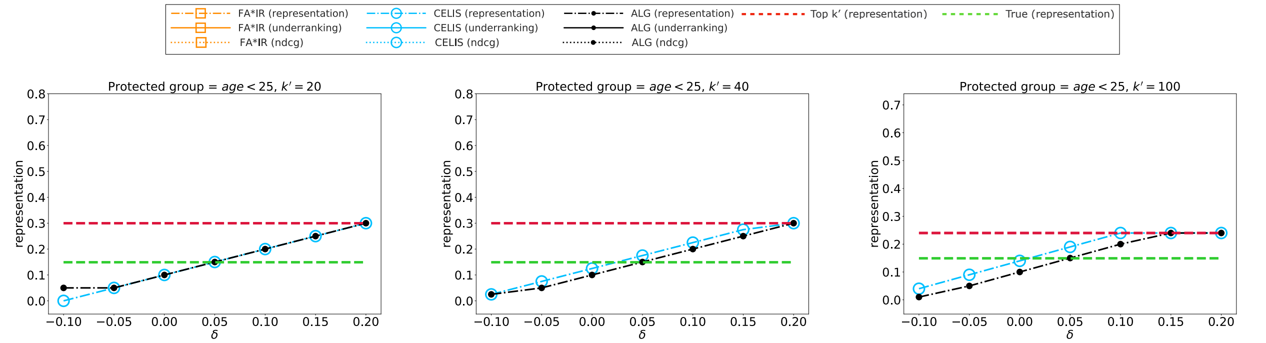
Table 2: True representation of the groups in the dataset.

B.1. Reverse Score based Ranking as True Ranking

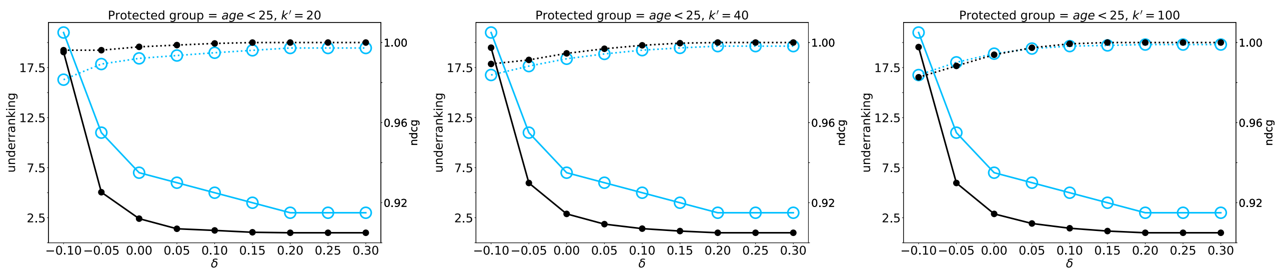
In Figure 12 to Figure 14, the true ranking is based on the negative score (or relavance) of the items. In the German Credit dataset, we use negative Schufa score for ranking, i.e., the individual with highest negative Schufa score will be ranked at the top and so on. When using the negative scores to obtain the true ranking, we observe that the individuals from the protected group are overrepresented in the top few ranks. In the plots (a) - (c) in these figures, the dahsed red line (representation of the group in the top k' ranks) is significantly above the dashed green line (representation of the group in the entire dataset). In this case, we can achieve proportional representation by placing upper bound constraints on the representation of the candidates from the protected group. We run Celis et al.'s DP algorithm with the constraints $\forall k' \in [k], L_{1,k'} = 0, L_{2,k'} = 0, U_{1,k'} = \lfloor (p_1^* + \delta) \cdot k' \rfloor, U_{2,k'} = k'$ and $k = 100$, where subscript 1 represents protected group and subscript 2 represents non-protected group. We run ALG with group fairness requirements $(\boldsymbol{\alpha} = (p_1^* + \delta, 1), \boldsymbol{\beta} = (0, 0), k = 100)$ In this COMPAS dataset, we use recidivism risk score to obtain true ranking as opposed to the negative recidivism score in Figure 3 and Figure 4. We observe here also that the protected groups *African American* and *female* is overrepresented in the top k' ranks. Hence we use upper bound constraints on these groups and all other constraints are removed.

We again observe a trade-off between group fairness and underranking and notice that in all the plots, ALG achieves better underranking than Celis et al.'s DP algorithm and also achieves very good group fairness.

Figures 15 to 17 are the evaluation of the algorithms for consecutive ranks. We again observe trade-off between representation and underranking. ALG achieves good representation in the consecutive ranks while achieving better underranking than the baselines.

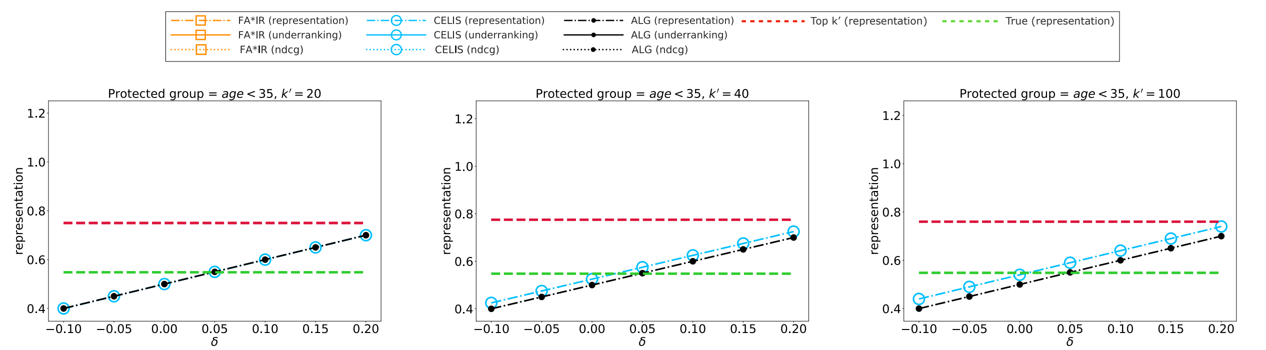


(a) Representation at top 20 ranks. (b) Representation at top 40 ranks. (c) Representation at top 100 ranks.

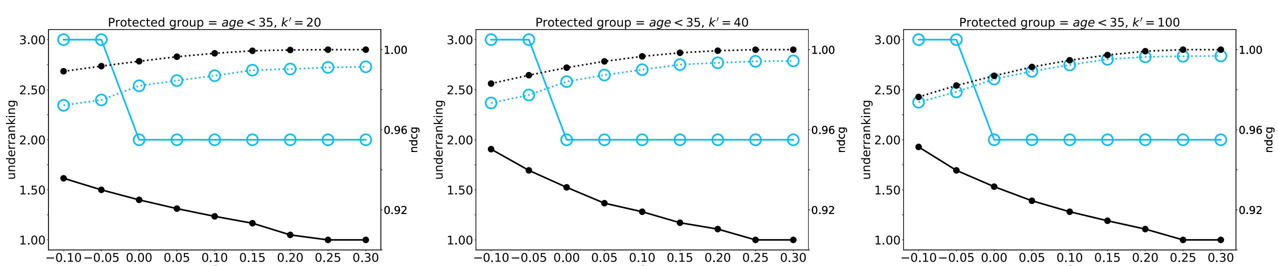


(d) Underranking, nDCG at top 20 ranks. (e) Underranking, nDCG at top 40 ranks. (f) Underranking, nDCG at top 100 ranks.

Figure 12: Results on the German Credit Risk dataset with *age* <25 as the protected group (reverse score-based ranking).

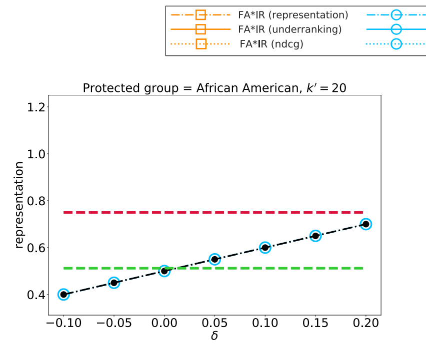


(a) Representation at top 20 ranks. (b) Representation at top 40 ranks. (c) Representation at top 100 ranks.

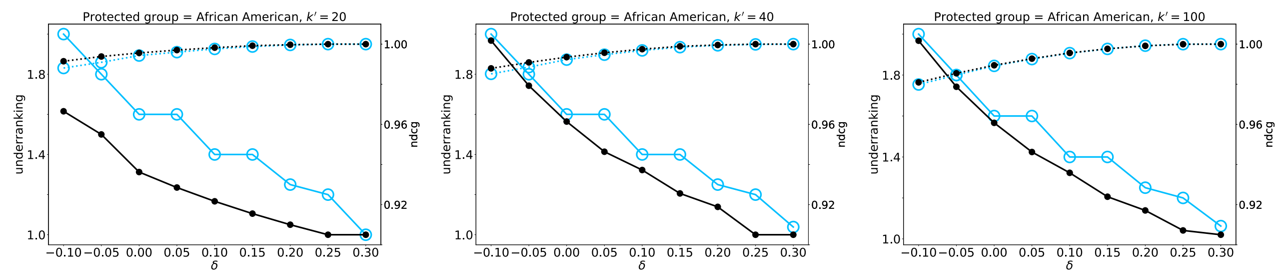


(d) Underranking, nDCG at top 20 ranks. (e) Underranking, nDCG at top 40 ranks. (f) Underranking, nDCG at top 100 ranks.

Figure 13: Results on the German Credit Risk dataset with *age* <35 as the protected group (reverse score-based ranking).

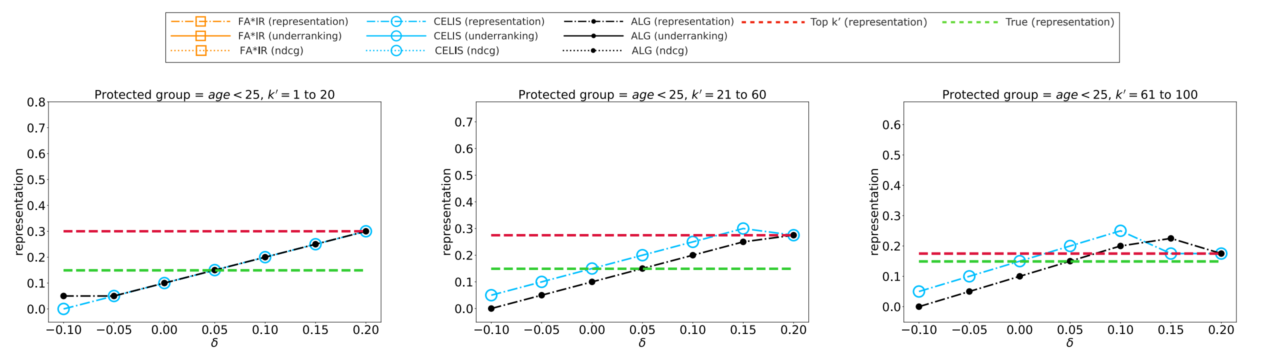


(a) Representation at top 20 ranks. (b) Representation at top 40 ranks. (c) Representation at top 100 ranks.

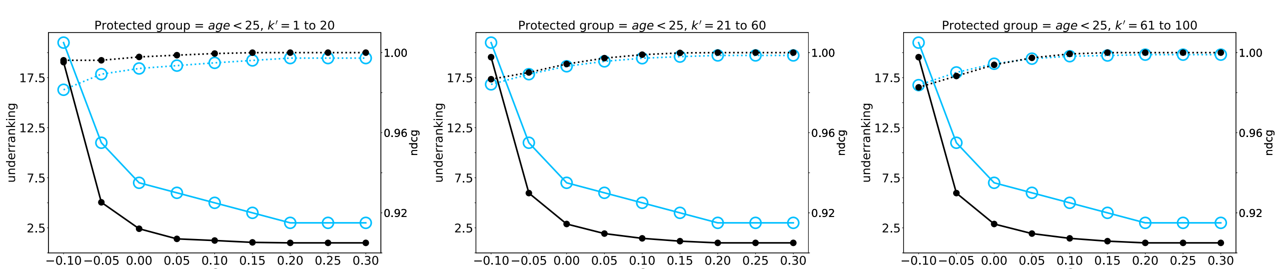


(d) Underranking, nDCG at top 20 ranks. (e) Underranking, nDCG at top 40 ranks. (f) Underranking, nDCG at top 100 ranks.

Figure 14: Results on the COMPAS Recidivism dataset with *African American* as the protected group (reverse score-based ranking).

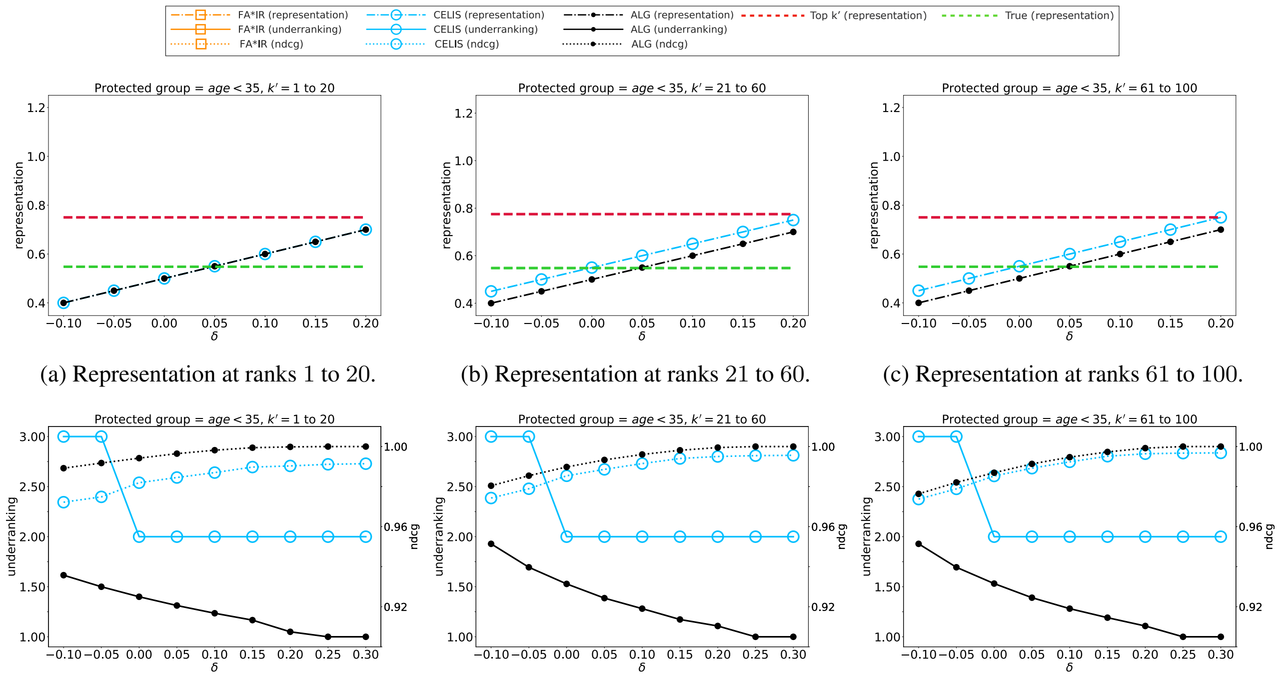


(a) Representation at ranks 1 to 20. (b) Representation at ranks 21 to 60. (c) Representation at ranks 61 to 100.



(d) Underranking, nDCG at top 20 ranks. (e) Underranking, nDCG at top 60 ranks. (f) Underranking, nDCG at top 100 ranks.

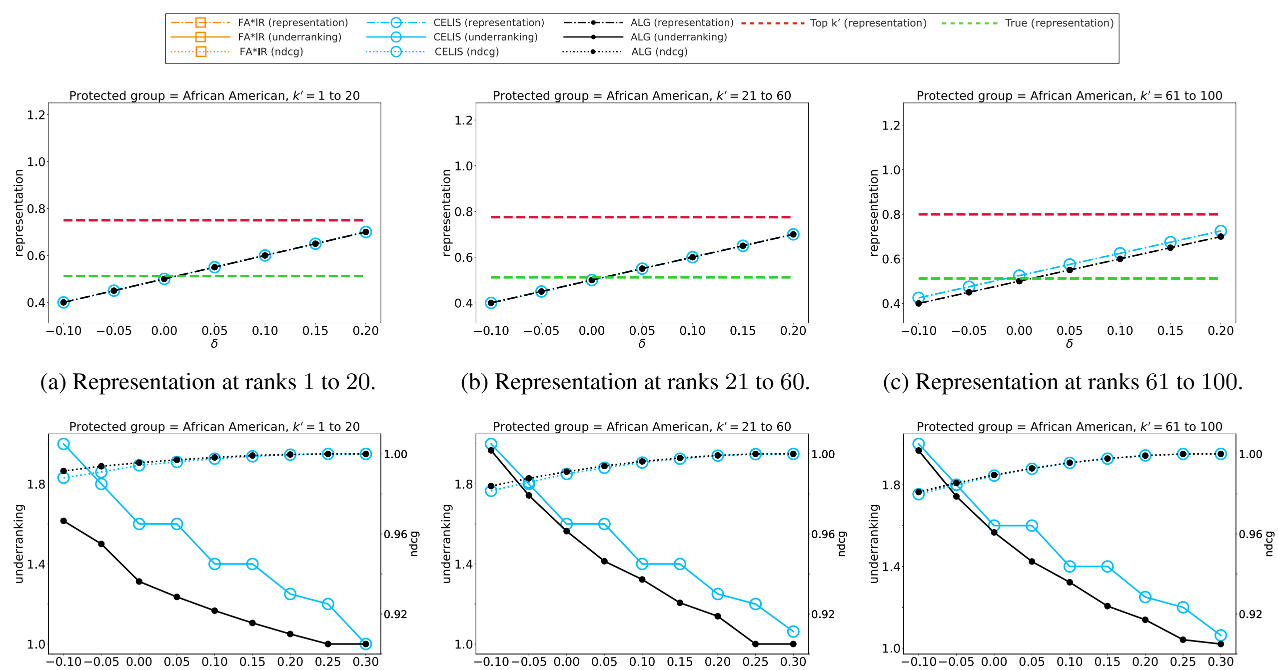
Figure 15: Results on the German Credit Risk dataset with *age*$<$ 25 as the protected group (reverse score-based ranking).



(a) Representation at ranks 1 to 20. (b) Representation at ranks 21 to 60. (c) Representation at ranks 61 to 100.

(d) Underranking, nDCG at top 20 ranks. (e) Underranking, nDCG at top 60 ranks. (f) Underranking, nDCG at top 100 ranks.

Figure 16: Results on the German Credit Risk dataset with *age*< 35 as the protected group (reverse score-based ranking).



(a) Representation at ranks 1 to 20. (b) Representation at ranks 21 to 60. (c) Representation at ranks 61 to 100.

(d) Underranking, nDCG at top 20 ranks. (e) Underranking, nDCG at top 60 ranks. (f) Underranking, nDCG at top 100 ranks.

Figure 17: Results on the COMPAS Recidivism dataset with *African American* as the protected group (reverse score-based ranking).