A. Competing Concentration Bounds

Theorem 5 (Hoeffding; Theorem 3.1.2 of Giné & Nickl (2016))**.** *If $X_1, \dots, X_n$ are independent mean-zero random variables satisfying $\mathbb{P}(\underline{B} \leqslant X_i \leqslant B) = 1$, then*

$$\mathbb{P}\left(S_n \geqslant \sqrt{\frac{1}{2}n(B-\underline{B})^2 \log\left(\frac{1}{\delta}\right)}\right) \leqslant \delta, \quad \forall \delta \in [0,1].$$

(There is a generalization of Hoeffding's inequality that relaxes the boundedness assumption by a sub-Gaussian assumption; see Zhao et al. (2016) for details.)

Theorem 6 (Adaptive Hoeffding; Corollary 1 of Zhao et al. (2016))**.** *If $X_1, \dots, X_n$ are independent mean-zero random variables satisfying $\mathbb{P}(\underline{B} \leqslant X_i \leqslant B) = 1$, then*

$$\mathbb{P}\left(\exists n \geqslant 1: \, S_n \geqslant (B-\underline{B})\sqrt{0.6n\log(\log_{1.1} n + 1) + \frac{\log(12/\delta)}{1.8}n}\right) \leqslant \delta, \quad \forall \delta \in [0,1].$$

Theorem 7 (Bernstein; Theorem 3.1.7 of Giné & Nickl (2016))**.** *If $X_1, \dots, X_n, \dots$ are independent random variables satisfying (2), then*

$$\mathbb{P}\left(S_n \geqslant \sqrt{2\sum_{i=1}^n A_i^2 \log\left(\frac{1}{\delta}\right) + \frac{1}{9}B^2\log^2\left(\frac{1}{\delta}\right)} + \frac{1}{3}B\log\left(\frac{1}{\delta}\right)\right) \leqslant \delta, \quad \forall \delta \in [0,1].$$

Theorem 8 (Empirical Bernstein; Eq. (5) of Mnih et al. (2008))**.** *If $X_1, X_2, \dots$ are independent mean zero random variables satisfying (2) with $A_1 = A_2 = \dots = A$, then*

$$\mathbb{P}\left(\exists n \geqslant 1: \, S_n \geqslant \sqrt{2n\eta\widehat{A}_n^2\log(3h(k_n)/(2\delta))} + 3B\eta\log(3h(k_n)/(2\delta))\right) \leqslant \delta,$$

where $\widehat{A}_n^2$ is the sample variance and k_n is the constant defined in Theorem 2.

B. More Simulations

B.1. Hyperparameters of Stitching

In Section 3, we mentioned that there are two hyperparameters of our stitching methods: (1) the spacing parameter $\eta > 1$ and (2) the power parameter $c > 1$ for the stitching function $h_c(k) = \zeta(c)(k+1)^c$ where $\zeta(\cdot)$ is the Riemann zeta function.

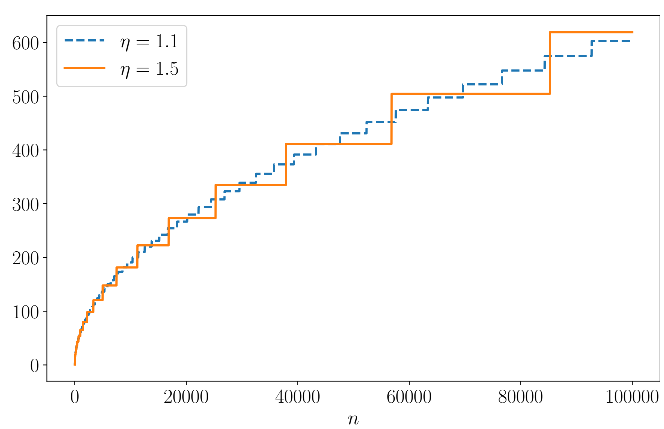


Figure 7: The upper bound of S_n obtained by adaptive Bentkus bound in Theorem 2 for different values of η. Both the variance $A = \sqrt{3}/4$ and the upper bound $B = 3/4$ is known.

Figure 7 illustrates that the choice of η determines how the budget δ is distributed across different sample sizes.

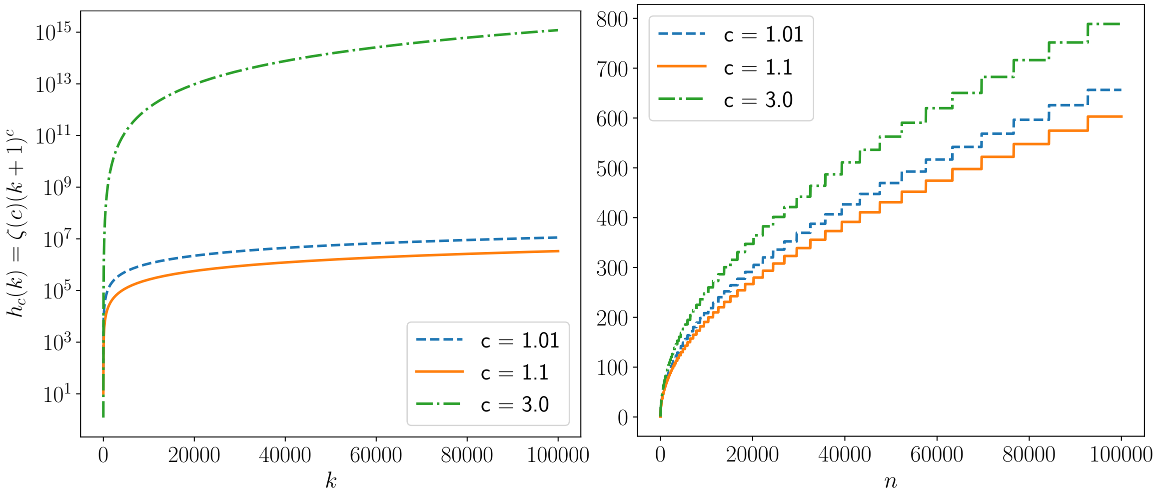
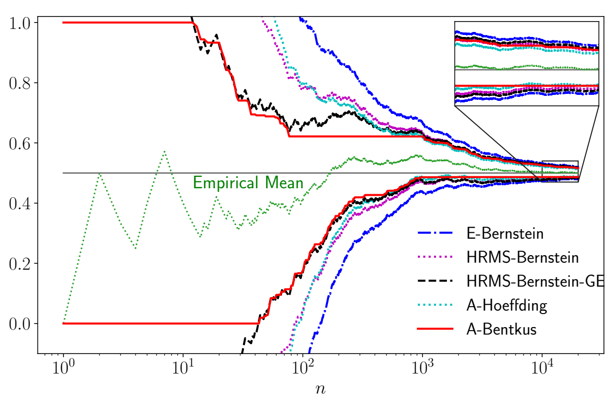


Figure 8: **Left:** The stitching function $h_c(\cdot)$ for different values of c. **Right:** The upper bound of S_n obtained by `A-Bentkus` with different values of c. Both the variance $A^2 = 3/16$ and the upper bound $B = 3/4$ is known.

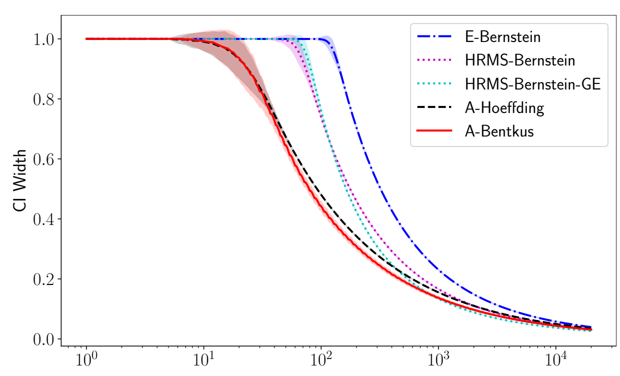
Figure 8 shows both the stitching function $h_c(\cdot)$ and corresponding upper bound `A-Bentkus` obtains. For a fixed sample size n, the bigger $h_c(k_n)$ is, the smaller budget $\delta/h_c(k_n)$ it obtains and hence it needs a larger upper bound. Hence, the faster $h_c(\cdot)$ grows, the more conservative upper bound (and corresponding, wider confidence interval) one will get.

B.2. Confidence Sequence for Bernoulli(0.5)

In this section, we present a comparison of our confidence sequence with `A-Hoeffding`, `E-Bernstein`, `HRMS-Bernstein`, and `HRMS-Bernstein-GE` on synthetic data from Bernoulli(0.5). In this case, $Y_1, Y_2, \ldots \sim$ Bernoulli(0.5) and the variance is $1/4$. Hence in this case Hoeffding's inequality is sharp and nothing can be gained by variance exploitation. We observe this very fact in our experiment, where our method behaves as well as `A-Hoeffding` for moderate to large sample sizes. Figures 9a and 9b show the comparison of confidence sequences in one replication and comparison of average width over 1000 replications. As in the case of Bernoulli(0.1) (Section 4.1), for small sample sizes, `A-Hoeffding` and `A-Bentkus` behave very closely and are better than all other methods but for n moderately large, the sharpness of `A-Bentkus` clearly pays off by outperforming `A-Hoeffding` and all other methods.



(a) One Replication



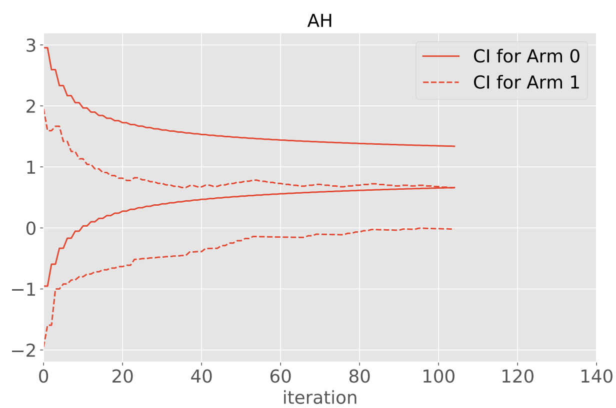
(b) Average Width over 1000 Replications

Figure 9: Comparison of the 95% confidence sequences for the mean when $Y_i \sim$ Bernoulli(0.5). Except `A-Hoeffding`, all other methods estimate the variance. `A-Bentkus` is the confidence sequence in (17). `HRMS-Bernstein-GE` involves a tuning parameter ρ which is chosen to optimize the boundary at $n = 500$. **(a)** shows the confidence sequences from a single replication. **(b)** shows the average widths of the confidence sequences over 1000 replications. The upper and lower bounds for all the other methods are cut at 1 and 0 for a fair comparison. The failure frequency is 0.001 for `HRMS-Bernstein-GE` and 0 for the others.

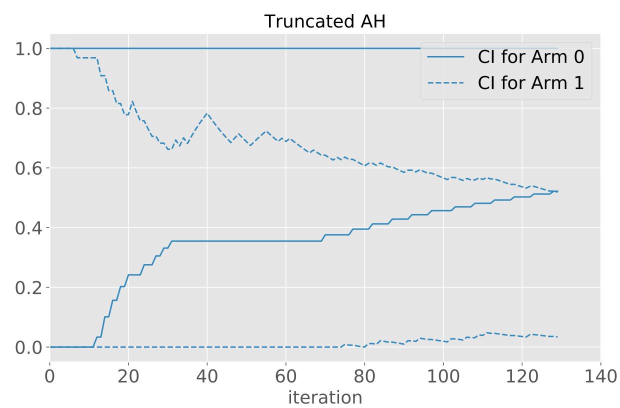
B.3. Discussion for the Best Arm Identification Problem

In Section 4.3, we mentioned that a confidence sequence for which the radius R_α stays constant for a stretch of samples yields a larger sample complexity. We present here more experimental details regarding this behavior.

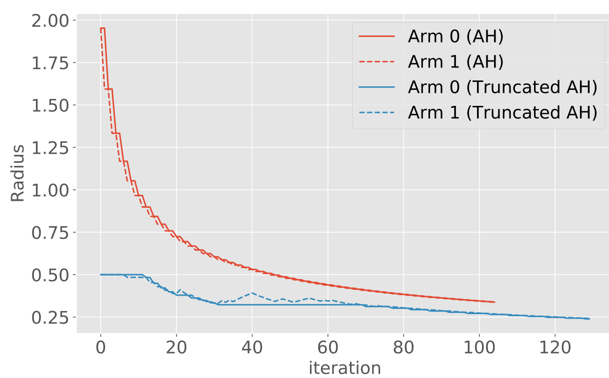
In the following, we experiment with a single instance of best arm identification problem where the number of arms is 2 (i.e., $K = 2$). The expected rewards are generated as the same as in Section 4.3, so that Arm 0 has mean $\mu_0 = 1$ is the best arm, and Arm 1 has mean $\mu_1 \approx 0.34$. For all the methods, we use the same data.



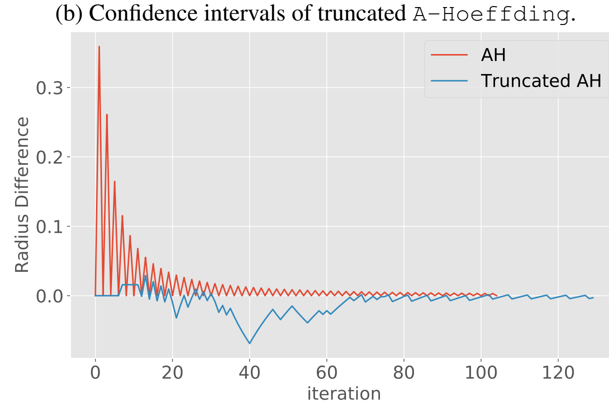
(a) Confidence intervals of A-Hoeffding for two arms.



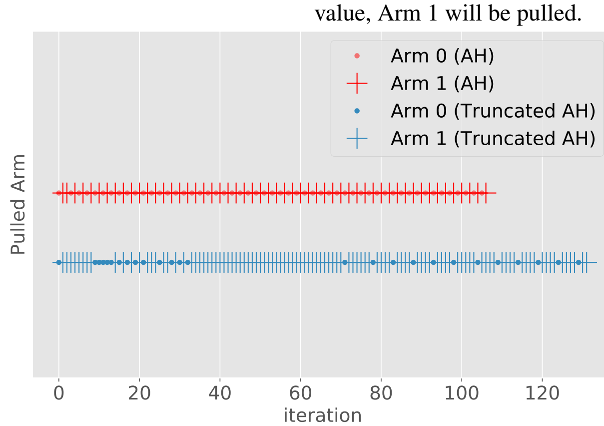
(b) Confidence intervals of truncated A-Hoeffding.



(c) Radius of A-Hoeffding (original and truncated) for two arms.



(d) The difference of the radius for two arms: $R_0 - R_1$. For positive difference value, Arm 0 will be pulled. For negative difference value, Arm 1 will be pulled.



(e) The arms pulled at each iteration. '·': Arm 0 is pulled. '+': Arm 1 is pulled. A-Hoeffding and truncated A-Hoeffding are marked in red and blue, respectively.

Figure 10: Identify the best arm out of two using A-Hoeffding and its truncated variant.

We first explain this phenomenon using `A-Hoeffding` and its truncated variant. `A-Hoeffding` can result in confidence intervals that are larger than $[0, 1]$. In the truncated version of `A-Hoeffding`, the upper confidence term of a confidence interval will be capped at 1, and the lower confidence term will be cut at 0, so that all the confidence intervals stay in $[0, 1]$ throughout the experiment. We shall see that the truncated variant would result in stationary radius and yield larger sample complexity compared with `A-Hoeffding`.

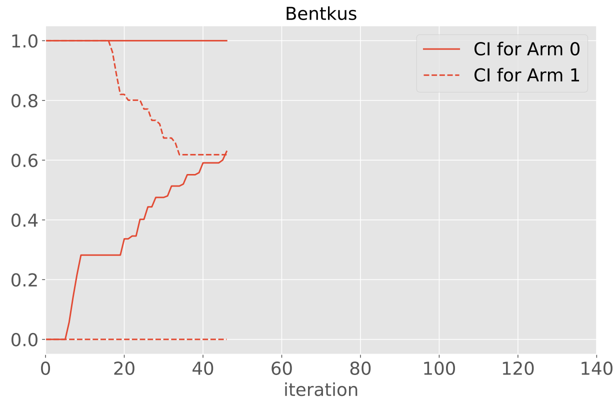
Figures 10a and 10b show the confidence intervals of each arm at each iteration, when `A-Hoeffding` and truncated `A-Hoeffding` are plugged into Algorithm 2. The algorithm will stop when the confidence intervals of the two arms completely separate (i.e., the lower bound of Arm 0 goes above the upper bound of Arm 1). Figure 10a and 10b show that `A-Hoeffding` used 107 iterations, while the truncated `A-Hoeffding` used 132 iterations. One can observe that in the initial stage of the algorithm, the confidence interval, without truncation, will likely get updated once a sample adds in, which does not hold for the truncated version; compare the first 15 iterations in Figures 10a and 10b. Therefore, the radius will not get updated for truncated `A-Hoeffding`, as shown in Figure 10c. Recall that Algorithm 2 samples an arm with largest radius; when both radii are same, we sample the arm with smaller empirical mean. Due to the stationary radius, in those iterations, truncated `A-Hoeffding` keeps sampling the same arm till an update happens.

In Figure 10d, we plot the difference between the radius for Arm 0 and Arm 1: $R_0 - R_1$. Arm 0 will be sampled if this value is positive and vice versa. Again, if R_0 is equal to R_1, we shall sample the arm with lower empirical mean. We can see the difference fluctuates evenly for `A-Hoeffding`, so that `A-Hoeffding` almost alternatively samples each arm, and the confidence intervals of both arms gets updated alternatively as shown in Figure 10a. In contrast, for truncated `A-Hoeffding`, the difference consistently stays above or below zero for some time, which means the same arm gets sampled. See Figure 10e for the arms pulled at each iteration; the '+' and '·' appear almost side-by-side with `A-Hoeffding` and they appear disproportionately with truncated `A-Hoeffding`.

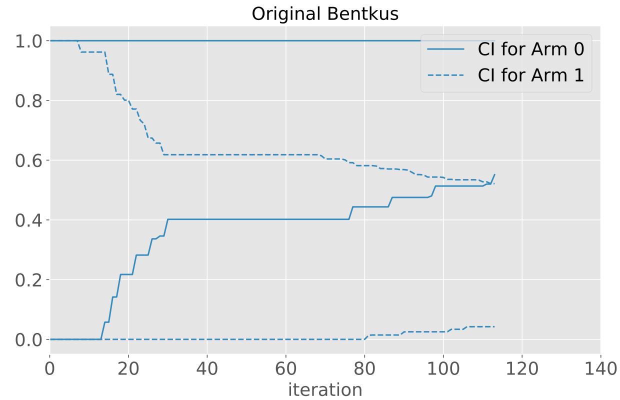
As mentioned, Algorithm 2 stops when the two confidence intervals separate, and it is not crucial for those intervals to be shorter. Hence, it will stop fast if (i) the confidence interval gets updated by every sample and (ii) the updates are significant for small number of samples (the early stage). Truncated `A-Hoeffding` underperforms in both aspects. This is also the reason why the Berstein type of confidence sequences underperforms `A-Hoeffding` in this problem (c.f. Section 4.3). Even though they are shorter for larger samples; `A-Hoeffding` is better with smaller samples.

Next, we investigate the performance for Bentkus type of methods. We write `A-Bentkus` to be the variant from Section 4.3, that is, we output confidence interval $\{[\mu_n^{\text{low}*}, \mu_n^{\text{up}*}], n \geqslant 1\}$ as in Theorem 4, but output radius $R_n = \mu_n^{\text{up}} - \mu_n^{\text{low}}$.

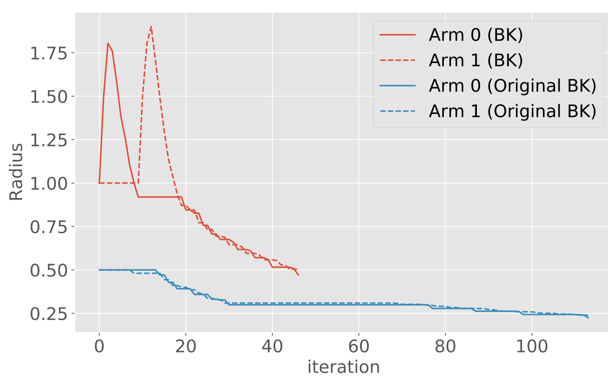
We write original `A-Bentkus` to be the one directly from Theorem 4, i.e., we output confidence interval $\{[\mu_n^{\text{low}*}, \mu_n^{\text{up}*}], n \geqslant 1\}$ and radius $R_n = \mu_n^{\text{up}*} - \mu_n^{\text{low}*}$. Note that $\mu_n^{\text{up}*} = \min_{1\leqslant i\leqslant n} \mu_i^{\text{up}}$ is the cumulative minimum, which essentially serves as the truncation of the upper confidence term, and similarly does the $\mu_n^{\text{low}*}$. We refer the readers to Theorem 4 for the details. Similar to the previous experiment, we shall see that the original `A-Bentkus` results in a larger sample complexity than `A-Bentkus`. Figure 11a presents the results.



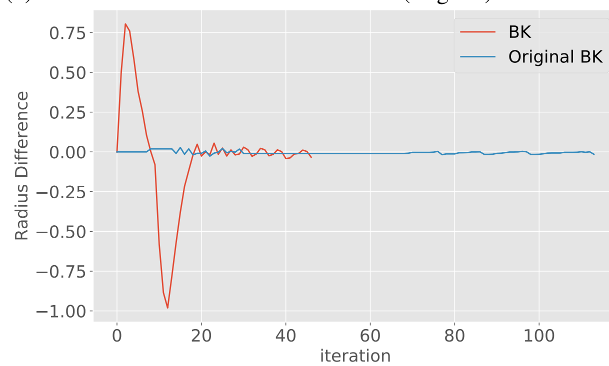
(a) Confidence intervals of `A-Bentkus` (variant) for two arms.



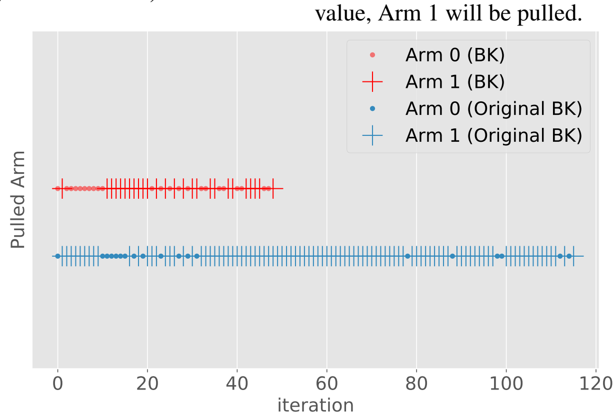
(b) Confidence intervals of `A-Bentkus` (original) for two arms.



(c) Radius of `A-Bentkus` (original and variant) for two arms.



(d) The difference of the radius for two arms: $R_0 - R_1$. For positive difference value, Arm 0 will be pulled. For negative difference value, Arm 1 will be pulled.



(e) The arms pulled at each iteration. '$\cdot$': Arm 0 is pulled. '+': Arm 1 is pulled. `A-Bentkus` and original `A-Bentkus` are marked in red and blue, respectively.

Figure 11: Identify the best arm out of two using original `A-Bentkus` and the variant introduced in Section 4.3.

Patterns similar to the `A-Hoeffding` and its truncated version happen here too. Although `A-Bentkus` keeps sampling the same arm in the beginning phase, it alternates the samples in the later stage. Comparing Figures 10e (`A-Hoeffding`) and 11e (`A-Bentkus`), the sampling pattern of `A-Hoeffding` is more uniform, however, `A-Bentkus` still outperforms `A-Hoeffding` due to its fast convergence.

C. Computation of $q(\delta; n, \mathcal{A}, B)$

In this section we provide some details on the computation of $q(\delta; n, \mathcal{A}, B)$ based on Bentkus (2004) and Pinelis (2009). We will restrict to the case where $A_1 = A_2 = \cdots = A_n = \cdots = A$.

For any random variable η, define

$$P_2(u; \eta) \ := \ \inf_{x \leqslant u} \frac{\mathbb{E}[(\eta - x)_+^2]}{(u-x)_+^2}.$$

For any A, B, set $p_{AB} = A^2/(A^2 + B^2)$. Define Bernoulli random variables $R_1, R_2, \ldots, R_n$ as

$$\mathbb{P}(R_i = 1) \ = \ p_{AB} \ = \ 1 - \mathbb{P}(R_i = 0).$$

Set $Z_n = \sum_{i=1}^n R_i$. Z_n is a binomial random variables with n trials and success probability p_{AB}: $Z_n \sim \text{Bi}(n, p_{AB})$. For $0 \leqslant k \leqslant n$, define

$$p_k \ := \ \mathbb{P}\left(Z_n \geqslant k\right), \quad e_k \ := \ \mathbb{E}\left[Z_n \mathbb{1}\left\{Z_n \geqslant k\right\}\right], \quad v_k \ := \ \mathbb{E}\left[Z_n^2 \mathbb{1}\left\{Z_n \geqslant k\right\}\right].$$

Proposition 2. *For all $u \in \mathbb{R}$,*

$$P_2\left(u; \sum_{i=1}^n G_i\right) \ = \ P_2\left(\frac{Bu + nA^2}{A^2 + B^2}; \, Z_n\right) \ = \ P_2\left(\frac{Bu + nA^2}{A^2 + B^2}; Z_n\right).$$

Furthermore, for any $x \geqslant 0$ and $1 \leqslant k \leqslant n-1$,

$$P_2\left(x; Z_n\right) \ = \ \begin{cases} 1, & \text{if } x \ \leqslant \ np_{AB}, \\ \frac{np_{AB}(1-p_{AB})}{(x - np_{AB})^2 + np_{AB}(1-p_{AB})}, & \text{if } np_{AB} < x \leqslant \frac{v_0}{e_0}, \\ \frac{v_k p_k - e_k^2}{x^2 p_k - 2xe_k + v_k}, & \text{if } \frac{v_{k-1} - (k-1)e_{k-1}}{e_{k-1} - (k-1)p_{k-1}} < x \leqslant \frac{v_k - ke_k}{e_k - kp_k}, \\ \mathbb{P}\left(Z_n = n\right) = p_{AB}^n, & \text{if } x \ \geqslant \ \frac{v_{n-1} - (n-1)e_{n-1}}{e_{n-1} - (n-1)p_{n-1}} = n. \end{cases}$$

Formally, we can set $P_2(x; Z_n) = 0$ for all $x > n$ because $\mathbb{P}(Z_n > n) = 0$.

Proof. The result is mostly an implication of Proposition 3.2 of Pinelis (2009). It is clear that

$$M_n := \sum_{i=1}^n G_i \ \stackrel{d}{=} \ \frac{A^2 + B^2}{B}\left(\sum_{i=1}^n R_i - \frac{nA^2}{A^2 + B^2}\right),$$

where $R_i \sim \text{Bernoulli}(A^2/(A^2 + B^2))$, that is,

$$\mathbb{P}\left(R_i = 1\right) \ = \ p_{AB} \ = \ 1 - \mathbb{P}(R_i = 0).$$

Proposition 3.2(vi) of Pinelis (2009) implies that

$$P_2(u; \, M_n) \ := \ P_2\left(\frac{Bu + nA^2}{A^2 + B^2}; \, Z_n\right).$$

Hence it suffices to find $P_2(x; \, Z_n)$ for all $x \in \mathbb{R}$. The support of Z_n is given by

$$\text{supp}(Z_n) = \{0, 1, 2, \ldots, n\}.$$

Proposition 3.2(iv) of Pinelis (2009) (with $\alpha = 2$) implies that

$$P_2\left(x; Z_n\right) \ = \ \begin{cases} 1, & \text{if } x \leqslant np_{AB}, \\ \mathbb{P}\left(Z_n = n\right), & \text{if } x \geqslant n. \end{cases}$$

Furthermore, $x \mapsto P_2(x; \sum_{i=1}^n R_i)$ is strictly decreasing on (np_{AB}, n). Define function $F(h) : \mathbb{R} \to \mathbb{R}$ such that

$$F(h) \ := \ \frac{\mathbb{E}[Z_n\left(Z_n - h\right)_+]}{\mathbb{E}\left(Z_n - h\right)_+}. \tag{18}$$

For any $np_{AB} < x < n$, let h_x be the unique solution of

$$F(h) \ = \ x \tag{19}$$

(Uniqueness here is established by Proposition 3.2(ii) of Pinelis (2009).) Then by Proposition 3.2(iii) of Pinelis (2009),

$$\begin{aligned} P_2\left(x; Z_n\right) &= \frac{\mathbb{E}[(Z_n - h_x)_+^2]}{(x - h_x)_+^2} \\ &= \frac{\mathbb{E}[Z_n\left(Z_n - h_x\right)_+] - h_x\mathbb{E}[(Z_n - h_x)_+]}{(x - h_x)_+^2} \\ &= \frac{(x - h_x)\mathbb{E}[(Z_n - h_x)_+]}{(x - h_x)_+^2} \\ &= \frac{\mathbb{E}[(Z_n - h_x)_+]}{(x - h_x)_+}. \end{aligned} \tag{20}$$

This holds for all $nA^2/(A^2 + B^2) < x < n$. We will now discuss solving (19).

Proposition 3.2(i) of Pinelis (2009) implies that $h \mapsto F(h)$ is continuous and increasing.

If $h \leqslant 0$,

$$F(h) = \frac{\mathbb{E}[Z_n(Z_n - h)]}{\mathbb{E}[Z_n - h]} = \frac{np_{AB}(1 - p_{AB}) + n^2p_{AB}^2 - hnp_{AB}}{np_{AB} - h} = np_{AB} + \frac{np(1 - p_{AB})}{np - h}.$$

This is strictly increasing on $(-\infty, 0]$, and $F(0) = np_{AB} + (1 - p_{AB})$. We get that for any $np_{AB} < x \leqslant np_{AB} + (1 - p_{AB})$,

$$F(h) = x \quad \Leftrightarrow \quad h_x = np_{AB} - \frac{np_{AB}(1 - p_{AB})}{x - np_{AB}}.$$

This further implies (from (20)) that

$$\begin{aligned} P_2(x; Z_n) &= \frac{\mathbb{E}[Z_n - h_x]}{x - h_x} \\ &= \frac{np_{AB}(1 - p_{AB})}{(x - np_{AB})^2 + np_{AB}(1 - p_{AB})}, \quad \text{for} \quad np_{AB} \leqslant x \leqslant np_{AB} + (1 - p_{AB}). \end{aligned}$$

If $0 < h < n - 1$, set $k = \lceil h \rceil$, in other words, $k - 1 < h \leqslant k$. Since $\{Z_n \geqslant h\} \Leftrightarrow \{Z_n \geqslant k\}$, hence

$$\begin{aligned} \mathbb{E}[Z_n\left(Z_n - h\right)_+] &= \mathbb{E}[Z_n^2 \mathbb{1}\{Z_n \geqslant h\}] - h\mathbb{E}[Z_n \mathbb{1}\{Z_n \geqslant h\}] \\ &= \mathbb{E}[Z_n^2 \mathbb{1}\{Z_n \geqslant k\}] - h\mathbb{E}[Z_n \mathbb{1}\{Z_n \geqslant k\}], \\ \mathbb{E}[(Z_n - h)_+] &= \mathbb{E}[Z_n \mathbb{1}\{Z_n \geqslant k\}] - h\mathbb{P}(Z_n \geqslant k). \end{aligned}$$

Therefore,

$$\begin{aligned} F(h) &= \frac{\mathbb{E}[Z_n^2 \mathbb{1}\{Z_n \geqslant k\}] - h\mathbb{E}[Z_n \mathbb{1}\{Z_n \geqslant k\}]}{\mathbb{E}[Z_n \mathbb{1}\{Z_n \geqslant k\}] - h\mathbb{P}(Z_n \geqslant k)} \\ &= \frac{v_k - he_k}{e_k - hp_k}. \end{aligned}$$

It is not difficult to verify that $F(\cdot)$ is strictly increasing in $(k - 1, k]$ and hence

$$h_x \ = \ \frac{v_k - xe_k}{e_k - xp_k}, \quad \text{if} \quad F(k - 1) < x \leqslant F(k).$$

Substituting this h_x in (20) yields the value of $P_2(x;\, Z_n)$, that is,

$$\begin{aligned} P_2(x;\, Z_n) &= \left(x - \frac{v_k - xe_k}{e_k - xp_k}\right)^{-1} \left(e_k - \frac{v_k - xe_k}{e_k - xp_k} p_k\right) \\ &= \left(\frac{e_k - xp_k}{2xe_k - x^2p_k - v_k}\right) \left(\frac{e_k^2 - v_kp_k}{e_k - xp_k}\right) \\ &= \frac{e_k^2 - v_kp_k}{2xe_k - x^2p_k - v_k}, \quad \text{whenever} \quad F(k - 1) < x \leqslant F(k), \end{aligned}$$

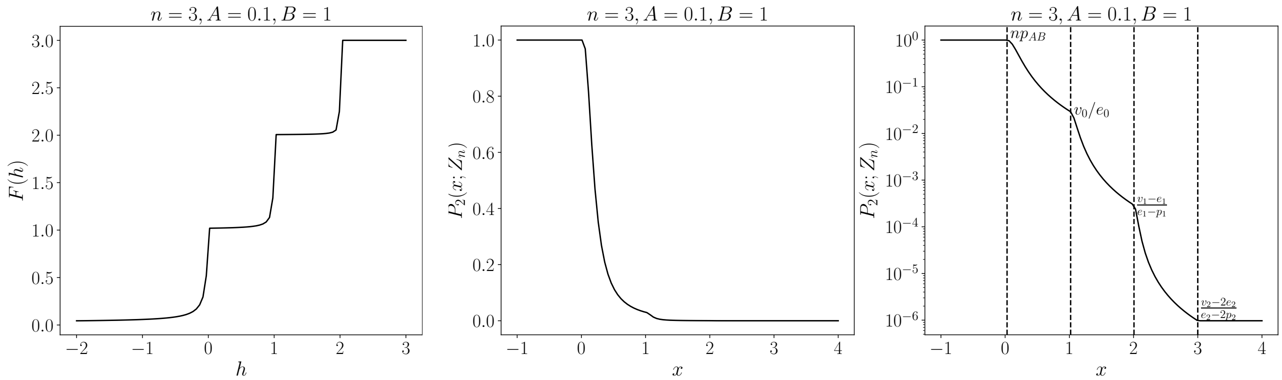


Figure 12: Examples functions $F(h)$ and $P_2(x; Z_n)$ when $n = 3$, $A = 0.1$ and $B = 1.0$. We plot $P_2(x; Z_n)$ in both linear (second plot) and log (third plot) scales on the y-axis.

where $F(k) = \frac{v_k - ke_k}{e_k - kp_k}$, $1 \leqslant k \leqslant n-1$. Hence for $1 \leqslant k \leqslant n-1$,

$$P_2(x; Z_n) \;=\; \frac{v_k p_k - e_k^2}{x^2 p_k - 2xe_k + v_k}, \quad \text{whenever} \quad \frac{v_{k-1} - (k-1)e_{k-1}}{e_{k-1} - (k-1)p_{k-1}} < x \leqslant \frac{v_k - ke_k}{e_k - kp_k}.$$

Finally, we prove that $F(\cdot)$ is a constant on $[n-1, n]$. It is clear that

$$\begin{aligned} F(n-1) &= \frac{v_{n-1} - (n-1)e_{n-1}}{e_{n-1} - (n-1)p_{n-1}} \\ &= \frac{\mathbb{E}[Z_n^2 \mathbb{1}\{Z_n \geqslant n-1\}] - (n-1)\mathbb{E}[Z_n \mathbb{1}\{Z_n \geqslant n-1\}]}{\mathbb{E}[Z_n \mathbb{1}\{Z_n \geqslant n-1\}] - (n-1)\mathbb{P}(Z_n \geqslant n-1)} \\ &= \frac{(n^2 - n(n-1))\mathbb{P}(Z_n = n)}{(n - (n-1))\mathbb{P}(Z_n = n)} = n. \end{aligned}$$

Further if $h > n-1$, then $(Z_n - h)_+ > 0$ if and only if $Z_n = h$ and hence from (18)

$$F(h) = \frac{\mathbb{E}[Z_n (Z_n - h)_+]}{\mathbb{E}[(Z_n - h)_+]} = \frac{n(n-h)\mathbb{P}(Z_n = n)}{(n-h)\mathbb{P}(Z_n = n)} = n.$$

Therefore, the function $F(h)$ is constant on $[n-1, n]$.

For $h > n$, we set $F(h) = n$ since $\mathbb{P}(Z_n > h) = 0$. To put all the pieces together, we have

$$F(h) = \begin{cases} np_{AB} + \frac{np(1-p_{AB})}{np-h} & \text{if} \quad h <= 0, \\ \dfrac{v_{\lceil h \rceil} - he_{\lceil h \rceil}}{e_{\lceil h \rceil} - hp_{\lceil h \rceil}} & \text{if} \quad 0 < h \leqslant n-1, \\ n & \text{if} \quad h > n-1. \end{cases}$$

Consequently, for $np_{AB} < x < n$,

$$h_x = F^{-1}(x) = \begin{cases} np_{AB} - \frac{np_{AB}(1-p_{AB})}{x - np_{AB}}, & \text{if} \quad np_{AB} < x \leqslant np_{AB} + (1 - p_{AB}), \\ \frac{v_k - xe_k}{e_k - xp_k}, & \text{if} \quad F(k-1) < x \leqslant F(k),\, 1 \leqslant k \leqslant n-1. \end{cases}$$

□

As a graphical example, Figure 12 plots $F(h)$ and $P_2(x; Z_n)$ when $n = 3$, $A = 0.1$ and $B = 1.0$.

C.1. Computation of the Quantile

Recall $p_{AB} = A^2/(A^2+B^2)$, $Z_n = \sum_{i=1}^n R_i$, and $\sum_{i=1}^n G_i$ is identically distributed as $B^{-1}(A^2+B^2)(Z_n - np_{AB})$. We will compute x_δ such that

$$P_2(x_\delta; Z_n) = \delta. \tag{21}$$

This implies that

$$P_2\left(\frac{(A^2+B^2)x_\delta - nA^2}{B}; \sum_{i=1}^n G_i\right) = \delta, \quad \text{or equivalently,} \quad q(\delta; n, A, B) = \frac{(A^2+B^2)x_\delta - nA^2}{B}.$$

Hence we concentrate on solving (21). Recall that for any $x \geqslant 0$ and $1 \leqslant k \leqslant n-1$,

$$P_2(x; Z_n) = \begin{cases} 1, & \text{if } x \leqslant np_{AB}, \\ \frac{np_{AB}(1-p_{AB})}{(x-np_{AB})^2 + np_{AB}(1-p_{AB})}, & \text{if } np_{AB} < x \leqslant \frac{v_0}{e_0} = np_{AB} + (1-p_{AB}), \\ \frac{v_k p_k - e_k^2}{x^2 p_k - 2xe_k + v_k}, & \text{if } \frac{v_{k-1} - (k-1)e_{k-1}}{e_{k-1} - (k-1)p_{k-1}} < x \leqslant \frac{v_k - ke_k}{e_k - kp_k}, \\ \mathbb{P}(Z_n = n) = p_{AB}^n, & \text{if } x \geqslant \frac{v_{n-1} - (n-1)e_{n-1}}{e_{n-1} - (n-1)p_{n-1}} = n. \end{cases} \tag{22}$$

The function $P_2(\cdot; Z_n)$ is a non-increasing function and hence if $\delta \leqslant p_{AB}^n$, then we get $x_\delta = n + 10^{-8}$; this corresponds to the last case in (22). If $P_2(v_0/e_0; Z_n) \leqslant \delta \leqslant 1$, then

$$x_\delta = np_{AB} + \sqrt{\frac{(1-\delta)np_{AB}(1-p_{AB})}{\delta}};$$

this corresponds to the first and second case in (22); note that $P_2(v_0/e_0; Z_n) = np_{AB}(1-p_{AB})/[(1-p_{AB})^2 + np_{AB}(1-p_{AB})]$. For the remaining cases, note that if there exists a $1 \leqslant k \leqslant n-1$ such that

$$P_2\left(\frac{v_k - ke_k}{e_k - kp_k}; Z_n\right) \leqslant \delta \leqslant P_2\left(\frac{v_{k-1} - (k-1)e_{k-1}}{e_{k-1} - (k-1)p_{k-1}}; Z_n\right),$$

then

$$\frac{v_{k-1} - (k-1)e_{k-1}}{e_{k-1} - (k-1)p_{k-1}} \leqslant x_\delta \leqslant \frac{v_k - ke_k}{e_k - kp_k}, \tag{23}$$

and using the closed form expression of $P_2(\cdot; Z_n)$ on this interval, we get

$$x_\delta = \frac{e_k + \sqrt{e_k^2 - p_k(v_k - (v_k p_k - e_k^2)/\delta)}}{p_k}. \tag{24}$$

Using these calculations, one can find k looping over $1 \leqslant k \leqslant n-1$ such that (23) holds. This approach has a complexity of $O(n)$, assuming the availability of p_k, e_k, and v_k.

We now describe an approach that reduces the complexity by finding quick-to-compute upper and lower bounds on x_δ. Lemmas 1.1 and 3.1 of Bentkus et al. (2006) show that

$$\mathbb{P}(Z_n \geqslant x) \leqslant P_2(x; Z_n) \leqslant \frac{e^2}{2}\mathbb{P}^\circ(Z_n \geqslant x), \tag{25}$$

where $\mathbb{P}^\circ(Z_n \geqslant x)$ represents the log-linear interpolation of $P(Z_n \geqslant x)$, that is, for $x \in \{0, 1, \ldots, n\}$

$$\mathbb{P}^\circ(Z_n \geqslant x) = \mathbb{P}(Z_n \geqslant x), \tag{26}$$

and for $x \in (k-1, k)$ such that $x = (1-\lambda)(k-1) + \lambda k$,

$$\mathbb{P}^\circ(Z_n \geqslant x) = (\mathbb{P}(Z_n \geqslant k-1))^{1-\lambda}(\mathbb{P}(Z_n \geqslant k))^\lambda.$$

Equation (2) of Bentkus (2002) further shows that

$$\mathbb{P}^\circ(Z_n \geqslant x) \leqslant (1-\lambda)\mathbb{P}(Z_n \geqslant k-1) + \lambda\mathbb{P}(Z_n \geqslant k). \tag{27}$$

Hence, to find $x = x_\delta$ satisfying $P_2(x; Z_n) = \delta$, find $k_1 \in \{0, 1, \ldots, n\}$ such that

$$\mathbb{P}(Z_n \geqslant k_1) \geqslant \delta.$$

This implies (from (25)) that $P_2(k_1; Z_n) \geqslant \delta$ and because $x \mapsto P_2(x; Z_n)$ is decreasing, $x_\delta \geqslant k_1$. Further, find $k_2 \in \{0, 1, \ldots, n\}$ such that

$$\mathbb{P}(Z_n \geqslant k_2) \leqslant 2\delta/e^2.$$

This implies (from (27)) that $\mathbb{P}^o(Z_n \geqslant k_2) = \mathbb{P}(Z_n \geqslant k_2) \leqslant 2\delta/e^2$. Hence using (26), we get $P_2(k_2; Z_n) \leqslant \delta$ which implies that $x_\delta \leqslant k_2$. Summarizing this discussion, we get that x_δ satisfying $P_2(x_\delta; Z_n) = \delta$ also satisfies

$$k_1 \leqslant x_\delta \leqslant k_2, \tag{28}$$

where

$$\mathbb{P}(Z_n \geqslant k_1) \geqslant \delta \quad \text{and} \quad \mathbb{P}(Z_n \geqslant k_2) \leqslant 2\delta/e^2.$$

The bounds in (28) are not very useful because the closed form experssion (24) of x_δ requires finding upper and lower bounds for x_δ in terms of $(v_k - ke_k)/(e_k - kp_k)$'s.

Now we note that

$$v_k \geqslant ke_k \geqslant k^2 p_k \quad \Rightarrow \quad \frac{v_{k_2} - k_2 e_{k_2}}{e_{k_2} - k_2 p_{k_2}} \geqslant k_2.$$

This combined with (28) proves that

$$k_1 \leqslant x_\delta \leqslant k_2 \leqslant \frac{v_{k_2} - k_2 e_{k_2}}{e_{k_2} - k_2 p_{k_2}}.$$

The lower bound here is still not in terms of the ratios $(v_k - ke_k)/(e_k - kp_k)$. But given the upper bound, we can search for $k \leqslant k_2$ (by running a loop from k_2 to 0) such that

$$\frac{v_{k-1} - (k-1)e_{k-1}}{e_{k-1} - (k-1)p_{k-1}} \;\leqslant\; x_\delta \;\leqslant\; \frac{v_k - ke_k}{e_k - kp_k}. \tag{29}$$

Another approach is to make use of the lower bound in (28). Because $k_1 \leqslant (v_{k_1} - k_1 e_{k_1})/(e_{k_1} - k_1 p_{k_1})$, there are two possibilities:

1. $k_1 \leqslant x_\delta \leqslant (v_{k_1} - k_1 e_{k_1})/(e_{k_1} - k_1 p_{k_1})$;
2. $k_1 \leqslant (v_{k_1} - k_1 e_{k_1})/(e_{k_1} - k_1 p_{k_1}) < x_\delta$.

In the first case, it suffices to search for $k \leqslant k_1$ such that (29). In the second case, we can search over $k_1 + 1 \leqslant k \leqslant k_2$ as before.

D. Proof of Theorem 1

It is clear that $(S_t, \mathcal{F}_t)_{t=1}^n$ with $\mathcal{F}_t = \sigma\{X_1, \ldots, X_t\}$ is a martingale because

$$\mathbb{E}\left[S_t | \mathcal{F}_{t-1}\right] = S_{t-1} + \mathbb{E}[X_t] = S_{t-1}.$$

Consider now the process

$$D_t := (S_t - x)_+^2 \quad \text{for a fixed} \quad x > 0.$$

The function $f : y \mapsto (y - x)_+^2$ is continuous and satisfies

$$f'(y) = \begin{cases} 0, & \text{if } y \leqslant x, \\ 2(y - x), & \text{if } y > x, \end{cases} \quad \text{and} \quad f''(y) = \begin{cases} 0, & \text{if } y \leqslant x, \\ 2, & \text{if } y > x. \end{cases}$$

Therefore, $f(\cdot)$ is a convex function. This implies by Jensen's inequality that

$$\mathbb{E}[D_t | \mathcal{F}_{t-1}] = \mathbb{E}[f(S_t) | \mathcal{F}_{t-1}] \geqslant f(S_{t-1}).$$

Hence $(D_t, \mathcal{F})_{t=1}^n$ is a submartingale. Doob's inequality now implies that

$$\begin{aligned}\mathbb{P}\left(\max_{1\leqslant t\leqslant n} S_t \geqslant u\right) &\overset{(a)}{=} \mathbb{P}\left(\max_{1\leqslant t\leqslant n} (S_t - x)_+^2 \geqslant (u-x)_+^2\right)\\ &= \mathbb{P}\left(\max_{1\leqslant t\leqslant n} D_t \geqslant (u-x)_+^2\right)\\ &\overset{(b)}{\leqslant} \frac{\mathbb{E}[D_n]}{(u-x)_+^2} \leqslant \frac{\mathbb{E}[(S_n - x)_+^2]}{(u-x)_+^2}.\end{aligned}$$

Here equality (a) holds for every $x \leqslant u$ and inequality (b) holds because of Doob's inequality. Because $x \leqslant u$ is arbitrary, we get

$$\mathbb{P}\left(\max_{1\leqslant t\leqslant n} S_t \geqslant u\right) \leqslant \inf_{x\leqslant u} \frac{\mathbb{E}[(S_n - x)_+^2]}{(u-x)_+^2},$$

and condition (2) along with Theorem 2.1 of Bentkus et al. (2006) (or Pinelis (2006)) imply that

$$\mathbb{P}\left(\max_{1\leqslant t\leqslant n} S_t \geqslant u\right) \leqslant \inf_{x\leqslant u} \frac{\mathbb{E}[(\sum_{i=1}^n G_i - x)_+^2]}{(u-x)_+^2}.$$

The definition (10) of $q(\delta; n, \mathcal{A}, B)$ readily implies

$$\mathbb{P}\left(\max_{1\leqslant t\leqslant n} S_t \geqslant q(\delta; n, \mathcal{A}, B)\right) \leqslant \delta.$$

This completes the proof of (11). We now prove the sharpness. Note that the condition

$$\mathbb{P}\left(\max_{1\leqslant t\leqslant n} S_t \geqslant n\tilde{q}(\delta^{1/n}; A, B)\right) \leqslant \delta \quad \text{for all} \quad \delta \in [0,1],$$

is equivalent to the existence of a function $x \mapsto H(x; A, B)$ such that

$$\mathbb{P}\left(\max_{1\leqslant t\leqslant n} S_t \geqslant nu\right) \leqslant H^n(u; A, B), \quad \text{for all} \quad u.$$

(The function $\delta \mapsto \tilde{q}(\delta^{1/n}; A, B)$ is the inverse of $u \mapsto H^n(u; A, B)$.) In particular, this implies that

$$\mathbb{P}\left(S_n \geqslant nu\right) \leqslant H^n(u; A, B) \quad \text{for all} \quad u.$$

Now, Lemma 4.7 of Bentkus (2004) (also see Eq. (2.8) of Hoeffding (1963)) implies that

$$\begin{aligned}H^n(u; A, B) &\geqslant \left\{\left(1 + \frac{Bu}{A^2}\right)^{-(A^2+Bu)/(A^2+B^2)} \left(1 - \frac{u}{B}\right)^{-(B^2-Bu)/(B^2+A^2)}\right\}^n\\ &= \inf_{h\geqslant 0} e^{-nhu}\mathbb{E}\left[e^{h\sum_{i=1}^n G_i}\right],\end{aligned}$$

where $G_1, \ldots, G_n$ are independent random variables constructed through (6). Proposition 3.5 of Pinelis (2009) implies that

$$\inf_{h\geqslant 0} e^{-nhu}\mathbb{E}\left[e^{h\sum_{i=1}^n G_i}\right] \geqslant \inf_{x\leqslant nu} \frac{\mathbb{E}[(\sum_{i=1}^n G_i - x)_+^2]}{(nu-x)_+^2}.$$

Summarizing the inequalities, we conclude

$$\mathbb{P}\left(S_n \geqslant nu\right) \leqslant \inf_{x\leqslant nu} \frac{\mathbb{E}[(\sum_{i=1}^n G_i - x)_+^2]}{(nu-x)_+^2} \leqslant \inf_{h\geqslant 0} \mathbb{E}\left[e^{h\sum_{i=1}^n G_i - h(nu)}\right] \leqslant H^n(u; A, B) \quad \forall\, u.$$

This proves that $q(\delta; n, A, B) \leqslant n\tilde{q}(\delta^{1/n}; A, B)$ for any valid $\tilde{q}(\cdot; A, B)$.

E. Proof of Theorem 2

The proof is based on (11) and a union bound. It is clear that

$$\begin{aligned}
&\mathbb{P}\left(\exists\, t \geqslant 1:\ \sum_{i=1}^{t} X_i \geqslant q(\delta/h(k_t); c_t, \mathcal{A}, B)\right) \\
&= \mathbb{P}\left(\bigcup_{k=0}^{\infty}\left\{\exists\, \lceil \eta^k \rceil \leqslant t \leqslant \lfloor \eta^{k+1} \rfloor :\ \sum_{i=1}^{t} X_i \geqslant q(\delta/h(k_t); c_t, \mathcal{A}, B)\right\}\right) \\
&= \mathbb{P}\left(\bigcup_{k=0}^{\infty}\left\{\exists\, \lceil \eta^k \rceil \leqslant t \leqslant \lfloor \eta^{k+1} \rfloor :\ \sum_{i=1}^{t} X_i \geqslant q(\delta/h(k); \lfloor \eta^{k+1} \rfloor, \mathcal{A}, B)\right\}\right) \\
&\leqslant \sum_{k=0}^{\infty} \mathbb{P}\left(\max_{\lceil \eta^k \rceil \leqslant t \leqslant \lfloor \eta^{k+1} \rfloor} \sum_{i=1}^{t} X_i \geqslant q(\delta/h(k); \lfloor \eta^{k+1} \rfloor, \mathcal{A}, B)\right) \\
&\leqslant \sum_{k=0}^{\infty} \frac{\delta}{h(k)} \leqslant \delta.
\end{aligned}$$

F. Proof of Theorem 3

Theorem 2 implies that

$$\mathbb{P}\left(\exists n \geqslant 1:\ S_n \geqslant q\left(\frac{\delta_1}{h(k_n)}; c_n, A, B\right)\right) \leqslant \delta_1.$$

Lemma F.1 (below) proves

$$\mathbb{P}\left(\exists n \geqslant 1:\ A \geqslant \bar{A}_n(\delta_2)\right) \leqslant \delta_2.$$

In particular this implies that

$$\mathbb{P}\left(\exists n \geqslant 1:\ A \geqslant \min_{1 \leqslant s \leqslant n} \bar{A}_s(\delta_2)\right) \leqslant \delta_2.$$

Combining the inequalities above with a union bound (and Lemma H.2) proves the result.

Lemma F.1. *Under the assumptions of Theorem 3, we have for any $\delta \in [0,1]$,*

$$\mathbb{P}\left(\exists t \geqslant 1:\ V_{2\lfloor t/2 \rfloor} - \lfloor t/2 \rfloor A^2 \leqslant -\frac{\sqrt{\lfloor c_t/2 \rfloor}(B - \underline{B})A}{\sqrt{2}} \Phi^{-1}\left(1 - \frac{2\delta}{e^2 h(k_t)}\right)\right) \leqslant \delta, \tag{30}$$

where $W_i = (X_{2i} - X_{2i-1})^2/2$ and $V_t := \sum_{i=1}^{\lfloor t/2 \rfloor} W_i$.

Proof. Fix $x \geqslant 0$. Note that for any $u \geqslant -x$,

$$\begin{aligned}
\mathbb{P}\left(\max_{1 \leqslant t \leqslant n} \{V_{2t} - tA^2\} \leqslant -x\right) &= \mathbb{P}\left(\max_{1 \leqslant t \leqslant n} (u - \{V_{2t} - tA^2\})_+ \geqslant (u + x)_+\right), \\
&\leqslant \frac{\mathbb{E}[(u - \{V_{2n} - 2nA^2\})_+^2]}{(u + x)_+^2}.
\end{aligned}$$

where the last inequality follows from the fact that $\{(u - \{V_{2t} - tA^2\}\}_{t \geqslant 1}$ is a submartingale. Therefore,

$$\begin{aligned}
\mathbb{P}\left(\max_{1 \leqslant t \leqslant n} \{V_{2t} - tA^2\} \leqslant -x\right) &\leqslant \inf_{u \geqslant -x} \frac{\mathbb{E}[(u - \{V_{2n} - nA^2\})_+^2]}{(u + x)_+^2} \\
&= \inf_{u \geqslant -x} \frac{\mathbb{E}[(u + nA^2 - V_{2n})_+^2]}{(u + x)_+^2} \\
&= \inf_{u \geqslant nA^2 - x} \frac{\mathbb{E}[(u - V_{2n})_+^2]}{(u - nA^2 + x)^2}.
\end{aligned}$$

Corollary 2.7 (Eq. (2.24)) of Pinelis (2016) implies that

$$\inf_{u\geqslant nA^2-x}\frac{\mathbb{E}[(u-V_{2n})_+^2]}{(u-nA^2+x)^2}\leqslant P_2(E_{1,n}+Z\sqrt{E_{2,n}};nA^2-x)=P_2(E_{1,n}+Z\sqrt{E_{2,n}};E_{1,n}-x),\tag{31}$$

where $E_{j,t}=\sum_{i=1}^{\lfloor t/2\rfloor}\mathbb{E}[W_i^j]$ for $j=1,2$ and Z stands for a standard normal distribution. Inequality (31) is *not* the best inequality to use and there is a more precise version; see Theorem 2.4(I) and Corollary 2.7 of Pinelis (2016). With the more precise version, the following steps will lead to a refined upper bound on A; we will not pursue this direction here.

It now follows from Bentkus (2008) that

$$P_2(E_{1,n}+Z\sqrt{E}_{2,n};E_{1,n}-x)\leqslant\frac{e^2}{2}\mathbb{P}\left(Z\leqslant-\frac{x}{\sqrt{E_{2,n}}}\right).$$

Because $X_i\in[\underline{B},B]$ with probability 1, $W_i\leqslant(B-\underline{B})^2/2$ and hence

$$E_{2,n}=\mathbb{E}\sum_{i=1}^{n}\mathbb{E}[W_i^2]\leqslant\frac{(B-\underline{B})^2}{2}\sum_{i=1}^{n}\mathbb{E}[W_i]=(B-\underline{B})^2E_{1,n}/2=n(B-\underline{B})^2A^2/2.$$

This implies that

$$\mathbb{P}\left(\max_{1\leqslant t\leqslant n}\{V_{2t}-tA^2\}\leqslant-x\right)\leqslant\frac{e^2}{2}\mathbb{P}\left(Z\leqslant-\frac{\sqrt{2}x}{\sqrt{n}(B-\underline{B})A}\right).$$

Equating the right hand side to δ yields

$$\mathbb{P}\left(\max_{1\leqslant t\leqslant n}\{V_{2t}-tA^2\}\leqslant-\frac{\sqrt{n}(B-\underline{B})A}{\sqrt{2}}\Phi^{-1}\left(1-\frac{2\delta}{e^2}\right)\right)\leqslant\delta.\tag{32}$$

Because of this maximal inequality, we can apply stitching and get (30). Note that

$$\begin{aligned}
&\mathbb{P}\left(\exists t\geqslant 1:\,V_{2\lfloor t/2\rfloor}-\lfloor t/2\rfloor A^2\leqslant-\frac{\sqrt{\lfloor c_t/2\rfloor}(B-\underline{B})A}{\sqrt{2}}\Phi^{-1}\left(1-\frac{2\delta}{e^2h(k_t)}\right)\right)\\
&=\mathbb{P}\left(\exists t\geqslant 2:\,V_{2\lfloor t/2\rfloor}-\lfloor t/2\rfloor A^2\leqslant-\frac{\sqrt{\lfloor c_t/2\rfloor}(B-\underline{B})A}{\sqrt{2}}\Phi^{-1}\left(1-\frac{2\delta}{e^2h(k_t)}\right)\right)\\
&=\mathbb{P}\left(\bigcup_{k=0}^{\infty}\left\{\exists\lceil\eta^k\rceil\leqslant t\leqslant\lfloor\eta^{k+1}\rfloor:\,V_{2\lfloor t/2\rfloor}-\lfloor t/2\rfloor A^2\leqslant-\frac{\sqrt{\lfloor c_t/2\rfloor}(B-\underline{B})A}{\sqrt{2}}\Phi^{-1}\left(1-\frac{2\delta}{e^2h(k_t)}\right)\right\}\right)\\
&\leqslant\sum_{k=0}^{\infty}\mathbb{P}\left(\exists\lceil\eta^k\rceil\leqslant t\leqslant\lfloor\eta^{k+1}\rfloor:\,V_{2\lfloor t/2\rfloor}-\lfloor t/2\rfloor A^2\leqslant-\frac{\sqrt{\lfloor c_t/2\rfloor}(B-\underline{B})A}{\sqrt{2}}\Phi^{-1}\left(1-\frac{2\delta}{e^2h(k_t)}\right)\right)\\
&\leqslant\sum_{k=0}^{\infty}\frac{\delta}{h(k)}\leqslant\delta,
\end{aligned}$$

where the last inequality follows from (32) applied to $\{1\leqslant t\leqslant\lfloor c_t/2\rfloor\}$.

Inequality (30) yields

$$\mathbb{P}\left(tA^2-\frac{\sqrt{\lfloor c_t/2\rfloor}(B-\underline{B})A}{\sqrt{2}}\Phi^{-1}\left(1-\frac{2\delta}{e^2h(k_t)}\right)-V_{2t}\leqslant 0\quad\forall\,t\geqslant 1\right)\geqslant 1-\delta.$$

Inequality

$$tA^2-\frac{\sqrt{\lfloor c_t/2\rfloor}(B-\underline{B})A}{\sqrt{2}}\Phi^{-1}\left(1-\frac{2\delta}{e^2h(k_t)}\right)-V_{2t}\leqslant 0$$

holds for $A>0$ if and only if

$$A\leqslant g_{2,t}+\sqrt{g_{2,t}^2+g_{3,t}},$$

where
$$g_{2,t} = \frac{\sqrt{\lfloor c_t/2 \rfloor}(B - \underline{B})A}{2\sqrt{2}t}\Phi^{-1}\left(1 - \frac{2\delta}{e^2 h(k_t)}\right) \quad \text{and} \quad g_{3,t} = \frac{V_{2\lfloor t/2 \rfloor}}{\lfloor t/2 \rfloor}.$$
Hence a rewriting of (30) is
$$\mathbb{P}\left(A \geqslant g_{2,t} + \sqrt{g_{2,t}^2 + g_{3,t}} \quad \forall\, t \geqslant 1\right) \geqslant 1 - \delta.$$
It is clear that $g_{2,t} = O(1/\sqrt{t})$ and $\mathbb{E}[V_{2\lfloor t/2 \rfloor}/\lfloor t/2 \rfloor] = A^2$ and hence the upper bounds above grows like $A + O(\sqrt{\log(h(k_t))/t})$. $\square$

G. Proof of Theorem 4

The assumption $\mathbb{P}(L \leqslant X_i \leqslant U) = 1$ implies that $\mathbb{P}(L - \mu \leqslant X_i - \mu \leqslant U - \mu) = 1$ and hence applying Theorem 2 with $X_i - \mu$ and its upper bound $U - \mu$ yields
$$\mathbb{P}\left(\exists n \geqslant 1:\ \sum_{i=1}^{n}(X_i - \mu) \geqslant q\left(\frac{\delta_1/2}{h(k_n)};\, c_n,\, A,\, U - \mu\right)\right) \leqslant \frac{\delta_1}{2}. \tag{33}$$
Similarly applying Theorem 2 with $\mu - X_i$ and its upper bound $\mu - L$ yields
$$\mathbb{P}\left(\exists n \geqslant 1:\ \sum_{i=1}^{n}(\mu - X_i) \geqslant q\left(\frac{\delta_1/2}{h(k_n)};\, c_n,\, A,\, \mu - L\right)\right) \leqslant \frac{\delta_1}{2}. \tag{34}$$
Finally Lemma F.1 implies that
$$\mathbb{P}\left(\exists n \geqslant 1:\ A \geqslant \bar{A}_n^*(\delta_2; U, L)\right) \leqslant \delta_2. \tag{35}$$
Now combining inequalities (33), (34), and (35) yields with probability $\geqslant 1 - \delta_1 - \delta_2$, for all $n \geqslant 1$
$$-\frac{1}{n}q\left(\frac{\delta_1/2}{h(k_n)};\, c_n,\, A,\, \mu - L\right) \leqslant \frac{S_n}{n} - \mu \leqslant \frac{1}{n}q\left(\frac{\delta_1/2}{h(k_n)};\, c_n,\, A,\, U - \mu\right),\ \text{and } A \leqslant \bar{A}_n^*(\delta_2).$$
On this event, we get by using $U - \mu \leqslant U - L$ and $\mu - L \leqslant U - L$,
$$\mu_0^{\text{low}} \leqslant \mu \leqslant \mu_0^{\text{up}},$$
and then recursively using $\mu_{n-1}^{\text{low}} \leqslant \mu \leqslant \mu_{n-1}^{\text{up}}$,
$$-\frac{1}{n}q\left(\frac{\delta_1/2}{h(k_n)};\, c_n,\, \bar{A}_n^*(\delta_2),\, \mu_{n-1}^{\text{up}} - L\right) \ \leqslant\ \frac{S_n}{n} - \mu \ \leqslant\ \frac{1}{n}q\left(\frac{\delta_1/2}{h(k_n)};\, c_n,\, \bar{A}_n^*(\delta_2),\, U - \mu_{n-1}^{\text{low}}\right).$$
This proves the result.

H. Auxiliary Results

Define $M_t, t \geqslant 1$ as $M_t := \sum_{i=1}^{t} G_i$, with
$$\mathbb{P}\left(G_i = -A_i^2/B\right) = \frac{B^2}{A_i^2 + B^2} \quad \text{and} \quad \mathbb{P}\left(G_i = B\right) = \frac{A_i^2}{A_i^2 + B^2}.$$

Lemma H.1. *For any $t \geqslant 1$ and $x \in \mathbb{R}$, the map $(A_1, \ldots, A_t) \mapsto \mathbb{E}[(M_t - x)_+^2]$ is non-decreasing.*

Proof. Suppose we prove that for every $y \in \mathbb{R}$,
$$A_1 \mapsto \mathbb{E}[(G_1 - y)_+^2] \text{ is non-decreasing,} \tag{36}$$
then by conditioning on $G_2, \ldots, G_t$ and taking $y = x + G_2 + \cdots + G_t$, we get for $A_1 \leqslant A_1'$
$$\mathbb{E}[(G_1(A_1) - y)_+^2] \leqslant \mathbb{E}[(G_1(A_1') - y)_+^2].$$

Now taking expectations on both sides with respect to $G_2, \ldots, G_t$ implies non-decreasingness of $A_1 \mapsto \mathbb{E}[(M_t - x)_+^2]$. This implies the result.

To prove (36),

$$\mathbb{E}[(G_1 - y)_+^2] = \frac{B^2}{A_1^2 + B^2}\left(-\frac{A_1^2}{B} - y\right)_+^2 + \frac{A_1^2}{A_1^2 + B^2}(B - y)_+^2.$$

Because $A_1 \to A_1^2/B^2$ is increasing, it suffices to show $A_1^2/B^2 \mapsto \mathbb{E}[(G_1 - y)_+^2]$ is non-decreasing with respect to A_1^2/B^2. Set $p = A_1^2/B^2$ and define

$$g(p) = \frac{1}{1+p}(-Bp - y)_+^2 + \frac{p}{1+p}(B - y)_+^2.$$

Differentiating with respect to p yields

$$\begin{aligned}\frac{\partial g(p)}{\partial p} &= -\frac{(-Bp - y)_+^2}{(1+p)^2} - \frac{2B(-Bp - y)_+}{1+p} + \frac{(B - y)_+^2}{(1+p)^2} \\ &= \frac{-(-Bp - y)_+^2 - 2B(1+p)(-Bp - y)_+ + (B - y)_+^2}{(1+p)^2}.\end{aligned}$$

If $y \leqslant -Bp$ then $y + Bp < 0$ and $B - y > B(1+p) > 0$ and hence

$$\frac{\partial g(p)}{\partial p} = \frac{-(Bp + y)^2 + 2B(1+p)(Bp + y) + (B - y)^2}{(1+p)^2} = \frac{B^2 + B^2p^2 + 2B^2p}{(1+p)^2} > 0.$$

If $-Bp < y < B$ then $y + Bp > 0$ and $B - y > 0$ and hence

$$\frac{\partial g(p)}{\partial p} = \frac{(B - y)^2}{(1+p)^2} > 0.$$

If $y > B$, then $\partial g(p)/\partial p = 0$. Hence $\partial g(p)/\partial p \geqslant 0$ for all p. This proves (36). ☐

Recall the definition of $q(\delta; t, \mathcal{A}, B)$ from (10). In the case of equal variances, that is, $A_1 = A_2 = \ldots = A$, we write $A, q(\delta; t, A, B)$ for $\mathcal{A}, q(\delta; t, \mathcal{A}, B)$, respectively. We now prove that $A \mapsto q(\delta; t_2, A, B)$ is an non-decreasing function.

Lemma H.2. *For any $t \geqslant 1$, the function $A \mapsto q(\delta; t, A, B)$ is an non-decreasing function.*

Proof. Lemma H.1 proves that $A \mapsto \mathbb{E}[(M_t - x)_+^2]$ is non-decreasing. This implies that $I(A; u)$ is also non-decreasing in A, where

$$I(A; u) \ := \ \inf_{x \leqslant u} \frac{\mathbb{E}[(M_t - x)_+^2]}{(u - x)_+^2}.$$

Lemma 3.1 of Bentkus et al. (2006) proves that $I(A; u)$ is also non-increasing in u. Fix $A_1 \leqslant A_2$. From the definition of δ,

$$I(A_1, q(\delta; t, A_1, B)) = \delta \quad \text{and} \quad I(A_2, q(\delta; t, A_2, B)) = \delta.$$

Because $I(A; u)$ is non-decreasing in A,

$$I(A_2; q(\delta; t, A_2, B)) = \delta = I(A_1; q(\delta; t, A_1, B)) \leqslant I(A_2; q(\delta; t, A_1, B))$$

Hence $I(A_2; q(\delta; t, A_2, B)) \leqslant I(A_2; q(\delta; t, A_1, B))$ and because $I(A; u)$ is non-increasing in u, we conclude that $q(\delta; t, A_1, B) \leqslant q(\delta; t, A_2, B)$. This proves the result modulo the condition $A \mapsto \mathbb{E}[(M_t - x)_+^2]$ is non-decreasing. ☐

Lemma H.3. *For any $\delta \in [0, 1]$, $q(\delta; t, AB, B^2) = Bq(\delta; t, A, B)$.*

Proof. Recall that $q(\delta; t, AB, B^2)$ is defined as the solution of

$$\inf_{x \leqslant u} \frac{\mathbb{E}[(M_t' - x)_+^2]}{(u - x)_+^2} = \delta,$$

where M'_t is defined as $M'_t = \sum_{i=1}^t G'_i$ with

$$\mathbb{P}\left(G'_i = -(A^2B^2)/B^2\right) = \frac{B^4}{A^2B^2 + B^4} = \frac{B^2}{A^2 + B^2} \quad \text{and,}$$
$$\mathbb{P}\left(G'_i = B^2\right) = \frac{A^2B^2}{A^2B^2 + B^4} = \frac{A^2}{A^2 + B^2}.$$

This implies that $G'_i \overset{d}{=} BG_i$ and hence $M'_t \overset{d}{=} BM_t$. Therefore,

$$\mathbb{E}[(M'_t - x)_+^2] = \mathbb{E}[(BM_t - x)_+^2] = B^2\mathbb{E}[(M_t - x/B)_+^2],$$

and

$$\inf_{x\leqslant u} \frac{\mathbb{E}[(M'_t - x)_+^2]}{(u-x)_+^2} = B^2 \inf_{x\leqslant u} \frac{\mathbb{E}[(M_t - x/B)_+^2]}{B^2(u/B - x/B)_+^2} = \inf_{x\leqslant u/B} \frac{\mathbb{E}[(M_t - x)_+^2]}{(u/B - x)_+^2}.$$

The right hand side above equals δ, when $u = Bq(\delta; t, A, B)$ because the definition of $q(\delta; t, A, B)$ implies that

$$\inf_{x\leqslant q(\delta;t,A,B)} \frac{\mathbb{E}[(M_t - x)_+^2]}{(q(\delta; t, A, B) - x)_+^2} = \delta.$$

This completes the proof. □

I. Alternative Empirical Bentkus Confidence Sequences with Estimated Variance

In Section 3.5, we presented one actionable version of Theorem 2, where we used an analytical upper bound on the variance A^2. In this section, we present an alternative empirical Bentkus confidence sequence that requires numerical computation. In our initial experiments, we found solving for the upper bound of A in this way to be unstable. Because the proof technique here is very analogues to that of the empirical Bernstein bound in Audibert et al. (2009, Eq. (48)-(50)), we present the alternative bound below.

Define the empirical variance as

$$\widehat{A}_n^2 \;:=\; n^{-1}\textstyle\sum_{i=1}^n (X_i - \bar{X}_n)^2, \quad \text{where} \quad \bar{X}_n = n^{-1}\sum_{i=1}^n X_i.$$

For any $\delta_1, \delta_2 \in [0,1]$, define

$$\bar{A}_n := \sup\left\{a \geqslant 0:\; \widehat{A}_n^2 \;\geqslant\; a^2 - \tfrac{B}{n} q\left(\tfrac{\delta_1}{h(k_n)}; c_n, a, B\right) - \tfrac{1}{n^2} q^2\left(\tfrac{\delta_2}{2h(k_n)}; c_n, a, B\right)\right\}.$$

Lemma I.1 shows that $\bar{A}_n$ is an over-estimate of A uniformly over n and yields the following actionable bound. Recall that $S_n = \sum_{i=1}^n X_i = n\bar{X}_n$.

Theorem 9. *If $X_1, X_2, \ldots$ are mean-zero independent random variables satisfying $\mathrm{Var}(X_i) = A^2$ and $\mathbb{P}(|X_i| > B) = 0$ for all $i \geqslant 1$, then for any $\delta_1, \delta_2 \in [0,1]$,*

$$\mathbb{P}\left(\exists n \geqslant 1:\; |S_n| \geqslant q\left(\frac{\delta_2}{2h(k_n)}; c_n, \bar{A}_n^*, B\right) \quad \textit{or} \quad A \geqslant \bar{A}_n^*(\delta_1)\right) \leqslant \delta_1 + \delta_2,$$

where $\bar{A}_n^* := \min_{1\leqslant s\leqslant n} \bar{A}_s$. Here k_n and c_n are same as those defined in Theorem 2.

This theorem is an analogue of the empirical Bernstein inequality Mnih et al. (2008, Eq. (5)). Furthermore, the upper bound $\bar{A}_n$ on A is better than that in the Bernstein version Audibert et al. (2009, Eq. (49)-(50)); see Lemma I.2.

I.1. Proof of Theorem 9 and Comparison of Standard Deviation Estimation from Other Inequalities

Lemma I.1. *If $X_1, X_2, \ldots$ are mean-zero independent random variables satisfying*

$$\mathrm{Var}(X_i) = A^2 \quad \textit{and} \quad \mathbb{P}(|X_i| > B) = 0, \quad \textit{for all} \quad i \geqslant 1,$$

then for any $\delta \in [0,1]$

$$\mathbb{P}\left(\exists\, t \geqslant 1:\; \widehat{A}_t^2 \;\leqslant\; A^2 - \frac{B}{t} q\left(\frac{\delta}{h(k_t)}; c_t, A, B\right) - \frac{1}{t^2}\left|\sum_{i=1}^t X_i\right|^2\right) \leqslant \delta.$$

Proof. Consider the random variable $X_i^2 - \mathbb{E}[X_i^2]$. These are mean zero and are bounded in absolute value by B^2. Further the variance can be bounded as

$$\mathrm{Var}(X_i^2 - \mathbb{E}[X_i^2]) = \mathbb{E}[(X_i^2 - \mathbb{E}[X_i^2])^2] \leqslant B^2\mathbb{E}[|X_i|^2] = B^2A^2.$$

Applying Theorem 2 with variables $X_i^2 - \mathbb{E}[X_i^2]$ implies

$$\mathbb{P}\left(\exists t \geqslant 1 : \sum_{i=1}^{t} -(X_i^2 - \mathbb{E}[X_i^2]) \geqslant q\left(\frac{\delta}{h(k_t)}; c_t, AB, B^2\right)\right) \leqslant \delta.$$

Lemma H.3 proves that

$$q\left(\frac{\delta}{h(k_t)}; c_t, AB, B^2\right) = Bq\left(\frac{\delta}{h(k_t)}; c_t, A, B\right).$$

Hence we get with probability at least $1-\delta$, simultaneously for all $t \geqslant 1$

$$\begin{aligned}\sum_{i=1}^{t}(X_i - \bar{X}_t)^2 &= \sum_{i=1}^{t} X_i^2 - \frac{1}{t}\left(\sum_{i=1}^{t} X_i\right)^2 \\ &\geqslant \sum_{i=1}^{t}\mathbb{E}[X_i^2] - Bq\left(\frac{\delta}{h(k_t)}; c_t, A, B\right) - \frac{1}{t}\left|\sum_{i=1}^{t} X_i\right|^2.\end{aligned}$$

Hence for any $\delta \in [0,1]$,

$$\mathbb{P}\left(\exists\, t \geqslant 1 :\, t\widehat{A}_t^2 \leqslant tA^2 - Bq\left(\frac{\delta}{h(k_t)}; c_t, A, B\right) - \frac{1}{t}\left|\sum_{i=1}^{n} X_i\right|^2\right) \leqslant \delta.$$

This completes the proof. ∎

We will now prove Theorem 9. Theorem 2 implies that

$$\mathbb{P}\left(\exists\, t \geqslant 1 : \left|\sum_{i=1}^{t} X_i\right| \geqslant q\left(\frac{\delta_2}{2h(k_t)}; c_t, \mathcal{A}, B\right)\right) \leqslant \delta_2, \tag{37}$$

Lemma I.1 implies that

$$\mathbb{P}\left(\exists\, t \geqslant 1 :\, \widehat{A}_t^2 \leqslant \frac{t}{t-1}A^2 - \frac{B}{t-1}q\left(\frac{\delta_1}{h(k_t)}; c_t, A, B\right) - \frac{1}{t(t-1)}\left|\sum_{i=1}^{t} X_i\right|^2\right) \leqslant \delta_1.$$

Hence with probability at least $1-\delta_1-\delta_2$, simultaneously for all $t \geqslant 1$,

$$\begin{aligned}\left|\sum_{i=1}^{t} X_i\right| &\leqslant q\left(\frac{\delta_2}{2h(k_t)}; c_t, A, B\right), \\ \widehat{A}_t^2 &\leqslant \frac{t}{t-1}A^2 - \frac{B}{t-1}q\left(\frac{\delta_1}{h(k_t)}; c_t, A, B\right) - \frac{1}{t(t-1)}\left|\sum_{i=1}^{t} X_i\right|^2\end{aligned}$$

On this event, $A \leqslant \bar{A}_t$ simultaneously for all $t \geqslant 1$ which in turn implies that $A \leqslant \min_{1\leqslant s\leqslant t}\bar{A}_s$ also holds simultaneously for all $t \geqslant 1$. Substituting this in (37) (along with Lemma H.2) implies the result.

Lemma I.2. *Suppose $\delta \mapsto \tilde{q}(\delta^{1/n}; A, B)$ is a function such that*

$$\mathbb{P}\left(\max_{1\leqslant t\leqslant n} S_t \geqslant n\tilde{q}(\delta^{1/n}; A, B)\right) \leqslant \delta, \tag{38}$$

for all $\delta \in [0,1]$ and independent random variables $X_1, \ldots, X_n$ satisfying (2). Define the (over)-estimator of A as

$$\tilde{A}_t := \sup\left\{a \geqslant 0 :\, \widehat{A}_t^2 \geqslant a^2 - \frac{Bc_t}{t}\tilde{q}\left((\delta/(3h(k_t)))^{1/c_t}; a, B\right) - \frac{c_t^2}{t^2}\tilde{q}^2\left((\delta/(3h(k_t)))^{1/c_t}; a, B\right)\right\}.$$

Then $\bar{A}_n \leqslant \tilde{A}_n$.

Proof. We have proved in Appendix D that (38) implies

$$q\left(\delta; n, a, B\right) \leqslant n\tilde{q}\left(\delta^{1/n}; a, B\right),$$

for all n, a, and B. Hence if a satisfies

$$\widehat{A}_t^2 \geqslant a^2 - \frac{B}{t} q\left(\frac{\delta}{3h(k_t)}; c_t, a, B\right) - \frac{1}{t^2} q^2\left(\frac{\delta}{3h(k_t)}; c_t, a, B\right),$$

then

$$\widehat{A}_n^2 \geqslant a^2 - \frac{Bc_t}{t}\tilde{q}\left((\delta/(3h(k_t)))^{1/c_t}; a, B\right) - \frac{c_t^2}{t^2}\tilde{q}^2\left((\delta/(3h(k_t)))^{1/c_t}; a, B\right),$$

which implies the result. □