A. Proof of Theorem 3.1

In this section, we present the proof of our main technical result – Theorem 3.1 – an algorithm for partitioning a given metric space (X, d) into a number of clusters $\mathcal{P} = (P_1, \dots, P_k)$ (where k is not fixed).

Recall our iterative process for obtaining this partitioning – Algorithm 2 – which makes use of Theorem 5.1 in each iteration to select a cluster from the set of not-yet clustered vertices.

The proof of Theorem 5.1 is presented in Appendix B. We now present the proof of Theorem 3.1 assuming Theorem 5.1.

Proof of Theorem 3.1. Property (a) of Theorem 5.1 guarantees that $\operatorname{diam}(P_i) \le 2R$ for every $i \in [k]$ and thus property (1) of Theorem 3.1 holds.

We now show that property (2) holds. Fix $u \in X$. Consider iteration $i \in [k]$. Note that set P_i satisfies property (b) of Theorem 5.1 regardless of what set X_i we have in the beginning of iteration i. That is, for every set $Y \subset X$ and $u \in Y$, we have

$$\sum_{v \in \mathrm{Ball}(u,R)\cap Y} \Big(\Pr\left\{\delta_{P_i}(u,v) = 1 \,|\, X_i = Y\right\} - D_\beta \frac{d(u,v)}{R} \Pr\{\vee_{P_i}(u,v) = 1 \,|\, X_i = Y\}\Big)^+ \lesssim \beta^q \cdot \mathbf{E}\Big[\sum_{v \in \mathrm{Ball}(u,2R)\cap Y} \frac{d(u,v)}{R} \cdot \vee_{P_i}(u,v) \,|\, X_i = Y \Big]. \tag{A.1}$$

We observe that inequality (A.1) can be written as follows (for all $u \in X$).

$$\sum_{v \in \mathrm{Ball}(u,R)} \Big(\Pr\left\{\delta_{P_i}(u,v) = 1 \text{ and } u, v \in X_i \,|\, X_i = Y\right\} - D_\beta \frac{d(u,v)}{R} \Pr\{\vee_{P_i}(u,v) = 1 \text{ and } u, v \in X_i \,|\, X_i = Y\}\Big)^+ \lesssim \beta^q \cdot \mathbf{E}\Big[\sum_{v \in \mathrm{Ball}(u,2R)} \frac{d(u,v)}{R} \cdot \vee_{P_i}(u,v) \cdot \mathbf{1}\left\{u, v \in X_i\right\} \,|\, X_i = Y \Big]. \tag{A.2}$$

If $u \notin Y$, then all terms in (A.2) are equal to 0, and the inequality trivially holds. If $u \in Y$, then corresponding terms in (A.1) and (A.2) with $v \in Y$ are equal to each other; all terms in (A.2) with $v \notin Y$ are equal to 0. Denote the event that $u, v \in X_i$ by $\mathcal{E}_{vi}$ (that is, $\mathcal{E}_{vi}$ happens if both points u and v are not clustered at the beginning of iteration i). We take the expectation of (A.2) over $X_i = Y$ and add up the inequalities over all $i \in [k]$. Using the subaddivity of function $x \mapsto x^+$, we obtain

$$\begin{aligned} &\sum_{v \in \mathrm{Ball}(u,R)} \Big(\sum_{i \in [k]} \Pr\left\{\delta_{P_i}(u,v) = 1 \text{ and } \mathcal{E}_{vi}\right\} - D_\beta \frac{d(u,v)}{R} \Pr\{\vee_{P_i}(u,v) = 1 \text{ and } \mathcal{E}_{vi}\}\Big)^+ \\ &\le \sum_{\substack{v \in \mathrm{Ball}(u,R)\\ i\in[k]}} \Big(\Pr\left\{\delta_{P_i}(u,v) = 1 \text{ and } \mathcal{E}_{vi}\right\} - D_\beta \frac{d(u,v)}{R} \Pr\{\vee_{P_i}(u,v) = 1 \text{ and } \mathcal{E}_{vi}\}\Big)^+ \\ &\lesssim \beta^q \cdot \mathbf{E}\Big[\sum_{\substack{v \in \mathrm{Ball}(u,2R)\\ i\in[k]}} \frac{d(u,v)}{R} \cdot \vee_{P_i}(u,v) \cdot \mathbf{1}\{\mathcal{E}_{vi}\} \Big]. \end{aligned} \tag{A.3}$$

Now consider any $v \in X \setminus \{u\}$. If u and v are separated by the partitioning $\mathcal{P}$, then they are separated at some iteration i. That is, for some $i \in [k]$:

- $\mathcal{E}_{vi}$ happens (in other words, u and v are not clustered at the beginning of iteration i)
- $\delta_{P_i}(u,v) = 1$ (exactly one of them gets clustered in iteration i)

Further, there is exactly one i such that both events above happen. On the other hand, if u and v are not separated by $\mathcal{P}$ then $\delta_{P_i}(u,v) = 0$ for all $i \in [k]$. We conclude that

$$\mathbf{1}\{\mathcal{P}(u) \neq \mathcal{P}(v)\} = \sum_{i \in [k]} \mathbf{1}\left\{\delta_{P_i}(u,v) = 1 \text{ and } \mathcal{E}_{vi}\right\}. \tag{A.4}$$

In particular, the expectations of the expressions on both sides of (A.4) are equal:

$$\Pr\{\mathcal{P}(u) \neq \mathcal{P}(v)\} = \sum_{i \in [k]} \Pr\left\{\delta_{P_i}(u,v) = 1 \text{ and } \mathcal{E}_{vi}\right\}. \tag{A.5}$$

Now consider the first iteration i at which at least one of the vertices u and v gets clustered. Note that (i) event $\mathcal{E}_{vi}$ happens and (ii) $\vee_{P_i}(u,v) = 1$ (that is, (i) both points u and v are not clustered at the beginning of iteration i; (ii) but at least one of them gets clustered in iteration i). Further, for $j < i$, $\vee_{P_i}(u,v) = 0$ and for $j > i$, $\mathcal{E}_{vj}$ does not happen. We conclude that event "$\vee_{P_i}(u,v) = 1$ and $\mathcal{E}_{vi}$" happens exactly for one value of $i \in [k]$. Therefore,

$$\sum_{i \in k} \vee_{P_i}(u,v) \cdot \mathbf{1}\{\mathcal{E}_{vi}\} = 1 \tag{A.6}$$

and

$$\sum_{i\in[k]} \Pr\{\vee_{P_i}(u,v)=1 \text{ and } \mathcal{E}_{vi}\} \tag{A.7}$$
$$= \sum_{i\in[k]} \mathbf{E}[\vee_{P_i}(u,v)\mathbf{1}\{\mathcal{E}_{vi}\}] = 1.$$

Plugging (A.5) and (A.7) into (A.3), we obtain

$$\sum_{v\in\mathrm{Ball}(u,R)} \left(\Pr\{\mathcal{P}(u)=\mathcal{P}(v)\} - D_\beta \frac{d(u,v)}{R}\right)^+$$
$$\lesssim \beta^q \cdot \mathbf{E}\Big[\sum_{v\in\mathrm{Ball}(u,2R)} \frac{d(u,v)}{R}\Big].$$

We conclude that property (2) holds. Next, we show that property (3) holds for every $u \in X$. As in the analysis of property (2), we consider some iteration i. Then property (c) of Theorem 5.1 guarantees that if $u \in X_i$ then

$$\sum_{i\in[k]} \sum_{v\in\mathrm{Ball}(u,r)\cap X_i} \delta_{P_i}(u,v) \lesssim \tag{A.8}$$
$$\lesssim \sum_{i\in[k]} \beta\cdot D_\beta^2 \cdot \left(\sum_{v\in\mathrm{Ball}(u,2R)\cap X_i} \frac{d(u,v)}{R}\cdot \vee_{P_i}(u,v)\right)$$

We rewrite (A.8) as follows:

$$\sum_{i\in[k]} \sum_{v\in\mathrm{Ball}(u,r)} \delta_{P_i}(u,v)\cdot \mathbf{1}\{\mathcal{E}_{vi}\} \lesssim$$
$$\lesssim \sum_{i\in[k]} \beta\cdot D_\beta^2\cdot \left(\sum_{v\in\mathrm{Ball}(u,2R)} \frac{d(u,v)}{R}\cdot \vee_{P_i}(u,v)\cdot \mathbf{1}\{\mathcal{E}_{vi}\}\right).$$

Note that this inequality holds for all $u \in X$: if $u \in X_i$, it is equivalent to (A.8); if $u \notin X_i$, then both sides are equal to 0, and the inequality trivially holds. Using formulas (A.4) and (A.6), we get

$$\sum_{v\in\mathrm{Ball}(u,r)} \mathbf{1}\{\mathcal{P}(u)\neq\mathcal{P}(v)\} \lesssim \beta\cdot D_\beta^2 \cdot \sum_{v\in\mathrm{Ball}(u,2R)} \frac{d(u,v)}{R}.$$

Therefore, property (3) holds. □

B. Proof of Theorem 5.1

In Section 5.1, we describe an iterative approach to finding a probabilistic metric decomposition for Theorem 3.1. In this section, we show how to find one cluster P of the partitioning. Given a metric space (X,d) and positive numbers r and R, our algorithm selects a subset $P \subseteq X$ that satisfies the three properties listed in Theorem 5.1. Recall that $\beta = r/R$, $D_\beta = 2(q+1)\ln {}^1\!/\!_\beta$, $R_0 = {}^R\!/\!_{D_\beta}$, $R_1 = R - R_0$ and $\rho_q(\beta) = ({}^1\!/\!_\beta)^{q+1}$ (see Figure 3). In

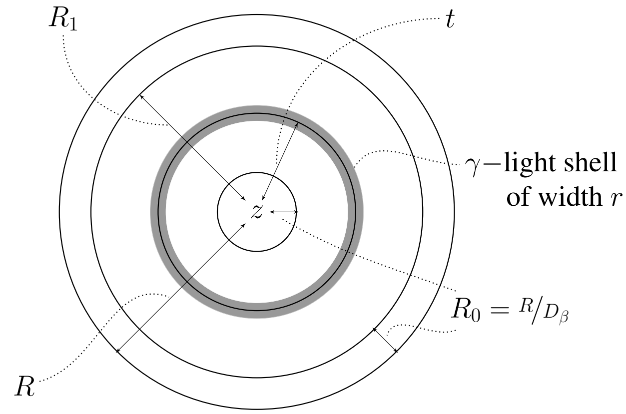

Figure 3. Light Ball
$R > r > 0$, $q \geq 1$, $\beta = r/R$, $D_\beta = 2(q+1)\ln {}^1\!/\!_\beta$,
$R_0 = R/D_\beta$, $R_1 = R - R_0$.

this proof, we assume that $\beta = r/R$ is sufficiently small (i.e, $\beta \leq \beta_q^*$ for some small $\beta_q^* = \Theta\big(\frac{1}{(q\ln(q+1)}\big)$ and, consequently, $R_0 = {}^R\!/\!_{D_\beta}$ is also small. Specifically, we assume that $r < R_0 < R_1 < R$ and $R_0 + r < R_1/100$.

Our algorithm for selecting the cluster P starts by picking a pivot point z that has the most points within a ball of small radius R_0. That is, z is the optimizer to the following expression:

$$z = \arg\max_{u\in X} |\mathrm{Ball}(u,R_0)|. \tag{B.1}$$

The algorithm then checks if the ball of a larger radius, R_1, around z has significantly more points in it in comparison to the ball of radius R_0 around z. If the ratio of the number of points in these two balls exceeds $\rho_q(\beta)$, the algorithm selects the set of points $\mathrm{Ball}(z,R_1)$ as our cluster P. We refer to this case as the "Heavy Ball" case. In Section B.2, we show that this set P satisfies the properties of Theorem 5.1.

If, however, $|\mathrm{Ball}(z,R_1)| < \rho_q(\beta)\cdot|\mathrm{Ball}(z,R_0)|$, then the algorithm outputs cluster $P = \mathrm{Ball}(z,t)$ where $t \in (0,R]$ is chosen as follows. First, the algorithm finds the set S of all radii $s \in (3R_0, R_1]$ for which the set $P = \mathrm{Ball}(z,s)$ satisfies Definition 5.2. Then, it chooses a random radius t in S (non-uniformly) so that random set $P = \mathrm{Ball}(z,t)$ satisfies property (b) of Theorem 5.1. In Appendix C.1, we discuss how to find the set S and show that $\mu(S) \geq R/2$ (where $\mu(S)$ is the Lebesgue measure of set S). Finally, in Appendix C.2, we describe a procedure for choosing a random radius t in S.

B.1. Useful Observations

In this section, we prove several lemmas which we will use for analyzing both the "Heavy Ball" and "Light Ball" cases. First, we show an inequality that will help us lower bound the right hand sides in inequalities (b) and (c) of Theorem 5.1.

Lemma B.1. *Assume that z is chosen according to (B.1). Consider t in $(3R_0, R_1]$ and u in X with $d(z,u) \in [2R_0, R]$. Let $P = \mathrm{Ball}(z,t)$. Denote:*

$$Y_P = \sum_{v \in \mathrm{Ball}(u,2R)} \frac{d(u,v) \vee_P (u,v)}{R}.$$

Then, $|P| \leq 2D_\beta Y_P$.

Remark: Note that in Theorem 5.1, the right side of inequality (b) equals $\beta^q \mathbf{E}[Y_P]$, and the right side of inequality (c) equals $\beta \cdot D_\beta^2 \cdot Y_p$.

Proof. Observe that $P \subset \mathrm{Ball}(u, 2R)$. Hence,

$$Y_P = \sum_{v \in \mathrm{Ball}(u,2R)} \frac{d(u,v) \vee_P (u,v)}{R} \geq \sum_{v \in P} \frac{d(u,v)}{R}.$$

For every $v \in P \setminus \mathrm{Ball}(u, R_0)$, we have $\frac{d(u,v)}{R} \geq \frac{R_0}{R} = D_\beta^{-1}$. Thus,

$$Y_P \geq D_\beta^{-1} \, |P \setminus \mathrm{Ball}(u, R_0)|.$$

We need to lower bound the size of $P \setminus \mathrm{Ball}(u, R_0)$. On the one hand, we have

$$\begin{aligned} |P \setminus \mathrm{Ball}(u, R_0)| &\geq |P| - |\,\mathrm{Ball}(u, R_0)| \\ &\geq |P| - |\,\mathrm{Ball}(z, R_0)|. \end{aligned}$$

Here, we used that $\mathrm{Ball}(z, R_0)$ is the largest ball of radius R_0 in X. On the other hand, $\mathrm{Ball}(z, R_0) \subset P \setminus \mathrm{Ball}(u, R_0)$, since $d(z,u) \geq 2R_0$. Thus, $|P \setminus \mathrm{Ball}(u, R_0)| \geq |\,\mathrm{Ball}(z, R_0)|$. Combining two bounds on $|P \setminus \mathrm{Ball}(u, R_0)|$, we get the desired inequality $|P \setminus \mathrm{Ball}(u, R_0)| \geq |P|/2$. □

We now provide a lemma that will help us verify property (b) of Theorem 5.1 for that point u.

Lemma B.2. *Consider an arbitrary probability distribution of t in $(3R_0, R_1]$. Let $P = \mathrm{Ball}(z,t)$, where z is chosen according to (B.1). If for each point $u \in \mathrm{Ball}(z, R)$ at least one of the following two conditions holds, then P satisfies property (b) of Theorem 5.1 for all points u in X.*
Condition I:

$$\Pr\{t \geq d(z,u) - R_0\} \lesssim \beta^{q+1} \cdot \frac{\mathbf{E}|\,\mathrm{Ball}(z,t)|}{|\,\mathrm{Ball}(z, R_0)|}. \tag{B.2}$$

Condition II: For every $v \in \mathrm{Ball}(u, R_0)$, we have

$$\Pr\left\{\delta_P(u,v) = 1\right\} - D_\beta \frac{d(u,v)}{R} \Pr\{\vee_P(u,v) = 1\} \lesssim \\ \lesssim \beta^{q+1}. \tag{B.3}$$

Remark: This lemma makes the argument about the distribution of t from the proof overview section (Section 5) more precise. As we discuss in subsection Light Ball 5.2.2, we have chosen the distribution of t (Cumulative distribution function F, Definition 5.4) to satisfy two properties: (Property I) the probability that u and v are separated by P is upper bounded by the probability that u or v is in P times $O(D_\beta)$; and (Property II) The probability that t is close to $\pi_S^{inv}(R/2)$ is small. Thus, Condition I of Lemma B.2 holds for u with $d(z,u)$ that are sufficiently close to $\pi_S^{inv}(R/2)$, and Condition II holds for u with for smaller values of $d(z,u)$.

Proof. Consider one term from the left hand side of property (b) of Theorem 5.1 for some u in X:

$$\Big(\Pr\left\{\delta_P(u,v) = 1\right\} - D_\beta \frac{d(u,v)}{R} \cdot \Pr\{\vee_P(u,v) = 1\}\Big)^+.$$

Note that $\{\delta_P(u,v) = 1\}$ denotes the event that *exactly one of the points u and v* lies in P; whereas $\{\vee_P(u,v) = 1\}$ denotes the event that *at least* one of u and v lies in P. Thus, $\Pr\{\vee_P(u,v) = 1\} \geq \Pr\{\delta_P(u,v) = 1\}$. Hence, this expression is positive only if $D_\beta \cdot d(u,v)/R < 1$, which is equivalent to

$$d(u,v) < R/D_\beta = R_0.$$

Thus, in the left hand side of property (b), we can consider only v in $\mathrm{Ball}(u, R_0)$ (rather than $\mathrm{Ball}(u, R)$). Moreover, if $d(z,u) > R$, then for all $v \in \mathrm{Ball}(u, R_0)$, we have $d(z,v) > R - R_0 = R_1$ and, consequently, $\delta_P(u,v) = 0$. Therefore, for such u, the left hand side of property (b) equals 0, and the inequality (b) holds trivially. We will thus assume that $d(z,u) \leq R$ (which is equivalent to $u \in \mathrm{Ball}(z,R)$). Similarly, since $t > 3R_0$, we will assume that $d(z,u) \geq 2R_0$ (otherwise, $u \in P$ and every $v \in \mathrm{Ball}(u, R_0)$ is in P, and thus $\delta_P(u,v) = 0$).

We now show that if Condition I or II of Lemma B.2 holds for $u \in \mathrm{Ball}(z,R)$ then property (b) is satisfied for that u.

I. Suppose, the first condition holds for $u \in \mathrm{Ball}(z,R)$. If $\delta_P(u,v) = 1$ then either $u \in P,\ v \notin P$ or $v \in P,\ u \notin P$. In the former case, $t \geq d(z,u)$; in the latter case, $t \geq d(z,v) \geq d(z,u) - R_0$. In either case, $t \geq d(z,u) - R_0$. Using that $|\,\mathrm{Ball}(u, R_0)| \leq |\,\mathrm{Ball}(z, R_0)|$ by our choice of z (see (B.1)), we bound the left hand side of (b) as follows

$$\begin{aligned} &\sum_{v \in \mathrm{Ball}(u,R_0)} \Pr\{\delta_P(u,v) = 1\} \\ &\qquad \leq |\,\mathrm{Ball}(u, R_0)| \Pr\{t \geq d(z,u) - R_0\} \\ &\qquad \leq |\,\mathrm{Ball}(z, R_0)| \Pr\{t \geq d(z,u) - R_0\} \end{aligned}$$

We now use the inequality from Condition I to get the bound

$$\sum_{v\in \mathrm{Ball}(u,R_0)} \Pr\{\delta_P(u,v)=1\} \lesssim \beta^{q+1}\mathbf{E}|\,\mathrm{Ball}(z,t)| = \beta^{q+1}\mathbf{E}|P|.$$

Finally, by Lemma B.1, we have the following bound on $\beta^{q+1}\mathbf{E}|P|$:

$$\beta^{q+1}\mathbf{E}|P| \le \beta^{q+1}\cdot 2D_\beta \mathbf{E}[Y_P] \le \beta^q \mathbf{E}[Y_P]. \qquad \text{(B.4)}$$

Here, we used that $2\beta D_\beta = 2r/R_0 < 1$ by our choice of β_q^*. The right hand side of the inequality in property (b) equals $\beta^q\mathbf{E}[Y_P]$. Thus, property (b) holds.

II. Suppose now that the second condition holds for $u \in \mathrm{Ball}(z,R)$. Then, each term in the left hand side of (b) is upper bounded by $O(\beta^{q+1})$. Hence, the entire sum is upper bounded by $O(\beta^{q+1}|\,\mathrm{Ball}(u,R_0)|)$, which in turn is upper bounded by $O(\beta^{q+1}|\,\mathrm{Ball}(z,R_0)|)$. Then,

$$\beta^{q+1}|\,\mathrm{Ball}(z,R_0)| \le \beta^{q+1}|\,\mathrm{Ball}(z,t)| = \beta^{q+1}\mathbf{E}|P| \le \beta^q\mathbf{E}[Y_P].$$

The last inequality follows from (B.4). We conclude that property (b) of Theorem 5.1 holds. □

B.2. Heavy Ball Case

In this subsection, we analyze the case when $|\,\mathrm{Ball}(z,R_1)| \ge \rho_q(\beta)\cdot|\,\mathrm{Ball}(z,R_0)|$. If this condition is met, then the algorithm outputs $P=\mathrm{Ball}(z,R_1)$. We will show that Theorem 5.1 holds for such a cluster P.

We first prove properties (a) and (b). Since the radius of P is $R_1 \le R$, its diameter is at most $2R$. So property (a) of Theorem 5.1 holds. To show property (b), we apply Lemma B.2 (item I) with $t = R_1$. Trivially, $\Pr\{t \ge d(z,u)-R_0\}\le 1$ and $\mathbf{E}|\,\mathrm{Ball}(z,t)| = |\,\mathrm{Ball}(z,R_1)| \ge \rho_q(\beta)|\,\mathrm{Ball}(z,R_0)|$. Thus, (B.2) is satisfied and property (b) also holds.

We now show property (c) of Theorem 5.1. Observe that if $d(z,u)\notin[R_1-r,R_1+r]$, then $\delta_P(u,v)=0$ for all $v\in\mathrm{Ball}(u,r)$ (because $P=\mathrm{Ball}(z,R_1)$). Hence, for such u, property (c) holds. Thus, we assume that $u\in[R_1-r,R_1+r]\subseteq[2R_0,R]$.

We bound the left hand side of (c) as follows:

$$\sum_{v\in\mathrm{Ball}(u,r)}\delta(u,v) \le |\,\mathrm{Ball}(u,r)| \le |\,\mathrm{Ball}(u,R_0)| \le |\,\mathrm{Ball}(z,R_0)|,$$

here we first use that $r\le R_0$ and then that z satisfies (B.1). Since we are in the Heavy Ball Case, we have $|P|\ge\rho_q(\beta)|\,\mathrm{Ball}(z,R_0)|$. Therefore,

$$\sum_{v\in\mathrm{Ball}(u,r)}\delta(u,v)\le |P|/\rho_q(\beta).$$

By Lemma B.1, the right hand side is upper bounded by

$$2D_\beta Y_P/\rho_q(\beta) = 2D_\beta\,\beta^{q+1}\,Y_P \lesssim \beta D_\beta^2\,Y_P.$$

The right hand side of (c) equals $\beta D_\beta^2\,Y_P$. Hence, property (c) is satisfied.

Thus we have shown that Theorem 5.1 holds for the case of Heavy Balls. To complete the proof, we show that Theorem 5.1 also holds for the case of Light Balls – we give this proof in Appendix C.

C. Light Ball Case

We now consider the case when $|\,\mathrm{Ball}(z,R_1)|\le\rho_q(\beta)\cdot|\,\mathrm{Ball}(z,R_0)|$. Recall that S is the set of all radii $s\in(3R_0,R_1]$ for which property (c) of Theorem 5.1 holds (Definition 5.2). The set S can be found in polynomial time since the number of distinct balls $\mathrm{Ball}(z,s)$ is upper bounded by the number of points in the metric space. We now recall map π_S used in Algorithm 3.

Map π_S. In Section 5.2, we define a measure preserving transformation π_S that maps a given measurable set $S\subset[0,R]$ to the interval $[0,\mu(S)]$ (Definition 5.3). We need this transformation in Algorithm 3. If S is the union of several disjoint intervals (as in our algorithm) then π_S simply pushes all intervals to the left so that every two consecutive intervals touch each other. We show the following lemma.

Lemma C.1. *For any measurable set S, π_S is a continuous non-decreasing 1-Lipschitz function, and π_S^{inv} is a strictly increasing function defined for all y in $[0,\mu(S)]$. Moreover, there exists a set Z_0 of measure zero such that for all $y\in[0,\mu(S)]\setminus Z_0$, we have $\pi_S^{inv}(y)\in S$.*

Proof. Note that $\pi_S^{inv}(y)$ is a right inverse for $\pi_S(x)$: $\pi_S(\pi_S^{inv}(y))=y$ (but not necessarily a left inverse). Let

$$I_S(x)=\begin{cases}1, & \text{if } x\in S\\ 0, & \text{otherwise}\end{cases}$$

be the indicator function of set S. Then $\pi_S(x)=\int_0^x I_S(t)dt$ (we use Lebesgue integration here). Since $0\le I_S(t)\le 1$, function π_S is non-decreasing, 1-Lipschitz, and absolutely continuous. By the Lebesgue differentiation theorem, $\pi_S(x)$ is almost everywhere differentiable and $\frac{d\pi_S(x)}{dx}=I_S(x)$ almost everywhere. Let $X_0=[0,R]\setminus S$ and $Z_0=\pi_S(X_0)$. Since π_S is absolutely continuous and $I_S(x)=0$ for $x\in X_0$, we have

$$\mu(Z_0)\le\int_{X_0}\frac{d\pi_S(x)}{dx}dx=\int_{X_0}I_S(x)dx=0.$$

Now if $y\notin Z_0$, then $\pi_S(\pi_S^{inv}(y))=y\notin Z_0$, thus $\pi_S^{inv}(y)\notin X_0$ or, equivalently, $\pi_S^{inv}(y)\in S$, as required.

Finally, we verify that π_S^{inv} is strictly increasing. Consider $a, b \in [0, \mu(S)]$ with $a < b$. Note that $a = \pi_S(\pi_S^{inv}(a))$ and $b = \pi_S(\pi_S^{inv}(b))$. Thus, $\pi_S(\pi_S^{inv}(a)) < \pi_S(\pi_S^{inv}(b))$. Since π_S is non-decreasing, $\pi_S^{inv}(a) < \pi_S^{inv}(b)$. □

Note that if S is a union of finitely many disjoint open intervals, then Z_0 is the image of the endpoints of those intervals under π_S.

C.1. Clusters Satisfying Property (c) of Theorem 5.1

We first show that if $|\operatorname{Ball}(z, R_1)| < \rho_q(\beta) \cdot |\operatorname{Ball}(z, R_0)|$, then $\mu(S) \geq R/2$. To this end, we define a ball with a γ-light shell of width r.

Definition C.2. *We say that the ball of radius $t \geq r$ around z has a γ-light shell of width r if*

$$|\operatorname{Ball}(z, t+r)| - |\operatorname{Ball}(z, t-r)| \leq \gamma \int_0^{t-r} |\operatorname{Ball}(z, x)|\, dx.$$

We let S_γ be the set of all radii t in the range $(3R_0, R_1]$ such that $\operatorname{Ball}(z, t)$ has a γ-light shell of width r. We now show that (a) $S_\gamma \subset S$ and (b) $\mu(S_\gamma) \geq R/2$ for $\gamma = 25r/R_0^2$. and, therefore, $\mu(S) \geq R/2$.

Lemma C.3. *We have $S_\gamma \subset S$.*

Proof. Consider a number t from S_γ and the ball of radius t around z: $P = \operatorname{Ball}(z, t)$. Let us pick an arbitrary point u. We are going to prove that inequality (5.2) holds and therefore $t \in S$. Consider $v \in \operatorname{Ball}(u, r)$. Observe that $\delta_P(u, v) = 1$ only if both u and v belong to the r neighborhoods of P and $X \setminus P$. Thus, if $\delta_P(u, v) = 1$, we must have $d(z, u), d(z, v) \in [t - r, t + r]$. If $d(z, u) \notin [t - r, t + r]$, then the left side of (5.2) equals 0, and we are done. Hence, we can assume that $d(z, u) \in [t - r, t + r]$.

Using the observation above, we bound the left hand side of (5.2) as

$$\sum_{v \in \operatorname{Ball}(u,r)} \delta_P(u, v) \leq |\operatorname{Ball}(z, t + r)| - |\operatorname{Ball}(z, t - r)|.$$

We now need to lower bound the right hand side of (5.2). Note that $\operatorname{Ball}(u, 2R)$ contains $\operatorname{Ball}(z, t)$, since

$$d(z, u) \leq t + r \leq R_1 + r = R - R_0 + r < R,$$

and $t < R$. Thus,

$$\sum_{v \in \operatorname{Ball}(u,2R)} \frac{d(u, v)}{R} \vee_p(u, v) \geq \frac{1}{R} \sum_{v \in \operatorname{Ball}(z,t)} d(u, v) \vee_p(u, v).$$

For all $v \in \operatorname{Ball}(z, t) \equiv P$, we have $\vee_P(u, v) = 1$. Hence,

$$\sum_{v \in \operatorname{Ball}(z,t)} d(u, v) \vee_p (u, v) = \sum_{v \in \operatorname{Ball}(z,t)} d(u, v) \qquad \text{(C.1)}$$

By the triangle inequality, we have

$$d(u, v) \geq (d(z, u) - d(z, v))^+ \geq ((t - r) - d(z, v))^+.$$

Observe that

$$((t - r) - d(z, v))^+ = \int_0^{t-r} \mathbf{1}\{d(z, v) \leq x\}\, dx.$$

Hence, (C.1) is lower bounded by

$$\sum_{v \in \operatorname{Ball}(z,t)} \int_0^{t-r} \mathbf{1}\{d(z, v) \leq x\}\, dx = \int_0^{t-r} \sum_{v \in \operatorname{Ball}(z,t)} \mathbf{1}\{d(z, v) \leq x\}\, dx = \int_0^{t-r} |\operatorname{Ball}(z, x)|\, dx.$$

Since the ball of radius t has a γ-light shell of width r, the expression above is, in turn, lower bounded by

$$\frac{|\operatorname{Ball}(z, t + r)| - |\operatorname{Ball}(z, t - r)|}{\gamma}.$$

Thus, the right hand side of inequality (5.2) is lower bounded by

$$\frac{25\beta D_\beta^2}{R} \cdot \frac{|\operatorname{Ball}(z, t + r)| - |\operatorname{Ball}(z, t - r)|}{\gamma}.$$

This completes the proof of Lemma C.3, since

$$\frac{25\beta D_\beta^2}{R} \cdot \frac{1}{\gamma} = \frac{25\beta D_\beta^2 R_0^2}{25R \cdot r} = \frac{(r/R)\, D_\beta^2\, (R/D_\beta)^2}{Rr} = 1.$$

□

Lemma C.4. *We have $\mu(S_\gamma) \geq R/2$.*

To prove this lemma, we use the following result from Appendix D.

Lemma C.5. *Consider a non-decreasing function $\Phi : [0, R] \to \mathbb{R}$ with $\Phi(0) = 1$ and $R > 0$. Let $r \in (0, R]$ and $\gamma \leq (0, 1/r]$. Then, for the subset S of numbers $t \in [0, R - r]$ for which inequality*

$$\Phi(t + r) \geq \Phi(t) + \gamma \int_0^t \Phi(x) dx \qquad \text{(C.2)}$$

holds, we have $\Phi(R) \geq e^{\eta\mu(S)-1}$, where $\eta = \sqrt{\gamma/(e-1)r}$, and $\mu(S)$ is the measure of set S.

Proof of Lemma C.4. We apply Lemma C.5 to the function

$$\Phi(t) = \frac{|\operatorname{Ball}(z, t + 3R_0)|}{|\operatorname{Ball}(z, 3R_0)|}$$

with parameters $r' = 2r$, $R' = R_1 - 3R_0 - r$, and $\gamma = 25r/R_0^2$. Note that to be able to apply Lemma C.5 we need $\gamma < 1/r'$ which is equivalent to $\beta D_\beta < 1/5\sqrt{2}$. The latter holds due to β being sufficiently small, i.e., $\beta \leq \Theta\big(\frac{1}{(q\ln(q+1)}\big)$. Observe that $\Phi(0) = 1$ and

$$\Phi(R') \leq \frac{|\operatorname{Ball}(z, R_1)|}{|\operatorname{Ball}(z, 3R_0)|} \leq \frac{\rho_q(\beta)|\operatorname{Ball}(z, R_0)|}{|\operatorname{Ball}(z, R_0)|} = \rho_q(\beta).$$

Here, we used that the $\operatorname{Ball}(z, R_1)$ is light. From Lemma C.5, we get that $\Phi(R') \geq e^{\eta'\mu(S')-1}$, where $\eta' = \sqrt{\gamma/(e-1)r'}$, and S' is the set of t for which Inequality (C.2) holds. Thus,

$$\begin{aligned}
\mu(S') &\leq \frac{1 + \ln \Phi(R')}{\eta'} \leq \frac{1 + \ln \rho_q(\beta)}{\eta'} = \frac{1 + D_\beta/2}{\eta'} \\
&= \sqrt{\frac{(e-1)r'}{\gamma}} \cdot (1 + D_\beta/2) \\
&= \sqrt{\frac{2(e-1)r}{25r}} \cdot R_0 \cdot (1 + D_\beta/2) \\
&= \sqrt{\frac{2(e-1)}{25}} \cdot (R_0 + R/2) < 0.4(R + R_0).
\end{aligned}$$

where we used $R_0 \cdot D_\beta = R$ and that $\sqrt{2(e-1)} < 2$. Therefore for the measure of the set $S'' = [0, R'] \setminus S'$ is at least $\mu(S'') \geq ((R - R_0) - 3R_0 - r) - 0.4(R + R_0) \geq R/2$. Here, we relied on our assumption that $R_0 + r < R_1/100$.

We claim that $S'' + 3R_0 + r \subset S_\gamma$. Consider an arbitrary $t \in S''$. First, observe that $t + 3R_0 + r \in (3R_0, R_1]$. Then,

$$\begin{aligned}
&\frac{|\operatorname{Ball}(z, t + 3R_0 + r')|}{|\operatorname{Ball}(z, 3R_0)|} - \frac{|\operatorname{Ball}(z, t + 3R_0)|}{|\operatorname{Ball}(z, 3R_0)|} \\
&\quad = \Phi(t + r') - \Phi(t) < \gamma \int_0^t \Phi(x)dx \\
&\quad\quad = \gamma \int_0^t \frac{|\operatorname{Ball}(z, x + 3R_0)|}{|\operatorname{Ball}(z, 3R_0)|}dx.
\end{aligned}$$

For $t' = t + 3R_0 + r$, we get

$$\begin{aligned}
&|\operatorname{Ball}(z, t' + r)| - |\operatorname{Ball}(z, t' - r)| < \\
&\quad < \gamma \int_0^{t'-3R_0-r} |\operatorname{Ball}(z, x + 3R_0)|dx \\
&\quad = \gamma \int_{3R_0}^{t'-r} |\operatorname{Ball}(z, x)|dx < \gamma \int_0^{t'-r} |\operatorname{Ball}(z, x)|dx.
\end{aligned}$$

Thus, $t' \in S_\gamma$. This finishes the proof. □

Lemma C.4 together with Lemma C.3 imply the following corollary.

Corollary C.6. *Let S be the set defined in Definition 5.2. Then, $\mu(S) \geq R/2$.*

C.2. Clusters Satisfying Property (b) of Theorem 5.1

We now show how to choose a random $t \in S$, so that the random cluster $P = \operatorname{Ball}(z, t)$ satisfies property (b) of Theorem 5.1. We first choose a random $x \in [0, R/2]$ with the cumulative distribution function $F(x)$ defined in Definition 5.4, and then let $t = \pi_S^{inv}(x)$, where $S \subset (3R_0, R_1]$ is the set obtained in the previous section. Note that by Lemma C.1, $t = \pi_S^{inv}(x) \in S$ with probability 1, since $\Pr\{x \in Z_0\} = 0$ (see Lemma C.1).

To show that property (b) is satisfied, we verify that for every u in $\operatorname{Ball}(z, R)$, Condition I or Condition II of Lemma B.2 holds.

Pick a point u in $\operatorname{Ball}(z, R)$. We consider two cases: $\pi_S(d(z, u)) > R/2 - R_0$ and $\pi_S(d(z, u)) \leq R/2 - R_0$. We prove that u satisfies Condition I of Lemma B.2 in the former case and Condition II in the latter case.

First case: $\pi_S(d(z, u)) > R/2 - R_0$. Write,

$$\Pr\{t \geq d(z, u) - R_0\} = \Pr\{x \geq \pi_S(d(z, u) - R_0)\}.$$

Since π_S is a 1-Lipschitz function, we have

$$\pi_S(d(z, u) - R_0) \geq \pi_S(d(z, u)) - R_0 \geq R/2 - 2R_0.$$

Therefore,

$$\Pr\{t \geq d(z, u) - R_0\} \leq 1 - F(R/2 - 2R_0).$$

We prove the following claim.

Claim C.7. *We have*

$$1 - F(R/2 - 2R_0) \lesssim \beta^{q+1}.$$

Proof. Write:

$$F(R/2 - 2R_0) = \frac{1 - e^{\frac{-R}{2R_0}} \cdot e^{\frac{2R_0}{R_0}}}{1 - e^{\frac{-R}{2R_0}}} = \frac{1 - e^2 e^{-D_\beta/2}}{1 - e^{-D_\beta/2}}.$$

Note that $e^{-D_\beta/2} = \beta^{q+1}$. Then,

$$1 - F(R/2 - 2R_0) = \frac{(e^2 - 1)}{1 - \beta^{q+1}} \cdot \beta^{q+1}.$$

Since the denominator of the right hand side is greater than $1/2$ (recall that we assume that β is sufficiently small), we have $1 - F(R/2 - 2R_0) \lesssim \beta^{q+1}$. □

Claim C.7 finishes the analysis of the first case, since $|\operatorname{Ball}(z, t)|/|\operatorname{Ball}(z, R_0)| \geq 1$ for every value of $t \geq R_0$.

Second case: $\pi_S(d(z, u)) \leq R/2 - R_0$. In this case, for every $v \in \operatorname{Ball}(u, R_0)$, we have

$$\pi_S(d(z, v)) \leq \pi_S(d(z, u) + R_0) \leq R/2.$$

Here, we used that π_S is a 1-Lipschitz function. We claim that inequality (B.3) holds for every two points $v_1, v_2 \in X$ with $\pi_S(d(z,v_1)), \pi_S(d(z,v_2)) \leq R/2$ and $d(v_1,v_2) \leq R_0$. In particular, it holds for $v_1 = u$ and $v_2 = v$. Without loss of generality assume, that $d(z,v_1) \leq d(z,v_2)$. Then,

$$\begin{aligned}\Pr\{\delta_P(v_1,v_2)=1\} &= \Pr\{d(z,v_1) \leq t < d(z,v_2)\}\\ &= \Pr\{\pi_S(d(z,v_1)) \leq x < \pi_S(d(z,v_2))\}\\ &= F(\pi_S(d(z,v_2))) - F(\pi_S(d(z,v_1))).\end{aligned}$$

Here, we used that random variable x has distribution function F. We show the following claim.

Claim C.8. *For all $x_1 \leq x_2$ in the range $[0, R/2]$, we have*

$$F(x_2) - F(x_1) \leq D_\beta \cdot \frac{(x_2 - x_1)}{R} \cdot \big(1 - F(x_1) + 2\beta^{q+1}\big).$$

Proof. We have

$$\begin{aligned}F(x_2) - F(x_1) &= \int_{x_1}^{x_2} F'(x)dx\\ &\leq (x_2 - x_1) \max_{x\in[x_1,x_2]} F'(x)\\ &= (x_2 - x_1)\cdot \frac{e^{-x_1/R_0}/R_0}{1-e^{-R/2R_0}}\\ &= D_\beta \cdot \frac{(x_2-x_1)}{R}\cdot\frac{e^{-x_1/R_0}}{1-e^{-R/2R_0}}.\end{aligned}$$

Here, we used that $R_0 = R/D_\beta$. We now need to upper bound the third term on the right hand side:

$$\begin{aligned}\frac{e^{-x_1/R_0}}{1-e^{-R/2R_0}} &= 1 - \frac{(1-e^{-R/2R_0}) - e^{-x_1/R_0}}{1-e^{-R/2R_0}}\\ &= 1 - F(x_1) + \frac{e^{-R/2R_0}}{1-e^{-R/2R_0}}.\end{aligned}$$

As in Claim C.7, let us use that $e^{-R/2R_0} = \beta^{q+1}$ and $1 - \beta^{q+1} \geq 1/2$ to get

$$\frac{e^{-x_1/R_0}}{1-e^{-R/2R_0}} \leq 1 - F(x_1) + 2\beta^{q+1}.$$

Combining the bounds above, we get the following inequality:

$$F(x_2) - F(x_1) \leq D_\beta \cdot \frac{x_2 - x_1}{R}\cdot\Big(1 - F(x_1) + 2\beta^{q+1}\Big).$$

□

Using Claim C.8 and the inequality

$$\begin{aligned}\pi_S(d(z,v_2))) - \pi_S(d(z,v_1) &\leq d(z,v_2) - d(z,v_1)\\ &\leq d(v_1,v_2),\end{aligned}$$

we derive the following upper bound

$$\begin{aligned}&\Pr\{\delta_P(v_1,v_2)=1\} \leq\\ &\leq D_\beta \frac{d(v_1,v_2)}{R}\cdot\big(1 - F(\pi_S(d(z,v_1)))) + 2\beta^{q+1}\big).\end{aligned}$$

Then,

$$\begin{aligned}\Pr\{\vee_P(v_1,v_2)=1\} &= \Pr\{d(z,v_1) \leq t\}\\ &= 1 - F(\pi_S(d(z,v_1))).\end{aligned}$$

Therefore,

$$\begin{aligned}&\Pr\{\delta_P(v_1,v_2)=1\} - D_\beta\frac{d(v_1,v_2)}{R}\cdot\Pr\{\vee_P(v_1,v_2)=1\} \leq\\ &\leq 2D_\beta\frac{d(v_1,v_2)}{R}\beta^{q+1}.\end{aligned}$$

Thus, the left hand side of (B.3) is upper bounded by

$$2D_\beta\cdot\beta^{q+1}\cdot\frac{d(v_1,v_2)}{R} \leq 2\beta^{q+1}.$$

Here, we use that $d(v_1,v_2) \leq R_0$ and $R_0 = R/D_\beta$.

D. Proof of Lemma C.5

We first prove Lemma C.5 for the case when S is a measure zero set. Specifically, we show the following lemma.

Lemma D.1. *Suppose, a non-decreasing function $\Phi : [0,R] \to \mathbb{R}$ with $\Phi(0) = 1$ satisfies the following inequality for all $t \in [0, R-r] \setminus Y_0$, where set Y_0 has measure zero:*

$$\Phi(t+r) \geq \Phi(t) + \gamma\int_0^t \Phi(x)dx, \tag{D.1}$$

for some $R > 0$, $r \in (0, R/2]$ and $\gamma \in (0, 1/r]$, then $\Phi(t) \geq \max\{e^{\eta t-1}, 1\}$ for all $t \in [0,R]$, where $\eta = \sqrt{\gamma/(e-1)r}$. Consequently, we have $\Phi(R) \geq e^{\eta R-1}$.

Proof. Since $\Phi(0) = 1$ and $\Phi(t)$ is non-decreasing, we have $\Phi(t) \geq 1$ for all $t \geq 0$. We now prove that $\Phi(t) \geq e^{\eta t-1}$. We establish this inequality by induction. The inductive hypothesis is that this inequality holds for $t \in [0, 1/\eta + ir] \cap [0,R]$ for integer $i \geq 0$. For $t \leq 1/\eta$, we have $\Phi(t) \geq 1 > e^{\eta t-1}$. Thus, the inductive hypothesis holds for $i = 0$. Suppose, it holds for i, we prove it for $i+1$.

First, consider an arbitrary $t^* \in [1/\eta, 1/\eta + (i+1)r] \cap [0,R] \setminus (Y_0 + r)$, where $Y_0 + r$ is the set Y_0 shifted right by r. Let $t = t^* - r$. Note that $t > 0$, since $r < 1/\eta$. Also, $t \notin Y_0$. Then, by the inductive hypothesis, we have $\Phi(x) \geq e^{\eta x-1}$ for all $x \in [1/\eta, t]$. Using Inequality (D.1), we obtain the following bound

$$\Phi(t^*) = \Phi(t+r)$$

$$
\begin{aligned}
&\geq \Phi(t) + \gamma \int_0^{1/\eta} \Phi(x)dx + \gamma \int_{1/\eta}^{t} \Phi(x)dx \\
&\geq e^{\eta t-1} + \gamma \int_0^{1/\eta} 1\, dx + \gamma \int_{1/\eta}^{t} e^{\eta x-1}\, dx \\
&= e^{\eta t-1} + \gamma/\eta + \gamma/\eta \cdot (e^{\eta t-1} - 1) \\
&= e^{\eta t-1}(1 + \gamma/\eta).
\end{aligned}
$$

Since $\eta = \sqrt{\gamma/(e-1)r}$, we have $\gamma/\eta = (e-1)\eta r$. Now, using the inequality $e^x \leq 1 + (e-1)x$ for $x \in [0,1]$, we get

$$
\begin{aligned}
\Phi(t^*) &\geq e^{\eta t-1}(1 + \gamma/\eta) = e^{\eta t-1}(1 + (e-1)\eta\delta) \\
&\geq e^{\eta t-1} \cdot e^{\eta r} = e^{\eta(t+r)-1} = e^{\eta t^*-1}.
\end{aligned}
$$

To finish the proof, we need to show that $\Phi(t^{**}) \geq e^{\eta t^{**}-1}$ for $t^{**} \in [1/\eta, 1/\eta + (i+1)r] \cap [0,R] \cap (Y_0 + r)$. Since $Y_0 + r$ has measure zero, there exists an increasing sequence t_k^* of numbers in $[0, 1/\eta + (i+1)r] \cap ([0,R] \setminus (Y_0 + r))$ that tends to t^{**} as $k \to \infty$. Using that Φ is a non-decreasing function and $e^{\eta t-1}$ is a continuous function, we have

$$
\Phi(t^{**}) \geq \lim_{k\to\infty} \Phi(t_k^*) \geq \lim_{k\to\infty} e^{\eta t_k^*-1} = e^{\eta t^{**}-1}.
$$

□

We now show that Lemma D.1 implies Lemma C.5. Loosely speaking, in the proof, we shift all intervals from the set S to the left to obtain a single interval $[0, \mu(S)]$. We then apply Lemma D.1 to the transformed function.

Proof of Lemma C.5. Let π_S and π_S^{inv} be the maps defined in Appendix C. Define $\Phi^*(t)$ as $\Phi^*(t) = \Phi(\pi_S^{inv}(t))$ and let Y_0 be a measure zero set as in Lemma C.5. We claim that $\Phi^*(t)$ satisfies (D.1) for all $t \in [0, \pi(S)] \setminus Y_0$. Fix $t \in [0, \pi(S)] \setminus Y_0$. Write

$$
\Phi^*(t+r) = \Phi(\pi_S^{inv}(t+r)) \geq \Phi(\pi_S^{inv}(t) + r).
$$

Here, we used that (a) $\pi_S^{inv}(t+r) \geq \pi_S^{inv}(t) + r$ and (b) Φ is a monotone function. By Lemma C.5, $\pi_S^{inv}(t) \in S$, thus

$$
\begin{aligned}
\Phi^*(t+r) &\geq \Phi(\pi_S^{inv}(t) + r) \\
&\geq \Phi(\pi_S^{inv}(t)) + \gamma \int_0^{\pi_S^{inv}(t)} \Phi(x)dx.
\end{aligned}
$$

We now observe that $\Phi^*(t) = \Phi(\pi_S^{inv}(t))$ and

$$
\begin{aligned}
\int_0^{\pi_S^{inv}(t)} \Phi(x)dx &\geq \int_0^{\pi_S^{inv}(t)} \Phi(x) \cdot \mathbf{1}(x \in S)dx \\
&= \int_0^{\pi_S^{inv}(t)} \Phi(x) d\pi_S(x) \\
&= \int_0^{t} \Phi^*(x)dx.
\end{aligned}
$$

Here, we used that $d\pi_S(x) = \mathbf{1}(x \in S)dx$ and $\pi_S(\pi_S^{inv}(t)) = t$. Thus, we showed that for all $t \in [0, \pi(S)] \setminus Y_0$, we have

$$
\Phi^*(t+r) \geq \Phi^*(t) + \int_0^t \Phi^*(x)dx.
$$

We now use Lemma D.1 with function Φ^* and $R' = \mu(S)$. We obtain the following inequality:

$$
\Phi^*(\mu(S)) \geq e^{\eta\mu(S)-1},
$$

which concludes the proof of Lemma C.5. □

E. Integrality Gap

In this section, we present an integrality gap example for the convex program (P) in Figure 1.

Construction. Let $n = 1 + \lceil\sqrt{1/\alpha}\rceil$. Consider a complete graph on n vertices. Let P be a path of length $n-1$. Denote its endpoints by s and t and the set of its edges by E_P. All edges in P are positive edges of weight 1. Edge (s,t) is a negative edge of weight 1. All other edges are positive edges of weight α.

The cost of the integral solution Clearly, every integral solution $\mathcal{P}$ should violate some edge $(u,v) \in P \cup \{(s,t)\}$ (since all these edges cannot be satisfied simultaneously). Thus, $\mathrm{dis}_u(\mathcal{P}, E^+, E^-) \geq 1$ and $\|\mathrm{dis}(\mathcal{P}, E^+, E^-)\|_p \geq 1$.

The cost of the CP solution. We define the CP solution as follows. Denote the distance between u and v along P by $\mathrm{dist}_P(u,v)$. Let $x_{uv} = \mathrm{dist}_P(u,v)/(n-1)$. Note that $x_{st} = 1$. The values of variables y_u are determined by constraints (P1) of the convex program.

Now we upper bound the contribution of every edge (u,v) (incident on u) to y_u in formula (P1). The contribution of $(u,v) \in E_P$ is $w_{uv}x_{uv} = 1 \cdot 1/(n-1)$; the contribution of edge (s,t) is $w_{st}(1 - x_{st}) = 0$ (whether or not it is incident on u), and the contribution of every other edge (u,v) is $w_{uv}x_{uv} \leq \alpha$. Since every vertex u is incident on at most 2 edges from E_P, we have $y_u \leq 2/(n-1) + \alpha n \lesssim \sqrt{\alpha}$. Now,

$$
\|y\|_p \leq n^{1/p} \cdot \max_u |y_u| \lesssim n^{1/p}\alpha^{1/2} \lesssim \alpha^{1/2-1/(2p)}.
$$

Integrality gap We conclude that the integrality gap is at least $\Omega((1/\alpha)^{1/2-1/(2p)})$.