A. Proofs

A.1. Proof of Theorem 1

Theorem 1. *Let $D = D_* + \delta D = \delta D, G = G_* + \delta G$. Let us denote $u(x) = \delta D(x), v(x) = (\delta G)(G_*^{-1}(x))$. We assume $u \in H^2(\mu)$ and $v \in H^1(\mu)$. Then, (2) can be approximated as*

$$f(D, G) = f_0 + g(u, v) + R_3(u, v),$$

where

$$g(u, v) = \alpha\langle u, u\rangle_\mu + \beta\langle \nabla_x u, v\rangle_\mu,$$

and

$$R_3(u, v) = O\left(\left(\|u\|_{H^2(\mu)} + \|v\|_{H^1(\mu)}\right)^3\right)$$

Proof. The proof is straightforward: we use Taylor expansion (Dieudonné, 1969, Theorem 8.14.3) in Banach spaces to approximate both terms in the objective up to second order. In order to obtain the statement of the theorem, we need the target functional to be differentiable up to the third order. Thus, we need to impose a certain restriction on our functional space. Namely, we require $\nabla_x^2 u$ and $\nabla_x v$ to exist. I.e., we need $u \in H^2(\mu)$ and $v \in H^1(\mu)$. The derivation of the approximation is then straightforward using standard calculus of variations arguments. We also use the fact that in the Nash equilibrium the first–order terms sum up to 0.

$$\begin{aligned}
&f(D_* + \delta D, G_* + \delta G) = f(\delta D, G_* + \delta G) = \\
&= \mathbb{E}_{x\sim\mu}\, \phi_1(\delta D(x)) + \mathbb{E}_{z\sim\mu_z}\, \phi_2(\delta D(G_*(z) + \delta G(z))) \\
&= f_0 + \alpha\, \mathbb{E}_{x\sim\mu}\, \delta D^2 + \beta\, \mathbb{E}_{z\sim\mu_z} \langle \nabla_x \delta D(G_*(z)), \delta G(z)\rangle + \\
&+ \text{higher order terms},
\end{aligned}$$

Since the functional variation (u, v) belongs to the product space $H^2(\mu) \times H^1(\mu)$, we obtain the remainder form in the required form by (Dieudonné, 1969, Theorem 8.14.3). Finally, the statement follows from the definitions of $\langle\cdot,\cdot\rangle_\mu$, u and v. ☐

A.2. Proof of Corollary 1

Corollary 1. *Let (u_0, v_0) be an eigenfunction with $\lambda = 0$. Then,*

$$u_0 = C, \quad \langle \nabla_x \hat{u}, v_0\rangle_\mu = 0, \forall \hat{u} \in H^2(\mu), \tag{10}$$

or in the strong form:

$$u_0 = C, \nabla_x \cdot (\rho v_0) = 0. \tag{11}$$

Here C is a constant such that $C\alpha = 0$. I.e., for $\alpha \neq 0$ we get $C = 0$, and $C \in \mathbb{R}$ otherwise.

Proof. From (8) we observe that the element (u_0, v_0) of the kernel satisfies the following equations $\forall \hat{u} \in H^2(\mu)$.

$$u_0 = C, -\alpha\langle C, \hat{u}\rangle_\mu - \beta\langle \nabla_x \hat{u}, v_0\rangle_\mu = 0. \tag{35}$$

Let us choose $\hat{u} = 1$. From the second equation it follows that $\alpha C = 0$ as desired. ☐

A.3. Proof of Theorem 3

Theorem 2. *The non-zero spectrum of (8) is described as follows.*

- *The eigenvalues are given by $\{\lambda_i^\pm\}_{i=1}^\infty$ where $\lambda_i^\pm$ are roots of the quadratic equation:*

$$\lambda^2 + \alpha\lambda + \beta^2\xi_i = 0. \tag{16}$$

- *The corresponding eigenfunctions are written in terms eigenfunctions of $-\Delta_\mu$ as follows.*

$$(u_{\lambda_i^\pm}, v_{\lambda_i^\pm}) = (w_{\xi_i}, \frac{\beta}{\lambda_i^\pm}\nabla_x w_{\xi_i}). \tag{17}$$

Proof. By putting $\hat{v} = \nabla_x \hat{u}$ into the second equation of (8), we get

$$\lambda(u_\lambda, \hat{u})_\mu = -\alpha(u_\lambda, \hat{u})_\mu - \frac{\beta^2}{\lambda}(\nabla_x \hat{u}, \nabla_x u_\lambda)_\mu,$$

which can be rewritten as

$$(\nabla_x \hat{u}, \nabla_x u_\lambda)_\mu = \frac{1}{\beta^2}\left((-\alpha\lambda - \lambda^2)(u_\lambda, \hat{u})_\mu\right) = \xi(u_\lambda, \hat{u})_\mu,$$

which means that ξ is an eigenvalue of $-\Delta_\mu$, and u_λ is its eigenfunction. The eigenvalue λ can be found from the solution of the quadratic equation (19)

$$\lambda = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta^2\xi}}{2}. \tag{36}$$

☐

A.4. Proof of Theorem 4

Theorem 4. *Let $u_0 \in H^2(\mu), v_0 \in H^1(\mu)$ and $\int u_0 d\mu = c_0$. Then, these functions can be written as*

$$\begin{aligned}
u_0 &= c_0 + \sum_{k=1}^{\infty}(c_k^+ + c_k^-)w_{\xi_k}, \\
v_0 &= \widetilde{v}_0 + \nabla_x V_0,\, V_0 = \sum_{k=1}^{\infty}\left(c_k^+\frac{\beta}{\lambda_i^+} + c_k^-\frac{\beta}{\lambda_i^-}\right)w_{\xi_k},
\end{aligned}$$

and $\widetilde{v}_0$ is divergence-free, i.e. $\langle \nabla_x \hat{u}, \widetilde{v}_0\rangle_\mu = 0$. The coefficients c_k^+ and c_k^- can be obtained as the solution of the linear systems:

$$\begin{pmatrix} 1 & 1 \\ \frac{\beta}{\lambda_i^+} & \frac{\beta}{\lambda_i^-}\end{pmatrix}\begin{pmatrix} c_k^+ \\ c_k^-\end{pmatrix} = \begin{pmatrix}\langle u_0, w_{\xi_k}\rangle_\mu \\ \langle V_0, w_{\xi_k}\rangle_\mu\end{pmatrix} \tag{20}$$

With this expansion, the solution to (6) is

$$u(t) = c_0 e^{-\alpha t} + \sum_{k=1}^{\infty} w_{\xi_k} \left(c_k^+ e^{\lambda_k^+ t} + c_k^- e^{\lambda_k^- t} \right),$$
$$v(t) = \widetilde{v}_0 + \nabla_x V(t), \tag{21}$$
$$V(t) = \sum_{k=1}^{\infty} w_{\xi_k} \left(c_k^+ \frac{\beta}{\lambda_i^+} e^{\lambda_k^+ t} + c_k^- \frac{\beta}{\lambda_i^-} e^{\lambda_k^- t} \right).$$

For $\alpha > 0$ the norms of $u(t)$ and $V(t)$ can be estimated as

$$\|u(t)\|_\mu \leq 2\|u_0\|_\mu e^{\eta t}, \|V(t)\|_\mu \leq C\|V_0\|_\mu e^{\eta t},$$

where $\eta = \mathrm{Re}\left(\frac{-\alpha + \sqrt{\alpha^2 - 4\beta^2 \xi_{\min}}}{2}\right) < 0$ is the maximal real part of the eigenvalues.

Proof. The decomposition of v_0 into a potential part and divergence-free part is a direct generalization of the classical result for the ordinary divergence and gradient, known as the Helmholtz decomposition (Griffiths, 2005). The divergence-free part $\widetilde{v}_0$ belongs to the kernel of the operator, thus it stays constant. The dynamics of u and v follows from the completeness of the eigenbasis of Δ_μ and the assumption that its spectrum is discrete, thus we can expand them in this basis. From Theorem 3 each component in the sum is an eigenfunction, thus its time dynamics is just $e^{\lambda_k^{\pm} t}$. For the constant term in $u(t)$, by substituting $\hat{u} = 1$ in (6), we obtain the following ODE $\langle u_t, 1 \rangle_\mu = -\alpha \langle u, 1 \rangle_\mu$, from which the statement follows. □

B. Example with known eigenfunctions

Consider a model example of the normal distribution, $\mu \sim \mathcal{N}(0,1)$. Then, μ has the density $\rho(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. The eigenfunctions and eigenvalues of $-\Delta_\mu$ can be computed explicitly. The strong form of the eigenproblem is

$$\frac{d}{dx} \rho \frac{dw_\xi}{dx} = -\xi \rho w_\xi,$$

i.e. w_ξ satisfies

$$\frac{d^2 w_\xi}{dx^2} - x \frac{dw_\xi}{dx} = -\xi w_\xi. \tag{37}$$

The solution of (37) exists for $\xi_k \in \mathbb{Z}_{\geq 0}$ and the corresponding eigenfunction is the Hermite polynomial:

$$w_{\xi_k} = H_k(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}.$$

The smallest non-zero eigenvalue is 1. Therefore, for the LSGAN model, we will have the discriminant in (19) always non-positive, and the convergence u and v will be exponential with the rate $e^{-\frac{t}{2}}$. The solution also will oscillate due to the presence of complex eigenvalues.

C. Examples of learned eigenfunctions.

Here we visualize the lowest eigenfunctions (corresponding to $\xi_{\min}$) obtained for two 2D distributions induced by images. See Figure 6. We observe that these eigenfunctions roughly perform a very basic 'clustering' of the data.

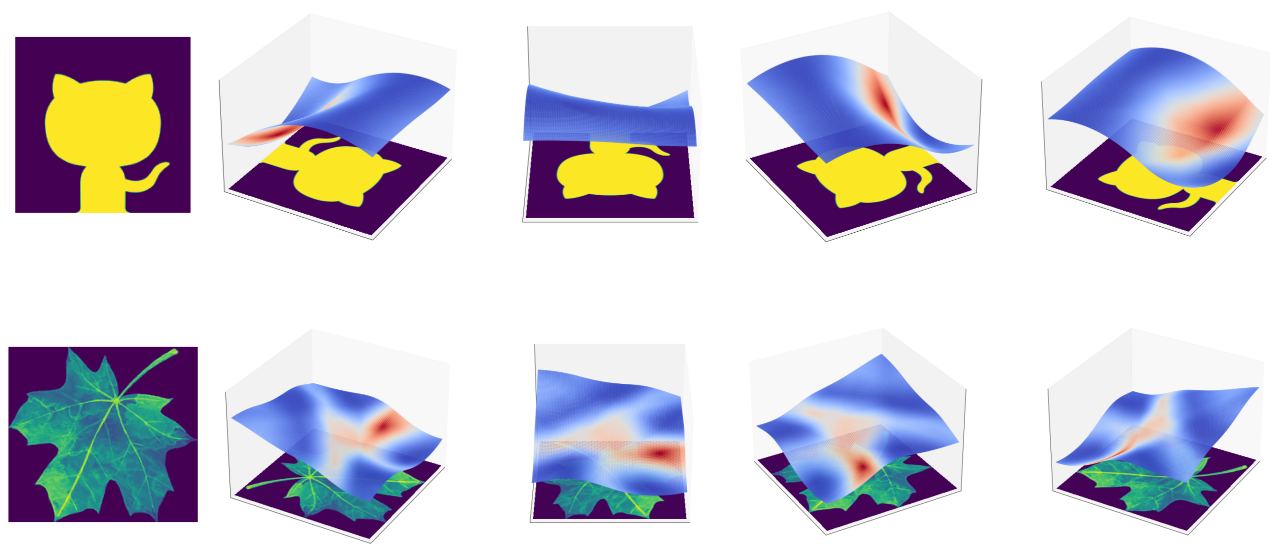

Figure 6. Examples of learned lowest eigenfunctions of $-\Delta_\mu$ for sample 2D distributions. Coloring is given by the norm of the function gradient.