Prior Image-Constrained Reconstruction using Style-Based Generative Models: Supplementary Information

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A. Theoretical Analysis

As described in the main manuscript, the theoretical analysis presented here provides a non-uniform recovery guarantee, which applies to *in-distribution objects* that are in the range of the StyleGAN G, further constrained by styles from the prior image. In contrast to the theoretical results presented in (Bora et al., 2017) where the Lipschitz constant of the generator network G is used to bound the number of measurements, the analysis presented here is in terms of the expected Frobenius norm of its Jacobian. Due to this, the presented analysis applies to generative networks having Lipschitz constants that are large as compared to the typical scaling of differences under the network, or generative networks that are not Lipschitz stable, such as StyleGAN2. The price paid is in terms of the guarantee being non-uniform, and allowing for an additional network-dependent term in the reconstruction error. Nevertheless, as we show, the proposed guarantees are useful in analyzing the behaviour of generative model-constrained reconstruction in general, and the PICGM method in particular.

We begin by defining the notation used.

Notation A.1.

1. *Let $p_{\mathbf{w}}$ denote the distribution of the extended latent space vector $\mathbf{w} = [\mathbf{u}_1^\top \ \mathbf{u}_2^\top \ \ldots \ \mathbf{u}_L^\top]^\top \in \mathcal{W}^+$, where $\mathbf{u}_i = g_{\text{mapping}}(\mathbf{z}_i)$, $\mathbf{z}_i \sim \mathcal{N}(0, I_k)$, $\mathbf{z}_i$'s are independently distributed, with I_k denoting the real $k \times k$ identity matrix.*

2. *Recall that as evidenced by (Wulff & Torralba, 2020), it can be assumed that if $\mathbf{w} \sim p_{\mathbf{w}}$, $\mathbf{v} = LReL_\alpha(\mathbf{w})$ is distributed as a multivariate Gaussian distribution. Let $\bar{\mathbf{v}}$ and Σ be its mean and covariance matrix respectively. Recall that $LReL_\alpha$ denotes the leaky-ReLU nonlinear activation, defined as*

$$LReL_\alpha(\mathbf{x})_i = \begin{cases} x_i, & x_i \geq 0, \\ \alpha x_i, & x_i < 0. \end{cases} \tag{1}$$

   The value of α is the reciprocal of the scaling value for negative numbers included in the last leakyReLU layer in the mapping network g_{mapping}.

3. *Let p_1, p_2 be positive integer multiples of k, with $1 \leq p_1 < p_2 \leq K$. Let $P = p_2 - p_1$. Let $\mathcal{W}^+_{p_1,p_2}$ be the P-dimensional subspace of $\mathcal{W}^+$ containing all $\mathbf{w}$ such that $\mathbf{w}_{1:p_1} = \mathbf{0}, \mathbf{w}_{p_2:K} = \mathbf{0}$.*

4. *Let*

$$B^K_{\mathbf{w}}(r) := \{\mathbf{w} \ s.t. \ \|\mathrm{LReL}_\alpha(\mathbf{w}) - \bar{\mathbf{v}}\|_\Sigma \leq r\},$$
$$B^K_{\mathbf{v}}(r) := \{\mathbf{v} \ s.t. \ \|\mathbf{v} - \bar{\mathbf{v}}\|_\Sigma \leq r\}.$$

   Similarly, let

$$B^{p_1,p_2}_{\mathbf{w}}(r) := \left\{\mathbf{w} \ s.t. \ \mathbf{w} \in B^K_{\mathbf{w}}(r), \mathbf{w}_{1:p_1} = \mathbf{w}^{(\mathrm{PI})}_{1:p_1}, \mathbf{w}_{p_2:K} = \mathbf{w}^{(\mathrm{PI})}_{p_2:K}\right\},$$
$$B^{p_1,p_2}_{\mathbf{v}}(r) := \left\{\mathbf{v} \ s.t. \ \mathbf{v} \in B^K_{\mathbf{v}}(r), \mathbf{v}_{1:p_1} = \mathbf{v}^{(\mathrm{PI})}_{1:p_1}, \mathbf{v}_{p_2:K} = \mathbf{v}^{(\mathrm{PI})}_{p_2:K}\right\},$$

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where for a vector $\mathbf{x} \in \mathbb{R}^K$, $\|\mathbf{x}\|_\Sigma^2 := \mathbf{x}^\top \Sigma^{-1}\mathbf{x}$. *Note that* $\alpha > 1$ *and* $1 \leq p_1 < p_2 \leq K$.

5. *Let* $J(\mathbf{w})$ *denote the Jacobian of the synthesis network* G *evaluated at* $\mathbf{w}$. *Let* $J_{p_1:p_2}(\mathbf{w})$ *denote the Jacobian of* G *with respect to* $\mathbf{w}_{p_1:p_2}$ *, evaluated at* $\mathbf{w}$.

We first prove the following series of lemmas.

Lemma A.1. *Let* $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_K$ *be the singular values of* $\sqrt{\Sigma}$. *Let* $\boldsymbol{\sigma} = [\sigma_1\ \sigma_2\ \dots\ \sigma_K]^\top$. *For* $r > 1$, *if* $\mathcal{N}_{\mathbf{w}}^{p_1,p_2}(\epsilon)$ *is an optimal* ϵ*-net of* $B_{\mathbf{w}}^{p_1,p_2}(r)$, *then*

$$\log|\mathcal{N}_{\mathbf{w}}^{p_1,p_2}| \leq P\log\left[\frac{6r}{\epsilon}\left(\epsilon + \frac{\|\boldsymbol{\sigma}\|_2}{\sqrt{K}}\right)\right].$$

Proof. First, note that if $\mathbf{w} \sim p_{\mathbf{w}}$, then the subsections of $\mathbf{w}$ corresponding to the different style inputs, i.e. $\mathbf{w}_{lk+1:(l+1)k}$, $l = 0,\dots,L-1$ are distributed such that $\mathbf{w}_{lk+1:(l+1)k}$ is independent to and identically distributed as $\mathbf{w}_{l'k+1:(l'+1)k}$ if $l \neq l'$. This implies that the singular values of $\sqrt{\Sigma}$ are degenerate to a certain degree. Specifically,

$$\sigma_{jL+1} = \sigma_{jL+2} = \dots = \sigma_{(j+1)L}, \quad j = 0, 2, \dots, k-1 \tag{2}$$

Let $\sigma'_1 \geq \sigma'_2 \geq \dots \geq \sigma'_P$ be the singular values of $\sqrt{\mathrm{Cov}(\mathbf{v}_{p_1:p_2})}$, and let $\boldsymbol{\sigma}' = [\sigma'_1\ \sigma'_2\ \dots\ \sigma'_P]^T$. Therefore, by Eq. (2),

$$\frac{\|\boldsymbol{\sigma}'\|}{\sqrt{P}} = \frac{\|\boldsymbol{\sigma}\|}{\sqrt{K}}. \tag{3}$$

Note that $B_{\mathbf{v}}^{p_1,p_2}(r)$ is an ellipsoid with center $\bar{\mathbf{v}}_{p_1:p_2}$ and principal radii of lengths $\sigma'_i r,\ i = 1, 2, \dots, P$. Assume for a moment, that there exists an integer p such that $\sigma'_p r > \epsilon \geq \sigma'_{p+1} r$. Let $\mathcal{N}_{\mathbf{v}}(\epsilon)$ be an optimum ϵ-net of $B_{\mathbf{v}}^{p_1,p_2}(r)$.

Therefore, by Theorem 2 in (Dumer, 2006),

$$\begin{aligned}
\log|\mathcal{N}_{\mathbf{v}}(\epsilon)| &\leq \sum_{i=1}^{p}\log\left(\frac{r\sigma'_i}{\epsilon}\right) + P\log 6, \\
&\leq \log\left[\left(\frac{r}{\epsilon}\right)^P \prod_{i=1}^{p}\sigma'_i \prod_{i=p+1}^{P}\epsilon\right] + P\log 6, && \text{(since } r > 1\text{,)} \\
&\leq P\log\left[\frac{6r}{\epsilon}\left(\epsilon + \frac{1}{P}\sum_{i=1}^{P}\sigma'_i\right)\right], && \text{(by AM-GM inequality,)} \\
&\leq P\log\left[\frac{6r}{\epsilon}\left(\epsilon + \frac{\|\boldsymbol{\sigma}'\|}{\sqrt{P}}\right)\right], && \text{(by AM-RMS inequality,)} \\
&\leq P\log\left[\frac{6r}{\epsilon}\left(\epsilon + \frac{\|\boldsymbol{\sigma}\|}{\sqrt{K}}\right)\right].
\end{aligned}$$

Observe that this bound is valid even if $\epsilon > \sigma'_1 r$ or $\epsilon < \sigma'_P r$. Since $\alpha > 1$, $\mathrm{LReL}_{1/\alpha}(.)$ is a bijective function with Lipschitz constant 1. Therefore, for every $\mathbf{v}_1 = \mathrm{LReL}_\alpha(\mathbf{w}_1)$, and $\mathbf{v}_2 = \mathrm{LReL}_\alpha(\mathbf{w}_2)$,

$$\|\mathbf{w}_1 - \mathbf{w}_2\| \leq \|\mathbf{v}_1 - \mathbf{v}_2\|.$$

Therefore,

$$\log|\mathcal{N}_{\mathbf{w}}^{p_1,p_2}(\epsilon)| \leq P\log\left[\frac{6r}{\epsilon}\left(\epsilon + \frac{\|\boldsymbol{\sigma}\|}{\sqrt{K}}\right)\right]. \tag{4}$$

□

Now, the following assumptions about StyleGAN are made.

1. **Path length regularity**:

$$\mathbb{E}_{\mathbf{w}\sim p_{\mathbf{w}}}\left(\|J(\mathbf{w})\|_F - a\right)^2 < b, \tag{AS1}$$

where $b > 0$ and $a = \mathbb{E}_{\mathbf{w}}\|J(\mathbf{w})\|_F$ are global constants. As described in the main manuscript, this assumption is inspired by the path-length regularization used in (Karras et al., 2020). Although during training a is implemented as $\mathbb{E}_{\mathbf{w}}\mathbb{E}_{\mathbf{y}\sim\mathcal{N}(0,I_n)}\left\|J(\mathbf{w})^\top\mathbf{y}\right\|$, as per the Hanson-Wright inequality, this concentrates to $\mathbb{E}_{\mathbf{w}}\|J(\mathbf{w})\|_F$, when $\mathbf{y}$ is high-dimensional (Vershynin, 2018). The value of a was estimated by empirically estimating $\mathbb{E}_{\mathbf{w}\sim p_{\mathbf{w}}}\mathbb{E}_{\mathbf{y}\sim\mathcal{N}(0,I_n)}\left\|J(\mathbf{w})^\top\mathbf{y}\right\|$ using 100 samples $\mathbf{w}$ drawn from $p_{\mathbf{w}}$ and 100 samples $\mathbf{y}$ drawn from $\mathcal{N}(0,I_n)$ for each sample of $\mathbf{w}$. b was empirically estimated over the same dataset of samples using Eq. (AS1). The values of a and $\sqrt{b}$ were estimated to be around 80.1 and 16.7 respectively for the specific StyleGAN2 trained and used in the inverse-crime study.

2. **Approximate local linearity**:

$$\mathbb{E}_{\mathbf{w}\sim p_{\mathbf{w}}} \max_{\substack{\mathbf{w}'\\ \|\mathbf{w}'-\mathbf{w}\|\le\epsilon}} \mathcal{L}(\mathbf{w}',\mathbf{w}) \le \beta^2(\epsilon), \tag{AS2}$$

where

$$\mathcal{L}(\mathbf{w}',\mathbf{w}) = \|G(\mathbf{w}') - G(\mathbf{w}) - J(\mathbf{w})(\mathbf{w}'-\mathbf{w})\|_2^2$$

This property essentially measures how close G is to its linear approximation in an ϵ-neighborhood around a point $\mathbf{w}$. For ease of notation, we will write

$$\phi^2_{p_1,p_2}(\epsilon;\mathbf{w}) := \max_{\substack{\mathbf{w}'\\ \|\mathbf{w}'-\mathbf{w}\|\le\epsilon\\ \mathbf{w}'-\mathbf{w}\in\mathcal{W}^+_{p_1,p_2}}} \mathcal{L}(\mathbf{w}',\mathbf{w}), \tag{5}$$

with $\phi_{p_1,p_2}(\epsilon;\mathbf{w}) \ge 0$. Approximate estimates of $\beta^2(\epsilon)$ were obtained for several values of ϵ by first computing the Jacobian at a point $\mathbf{w}\sim p_{\mathbf{w}}$, and then iteratively maximizing $\mathcal{L}(\mathbf{w}',\mathbf{w})$ using a projected gradient ascent-type algorithm. Figure 1 shows the plot of $\beta^2(\epsilon)$ versus ϵ estimated over a dataset of 100 samples $\mathbf{w}$ from $p_{\mathbf{w}}$ for the StyleGAN2 trained and used in the inverse-crime study.

Lemma 4.1. If $\mathbf{w}$ is a sample from $p_{\mathbf{w}}$, then it satisfies the following three properties with probability at least $1 - O(1/K)$:

$$\|J(\mathbf{w})\|_F \le \sqrt{K}a, \tag{P1}$$

$$\phi_{1,K}(\epsilon;\mathbf{w}) \le \sqrt{K}\beta(\epsilon) \tag{P2}$$

$$\left\|\Sigma^{-1/2}\mathrm{LReL}_\alpha(\mathbf{w})\right\|_2 \le \sqrt{K}(1+o(1)) \tag{P3}$$

Proof. For $\mathbf{w}$ sampled from $p_{\mathbf{w}}$, P1 and P2 are true with probability at least $1 - \frac{a^2+b}{Ka^2}$ and $1 - 1/K$ respectively, due to Markov's inequality (Vershynin, 2018). P3 is true with probability at least $1 - \Omega(e^{-cK})$ due to concentration of norm (Vershynin, 2018). Therefore, by union bound, $\mathbf{w}$ satisfies all the three with probability at least $1 - O(1/K)$. □

As a consequence of Lemma 4.1, for

$$\mathbf{w}_s = [\mathbf{w}_{1:p_1}^{(\mathrm{PI})\top} \quad \mathbf{w}_{p_1:p_2}^\top \quad \mathbf{w}_{p_2:K}^{(\mathrm{PI})\ \top}]^\top,$$

the following properties also hold with probability at least $1 - O(1/K)$ if $\mathbf{w}, \mathbf{w}^{(\mathrm{PI})} \sim p_{\mathbf{w}}$:

$$\|J_{p_1:p_2}(\mathbf{w}_s)\|_F \le \sqrt{K}a, \tag{TP1}$$

(Since $J_{p_1:p_2}(\mathbf{w}_s)$ is a sub-matrix of $J(\mathbf{w}_s)$)

$$\phi_{p_1,p_2}(\epsilon;\mathbf{w}_s) \le \sqrt{K}\beta(\epsilon) \tag{TP2}$$

If $\mathbf{w}$ satisfies properties P1, P2 and P3, then $G(\mathbf{w})$ will be referred to as an in-distribution image in the range of G, since these are the properties of a typical sample from the StyleGAN.

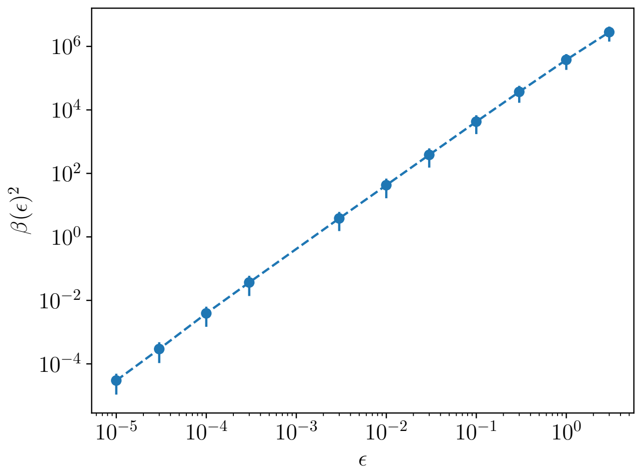

Figure 1: Plot of $\beta(\epsilon)^2$ versus ϵ

Notation A.2.
Let $\tilde{B}_{\mathbf{w}}^{p_1,p_2}(r)$ be the set of all points in $B_{\mathbf{w}}^{p_1,p_2}(r)$ satisfying properties P1 and P2.

Lemma A.2. *Let $0 < \delta \leq \frac{a\|\boldsymbol{\sigma}\|_2}{72\sqrt{K}}$. Let $\tilde{\mathcal{N}}_{\mathbf{f}}(\delta)$ be a discrete set in $G(\tilde{B}_{\mathbf{w}}^{p_1,p_2}(r))$ such that for every $\mathbf{f} \in G(\tilde{B}_{\mathbf{w}}^{p_1,p_2}(r))$, there exists an $\mathbf{f}_0 \in \mathcal{N}_{\mathbf{f}}(\delta)$ such that*

$$\|\mathbf{f} - \mathbf{f}_0\|_2 \leq \delta + \sqrt{K}\beta(\delta/a). \tag{6}$$

Then,

$$|\tilde{\mathcal{N}_{\mathbf{f}}}(\delta)| \leq P \log\left(\frac{145ar\,\|\boldsymbol{\sigma}\|}{\sqrt{P}\delta}\right). \tag{7}$$

Proof. The outline of this proof is as follows. First, an δ/a-covering over $\tilde{B}_{\mathbf{w}}^{p_1,p_2}(r)$ will be constructed. Then, each of the spherical balls covering $\tilde{B}_{\mathbf{w}}^{p_1,p_2}(r)$ is transformed into approximately an ellipsoid depending upon the Jacobian of G at the center of the spherical ball. Then, each of these ellipsoids will be approximately covered by a δ-net. The collection of all such approximate δ-nets covering the individual ellipsoids will give an approximate δ-net over $G(\tilde{B}_{\mathbf{w}}^{p_1,p_2}(r))$, which is the result required.

Let $\epsilon = \delta/a$. Let $\tilde{\mathcal{N}}_{\mathbf{w}}(\epsilon)$ be an optimal ϵ-covering of $\tilde{B}_{\mathbf{w}}^{p_1,p_2}(r)$. Also, let $\mathcal{N}_{\mathbf{w}}(\epsilon)$ denote an optimal ϵ-covering of $B_{\mathbf{w}}^{p_1,p_2}(r)$. Therefore, since $\tilde{B}_{\mathbf{w}}^{p_1,p_2}(r) \subseteq B_{\mathbf{w}}^{p_1,p_2}(r)$, (Vershynin, 2018),

$$\begin{aligned} \log|\tilde{\mathcal{N}}_{\mathbf{w}}(\epsilon)| &\leq \log|\mathcal{N}_{\mathbf{w}}(\epsilon/2)| \\ &\leq P\log\left[\left(\frac{12r}{\epsilon}\right)\left(\frac{\|\boldsymbol{\sigma}\|}{\sqrt{K}} + \frac{\epsilon}{2}\right)\right]. && \text{(using Lemma A.1.)} \end{aligned}$$

Now, consider a point $\mathbf{w}_0 \in \tilde{\mathcal{N}}_{\mathbf{w}}(\epsilon)$. Therefore, for every $\mathbf{w}' \in \tilde{B}_{\mathbf{w}}^{p_1,p_2}(r)$ such that $\|\mathbf{w}' - \mathbf{w}_0\| \leq \epsilon$, we have

$$\|G(\mathbf{w}') - G(\mathbf{w}_0)\| \leq \|J(\mathbf{w}_0)(\mathbf{w}' - \mathbf{w}_0)\| + \sqrt{K}\beta(\epsilon). \tag{8}$$

Therefore, upto an error of $\sqrt{K}\beta(\epsilon)$, $G(\mathbf{w}') - G(\mathbf{w}_0)$ lies in an ellipsoid $E(\mathbf{w}_0)$ with principal radii $\varsigma_1\epsilon, \varsigma_2\epsilon, \ldots, \varsigma_P\epsilon$, where $\varsigma_1 \geq \varsigma_2 \geq \cdots \geq \varsigma_P$ are the singular values of $J_{p_1:p_2}(\mathbf{w}_0)$.

Let $\mathcal{N}_\delta(\mathbf{w}_0)$ be a δ-covering of $E(\mathbf{w}_0)$. For a moment assume that there exists an integer p such that $\varsigma_p < a \leq \varsigma_{p+1}$. Therefore, from Theorem 2 in (Dumer, 2006), we have

$$
\begin{aligned}
\log |\mathcal{N}_\delta(\mathbf{w}_0)| &\leq \sum_{i=1}^{p} \log \left(\frac{\varsigma_i}{a}\right) + P \log 6, \\
&= \log \left[\frac{1}{a^P} \prod_{i=1}^{p} \varsigma_i \prod_{i=p+1}^{P} a\right] + P \log 6 \\
&\leq P \log \left[\frac{1}{a}\left(\frac{\|\boldsymbol{\varsigma}\|_2}{\sqrt{P}} + a\right)\right] + P \log 6, \\
&\leq P \log \left[\frac{1}{a}\left(\frac{\|\boldsymbol{\varsigma}\|_2}{\sqrt{P}} + a\sqrt{\frac{K}{P}}\right)\right] + P \log 6, \\
&\leq P \log 12 \sqrt{\frac{K}{P}}. \qquad \text{(using Property TP1.)}
\end{aligned}
$$

Note that this bound holds even when $a \geq \varsigma_1$, or $a \leq \varsigma_P$.

Therefore, for every $\mathbf{w}'$ such that $\|\mathbf{w}' - \mathbf{w}_0\| \leq \epsilon$, there exists a point $\mathbf{f}_1$ in $\mathcal{N}_\delta(\mathbf{w}_0)$ such that

$$
\begin{aligned}
&\|G(\mathbf{w}_0) + J(\mathbf{w}_0)(\mathbf{w}' - \mathbf{w}_0) - \mathbf{f}_1\| \leq \delta, \\
\Rightarrow &\|G(\mathbf{w}_0) + J(\mathbf{w}_0)(\mathbf{w}' - \mathbf{w}_0) - G(\mathbf{w}') + G(\mathbf{w}') - \mathbf{f}_1\| \leq \delta, \\
\Rightarrow &\|G(\mathbf{w}') - \mathbf{f}_1\| \leq \delta + \sqrt{K}\beta(\delta/a) \qquad (9)
\end{aligned}
$$

(by triangle inequality)

This holds for all $\mathbf{w}_0 \in \tilde{\mathcal{N}}_{\mathbf{w}}^K(r)$. Therefore, a suitable candidate set for $\tilde{\mathcal{N}}_{\mathbf{f}}(\delta)$ is

$$
\tilde{\mathcal{N}}_{\mathbf{f}}(\delta) = \{\mathbf{w}_1 \; s.t. \; \mathbf{w}_1 \in \mathcal{N}_\delta(\mathbf{w}_0), \; \mathbf{w}_0 \in \tilde{\mathcal{N}}_{\mathbf{w}}^K(r)\} \qquad (10)
$$

Therefore,

$$
\begin{aligned}
\log |\tilde{\mathcal{N}}_{\mathbf{f}}(\delta)| &\leq P \log 12 \sqrt{\frac{K}{P}} + P \log \left[\frac{12ar}{\delta}\left(\frac{\|\boldsymbol{\sigma}\|}{\sqrt{K}} + \frac{\delta}{2a}\right)\right], \\
&\leq P \log \left[\left(\frac{144ar\sqrt{K}}{\delta\sqrt{P}}\right)\left(\frac{\|\boldsymbol{\sigma}\|}{\sqrt{K}} + \frac{\delta}{2a}\right)\right], \\
&\leq P \log \left(\frac{145ar\,\|\boldsymbol{\sigma}\|}{\sqrt{P}\delta}\right).
\end{aligned}
$$

□

Proof of Theorem 4.1:

First, we note that the result of Lemma A.2 applies to a decreasing sequence of δ's: $\delta_i = \delta_0/2^i$. Also, from Fig. 1, we note that $\beta(\epsilon)$ goes polynomially with ϵ with $\beta(0) = 0$. Due to these, Lemma 8.2 in (Bora et al., 2017) can be reformulated as following, with the proof proceeding similarly as (Bora et al., 2017).

Lemma A.3. *Let $H \in \mathbb{R}^{m \times n}$ be a matrix with iid Gaussian random elements having mean 0 and variance $1/m$. Let $0 < \delta \leq \frac{a\|\boldsymbol{\sigma}\|_2}{72\sqrt{K}}$. For all $\delta' < \delta$, let $\beta(\delta'/a)$ go polynomially as δ'/a, with $\beta(0) = 0$. If*

$$
m = \Omega\left(P \log \frac{ar\,\|\boldsymbol{\sigma}\|}{\delta}\right), \qquad (11)
$$

then for any $\mathbf{f} \in G(\tilde{B}_{\mathbf{w}}^{p_1,p_2}(r))$, if $\mathbf{f}' = \arg\min_{\hat{x} \in \tilde{\mathcal{N}}_{\mathbf{f}}(\delta)} \left\|\mathbf{f} - \hat{\mathbf{f}}\right\|$, $\|H(\mathbf{f} - \mathbf{f}')\| \leq O(\delta + \sqrt{K}\beta(\delta/a))$ with probability $1 - e^{-\Omega(m)}$.

The proof goes similarly as the proof of Lemma 8.2 in (Bora et al., 2017). Furthermore, similar to Lemma 4.1 in (Bora et al., 2017), Lemma A.2 and Lemma A.3 give rise to the set-restricted eigenvalue condition of H on $G(\tilde{B}_{\mathbf{w}}^{p_1,p_2}(r))$ as follows:

Lemma A.4 (Set-restricted eigenvalue condition)**.** *Let $\tau < 1$. Let H be an matrix with iid Gaussian-distributed elements with mean 0 and variance $1/m$. Let $0 < \delta \leq \frac{a\|\boldsymbol{\sigma}\|_2}{72\sqrt{K}}$. For all $\delta' < \delta$, let $\beta(\delta'/a)$ go polynomially as δ'/a, with $\beta(0) = 0$. If*

$$m = \Omega\left(\frac{P}{\tau^2}\log\frac{ar\,\|\boldsymbol{\sigma}\|}{\delta}\right), \tag{12}$$

then H satisfies the S-REC$\big(G(\tilde{B}_{\mathbf{w}}^{p_1,p_2}(r)),\ 1-\tau,\ \delta + \sqrt{K}\beta(\delta/a)\big)$ with probability $1 - e^{-\Omega(\tau^2 m)}$.

Lemma A.4, Lemma 4.3 in (Bora et al., 2017) and Lemma 4.1 imply Theorem 4.1 which is restated here for convenience.

Theorem 4.1.

Let $H \in \mathbb{R}^{m\times n}$ satisfy S-REC$(G(\tilde{B}_{\mathbf{w}}^{p_1,p_2}(r)), \gamma, \delta + \sqrt{K}\beta(\delta/a))$. Let $\mathbf{n} \in \mathbb{R}^m$. Let $\mathbf{w}, \mathbf{w}^{(\mathrm{PI})} \sim p_{\mathbf{w}}$. Let $\mathbf{f}^{(\mathrm{PI})} = G(\mathbf{w}^{(\mathrm{PI})})$ be the known prior image. Let

$$\tilde{\mathbf{w}} = [\mathbf{w}_{1:p_1}^{(\mathrm{PI})\top} \quad \mathbf{w}_{p_1:p_2}^{\top} \quad \mathbf{w}_{p_2:K}^{(\mathrm{PI})\top}]^\top.$$

Let $\tilde{\mathbf{f}} = G(\tilde{\mathbf{w}})$ represent the object to-be-imaged. Let $\mathbf{g} = H\tilde{\mathbf{f}} + \mathbf{n}$ be the imaging measurements. Let

$$\hat{\mathbf{f}} = \underset{\mathbf{f}\in G(\tilde{B}_{\mathbf{w}}^{K}(r))}{\arg\min}\ \|\mathbf{g} - H\mathbf{f}\|_2^2. \tag{13}$$

Then,

$$\|\hat{\mathbf{f}} - \tilde{\mathbf{f}}\| \leq \frac{1}{\gamma}(2\,\|\mathbf{n}\| + \delta + \sqrt{K}\beta(\delta/a)) \tag{14}$$

with probability $1 - O(1/K)$.

B. Additional Figures

B.1. Inverse crime study: $n/m = 5$

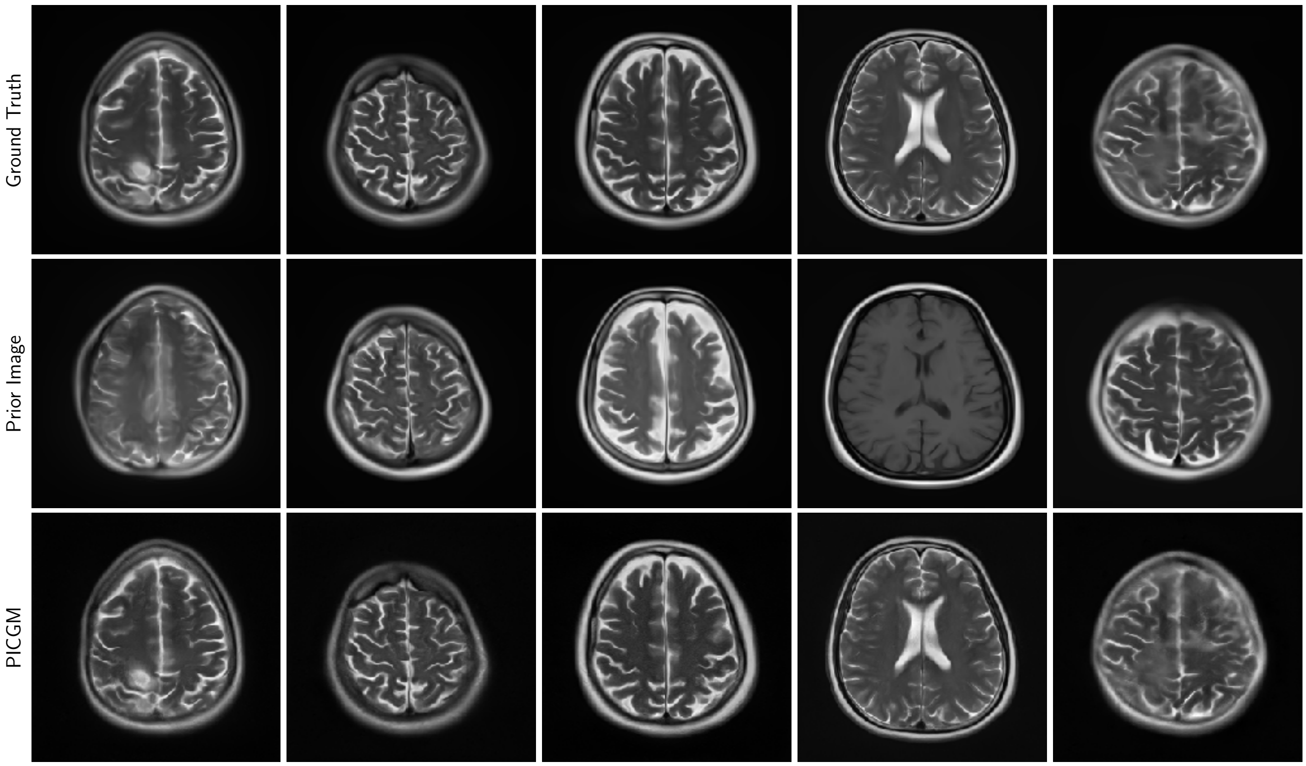

Figure 2: Ground truth, prior image and image estimated from Gaussian measurements with $n/m = 5$ using the proposed approach in the inverse crime case.

B.2. Inverse crime study: $n/m = 10$

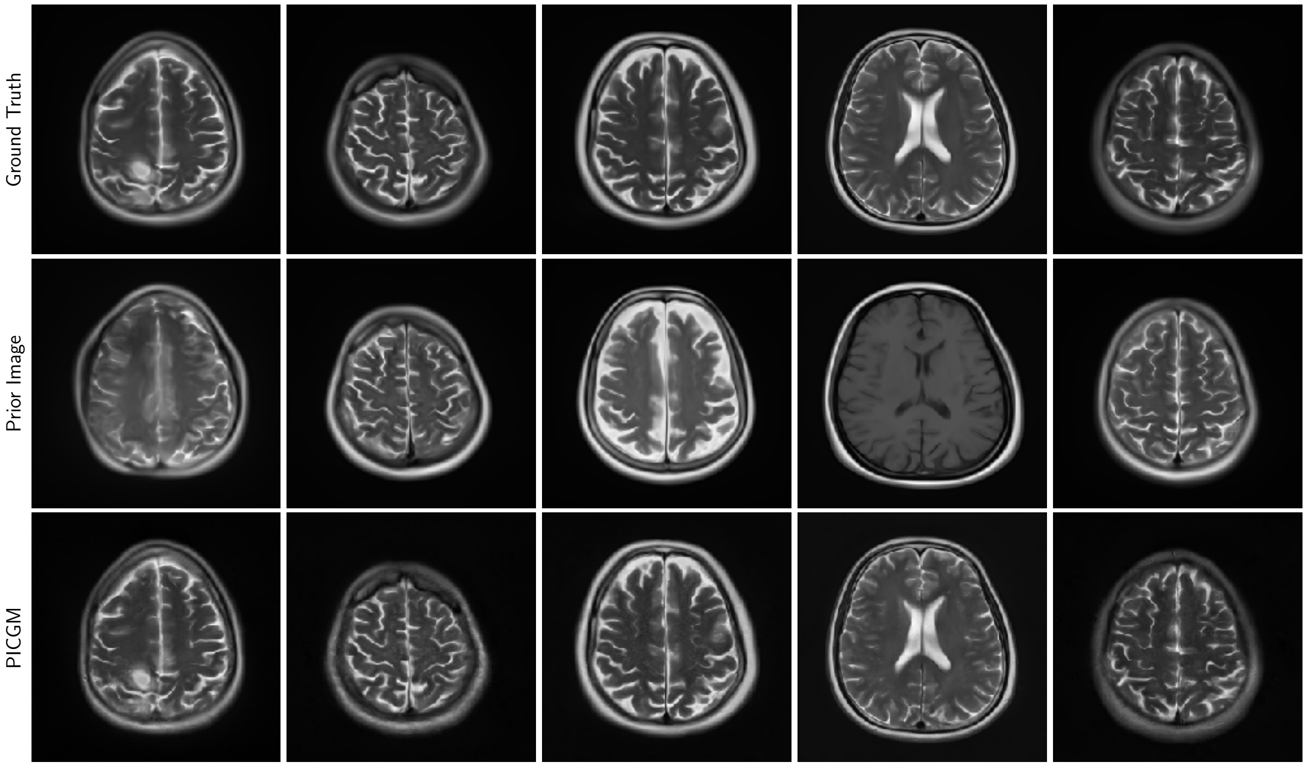

Figure 3: Ground truth, prior image and image estimated from Gaussian measurements with $n/m = 10$ using the proposed approach in the inverse crime case.

B.3. Inverse crime study: $n/m = 20$

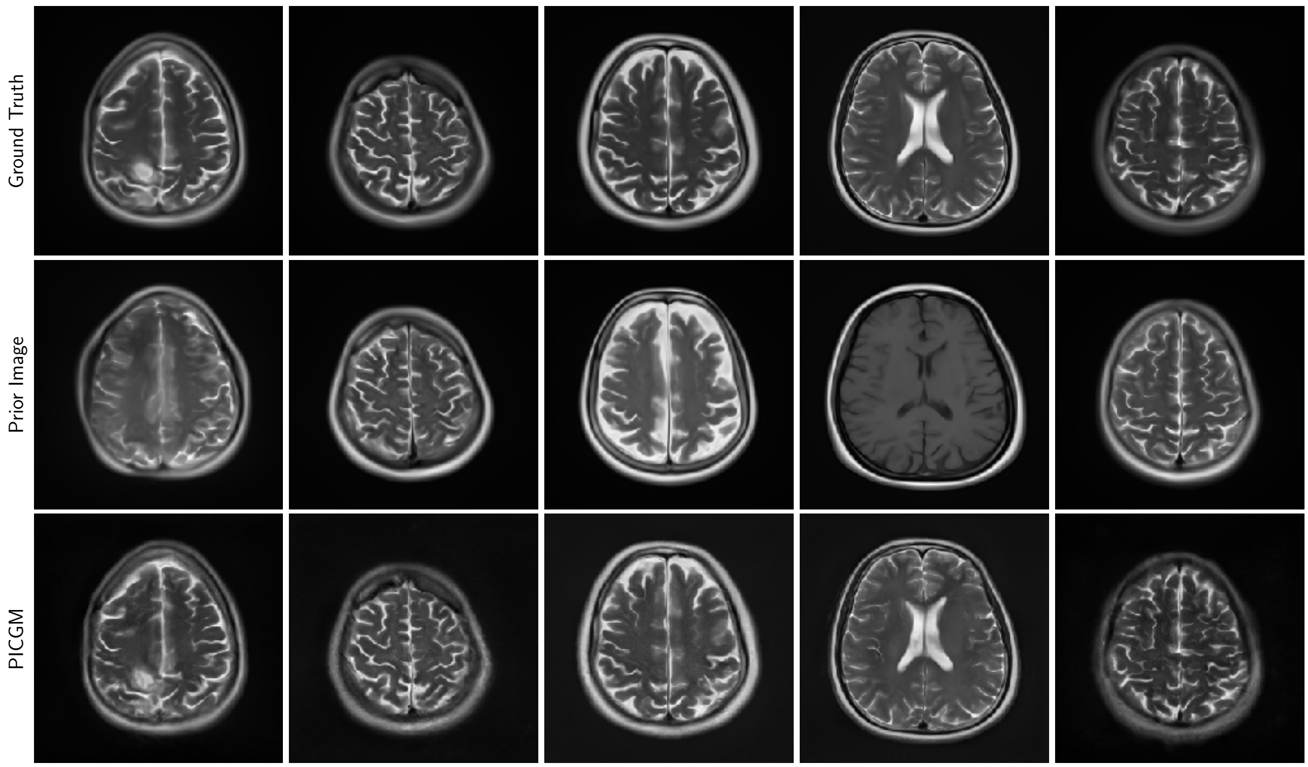

Figure 4: Ground truth, prior image and image estimated from Gaussian measurements with $n/m = 20$ using the proposed approach in the inverse crime case.

B.4. Inverse crime study: $n/m = 50$

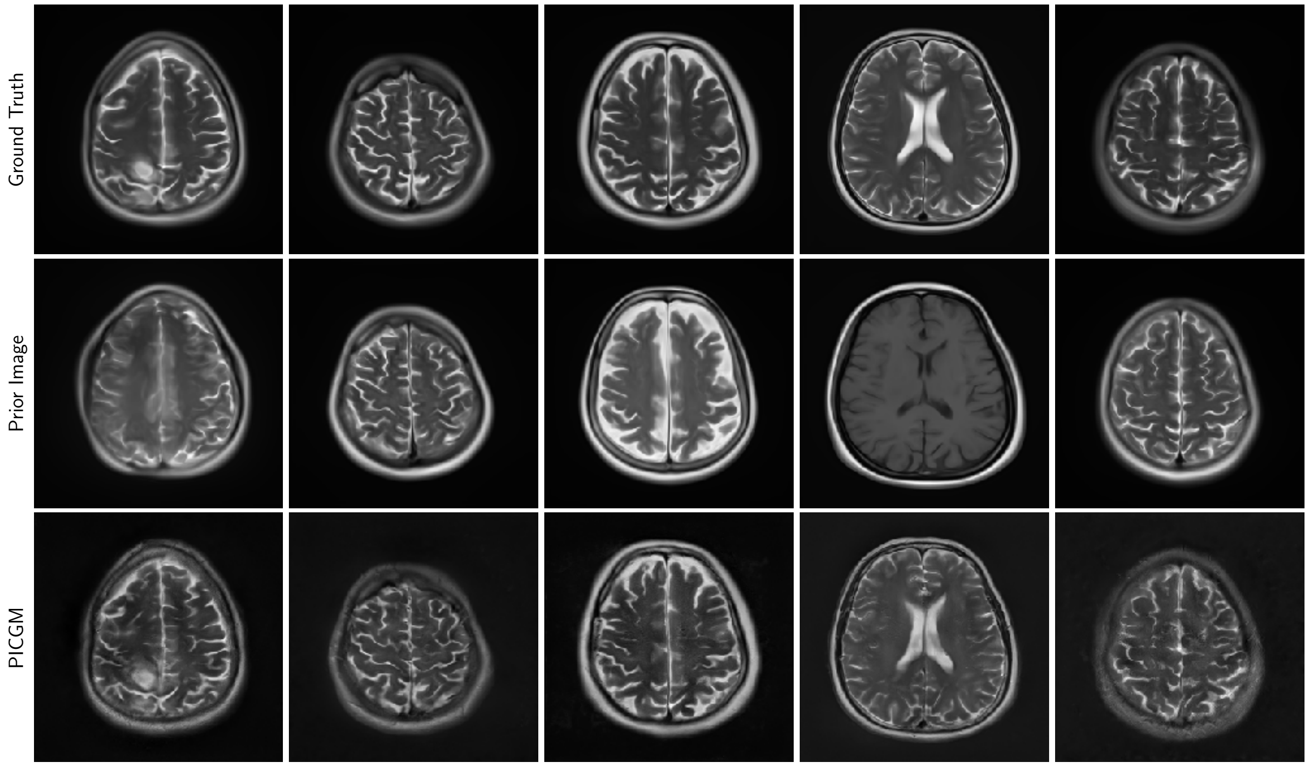

Figure 5: Ground truth, prior image and image estimated from Gaussian measurements with $n/m = 50$ using the proposed approach in the inverse crime case.

B.5. Face image study: $n/m = 50$

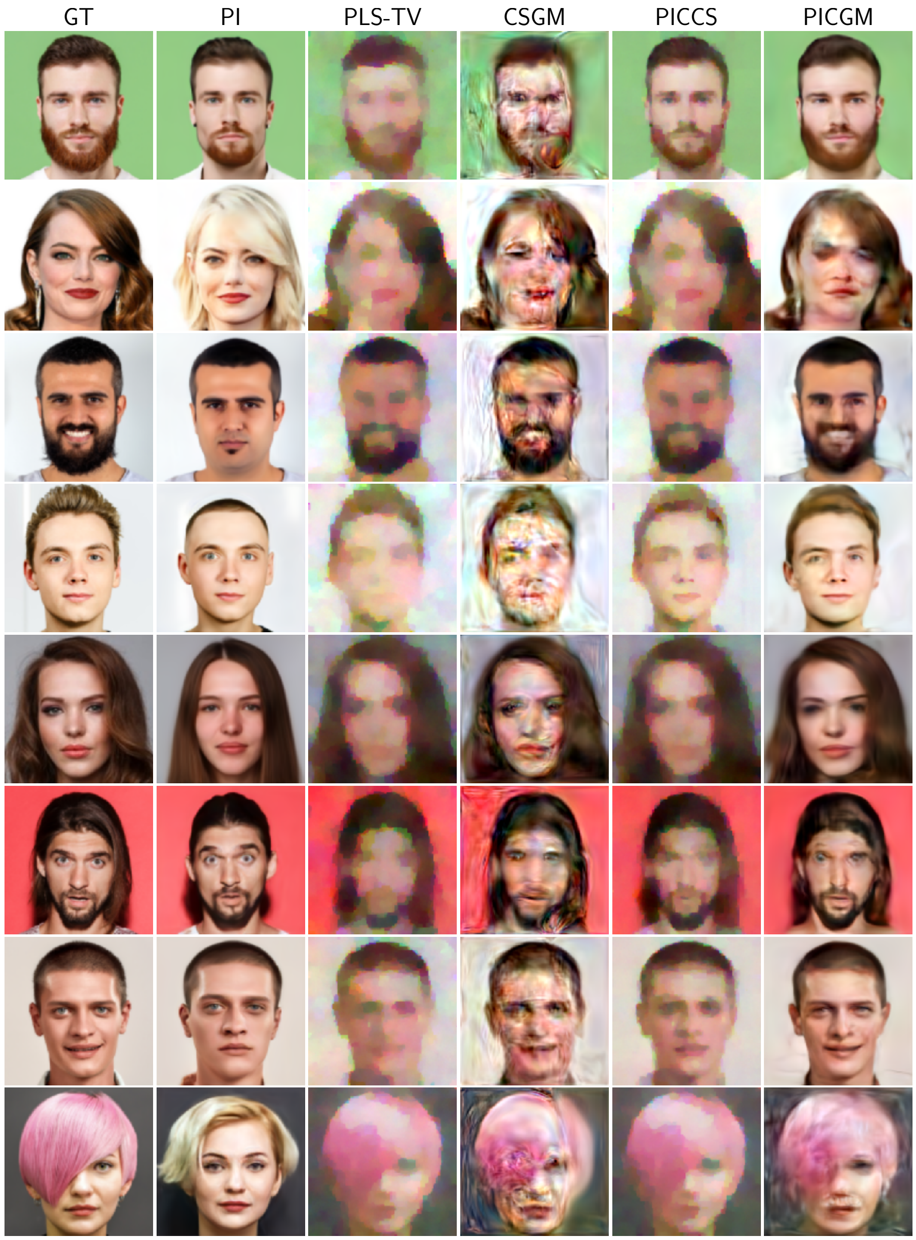

Figure 6: Ground truth, prior image and image estimated from Gaussian measurements with $n/m = 50$ using the proposed approach for the face image study.

B.6. MR image study: $n/m = 2$

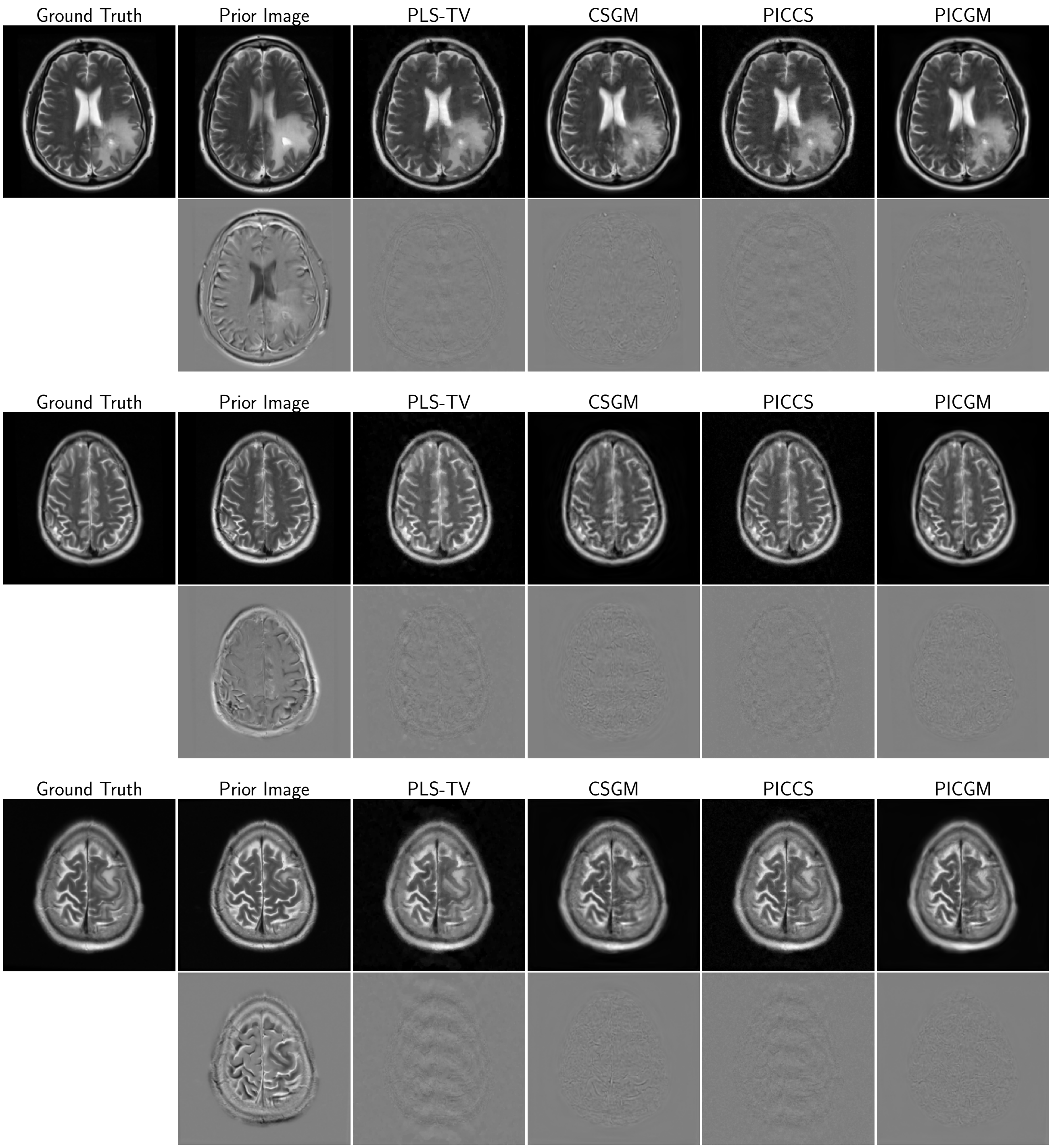

Figure 7: Ground truth, prior image, and images reconstructed from simulated MRI measurements with $n/m = 2$ along with difference images for the MR image study

B.7. MR image study: $n/m = 4$

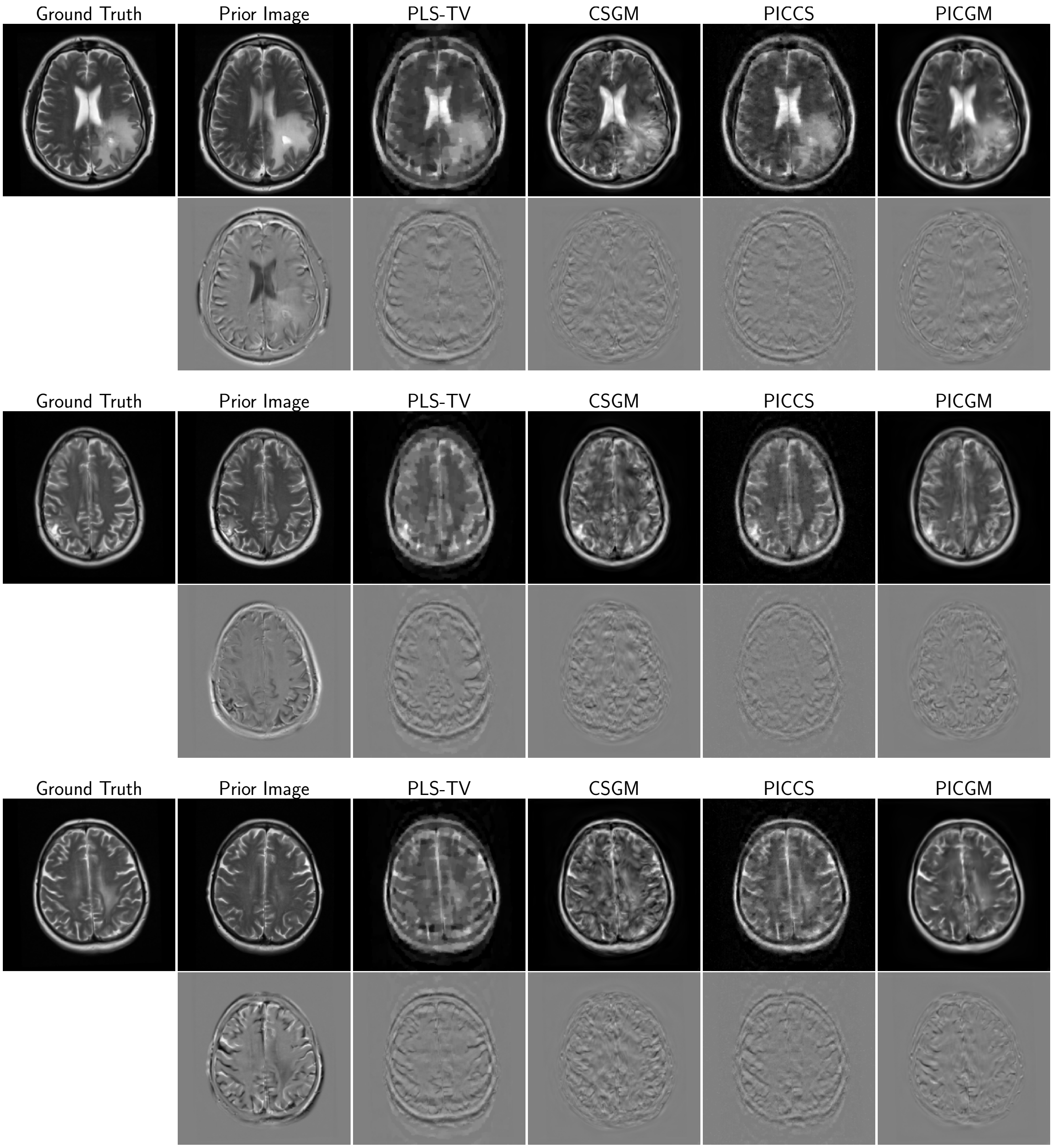

Figure 8: Ground truth, prior image, and images reconstructed from simulated MRI measurements with $n/m = 4$ along with difference images for the MR image study

B.8. MR image study: $n/m = 6$

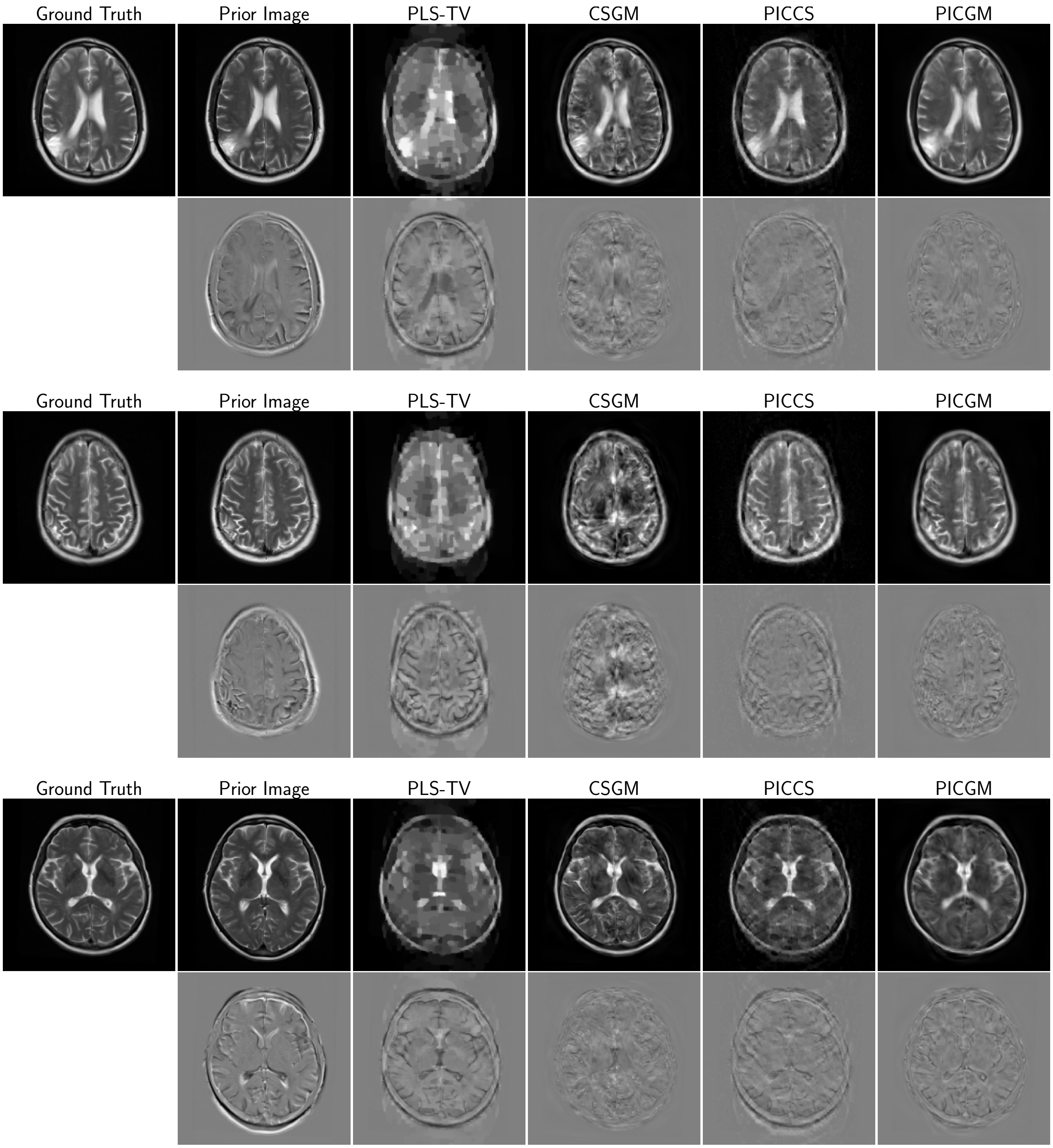

Figure 9: Ground truth, prior image, and images reconstructed from simulated MRI measurements with $n/m = 6$ along with difference images for the MR image study

B.9. MR image study: $n/m = 8$

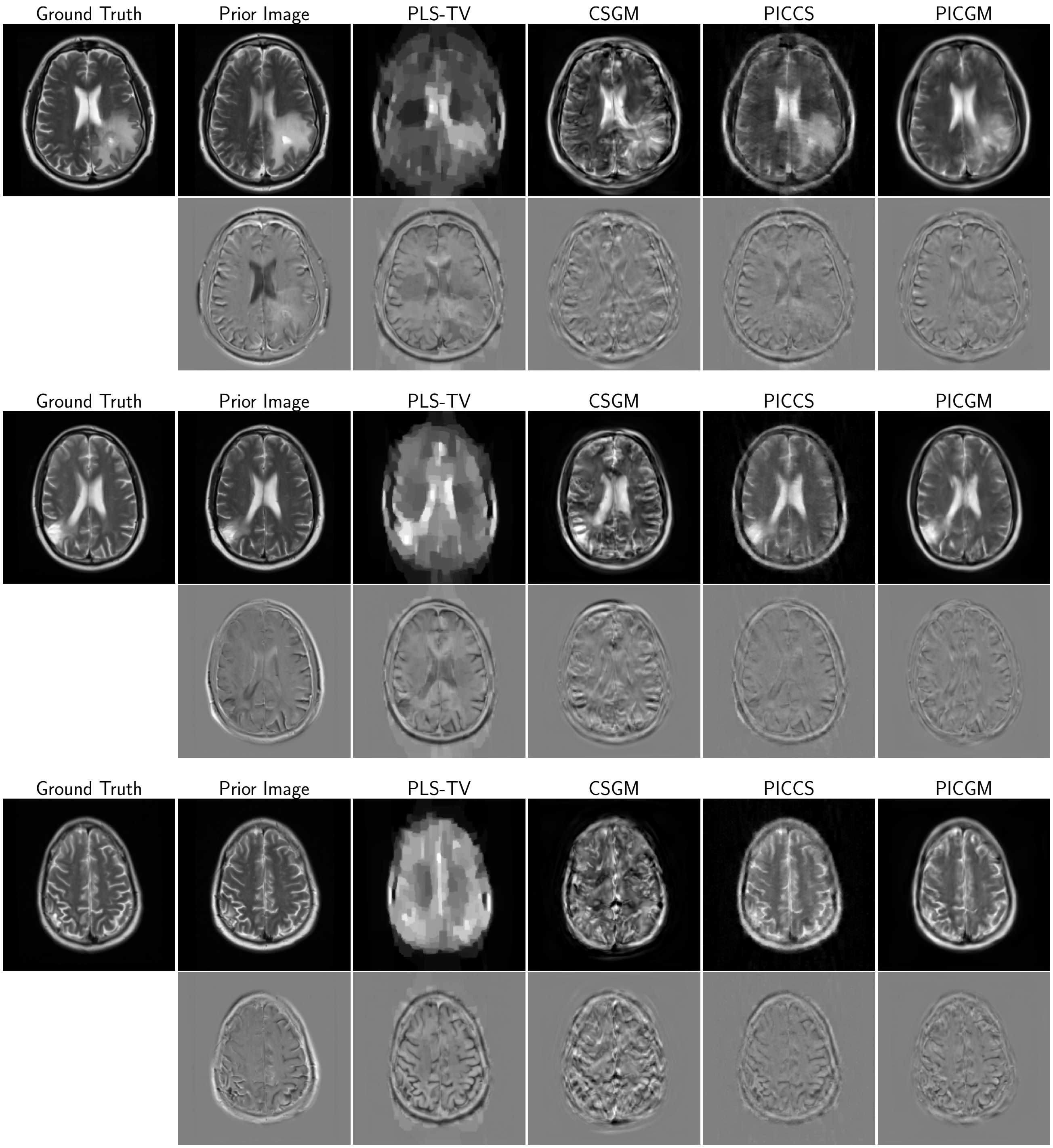

Figure 10: Ground truth, prior image, and images reconstructed from simulated MRI measurements with $n/m = 8$ along with difference images for the MR image study

B.10. MR image study: $n/m = 12$

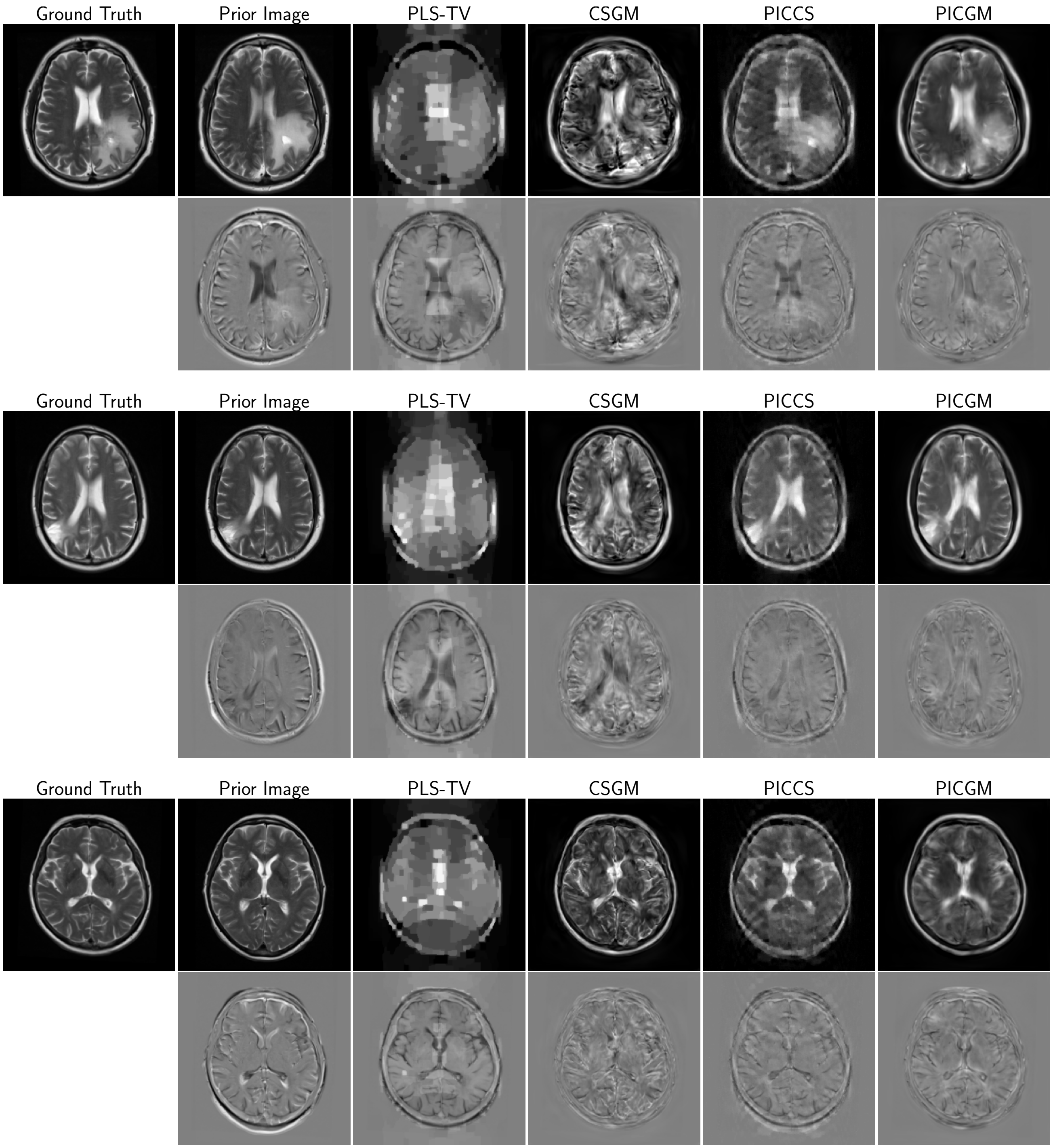

Figure 11: Ground truth, prior image, and images reconstructed from simulated MRI measurements with $n/m = 12$ along with difference images for the MR image study

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