Supplementary Materials: No-regret Algorithms for Capturing Events in Poisson Point Processes

In this appendix, we provide all proofs as well as additional results which did not fit to the main body of the paper. You can find details of numerical experiments.

A. Additional Results

A.1. Incompatibility of Log-Linear model

To demonstrate that the RKHS assumption and log-linear model are difficult to reconcile, suppose that λ belongs to a finite dimensional RKHS. Then $\mathbb{E}[N(A)] = \int_A \lambda(x)dx = \int_A \phi(x)^\top \theta dx = \varphi(A)^\top \theta$. With this linear structure, we obey that $\mathbb{E}[N(A \cup B)] = \mathbb{E}[N(A)] + \mathbb{E}[N(B)] = (\varphi(A) + \varphi(B))^\top \theta$.

Now imposing a log-linear model we say that there exists a $\psi(A)$ s.t. $\varphi(A)^\top \theta \overset{!}{=} \exp(\eta^\top \psi(A))$, where $\psi(A)$ represent the set A. The Poisson process structure stipulates that again that

$$\exp(\eta^\top \psi(A)) + \exp(\eta^\top \psi(B)) = \exp(\eta^\top \psi(A \cup B))$$

for all A, B disjoint, where feature representation $\psi(A \cup B)$ cannot be linear as is the representation φ. Given this constraint, it is unclear how to incorporate similarity between different sensing regions with RKHS assumption.

If we were to ignore this modeling issue, and sacrifice statistical efficiency by choosing a rich representation for ψ, then indeed one could model the response with a generalized linear model. In that case, one can either regress on the log of the counts or use likelihood principle. The log-Poisson noise is sub-Gaussian and hence standard results apply directly. Another option is to use generalized linear bandits Filippi et al. (2010); Jun et al. (2017) with likelihood. Unfortunately Poisson responses are not covered by the theory of the previous works since they assume bounded rewards. Apart from logarithmic transformation, another standard approach to count data, would warrant *variance stabilization* transformation such as Anscombe transformation, however while such transformation leads to better behavior in regression, the non-linear link function would break again the same linear structure associated with the Poisson process.

A.2. Information Directed Sampling (IDS)

As we saw in the main text, CAPTURE-UCB does not change its decision rule depending on the feedback mode. This seems unsatisfactory as one might expect that the semi-bandit feedback introduces a more complex trade-off in terms of reward versus the gained information. When one views the semi-bandit feedback of *count-record* as observing a collections of actions $\{\varphi_B^\top \theta\}_{B \subset A}$, but suffering the instantaneous regret on a different action, $\varphi_A^\top \theta$, we can see that this fits the so called *partial monitoring* framework.

Recently, Kirschner et al. (2020) proposed to use Information Directed Sampling (IDS) to solve linear partial monitoring problems including semi-bandit feedback. For semi-bandit feedback IDS has the same asymptotic regret as UCB, but the algorithm is more appealing due to its different treatment of *histogram* and *count-record* feedback. IDS optimizes a distribution P over actions $\mathcal{A}$ that minimizes the expected ratio of regret gap and information gain:

$$P_{\text{IDS}} = \underset{p \in [0,1]^{|\mathcal{A}|}, \sum_A p_A = 1}{\arg\min} \frac{\mathbb{E}_{A \sim p}\left[\text{gap}_t(A)^2\right]}{\mathbb{E}_{A \sim p}\left[w(A)^2 I_t(A)\right]}, \tag{12}$$

where by regret gap we mean the worst case instantenous regret which can be estimated as a convex program:

$$\begin{aligned} \text{gap}_t(A) = \max_{\theta} \max_{\bar{A} \in \mathcal{A}} \theta^\top (\varphi_{\bar{A}} - \varphi_A) \\ \text{subject to (10).} \end{aligned} \tag{13}$$

The information gain can be chosen from multiple sound choices, perhaps the most natural candidate is:

$$I_t(A) = \log\det(\mathbf{I} + \boldsymbol{\Upsilon}_A \mathbf{V}_t^{-1} \boldsymbol{\Upsilon}_A^\top), \tag{14}$$

where $\boldsymbol{\Upsilon}_A$ is the feature decomposition of A in terms of partition $\mathcal{B}$ as $(\boldsymbol{\Upsilon}_A)_B := \varphi_B$ for all $B \in \mathcal{B}$ and $B \subset A$. However, different efficient information gains are often better performing such as information gain on optimistic action. Let φ_{UCB} be the current optimistic action, then

$$I_t^{\text{UCB}}(A) = \log\det(1 + \varphi_{\text{UCB}}^\top \mathbf{V}_t^{-1}\varphi_{\text{UCB}}) - \log\det(1 + \varphi_{\text{UCB}}^\top(\mathbf{V}_t + \boldsymbol{\Upsilon}_A\boldsymbol{\Upsilon}_A^\top)^{-1}\varphi_{\text{UCB}}).$$

It is known that the optimal distribution p is supported only on two actions (Russo and Van Roy, 2014), and can be calculated efficiently. We show it in the experiments with the hope that it might perform better on count-record feedback, which we observe in *San Francisco* benchmark.

A.3. Generalized Representer Theorem

If we were to maintain the general form of the estimator (3), we could use representer theorem once the positivity constraint could be enforced only on finite number of points or over finite number of sets. Agrell (2019) and Aubin-Frankowski and Szabó (2020) show that one can enforce positivity in this manner heuristically, or even provably by tightening the finite number constraints as $\lambda(x_i) \geq \|\lambda\|_k \zeta_i$, where ζ_i is the tightening parameter associated to point x_i. In such provable case, one needs to create and ϵ-net of the whole domain and enforce it on each x_i forming ϵ-net. Such approach is similar in spirit to positive triangle basis but thorough evaluation of this is out of scope of this paper.

Having the above conditions fulfilled, one can show a simple corollary of the classical representer theorem (Schölkopf et al., 2001) that similar result still applies.

Corollary 1 (Generalized Representer Theorem)**.** *Let $\{(A_i, n_i)\}_{i=1}^n$ be data in form of Borel sets A_i and counts n_i. Let k be a real valued kernel. The optimization problem*

$$\arg\min_{f \in \mathcal{H}_k} c(f, \{A_i, y_i,\}_{i=1}^n, \{x_j\}_{j=1}^N) + g(\|f\|_k) \tag{15}$$

where c is a cost function where dependence on A_i enters via $\int_{A_i} f(x)dx$ and where $g : \mathbb{R} \to \mathbb{R}_+$ is monotone increasing, then any minimizer admins a representation s.t. $f(x) = \sum_{i=1}^n \alpha_i \int_{A_i} k(z,x)dz + \sum_{j=1}^N \beta_j k(x_j, x)$.

Proof. An element of $\mathcal{H}_k$ can be written in span of $\varphi(A_i) = \int_{A_i} k(z,\cdot)dz$ for $i \in [n]$ and $k(x_j,\cdot)$ for $j \in [N]$, and its complement v, $f = \sum_{i=1}^n \alpha_i \int_{A_i} k(z,\cdot)dz + \sum_{j=1}^N \beta_j k(x_j,\cdot) + v$. Then, for any $j \in [n]$,

$$\begin{aligned}
\int_{A_j} f(x)dx &= \int_{A_j} \langle \sum_{i=1}^n \alpha_i \int_{A_i} k(z,\cdot)dz + \sum_{k=1}^N \beta_k k(x_k,\cdot) + v, k(\cdot,x)\rangle\, dx \\
&= \langle \sum_{i=1}^n \alpha_i \int_{A_i} k(z,\cdot)dz + \sum_{k=1}^N \beta_k k(x_k,\cdot) + v, \int_{A_j} k(\cdot,x)dx\rangle \\
&= \langle \sum_{i=1}^n \alpha_i \int_{A_i} k(z,\cdot)dz + \sum_{k=1}^N \beta_k k(x_k,\cdot), \int_{A_j} k(\cdot,x)dx\rangle
\end{aligned}$$

where due to orthogonality the expression does not depends on v. Likewise,

$$\begin{aligned}
f(x)dx &= \langle \sum_{i=1}^n \alpha_i \int_{A_i} k(z,\cdot)dz + \sum_{k=1}^N \beta_k k(x_k,\cdot) + v, k(\cdot,x)\rangle \\
&= \langle \sum_{i=1}^n \alpha_i \int_{A_i} k(z,\cdot)dz + \sum_{k=1}^N \beta_k k(x_k,\cdot), k(\cdot,x)\rangle\,.
\end{aligned}$$

Secondly, due to orthogonality

$$\|f\|_k = \left\| \sum_{i=1}^n \alpha_i \int_{A_i} k(z,\cdot)dz + \sum_{k=1}^N \beta_k k(x_k,\cdot) + v \right\|_k$$

$$= \sqrt{\left\|\sum_{i=1}^{n} \alpha_i \int_{A_i} k(z,\cdot)dz\right\|_k^2 + \left\|\sum_{k=1}^{N} \beta_k k(x_k,\cdot)\right\|_k^2 + \|v\|_k^2} \geq \left\|\sum_{i=1}^{n} \alpha_i \int_{A_i} k(z,\cdot)dz + \sum_{k=1}^{N} \beta_k k(x_k,\cdot)\right\|_k .$$

where v can be chosen to be zero.

□

A.4. Optimality conditions: Linear Regression

For linear regression formulated in (3) with triangle basis, the KKT conditions for $\hat{\theta}$ can be written as there exists $\xi \in \mathbb{R}^{2m}$ s.t. $\xi \geq 0$, and

$$(\Phi\mathbf{\Sigma}^{-1}\Phi^\top + \mathbf{I}\gamma)\theta - \Phi^\top\mathbf{\Sigma}^{-1}n + \xi^\top\mathbf{\Lambda} = 0 \tag{16}$$

$$\xi^\top\mathbf{\Lambda}\theta = 0 \tag{17}$$

The equations can be rewritten as follows using the model assumption $n = \Phi^\top\theta^* + \epsilon$.

$$(\Phi\mathbf{\Sigma}^{-1}\Phi^\top + \mathbf{I}\gamma)\theta - \Phi^\top\mathbf{\Sigma}^{-1}(\Phi\theta^* + \epsilon) + \xi^\top\mathbf{\Lambda} = (\Phi\mathbf{\Sigma}^{-1}\Phi^\top + \mathbf{I}\gamma)(\theta - \theta^*) + \gamma\theta^* + \Phi^\top\mathbf{\Sigma}^{-1}\epsilon + \xi^\top\mathbf{\Lambda} = 0$$

and further,

$$(\Phi\mathbf{\Sigma}^{-1}\Phi^\top + \mathbf{I}\gamma)^{1/2}(\theta - \theta^*) = -(\Phi\mathbf{\Sigma}^{-1}\Phi^\top + \mathbf{I}\gamma)^{-1/2}(\gamma\theta^* + \Phi^\top\mathbf{\Sigma}^{-1}\epsilon + \xi^\top\mathbf{\Lambda})$$

$$\|\theta - \theta^*\|_{\mathbf{V}_t}^2 \leq \gamma^2 \left\|(\mathbf{V}_t)^{-1}\theta^*\right\|^2 + \left\|\xi^\top\mathbf{\Lambda}\right\|_{(\mathbf{V}_t)^{-1}}^2 + \left\|\Phi^\top\mathbf{\Sigma}^{-1}\epsilon\right\|_{(\mathbf{V}_t)^{-1}}^2 \tag{18}$$

$$\leq \gamma\|\theta^*\|_2^2 + \left\|\xi^\top\mathbf{\Lambda}\right\|_{(\mathbf{V}_t)^{-1}}^2 + \left\|\Phi^\top\mathbf{\Sigma}^{-1}\epsilon\right\|_{(\mathbf{V}_t)^{-1}}^2 \tag{19}$$

This is the way to derive ellipsoidal sets from the optimality conditions. Notice that if θ is in the interior of the constraint set, $\xi = 0$ and the solution boils down to standard ridge regularized least-squares error.

A.5. Optimality conditions: Trace Regression

The trace-regression formulation is a conic optimization problem where the solution lies in the intersection of second order as well as PSD cone. To give a KKT characterization we rather appeal to the semi-infinite quadratic formulation of the problem. After vectorization the problem becomes:

$$\min_{\mathrm{vec}\,\mathbf{\Theta} \in \mathbb{R}^{m^2}} \sum_{i=1}^{t} \frac{1}{\sigma_i^2}(\mathrm{vec}\,\mathbf{\Theta}^\top \,\mathrm{vec}\,\mathbf{\Psi}_i - n_i)^2 + \gamma\|\mathrm{vec}\,\mathbf{\Theta}\|_2^2$$

$$\text{subject to } \mathrm{vec}(xx^\top)^\top \,\mathrm{vec}\,\mathbf{\Theta} \geq 0 \;\forall\, x \in \mathbb{R}^{m^2}$$

Notice that, if $\mathbf{\Theta}$ was known to us the two follwoing constraint conditions were equivalent:

$$x^\top\mathbf{\Theta}x \geq 0 \;\forall\, x \in \mathbb{R}^{m^2} \;\Leftrightarrow\; v_i^\top\mathbf{\Theta}v_i \geq 0 \;\forall\, v_i \text{ s.t. } \mathbf{\Theta}v_i = \eta_i v_i \text{ and } \|v_i\|_2^2 = 1.$$

This means that the feasibility of the solution can be checked by verifying only finite number of constraint. Hence the optimality for $\mathbf{\Theta}$ can be written in the same form as above (16)

$$(\Phi\mathbf{\Sigma}^{-1}\Phi^\top + \mathbf{I}\gamma)\,\mathrm{vec}\,\mathbf{\Theta} - \Phi^\top\mathbf{\Sigma}^{-1}n + \xi^\top\mathbf{\Lambda} = 0 \tag{20}$$

$$\xi^\top\mathbf{\Lambda}\,\mathrm{vec}\,\mathbf{\Theta} = 0 \tag{21}$$

with the difference that $\Phi_{i:} = \mathrm{vec}(\mathbf{\Psi}_i)$ and $\mathbf{\Lambda}_{i:} = \mathrm{vec}(v_i v_i^\top)$, and $\xi \in \mathbb{R}^m$. This means, that for the trace-regression the same inequality holds,

$$\|\mathrm{vec}\,\mathbf{\Theta} - \mathrm{vec}\,\mathbf{\Theta}^*\|_{\mathbf{V}_t}^2 \leq \gamma\|\mathrm{vec}\,\mathbf{\Theta}^*\|_2^2 + \left\|\mu^\top\bar{\mathbf{\Lambda}}\right\|_{(\mathbf{V}_t)^{-1}}^2 + \left\|\Phi^\top\mathbf{\Sigma}^{-1}\epsilon\right\|_{(\mathbf{V}_t)^{-1}}^2 \tag{22}$$

B. Adaptive Confidence Sets and Proofs

We split the proof and analysis that eventually leads to the Theorem 1 into two parts. First, we identify a non-negative super-martingale for Poisson noise by scaling. Then having this in the following subsection, we use a method of pseudo-maximization to bound the normalized residuals.

B.1. Non-negative Super-martingales scaled for Poisson tails

Let us first define the necessary notation and then jump directly to the main result of this section in Lemma 2. We spit the Lemma 1 to two parts. First, we show existence of k and then we show that it can be found by a simple root finding program in Lemma 3.

Note that the results are stated in terms of general $B \geq 1$, but for the algorithm and correction we always use $B = 1$. The necessary generality in terms of B is only for sake of generality.

The necessary notation:

$$S_t := \sum_{i=1}^{t} \frac{x_i}{\sigma_i^2}\epsilon_i \quad \mathbf{V}_t := \sum_{i=1}^{t} \frac{x_i x_i^\top}{\sigma_i^2} + \gamma \mathbf{I} \tag{23}$$

and

$$M_t(x) := \exp(x^\top S_t - \frac{1}{2}\left\|x\right\|_{\mathbf{V}_t}^2). \tag{24}$$

Lemma 2 (Super-martingale)**.** *Let $\epsilon_i = z_i - \mu_i$, where $z_i \sim$ Poisson(μ_i), for $i \in [t]$, where $\mu > 0$, and $\left\|x_i\right\|_2 \leq U$ and $\left\|x\right\| \leq B$ ($B \geq 1$), then there exists a finite $k_i(BU, \mu_i) \geq 1$ such that, where $\sigma_i^2 = k\mu_i$ and $\mathbb{E}[\sigma_i^2|\mathcal{F}_{i-1}] = \sigma_i^2$, $M_t(x)$ is adapted super-martingale for each $\left\|x\right\|_2 \leq B$, with $M_0(x) = 1$.*

Proof.

$$\begin{aligned} \mathbb{E}[M_t(x)|\mathcal{F}_{t-1}] &= \mathbb{E}[\exp(\langle x, S_t\rangle - \frac{1}{2}\left\|x\right\|_{\mathbf{V}_t}^2)|\mathcal{F}_{t-1}] && (25)\\ &= M_t(x)\mathbb{E}\left[\exp\left(\frac{\epsilon_t}{\bar{\sigma}_t^2}x^\top x_t - \frac{1}{2\sigma_t^2}(x^\top x_t)^2\right)|\mathcal{F}_{t-1}\right] && (26)\end{aligned}$$

Let us calculate the first expectation i.e. moment generating function of Poisson distribution,

$$\begin{aligned} \mathbb{E}[\exp(\frac{\epsilon_t}{\sigma_t^2}x^\top x_t)] = \mathbb{E}[\exp(\frac{X_t - \mu_t}{\sigma_t^2}x^\top x_t)] &= \exp(-\frac{\mu_t}{\sigma_t^2}x^\top x_t)\mathbb{E}[\exp(\frac{X_t(x_t)^\top x}{\sigma_t^2})] && (27)\\ &= \exp(-\frac{\mu_t}{\sigma_t^2}x^\top x_t)\exp(\mu_t(e^{(x_t^\top x)/\sigma_t^2} - 1)). && (28)\end{aligned}$$

Combining the (26) and (28), we see that the process is super-martingale if the exponent of $\exp$ is negative in other words,

$$\begin{aligned} -\frac{1}{2}\frac{(x^\top x_t)^2}{\sigma_t^2} - \frac{(x^\top x_t)\mu_t}{\sigma_t^2} + \mu_t(e^{(x_t^\top x)/\sigma_t^2} - 1) &= -\frac{1}{2}\frac{(x^\top x_t)^2}{k_t\mu_t} - \frac{(x^\top x_t)}{k_t} + \mu_t(e^{(x_t^\top x)/k_t\mu_t} - 1) \leq 0\\ &\Leftrightarrow \underbrace{-\frac{1}{2}\frac{(x^\top x_t)^2}{k_t\mu_t^2} - \frac{(x^\top x_t)}{k_t\mu_t} + (e^{(x_t^\top x)/k_t\mu_t} - 1)}_{g(k, x_t^\top x/\mu_t)} \leq 0\end{aligned}$$

where the equivalence is by $\mu_t > 0$. If $\mu_t = 0$ the process is super-martingale trivially. We show that the g is increasing as $k \to \infty$, and the limit is of g is located at zero. This intuitively means that g as function of k has to approach zero from below. Hence there exists k^* s.t. any $k \geq k^*(Z/\mu)$, g is negative for a given Z and μ. Let us use shorthand $Z := \max_{x\in\mathcal{D}} x^\top x_t$, where $|Z| \leq \left\|x_t\right\|\left\|x\right\| \leq UB$ and drop the subscript t.

Firstly, note that $\lim_{k\to\infty} g(k, \mu_t, x_t^\top) = 0$. Secondly,

$$\partial_k g(k, Z/\mu) = \frac{Z^2}{2k^2\mu^2} + \frac{Z}{k^2\mu} - \frac{Z}{\mu k^2}\exp(Z/\mu k) = \frac{Z^2}{2\mu^2 k^2} + \mathcal{O}\left(\frac{1}{k^3}\right),$$

where the leading term is positive, which proves the result. Should $U = 0$, the result holds trivially with any k. $\square$

Corollary 2. *Under assumption of Lemma 2,*

$$\mathrm{P}(|S_t^\top x| \geq \log(1/\delta) + \frac{1}{2}\|x\|_{\mathbf{V}_t}) \leq \delta$$

for $\|x\|_2 \leq B$.

Proof. Follows from straightforward application of Ville's theorem. $\square$

The next lemma demonstrates that k can be solved as root finding problem. The proof splits the problem to two parts for positive and negative Z. For negative Z, g is always dominated by $Z = 0$. For positive, g first decreases as Z grows from zero, but then eventually increases, suggesting that the extremum occurs either at maximal positive Z or $Z = 0$.

Lemma 3. *Under assumptions of Lemma 2, the smallest* $k^*(BU/\mu)$ *for any* $x^\top x_t \in [-BU, BU]$*, where* $B \geq 1$ *and* $U \geq 0$*, can be calculated by solving the following optimization problem:*

$$k^*(BU/\mu) = \arg\min_{k \geq 1} k \text{ subject to } -\frac{1}{2}\frac{(UB)^2}{k\mu^2} - \frac{UB}{k\mu} + (e^{UB/k\mu} - 1) \leq 0 \tag{29}$$

Proof. Let g be as in the proof of Lemma 2. In fact, what we want to show that the optimization problem,

$$\arg\min_{k \geq 1} k \text{ subject to } -\frac{1}{2}\frac{(Z)^2}{k\mu^2} - \frac{Z}{k\mu} + (e^{Z/k\mu} - 1) \leq 0 \;\forall Z \in [-UB, UB]$$

is equivalent to (29). Let us use shorthand $h(k) = \sup_Z -\frac{1}{2}\frac{(Z)^2}{k\mu^2} - \frac{Z}{k\mu} + (e^{Z/k\mu} - 1)$.

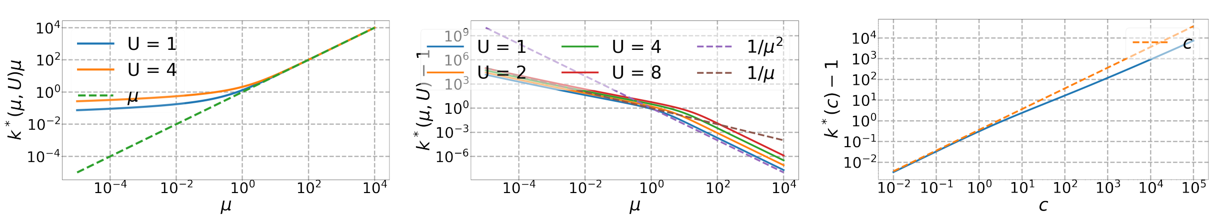
Let us split the problem by first showing that optimum occur either at $Z = 0$ or $Z = UB$. If $Z = 0$ is the optimum then trivially for any k, $h(k) = 0$ including $k = 1$. In that case plugging in $Z = UB$ instead of 0, would arrive also at $k = 1$ as well, since $g(k, 0) \geq g(k, UB)$.

For the maximum to occur at $Z \neq 0$, $h(k) \geq 0$. The first term $-\frac{1}{2}\frac{(Z)^2}{k\mu^2} - \frac{Z}{k\mu}$ is monotonically decreasing in Z, while the second $(e^{Z/k\mu} - 1)$ is monotonically increasing in Z, but asymptotically the second term prevails. As near $Z = 0 + \epsilon$, for arbitrarily small positive ϵ, the $\partial_Z g(k, \epsilon) < 0$ for $k > 1$, hence there exists Z^* s.t. $h(Z^*) = 0$, but $Z^* \neq 0$ and $\partial_Z g(k, Z) \geq 0$ for $Z \geq Z^*$. Thus if $UB \geq Z^*$, the optimum occurs at $Z = UB$; otherwise it occurs at $Z = 0$.

In the above we considered only positive domain. For negative Z, $Z = -|V|$, we show that $\partial_Z g(k, -|V|) \geq 0$. Hence the function is increasing as a function of $-|V|$, thus for all $Z \leq 0$, $g(k, Z) \leq g(k, 0) = 0$.

$\square$

Notice than the condition in Lemma 3 is specified for each value of μ_t separately. Hence we hope that values with already large variance μ do not need to be scaled by large constants. Only the small values for μ are unreliable and need to be scaled. For further values depending on the bound U, we provide in the Figure 3a, and in Figure 3b we show the blow-up in the denominator. Standard sub-Gaussian variable would follow the dashed line, while for Poisson we increase the variance accordingly. For large μ the standard sub-Gaussian treatment becomes valid.



(a) Scaling of the variance $k^*(U/\mu)\mu$ as a function of μ, we see that as $\mu \to \infty$ k is close to 1. In fact, this is not at all very asymptotic.

(b) The value of critical $k^* - 1$ s.t. $M(x)$ is super-martingale.

(c) The value of $k^*(c) - 1$, where $c = U/\mu$.

Figure 3. Empirical exploration of the quantities $k^*(\mu)$, $k^*(U/\mu)$ and $k^*(\mu)\mu$. $B = 1$.

B.2. Qualitative behavior of the variance scaling parameter

The following Lemmas demonstrate extreme behavior and monotonicity properties of $k^*(U/\mu)\mu$. For example, as $\mu \to 0$, the confidence sets do not blow-up to infinity, but instead they very slowly converge to zero variance; expected behavior.

Lemma 4 ($\mu \to 0$). *Under the assumptions of Lemma 2, $B = 1$, let k^* be such as in (29) then, $k^*(U/\mu)\mu$ is monotonically decreasing in μ and,*

$$k^*(U/\mu)\mu = \mathcal{O}\left(\frac{U}{\log(U/\mu)}\right) \quad \textit{as} \quad \mu \to 0$$

Proof. The monotonicity is proved as a special case of Lemma 7. Let $c = U/\mu$. We know that as a function of c, $k(c)$ is increasing due to Lemma 6. Let us use an ansatz $k(c) = c/f(c)$, where $f(c)$ has strictly smaller asymptotic growth than c to maintain monotonicity.

$$-c^2/2k - c/k + \exp(c/k) - 1 = -c\frac{f(c)}{2} - f(c) + \exp(f(c)) - 1 = 0$$

As $c \to \infty$, the dominating terms are $cf(c)$ and $\exp(f(c))$, hence they need to balance for the implicit relationship to hold as $\mathcal{O}(cf(c)) = \mathcal{O}(\exp(f(c)))$. Thus coming to conclusion that $f(c) = \mathcal{O}(\log(c))$.

Hence asymptotically as $c \to \infty$, $k(c)/c = \log(c)$. Rearranging and using $c = U/\mu$, $k(U/\mu)\mu = \frac{U}{\log(U/\mu)}$, which finishes the proof. □

Lemma 5 ($\mu \to \infty$). *Under the assumptions of Lemma 2, $B = 1$, let k^* be such as in (29) then, $\lim_{\mu\to\infty} k^*(\mu) = 1$.*

Proof. The asymptotic expansion of $g(k, U/\mu) = \frac{-U^2}{2k\mu^2} + \frac{U^2}{2k^2\mu^2} + \mathcal{O}(\frac{1}{\mu^3}) \leq 0$. Hence, in order to be negative, asymptotically, k can be chosen to be close 1. □

Lemma 6 (Monotonicity). *$k^*(c_1) \geq k^*(c_2)$ where $c_1 \geq c_2$ and in particular, $k^*(U/\mu_1) \geq k^*(U/\mu_2)$ when $\mu_1 \leq \mu_2$.*

Proof. The second statement follows from the more general statement $k^*(c_1) \geq k^*(c_2)$ where $c_1 \geq c_2$. We prove it by showing that $\partial_c k^*(c) \geq 0$ for all c.[5]

We use implicit definition of k^* as:

$$-\frac{1}{2}\frac{c^2}{k^*} - \frac{c}{k^*} + (e^{c/k^*} - 1) = 0 \tag{30}$$

By differentiation

$$-\frac{c}{k^*} + \frac{c^2}{2(k^*)^2}\frac{dk^*}{dc} - \frac{1}{k^*} + \frac{c}{(k^*)^2}\frac{dk^*}{dc} + \exp(c/k^*)\left(\frac{1}{k^*} - \frac{c}{(k^*)^2}\frac{dk^*}{dc}\right) = 0$$

reorganizing, we find that

$$\frac{dk^*}{dc} = \frac{k^*}{c}\left(\frac{c + 1 - \exp(c/k^*)}{c/2 + 1 - \exp(c/k^*)}\right).$$

Now we look at numerator and denominator separately and show that they are both negative. Multiplying both numerator and denominator by c/k^*, we get for denominator:

$$\begin{aligned}\frac{c^2}{2k} + \frac{c}{k} - c/k\exp(c/k) &\overset{(30)}{=} (\exp(c/k) - 1) - c/k\exp(c/k) \\ &= -1 + \underbrace{\exp(c/k)}_{\geq 1}(1 - c/k) \leq -1 + \underbrace{\exp(c/k)}_{\geq 1}\exp(-c/k) = 0.\end{aligned}$$

[5] The handling of individual cases becomes tedious. We are unfortunately not aware of better proof.

For numerator:

$$\begin{aligned}\frac{c^2}{k}+\frac{2c}{k}-\frac{c}{k}-c/k\exp(c/k) &\overset{(30)}{=} -\frac{c}{k}+2(\exp(c/k)-1)-c/k\exp(c/k)\\ &= \frac{1}{2}(-(1+\frac{c}{k})+\underbrace{\exp(c/k)}_{\geq 1}(1-c/2k)\\ &\leq \frac{1}{2}(-(1+\frac{c}{k})+\underbrace{\exp(c/k)}_{\geq 1}(1-c/2k)\end{aligned}$$

The above as function of c/k at $c/k=0$ is zero. Its first derivative in c/k is non-negative always:

$$2(-1/2+\exp(c/k)(1-c/2k)-\exp(c/k)/2)=2(-1/2+1/2\exp(c/k)(1-c/k)\leq 2(-1/2+1/2)=0.$$

Hence $\frac{c^2}{k}+\frac{2c}{k}-\frac{c}{k}-c/k\exp(c/k)\leq 0$ as well. This shows that $\frac{dk^*}{dc}=\frac{k^*}{c}\left(\frac{c+1-\exp(c/k^*)}{c/2+1-\exp(c/k^*)}\right)\geq 0$, which finished the proof.

□

Lemma 7 (Monotonicity II). *$k^*(c_1)/c_1\leq k^*(c_2)/c_2$ where $c_1\geq c_2$.*

Proof. Using the similar ideas as in Proof of Lemma 6,

$$\frac{d}{dc}(k^*(c)/c)=\frac{dk}{dc}\frac{1}{c}-\frac{k}{c^2}=\frac{k}{c^2}\left(\frac{c+1-\exp(c/k)}{c/2+1-\exp(c/k)}-1\right)=\frac{k}{c^2}\left(\frac{c/2}{c/2+1-\exp(c/k)}\right)\leq 0,$$

as the numerator is positive but the denominator is negative as in the proof of Lemma 6. □

Lemma 8 (Growth in terms of B). *Let $U\geq 0$ fixed, then for any $p>1$, there exist $\tilde{B}(U/\mu,p)$ such that for any $B\geq\tilde{B}$,*

$$\frac{B^{1/p}}{k^*(UB/\mu)}\leq\frac{1}{k^*(U/\mu)}. \tag{31}$$

Proof. Rearranging the statement it is equivalent to showing that there exists $\tilde{B}$ s.t. $B\geq\tilde{B}$.

$$k^*(U/\mu)\leq k^*(UB/\mu)B^{-1/p}.$$

We show this statement by showing that as $B\to\infty$, $k^*\geq\mathcal{O}(B)$. Hence for any $p>1$, there must be a point when this outgrows it.

Rearranging (29)

$$k^*(cB)=\frac{B^2c^2+2Bc}{2(\exp(Bc/k^*)-1)}\geq\frac{B^2c^2+2Bc}{2(\exp(Bc)-1)}=\frac{Bc^2+2c}{2\sum_{j=1}^{\infty}B^{j-1}\frac{c^j}{j!}}=Bc/2+o(B),$$

which proves the result. □

B.3. Confidence sets and pseudo-maximization

In the following proposition we combine techniques from Abbasi-Yadkori et al. (2011), de la Peña et al. (2009) and Faury et al. (2020) to bound the norm of the noise process S_t.

Surprisingly, the nearly concurrent work of Faury et al. (2020) implements very similar ideas which we employ for our purposes. We use the technique of pseudo-maximization, where we integrate $\int_x M_t(x)dx$ in such a way that it maintains martingale and dependence on $\|S_t\|_{\mathbf{V}_t^{-1}}$. After integration, we simply use Ville's maximal inequality for non-negative super-martingales. The challenge is that we cannot integrate $M_t(x)$ over the whole $\mathbb{R}^m$ instead only on $\|x\|_2\leq 1$.

Proposition 1. *Let $\epsilon_i = z_i - \mu_i$, where $z_i \sim Poisson(\mu_i)$, for $i \in [t]$, where $\mu > 0$, and $\|x_i\|_2 \leq U$, $x_i \in \mathbb{R}^m$ for all $i \in [t]$. For $\delta \in (0,1)$ and $k \in (0,1)$,*

$$\mathrm{P}\left(t \geq 0 : \|S_t\|_{\mathbf{V}_t^{-1}} \geq \sqrt{\gamma}k + \frac{1}{\sqrt{\gamma}k}\log\left(\frac{1}{\delta}\frac{\det(\mathbf{V}_t)^{1/2}}{\det(\gamma\mathbf{I})^{1/2}}\right) + \frac{m}{\sqrt{\gamma}k}\log\left(\frac{1}{1-k}\right)\right) \leq \delta \tag{32}$$

where $S_t = \sum_{i=1}^t \frac{x_i}{\sigma_i^2}\epsilon_i$, $\mathbf{V}_t := \sum_{i=1}^t \frac{x_i x_i^\top}{\sigma_i^2} + \gamma\mathbf{I}$, and $\mathbb{E}[\sigma_t|\mathcal{F}_{t-1}] = \sigma_t$, where also $\sigma_t^2 \geq k_t^* \mu_t$, where k_t^* satisfies (29).

Proof. Let $k \in (0,1)$. By Lemma Lemma 20.3 of Szepesvari and Lattimore (2019) the following process is also non-negative martingale:

$$\bar{M}_t = \int_{\|x\|_2 \leq 1} M(x) dh(x) \tag{33}$$

where h is probability density function with support on $\|x\|^2 \leq 1$. Now using Ville's inequality

$$\mathrm{P}\left(t \geq 0 : \sup \log(\bar{M}_t) \geq \log\left(\frac{1}{\delta}\right)\right) \leq \delta$$

Let us compute the integral (33), or in fact a lower bound for it. Let h be a truncated normal distribution with inverse variance γ, and $N(h)$ the normalization factor. Also let $\tilde{\mathbf{V}}_t = \mathbf{V}_t - \gamma\mathbf{I}$ be a covariance matrix without the regularization. The actual correction will be based on U as in the statement of the theorem.

Also, let us use shorthand $f(x) = x^\top S_t - x^\top \mathbf{V}_t x = f(x^*) + \nabla f(x^*)^\top (x - x^*) - (x - x^*)^\top \mathbf{V}_t (x - x^*)$, where $x^* = \arg\max_{\|x\|_2 \leq k} f(x)$ where $k \in (0,1)$.

$$\begin{aligned}
\bar{M}_t &= \frac{1}{N(h)} \int_{\|x\|_2 \leq 1} \exp(x^\top S_t - x^\top \tilde{\mathbf{V}}_t x) h(x) dx \\
&= \frac{\exp(f(x^*))}{N(h)} \int_{\|x\|_2 \leq 1} \exp((x - x^*)^\top \nabla f(x^*) - (x - x^*)^\top \mathbf{V}_t (x - x^*)) dx \\
&= \frac{\exp(f(x^*))}{N(h)} \int_{\|y + x^*\|_2 \leq 1} \exp(y^\top \nabla f(x^*) - y^\top \mathbf{V}_t y) dy \\
&\geq \frac{\exp(f(x^*))}{N(h)} \int_{\|y\|_2 \leq (1-k)} \exp(y^\top \nabla f(x^*) - y^\top \mathbf{V}_t y) dy \\
&= \frac{\exp(f(x^*))}{N(h)} \int_{\|y\|_2 \leq (1-k)} \exp(y^\top \nabla f(x^*)) \exp(-y^\top \mathbf{V}_t y) dy \\
&= \frac{\exp(f(x^*)) N(g)}{N(h)} \mathbb{E}_g[\exp(y^\top \nabla f(x^*))]
\end{aligned}$$

where $g \propto \exp(-\frac{1}{2} y^\top 2\mathbf{V}_t y)$ is truncated normal distribution on ball with the radius $(1-k)$ and $N(g)$ is its normalization constant.

$$\begin{aligned}
\bar{M}_t &\geq \frac{\exp(f(x^*)) N(g)}{N(h)} \mathbb{E}_g[\exp(y^\top \nabla f(x^*))] && (34) \\
&\overset{\text{Jensen}}{\geq} \frac{\exp(f(x^*)) N(g)}{N(h)} \exp(\mathbb{E}_g[y^\top \nabla f(x^*)]) && (35) \\
&= \frac{\exp(f(x^*)) N(g)}{N(h)} && (36)
\end{aligned}$$

Now using Ville's inequality,

$$\delta \geq P\left(t \geq 0 : \sup_t \log\left(\bar{M}_t\right) \geq \log(1/\delta)\right) \geq P\left(t \geq 0 : f(x^*) \log\left(\frac{N(g)}{N(h)}\right) \geq \log(1/\delta)\right) \tag{37}$$

$$= P\left(t \geq 0 : f(x^*) \geq \log\left(\frac{N(h)}{\delta N(g)}\right)\right) \tag{38}$$

$$= P\left(t \geq 0 : \max_{\|x\|_2 \leq k} x^\top S_t - x^\top \mathbf{V}_t x \geq \log\left(\frac{N(h)}{\delta N(g)}\right)\right) \tag{39}$$

$$\geq P\left(t \geq 0 : x^\top S_t - x^\top \mathbf{V}_t x \geq \log\left(\frac{N(h)}{\delta N(g)}\right)\right) \tag{40}$$

$$\tag{41}$$

In particular plugging in $x = \frac{\mathbf{V}_t^{-1} S_t}{\|S_t\|_{\mathbf{V}_t^{-1}}} \sqrt{\gamma} k$, which has bounded norm $\|x\|_2 \leq k$, leads to:

$$\delta \geq P\left(t \geq 0 : \|S_t\|_{\mathbf{V}_t^{-1}} \sqrt{\gamma} k - \frac{S_t^\top \mathbf{V}_t^{-1} \mathbf{V}_t \mathbf{V}_t^{-1} S_t}{\|S_t\|^2_{\mathbf{V}_t^{-1}}} \gamma k^2 \geq \log\left(\frac{N(h)}{\delta N(g)}\right)\right) \tag{42}$$

$$\geq P\left(t \geq 0 : \|S_t\|_{\mathbf{V}_t^{-1}} \sqrt{\gamma} k - \gamma k^2 \geq \log\left(\frac{N(h)}{\delta N(g)}\right)\right) \tag{43}$$

$$\geq P\left(t \geq 0 : \|S_t\|_{\mathbf{V}_t^{-1}} \geq \sqrt{\gamma} k + \frac{1}{\sqrt{\gamma} k} \log\left(((1-k)\sqrt{\gamma})^{-m} \det(\mathbf{V}_t)^{1/2} \delta^{-1}\right)\right) \tag{44}$$

$$= P\left(t \geq 0 : \|S_t\|_{\mathbf{V}_t^{-1}} \geq \sqrt{\gamma} k + \frac{1}{\sqrt{\gamma} k} \log\left(\frac{1}{\delta} \frac{\det(\mathbf{V}_t)^{1/2}}{\det(\gamma \mathbf{I})^{1/2}}\right) + \frac{m}{\sqrt{\gamma} k} \log\left(\frac{1}{1-k}\right)\right) \tag{45}$$

We use Lemma 9 in the second to last step to bound the $\log(N(h)/N(g))$. □

Lemma 9 (Generalized version of Lemma 6 of Faury et al. (2020))**.** *Let the normalization constants be as in the Proof of Proposition 1.*

$$\log\left(\frac{N(h)}{N(g)}\right) \leq \log\left((1-k)^{-m} \gamma^{-m/2} \det(\mathbf{V}_t)^{1/2}\right) \tag{46}$$

Proof.

$$\begin{aligned} N(h) &= \int_{\|x\|_2 \leq} \exp(-\gamma \|x\|^2) \\ &\leq (2\gamma)^{-m/2} \int_{\|x\|_2 \leq B\sqrt{2\gamma}} \exp(-\frac{1}{2} \|x\|^2) \\ N(g) &= \int_{\|x\|_2 \leq (1-k)} \exp\left(\frac{1}{2} x^\top 2\mathbf{V}_t x\right) dx \\ &= \det(2\mathbf{V}_t)^{-1/2} \int_{\left\|2^{-1/2} \mathbf{V}_t^{-1/2} x\right\|_2 \leq (1-k)} \exp\left(-\frac{1}{2} \|x\|_2^2\right) dx \\ &\geq \det(2\mathbf{V}_t)^{-1/2} \int_{\|x\|_2 \leq (1-k)\sqrt{2\gamma}} \exp\left(-\frac{1}{2} \|x\|_2^2\right) dx \end{aligned}$$

Now we have

$$\frac{N(h)}{N(g)} \leq \gamma^{-m/2} \det(\mathbf{V}_t)^{1/2} \frac{\int_{\|x\|_2 \leq \sqrt{2\gamma}} \exp(-\frac{1}{2} \|x\|^2)}{\int_{\|x\|_2 \leq (1-k)\sqrt{2\gamma}} \exp\left(\frac{1}{2} \|x\|_2^2\right) dx} \leq (1-k)^{-m} \gamma^{-m/2} \det(\mathbf{V}_t)^{1/2}$$

as

$$\frac{\int_{\|x\|_2 \leq \sqrt{2\gamma}} \exp(-\frac{1}{2} \|x\|^2)}{\int_{\|x\|_2 \leq (1-k)\sqrt{2\gamma}} \exp\left(\frac{1}{2} \|x\|_2^2\right) dx} \leq \frac{\text{volume}(\sqrt{2\gamma})}{\text{volume}((1-k)\sqrt{2\gamma})} \leq (1-k)^{-m}$$

□

These results can be combined to our main theorem.

Theorem 3 (Poisson Concentration, Theorem 1 in the main text). *Let $\delta \in (0,1)$, then under assumption of Poisson feedback and Assumptions 1, 2, the solution to (3) or (3) satisfies*

$$\|\theta - \theta^*\|_{\mathbf{V}_t} \leq \sqrt{\left(\sqrt{\gamma}/2 + \frac{2}{\sqrt{\gamma}}\log\left(\frac{1}{\delta}\frac{\det(\mathbf{V}_t)^{1/2}}{\det(\gamma\mathbf{I})^{1/2}}\right) + \frac{2m}{\sqrt{\gamma}}\log(2)\right)^2 + \|\mathbf{\Lambda}^\top\xi\|^2_{\mathbf{V}_t^{-1}} + \gamma\|\theta^*\|^2} \tag{47}$$

with $1-\delta$, where $\mathbf{V}_t = \sum_{i=1}^t \frac{x_i x_i}{\sigma_t^2}$ and σ_t^2 is such that $\sigma_t^2 = k^*\mu_t$ as in (29) and $\mathbb{E}[\sigma_t^2|\mathcal{F}_{t-1}] = \sigma_t^2$. The ξ is a dual variable associated with satisfying the constraints in the KKT conditions for the optimization problem (16).

Proof of Theorem 1. Using the optimality conditions 6:

$$\begin{aligned}
\left(\mathbf{X}^\top\mathbf{\Sigma}^{-1}\mathbf{X} + \mathbf{I}\gamma\right)\hat{\theta} + \mathbf{X}\mathbf{\Sigma}^{-1}n + \mathbf{\Lambda}^\top\xi &= 0 \\
\left(\mathbf{X}^\top\mathbf{\Sigma}^{-1}\mathbf{X} + \mathbf{I}\gamma\right)\hat{\theta} + \mathbf{X}^\top\mathbf{\Sigma}^{-1}(\mathbf{X}\theta^* + \epsilon) + \mathbf{\Lambda}^\top\xi &= 0 \qquad \text{(model)} \\
\left(\mathbf{X}^\top\mathbf{\Sigma}^{-1}\mathbf{X} + \mathbf{I}\gamma\right)(\hat{\theta} - \theta^*) &= \mathbf{X}^\top\mathbf{\Sigma}^{-1}\epsilon - \mathbf{\Lambda}^\top\xi - \gamma\theta^* \\
\left(\mathbf{X}^\top\mathbf{\Sigma}^{-1}\mathbf{X} + \mathbf{I}\gamma\right)^{1/2}(\hat{\theta} - \theta^*) &= \left(\mathbf{X}^\top\mathbf{\Sigma}^{-1}\mathbf{X} + \mathbf{I}\gamma\right)^{-1/2}\left(\mathbf{X}^\top\mathbf{\Sigma}^{-1}\epsilon - \mathbf{\Lambda}^\top\xi - \gamma\theta^*\right) \\
\mathbf{V}_t^{1/2}(\hat{\theta} - \theta^*) &= \mathbf{V}_t^{-1/2}\left(\mathbf{X}^\top\mathbf{\Sigma}^{-1}\epsilon - \mathbf{\Lambda}^\top\xi - \gamma\theta^*\right) \\
\left\|\hat{\theta} - \theta^*\right\|^2_{\mathbf{V}_t} &\leq \|S_t\|^2_{\mathbf{V}_t^{-1}} + \left\|\mathbf{\Lambda}^\top\xi\right\|^2_{\mathbf{V}_t^{-1}} + \gamma^2\theta^*\mathbf{V}_t^{-1}\theta^* \\
\left\|\hat{\theta} - \theta^*\right\|^2_{\mathbf{V}_t} &\leq \|S_t\|^2_{\mathbf{V}_t^{-1}} + \left\|\mathbf{\Lambda}^\top\xi\right\|^2_{\mathbf{V}_t^{-1}} + \gamma\|\theta^*\|^2 \\
\left\|\hat{\theta} - \theta^*\right\|^2_{\mathbf{V}_t} &\leq \left(\sqrt{\gamma}k + \frac{1}{\sqrt{\gamma}k}\log\left(\frac{1}{\delta}\frac{\det(\mathbf{V}_t)^{1/2}}{\det(\gamma\mathbf{I})^{1/2}}\right) + \frac{m}{\sqrt{\gamma}k}\log\left(\frac{1}{1-k}\right)\right)^2 + \left\|\mathbf{\Lambda}^\top\xi\right\|^2_{\mathbf{V}_t^{-1}} + \gamma\|\theta^*\|^2 \qquad \text{(Prop. 1)}
\end{aligned}$$

Using $k = 2$ in particular finished the proof.

$$\left\|\hat{\theta} - \theta^*\right\|^2_{\mathbf{V}_t} \leq \left(\sqrt{\gamma}/2 + \frac{2}{\sqrt{\gamma}}\log\left(\frac{1}{\delta}\frac{\det(\mathbf{V}_t)^{1/2}}{\det(\gamma\mathbf{I})^{1/2}}\right) + \frac{2m}{\sqrt{\gamma}}\log(2)\right)^2 + \left\|\mathbf{\Lambda}^\top\xi\right\|^2_{\mathbf{V}_t^{-1}} + \gamma\|\theta^*\|^2$$

$$\left\|\hat{\theta} - \theta^*\right\|^2_{\mathbf{V}_t} \leq \left(\sqrt{\gamma}/2 + \frac{1}{\sqrt{\gamma}}\log\left(\left(\frac{1}{\delta}\right)^2\frac{\det(\mathbf{V}_t)}{\det(\gamma\mathbf{I})}\right) + \frac{2m}{\sqrt{\gamma}}\log(2)\right)^2 + \left\|\mathbf{\Lambda}^\top\xi\right\|^2_{\mathbf{V}_t^{-1}} + \gamma\|\theta^*\|^2$$

$\square$

C. Regret bound and Proofs

C.1. Information Gain

Let us start with the definition of information gain:

Definition 1 (Approximate Variance scaled Information Gain)**.** *Given a finite dimensional representation of* $k(x,y) = \sum_{i=1}^m \Phi_i(x)\Phi_i(y)$*, the information gain can be expressed as*

$$\gamma_T(\gamma, k) := \log\det\left(\sum_{t=1}^T \frac{\Phi(z_t)\Phi(z_t)^\top}{\sigma_t^2} + \gamma\mathbf{I}\right) - \log\det(\gamma\mathbf{I}) = \log\det\left(\sum_{t=1}^T \frac{\Phi(z_t)\Phi(z_t)^\top}{\gamma\sigma_t^2} + \mathbf{I}\right), \tag{48}$$

which can be re-expressed as,

$$\gamma_T(\{\gamma\sigma_t^2\}, k) = \log\det\left(\mathbf{K} + \mathbf{D}(\sigma^2\gamma)\right) - \log\det(\mathbf{D}(\sigma^2\gamma))$$

where $\mathbf{K} \in \mathbb{R}^{T\times T}$ *and* $\mathbf{D}$ *is a diagonalization operator, where* $\mathbf{D}(\sigma\gamma)$ *contains* $\sigma_t^2\gamma$ *on the diagonal.*

The information gain studied here has been introduced in Srinivas et al. (2010), and was originally studied with $\gamma = 1$ due to connections with Bayesian framework (Kanagawa et al., 2018). Srinivas et al. (2010) provides bounds on information gain via greedy maximization perspective. Mutný and Krause (2018) uses Gaussian quadrature to derive bounds on this quantity for stationary kernels. Additional bounds can be found in works of Vakili et al. (2021).

First, we present a claim presented in the paper than for any finite dimensional bounded feature representation, there exist dimension dependent bound.

Lemma 10 (Linear basis bound)**.** *Let* $\tilde{k}(x,y) = \phi(x)^\top\phi(x)$ *where* $\|\phi(x)\|_\infty \leq u$ *and be* s *s.t. for all* t*,* $\sigma_t^2 \geq s^2$*. Then,* $\gamma_T(\{\sigma_t^2\}, k) \leq m\log\left(1 + \frac{Tu^2}{\gamma s}\right)$

Proof.

$$\gamma_T(\{\gamma\sigma_t^2\}, k) = \log\det\left(\sum_{t=1}^T \frac{\Phi(x_t)\Phi(x_t)^\top}{\gamma\sigma_t^2} + \mathbf{I}\right) \leq \log\det\left(\sum_{t=1}^T \frac{\Phi(x_t)\Phi(x_t)^\top}{\gamma s^2} + \mathbf{I}\right)$$
$$\leq \log\left(\prod_{j=1}^m (\sum_{t=1}^T \frac{(\Phi(x_t)_i)^2}{\gamma s^2} + 1)\right) \leq m\log(1 + \frac{u^2}{s^2\gamma})$$

□

C.2. Information Gain for Triangle Basis

In this section we show that as $m \to \infty$ grows for triangle basis, information gain converges to the information gain with kernel k defined on T points. By Lemma 3 in Mutný and Krause (2018), for $|k(x,y) - \tilde{k}(x,y)|_\infty \leq \epsilon$, then for $\mathbf{K} \in \mathbb{R}^{T\times T}$ (Lemma 3 in the above):

$$\mathbf{K} - \tilde{\mathbf{K}} \preceq \epsilon T,$$

Lemma 11 (Triangle basis)**.** *Let* $\tilde{k}(x,y) = \Phi(x)^\top\Phi(x)$ *be triangle basis approximation to kernel* k*, then* $\gamma_T(\tilde{k}) \leq \gamma_T(k) + \frac{T^2\epsilon}{\gamma s} + \mathcal{O}(\epsilon^2)$*, where* ϵ *denotes the maximum point-wise approximation and* $s \geq \max_t \sigma_t$*.*

Proof.

$$\gamma_T = \log\det\left(\sum_{t=1}^T \frac{\Phi(x_t)\Phi(x_t)^\top}{\gamma\sigma_t^2} + \mathbf{I}\right) = \log\det\left(\sum_{t=1}^T \frac{\mathbf{\Gamma}^{1/2}\phi(x_t)\phi(x_t)^\top\mathbf{\Gamma}^{1/2}}{\gamma\sigma_t^2} + \mathbf{I}\right)$$
$$= \log\left(\det\left(\Phi^\top\mathbf{\Gamma}\Phi + \mathbf{D}\right)\det(\mathbf{D}^{-1})\right) \leq \log\det\left(\mathbf{K} + \mathbf{I}T\epsilon + \mathbf{D}\right) - \log\det(\mathbf{D})$$
$$= \log(\det(\mathbf{K} + \mathbf{D})) - \log\det(\mathbf{D}) + \mathrm{Tr}((\mathbf{K} + \mathbf{D})^{-1})T\epsilon + \mathcal{O}(\epsilon^2)$$
$$\leq \log(\det(\mathbf{K} + \mathbf{D})) - \log\det(\mathbf{D}) + \frac{T^2\epsilon}{\gamma s} + \mathcal{O}(\epsilon^2)$$

□

As $\epsilon(m) \to 0$ as $m \to 0$, $\gamma_T(\tilde{k}) \to \gamma_T(k)$.

C.3. Information Gain for Trace Regression

The following Lemma shows that the trace-regression formulation for a fixed dimensionality is equivalent to RKHS space formed by squaring the kernel.

Lemma 12 ($\mathcal{H}_k = \mathcal{H}_{\tilde{k}^2}$). *If $\tilde{k}(x,y) = \sum_{i=1}^m \psi_i(x)\psi_i(y)$ then the spaces $f(\cdot) = \Psi(\cdot)^\top \mathbf{\Theta}\Psi(\cdot)$ with $\|\mathbf{\Theta}\|_F$ bounded and RKHS with $k = \tilde{k}^2$ are the same.*

Proof. We first prove one way:

$$f = \mathrm{Tr}(\mathbf{\Theta}\Psi\Psi^\top) = \mathrm{vec}(\mathbf{\Theta})^\top \mathrm{vec}(\Psi\Psi^\top) = \sum_{i=1}^{m^2} \mathrm{vec}(\Psi\Psi^\top)_i \,\mathrm{vec}(\mathbf{\Theta})_i^\top \tag{49}$$

$$= \sum_{i=1}^{m^2} (\mathbf{I}\otimes\Psi)_{i:}\, \mathrm{vec}(\Psi)\,\mathrm{vec}(\mathbf{\Theta})_i^\top = \sum_{i=1}^{m^2} \phi_i\theta_i \tag{50}$$

where the new kernel can be identified as $k(x,y) = \sum_{i=1}^{m^2} \phi_i(x)\phi_i(y) = \sum_{i=1}^{m^2} (\mathbf{I}\otimes\Psi(x))_{i:}\,\mathrm{vec}(\Psi(x))(\mathbf{I}\otimes\Psi(y))_{i:}\,\mathrm{vec}(\Psi(y)) = (\Psi(x)^\top\Psi(y))^2 = \tilde{k}(x,y)^2$

The other way

$$f = \sum_{i=1}^n \tilde{k}(x_i,x)^2\alpha_i = \sum_{i=1}^n \tilde{k}(x_i,x)\tilde{k}(x_i,x)\alpha_i = \sum_{j=1}^m\sum_{l=1}^m\sum_{i=1}^n \alpha_i\psi_j(x)\psi_j(x_i)\psi_l(x)\psi_l(x_i)$$

$$= \sum_{j=1}^m\sum_{l=1}^m \left(\sum_{i=1}^n \alpha_i\psi_j(x_i)\psi_l(x_i)\right)\psi_l(x)\psi_j(x) = \sum_{j=1}^m\sum_{l=1}^m \mathbf{\Theta}_{ij}\psi_l(x)\psi_j(x) = \Psi(x)^\top\mathbf{\Theta}\Psi(x).$$

$\square$

Proposition 2 (Trace Regression InfoGain). *Let $\tilde{k}(x,y) = \sum_{i=1}^m \psi_i(x)\psi_i(y)$ have the representation then the information gain associated with trace regression defined via basis of $\tilde{k}$, have the information gain equal to:*

$$\gamma_T(k) = \log\det\left(\sum_{t=1}^T \frac{1}{\sigma_i^2}\mathrm{vec}(\Psi(x_i)\Psi(x_i)^\top)\,\mathrm{vec}(\Psi(x_i)\Psi(x_i)^\top)^\top + \gamma\mathbf{I}\right) - \log\det(\gamma\mathbf{I})$$

satisfying

$$\gamma_T(k) = \gamma_T(\tilde{k}^2)$$

Proof. Note that $\mathrm{vec}(\Phi\Phi^\top) = \sum_{i=1}^m e_i\otimes\Phi\Phi^\top e_i$. Consequently,

$$\mathrm{vec}(\Psi(x_i)\Psi(x_i)^\top)\,\mathrm{vec}(\Psi(x_i)\Psi(x_i)^\top)^\top = \Psi(x_i)\Psi(x_i)^\top\otimes\Psi(x_i)\Psi(x_i)^\top$$

$$\gamma_T(k) = \log\det\left(\sum_{t=1}^T \frac{1}{\sigma_i^2}\mathrm{vec}(\Psi(x_i)\Psi(x_i)^\top)\,\mathrm{vec}(\Psi(x_i)\Psi(x_i)^\top)^\top + \gamma\mathbf{I}\right) - \log\det(\gamma\mathbf{I}) \tag{51}$$

$$= \log\det\left(\sum_{t=1}^T \frac{1}{\sigma_i^2}\Psi(x_i)\Psi(x_i)^\top\otimes\Psi(x_i)\Psi(x_i)^\top + \gamma\mathbf{I}_m\otimes\mathbf{I}_m\right) - \log\det(\gamma\mathbf{I}_m\otimes\mathbf{I}_m) \tag{52}$$

$$= \log\det\left(\sum_{t=1}^T \frac{1}{\sigma_i^2}(\Psi(x_i)\otimes\Psi(x_i))(\Psi(x_i)\otimes\Psi(x_i))^\top + \gamma\mathbf{I}_m\otimes\mathbf{I}_m\right) - \log\det(\gamma\mathbf{I}_m\otimes\mathbf{I}_m) \tag{53}$$

$$(*) = \log\left(\gamma^{m^2}\det\left(\mathit{\Psi}\mathbf{D}\mathit{\Psi}^\top + \mathbf{I}_m\otimes\mathbf{I}_m\right)\right) - m\log\det(\gamma\mathbf{I}_m) \tag{54}$$

where $\mathit{\Psi}\in\mathbb{R}^{d^2\times T}$ and $\mathbf{D}_{ii} = \frac{1}{\gamma\sigma_i^2}$ for $i\in[T]$. Using Weinstein–Aronszajn:

$$(*) = \log\left(\gamma^{m^2}\det\left(\mathbf{D}^{1/2}\Psi^\top\Psi\mathbf{D}^{1/2} + \mathbf{I}_T\right)\right) - m\log\det(\gamma\mathbf{I}_m) \tag{55}$$

$$= \log\left(\gamma^{m^2}\det\left(\mathbf{D}^{1/2}(\mathbf{K}\circ\mathbf{K})\mathbf{D}^{1/2} + \mathbf{I}_T\right)\right) - m\log\det(\gamma\mathbf{I}_m) \tag{56}$$

Alternatively the proof follows directly from Lemma 12. □

C.4. Information Gain via Approximations: QFF

This lemma shows in general that finite basis approximation can generate bounds for information gain. These results can be combined with trace-regression or imposing positivity on finite number of points in the context of this work:

Lemma 13. *Let $|k(x,y) - \tilde{k}(x,y)|_\infty \leq \epsilon$ where $\tilde{k}$ is the finite basis approximation,*

$$\gamma_T = \log\det(\mathbf{K}/\gamma + \mathbf{I}) \leq \log\det\left(\Phi^\top\Phi + \gamma\mathbf{I}_m\right) - \log\det\left(\gamma\mathbf{I}_m\right) + T\log((1+\frac{\epsilon T}{\gamma})\gamma^{m/T-1})$$
$$\leq m\log\left(1 + \frac{\max_{x,i}\Phi(x)_i^2 T}{\gamma}\right) + T\log((1+\frac{\epsilon T}{\gamma})\gamma^{m/T-1}))$$

Proof.

$$\gamma_T = \log\det\left(\frac{\mathbf{K}}{\gamma} + \mathbf{I}\right) \leq \log\det\left(\frac{\tilde{\mathbf{K}}}{\gamma} + \frac{\mathbf{I}\epsilon T}{\gamma} + \mathbf{I}\right) \overset{\text{concave}}{\leq} \log\det\left(\frac{\tilde{\mathbf{K}}}{\gamma} + \mathbf{I}\right) + \log\det\left(\frac{\mathbf{I}\epsilon T}{\gamma} + \mathbf{I}\right)$$
$$= \log\det\left(\frac{\Phi\Phi^\top}{\gamma} + \mathbf{I}\right) + \log\det\left(\frac{\mathbf{I}\epsilon T}{\gamma} + \mathbf{I}\right) = \log\det\left(\Phi^\top\Phi + \gamma\mathbf{I}_m\right) - \log\det\left(\gamma\mathbf{I}_T\right) + \log\det\left(\frac{\mathbf{I}\epsilon T}{\gamma} + \mathbf{I}\right)$$
$$= \log\det\left(\Phi^\top\Phi + \gamma\mathbf{I}_m\right) - m\log\det(\mathbf{I}_m\gamma) - (T-m)\log(\gamma) + T\log(1+\frac{\epsilon T}{\gamma})$$

□

Using QFF with squared exponential kernel, $\epsilon = C\exp(-m^{1/d})$, and the max-norm is equal to 1. Then choosing $m = (2\log T)^d$.

$$\gamma_T \leq (2\log T)^d\log(1+T/\gamma)\underbrace{-T\log((1+\frac{C}{T\gamma}))}_{\leq C\gamma^{-1}}\underbrace{-T\left(\frac{(2\log T)^d}{T} - 1\right)\log(\gamma)}_{\leq 0 \text{ for } \gamma\geq 1} \leq \mathcal{O}(\log(T)^{d+1})$$

or

$$\gamma_T \leq (2\log T)^d\log(1+T/\gamma)\underbrace{-T\log((1+\frac{C}{T\gamma}))}_{\leq C\gamma^{-1}}\underbrace{+\left((2\log T)^d - T^{-1}\right)\log(\gamma)}_{\leq 0 \text{ for } \gamma\leq 1} \leq \mathcal{O}(\log(T)^{d+1})$$

which is a classical result from Srinivas et al. (2010). Using $\epsilon = Cm^{-s/d}$ for kernels that generate s-times continuously differentiable functions gives bounds for Matérn kernels.

Remark 1: Bayesian version of information gain follows with $\gamma = 1$, and the *second terms vastly simplifies*. The frequentist setting is much more complicated since the strength of regularization in finite basis approximation does not have the same strength in the kernelized one and needs to be taken into account, by properly rescaling γ in order to have the same capacity. However, the overall asymptotic scaling as $T\to\infty$ does not change.

C.5. The Effect of Integration on Information Gain

Since we are using integral measurements in our problem, we are not using point functionals to evaluate members of RKHS via $\langle k(x,\cdot), f\rangle_k = f(x)$. Instead we are using different evaluation elements $z_A = \int_A k(a,\cdot)dx$ to evaluate $\int_A f(x)dx = \langle z_A, f\rangle_k$.

Hence formulating the kernelized version leads to kernel values $\mathbf{K}_{A_1,A_2} = \langle z_{A_1}, z_{A_2}\rangle_k$ (instead of the usual $\mathbf{K}_{ij} = \langle k(\cdot, x_i), k(\cdot, x_j)\rangle_k$).

Note that each

$$\mathbf{K}_{A_1,A_2} = \langle z_{A_1}, z_{A_2}\rangle_k = \int_{A_1}\int_{A_2} k(x,y)dxdy = k(\tilde{x},\tilde{y})\operatorname{vol}(A_1)\operatorname{vol}(A_2). \tag{57}$$

where $\tilde{x} \in A_1$ and $\tilde{y} \in A_2$ such that the equality holds and vol denotes the volume of the set. Since kernel k is continuous in both variable, this is choice is possible due to mean value theorem.

Consequently for positive kernels kernel defined for the integral measurements can be $\mathbf{K} = \mathbf{D}(\mathrm{vol})\tilde{\mathbf{K}}\mathbf{D}(\mathrm{vol})$, where $\tilde{\mathbf{K}}$ is the standard pointwise defined kernel on the elements in the respective sets.

Formally, we summarize this in the following Lemma.

Lemma 14. *For continuous kernel k, the information gain with integral actions can be upper-bounded by*

$$\gamma_T(\mathbf{K}) = \log\det(\mathbf{D}(\mathrm{vol})\tilde{\mathbf{K}}\mathbf{D}(\mathrm{vol}) + \mathbf{\Sigma}) - \log\det(\mathbf{\Sigma}) \leq \gamma_T(\{\gamma\sigma_t/\operatorname{vol}(A_t)^2\}, k)$$

as in Definition 1, where $\tilde{\mathbf{K}}$ is the classical pointwise kernel with $\tilde{\mathbf{K}}_{ij} = k(x_i, x_j)$, for specific $x_i \in A_i$ and $x_j \in A_j$, and $\mathbf{K}$ is integral evaluation as in Equation (57).

Due to the previous result, analyzing information gain γ_T can be related to the classical definition with new regularization parameter $\tilde{\gamma} = \max_{i\in[T]} \frac{\gamma}{\operatorname{vol}(A_i)^2}$. Additionally, any worst case bounds measuring worst case point allocation will hold also in this case. These bounds are usually formulated with $\gamma = 1$, which can be directly related by adjusting γ in the algorithm such that the overall $\tilde{\gamma} = 1$. Also, if $\tilde{\gamma} > 1$, the bounds are still valid. Most importantly, for stationary kernels, asymptotic dependence does not change with the regularization constant as we have demonstrated in Appendix **??**.

C.6. Auxiliary lemmas

Lemma 15 (Monotonicity of variances)**.** *For any t and any $A \in \mathcal{A}$, $\sigma_{t-1}^2(A) \geq \sigma_t^2(A)$.*

Proof.

$$\begin{aligned}
\sigma_{t-1}^2(A) &= \max_{\mu\in[\mathrm{lcb}_{t-1}(A),\mathrm{ucb}_{t-1}(A)]} \mu k^*(\tau_{t-1}(B)\mu) \\
&\overset{\text{Lemma 6}}{\geq} \max_{\mu\in[\mathrm{lcb}_{t-1}(A),\mathrm{ucb}_{t-1}(A)]} \mu k^*(\tau_t(B)\mu) \\
&\geq \max_{\mu\in[\mathrm{lcb}_{t}(A),\mathrm{ucb}_{t}(A)]} \mu k^*(\tau_t(B)\mu) = \sigma_t^2(A).
\end{aligned}$$

□

Lemma 16 (Lemma 11 in Hazan et al. (2007))**.** *Let x_t be $t = 1\ldots n$, then $\mathbf{V}_t = \sum_{i=1}^t x_i x_i^\top + \mathbf{V}_0$, where $\mathbf{V}_0 \succ 0$,*

$$\sum_{t=1}^n x_t^\top \mathbf{V}_t^{-1} x_t \leq \sum_{t=1}^n \log\left(\frac{\det(\mathbf{V}_t)}{\det(\mathbf{V}_{t-1})}\right) = \log\det(\mathbf{V}_n) - \log\det(\mathbf{V}_0). \tag{58}$$

C.7. Regret: Histogram feedback

Corollary 3 (Histogram Regret)**.** *Let $\delta \in (0,1)$, then under assumption of Poisson feedback, Assumption 1 and 2.* CAPTURE-UCB *with heteroscedasdic linear regression and histogram feedback suffers*

$$R_T \leq 2\sqrt{\rho T\Delta\beta_t(\delta,\mathcal{A})\gamma_T(\mathcal{A})}$$

with probability $1-\delta$, $\sqrt{\beta(\delta)} = \sqrt{\left(\sqrt{\gamma}/2 + \frac{2}{\sqrt{\gamma}}\log\left(\frac{1}{\delta}\frac{\det(\mathbf{V}_t)^{1/2}}{\det(\gamma\mathbf{I})^{1/2}}\right) + \frac{2m}{\sqrt{\gamma}}\log(2)\right)^2 + \|\mathbf{\Lambda}^\top\xi\|_{\mathbf{V}_t^{-1}}^2 + \gamma u}$, where Ξ denotes the maximum norm of dual variable $\|\xi_t\| \leq \Xi$ for all t, γ_T as in Definition (1), Δ is the fixed sensing duration, and $\rho = \max_{t\in T}\sigma_{t-1}^2(A_t)$.

Proof. In the following proof we use finite dimensional representation where $\varphi_{A_t} = \int_{A_t} \phi(x)dx$ and $\int_{A_t} \lambda(x)dx = \varphi_{A_t}^\top \theta^*$.

$$R(\{(A_t,\Delta)\}_{t=1}^T) = \sum_{t=1}^T \Delta w(A_t,\Delta)\frac{\varphi_{A^*}^\top \theta^* \Delta}{w(A^*,\Delta)} - \varphi_{A_t}^\top \theta^* \Delta \tag{59}$$

$$\leq \sum_{t=1}^T \Delta w(A_t)\frac{\mathrm{ucb}_t(A^*)}{w(A^*)} - \Delta\varphi_{A_t}^\top \theta^* \tag{60}$$

$$\overset{\text{optimism}}{\leq} \sum_{t=1}^T \Delta w(A_t)\frac{\mathrm{ucb}_t(A_t)}{w(A_t)} - \Delta\varphi_{A_t}^\top \theta^* \tag{61}$$

$$= \sum_{t=1}^T \Delta(ucb_t(A_t) - \varphi_{A_t}^\top \theta^*) \tag{62}$$

$$\leq \sum_{t=1}^T \Delta \max_{\theta\in C_t}(\theta-\theta^*)^\top \varphi_{A_t} \leq \sum_{t=1}^T \Delta \max_{\theta\in C_t} \|\theta-\theta^*\|_{\mathbf{V}_t} \|\varphi_{A_t}\|_{(\mathbf{V}_t)^{-1}} \tag{63}$$

$$\leq \sum_{t=1}^T \Delta \max_{\theta\in C_t} \left\|\theta - \hat\theta_t + \hat\theta_t - \theta^*\right\|_{\mathbf{V}_t} \|\varphi_{A_t}\|_{\mathbf{V}_t^{-1}} \tag{64}$$

$$\overset{(9)}{\leq} \sum_{t=1}^T 2\Delta\beta_t^{1/2} \|\varphi_{A_t}\|_{(\mathbf{V}_t)^{-1}} \leq \sqrt{\beta_T \Delta T}\sqrt{\sum_{t=1}^T \Delta \|\varphi_{A_t}\|^2_{(\mathbf{V}_t)^{-1}}} \tag{65}$$

$$\sum_{t=1}^T \Delta \|\varphi_{A_t}\|^2_{(\mathbf{V}_t)^{-1}} = \sum_{t=1}^T \mathrm{Tr}\left((\mathbf{V}_t)^{-1}\Delta\varphi_{A_t}\varphi_{A_t}^\top\right) \tag{66}$$

$$= \sum_{t=1}^T \mathrm{Tr}\left(\left(\sum_{j=1}^{t-1}\frac{\varphi_{A_j}\varphi_{A_tj}^\top}{\sigma_j^2(A_j)} + \gamma\mathbf{I}\right)^{-1} \Delta\varphi_{A_t}\varphi_{A_t}^\top\right) \tag{67}$$

$$\leq \sum_{t=1}^T \sigma_{t-1}^2(A_t)\,\mathrm{Tr}\left(\tilde{\mathbf{V}}_t^{-1}(\mathbf{V}_t - \mathbf{V}_{t-1})\right) \tag{68}$$

$$\overset{(58)}{\leq} \sum_{t=1}^T \sigma_{t-1}^2(A_t)\log\left(\frac{\det(\mathbf{V}_t)}{\det(\mathbf{V}_{t-1})}\right) \leq \sum_{t=1}^T \sigma_{t-1}^2(A_t)\log\left(\frac{\det(\mathbf{V}_t)}{\det(\mathbf{V}_{t-1})}\right) \tag{69}$$

$$\leq \rho(\log\det(\mathbf{V}_T) - \log\det(\gamma\mathbf{I})) = \rho\gamma_T \tag{70}$$

where the second to last step follows by Lemma 15, and $\rho = \max_{t\in T}\sigma_{t-1}^2(A_t)$, which is finite and depends only on u. □

C.8. Regret: Count-record feedback

Corollary 4 (Count-record Regret)**.** *Let $\delta \in (0,1)$, then under assumption of Poisson feedback, Assumptions 1, 2 and $\mathcal{B}$ be a fixed partition.* CAPTURE-UCB *with heteroscedasdic linear regression and count-record feedback suffers*

$$R_T \leq 2\sqrt{\rho T\Delta\beta_t(\delta,\mathcal{B})\gamma_T(\mathcal{B})}$$

with probability $1-\delta$, $\sqrt{\beta(\delta)} = \sqrt{\left(\sqrt{\gamma}/2 + \frac{2}{\sqrt{\gamma}}\log\left(\frac{1}{\delta}\frac{\det(\mathbf{V}_t)^{1/2}}{\det(\gamma\mathbf{I})^{1/2}}\right) + \frac{2m}{\sqrt{\gamma}}\log(2)\right)^2 + \|\mathbf{\Lambda}^\top\xi\|^2_{\mathbf{V}_t^{-1}} + \gamma u}$*, where Ξ denotes the maximum norm of dual variable $\|\xi_t\| \leq \Xi$ for all t, γ_T as in Definition (1), Δ is the fixed sensing duration, and* $\rho = \max_{t\in T}\max_{B\in A_t}\sigma_{t-1}^2(B)$.

Proof.

$$R(\{(A_t,\Delta_t)\}_{t=1}^T) = \sum_{t=1}^T \Delta w(A_t,\Delta)\frac{\varphi_{A^*}^\top\theta^*\Delta}{w(A^*,\Delta)} - \varphi_{A_t}^\top\theta^*\Delta \tag{71}$$

$$\leq \sum_{t=1}^{T} \Delta w(A_t) \frac{\mathrm{ucb}_t(A^*)}{w(A^*)} - \Delta_t \varphi_{A_t}^\top \theta^* \leq \sum_{t=1}^{T} \Delta w(A_t) \frac{\mathrm{ucb}_t(A_t)}{w(A_t)} - \Delta \varphi_{A_t}^\top \theta^* \tag{72}$$

$$\leq \sum_{t=1}^{T} \Delta(\mathrm{ucb}_t(A_t) - \varphi_{A_t}^\top \theta^*) \tag{73}$$

$$\leq \Delta \sum_{t=1}^{T} \max_{\theta \in C_t} (\theta - \theta^*)^\top \sum_{B \subset A_t} \varphi_B \leq \Delta \sum_{t=1}^{T} \sum_{B \subset A_t} \max_{\theta_B \in C_t} \|\theta_B - \theta^*\|_{\mathbf{V}_t} \|\varphi_B\|_{\mathbf{V}_t^{-1}} \tag{74}$$

$$\leq \sum_{t=1}^{T} \Delta \sum_{B \subset A_t} \max_{\theta \in C_t} \left\| \theta_B - \hat{\theta}_t + \hat{\theta}_t - \theta^* \right\|_{\mathbf{V}_t} \|\varphi_B\|_{\mathbf{V}_t^{-1}} \tag{75}$$

$$\leq \sum_{t=1}^{T} 2\Delta \beta_t^{1/2} \sum_{B \subset A_t} \|\varphi_B\|_{\mathbf{V}_t^{-1}} \overset{\text{Jensen}}{\leq} 2 \sum_{t=1}^{T} \Delta_t \beta_t^{1/2} \sqrt{\sum_{B \subset A_t} \varphi_B^\top \mathbf{V}_t^{-1} \varphi_B} \tag{76}$$

$$\overset{CS}{\leq} 2 \sqrt{\sum_{t=1}^{T} \beta_t \Delta} \sqrt{\sum_{t=1}^{T} \Delta \sum_{B \subset A_t} \varphi_B^\top \mathbf{V}_t^{-1} \varphi_B} \tag{77}$$

We analyze the two terms in square roots separately. First,

$$\sum_{t=1}^{T} \Delta \sum_{B \subset A_t} \varphi_B^\top \mathbf{V}_t^{-1} \varphi_B = \sum_{t=1}^{T} \Delta \,\mathrm{Tr} \left(\mathbf{V}_t^{-1} \sum_{B \subset A_t} \varphi_B \varphi_B^\top \right)$$

$$(*) = \sum_{t=1}^{T} \Delta \,\mathrm{Tr} \left(\left(\sum_{j=1}^{t-1} \sum_{B \subset A_j} \frac{\varphi_B \varphi_B^\top}{\sigma_j(B)^2} + \gamma \mathbf{I} \right)^{-1} \sum_{B \subset A_t} \varphi_B \varphi_B^\top \right)$$

Using a shorthand, $\mathbf{V}_{t+1} - \mathbf{V}_t = \sum_{B \subset A_t} \frac{1}{\sigma_t^2(B)} \varphi_B \varphi_B^\top \Delta$.

$$(*) \leq \sum_{t=1}^{T} \mathrm{Tr} \left(\max_{B \in A_t} \sigma_{t-1}^2(B) \mathbf{V}_{t+1}^{-1} (\mathbf{V}_{t+1} - \mathbf{V}_t) \right) \overset{(58)}{\leq} \sum_{t=1}^{T} \max_{B \in A_t} \sigma_{t-1}^2(B) \log \left(\frac{\det(\mathbf{V}_{t+1})}{\det(\mathbf{V}_t)} \right)$$

$$\overset{\text{Lemma (15)}}{\leq} \sum_{t=1}^{T} \max_{B \in A_t} \sigma_{t-1}^2(B) \log \left(\frac{\det(\mathbf{V}_{t+1})}{\det(\mathbf{V}_t)} \right) \leq \max_{t \in T} \max_{B \in A_t} \sigma_{t-1}^2(B) \gamma_T(\mathcal{B}).$$

□

C.9. Information Gain: Histogram vs Count-record Feedback

In the previous subsection we have identified that the regret of a method depends on the maximum information gain γ_T. Depending on the feedback form the value of maximum information gain differs. In general, $\gamma_T(\mathcal{B})$ tends to be larger than $\gamma_T(\mathcal{A})$ since the other contains potentially more independent vectors. To demonstrate this fact let us assume $\mathcal{B}$:

$$\gamma_T(\mathcal{A}) = \log\det \left(\sum_{i=1}^{t} \frac{1}{\sigma_t(A_t)^2 \gamma} \varphi_{A_t} \varphi_{A_t}^\top + \mathbf{I} \right) = \log\det \left(\sum_{i=1}^{t} \frac{1}{\sigma_t(A_t)^2 \gamma} \sum_{B' \subset A_t} \sum_{B \subset A_t} \varphi_B \varphi_{B'}^\top + \mathbf{I} \right). \tag{78}$$

Whereas playing the same actions,

$$\gamma_T(\mathcal{B}) = \log\det \left(\sum_{i=1}^{t} \sum_{B \subset A_t} \frac{1}{\sigma_t(B)^2 \gamma} \varphi_B \varphi_B^\top + \mathbf{I} \right). \tag{79}$$

Hence $\gamma_T(\mathcal{A})$ does not contain a lot of cross terms which $\gamma_T(\mathcal{A})$ contains, however the scaling of variances $\sigma_t(B^2$ and $\sigma_t(A)^2$ is different. Despite the fact that smaller sensing sets B often having a bigger correction k^*, it is still reasonable to assume to expect for most applications that $\sigma_t(A)^2 \geq \sigma_t(B)^2$. In fact, if we assume that $\sigma_t(A_t)^2$ is at least as big as multiple of the constituent $\sigma_t(B)^2$, we can formally relate them. If there was no correction k^*, this would always be satisfied, since Poisson responses have a variance growing with the mean value. For $k^* \approx 1$, this is satisfied approximately.

Lemma 17 (Count-record vs Histogram)**.** *If $\sigma_t(B)^2\#(A) \leq \sigma_t(A_t)^2$ for all $B \subset A_t$ and $t \in [T]$, where $\#(A)$ denotes the number of B that cover A_t.*

$$\gamma_T(\mathcal{A}) \leq \gamma_T(\mathcal{B})$$

Proof. Let us use a shorthand: $\boldsymbol{\Upsilon}_A$ is the feature decomposition of A in terms of partition $\mathcal{B}$ as $(\boldsymbol{\Upsilon}_A)_B := \varphi_B$ for all $B \in \mathcal{B}$ and $B \subset A$. Also, let $\mathbf{S}_A$ be a matrix full of ones with the size $\mathbf{S} \in \mathbb{R}^{\#A \times \#A}$

$$\begin{aligned}
\gamma_T(\mathcal{A}) &= \log\det\left(\sum_{i=1}^{t} \frac{1}{\sigma_t(A_t)^2\gamma} \sum_{B' \subset A_t} \sum_{B \subset A_t} \varphi_B \varphi_{B'}^\top + \mathbf{I}\right) \\
&= \log\det\left(\sum_{i=1}^{t} \frac{1}{\sigma_t(A_t)^2\gamma} \boldsymbol{\Upsilon}_{A_t}\mathbf{S}_{A_t}\boldsymbol{\Upsilon}_{A_t}^\top + \mathbf{I}\right) \\
&\leq \log\det\left(\sum_{i=1}^{t} \frac{\#(A)}{\sigma_t(A_t)^2\gamma} \boldsymbol{\Upsilon}_{A_t}\boldsymbol{\Upsilon}_{A_t}^\top + \mathbf{I}\right) \\
&\leq \log\det\left(\sum_{i=1}^{t} \frac{1}{\gamma}\frac{\boldsymbol{\Upsilon}_{A_t}\boldsymbol{\Upsilon}_{A_t}^\top}{\sigma_t(B)^2} + \mathbf{I}\right) = \log\det\left(\sum_{i=1}^{t} \frac{1}{\gamma} \sum_{B \subset A_t} \frac{1}{\sigma_t(B)^2} \varphi_B \varphi_B^\top + \mathbf{I}\right) = \gamma_T(\mathcal{B})
\end{aligned}$$

In the second line we used that the maximum eigenvalue of $\mathbf{S}_{A_t}$ is at most $\#(A)$, and in the third we used the assumption in the statement of the lemma. $\square$

D. Details of Numerical Experiments

D.1. Values of U for common kernels

We use two common feature representation in our work

Name	U	L_∞ error as function of m
QFF	$=1$	$\mathcal{O}(\exp(-m))$
Triangle basis	≤ 1	$\leq m^{-1}$ (Conjecture m^{-2})

The value of $\|\theta\|_2 \leq u$, and hence, the largest μ in the experiments is

$$\mu(A) = \varphi_A^\top \theta \Delta \leq \|\varphi_A\| \, \|\theta\| \, \Delta = \left\| \int_A \phi(x) dx \right\| u\Delta \leq \mathrm{vol}(A) U u \Delta.$$

D.2. Practical implementation tips

UCB While the calculation of UCB in (10) is a convex program and can be easily implemented, we employ a minor approximation, where we optimize ucb without the inequality constraints and then threshold the results if they are below or above the inequality constraints. Similarly, for $\Theta \succeq$ we only calculate the upper-bound without the psd constraint as we found this to be sufficiently good approximation not impacting the correctness of the algorithm.

Superimposability Poisson counts are superimposable – this means that sensing the same regions multiple times can be merged together as they contain the same amount of information. While the likelihood estimator automatically implements this feature, we need to enforce it with regression perspective, which can be done in connection with *tweaking*.

D.3. Further benchmarks

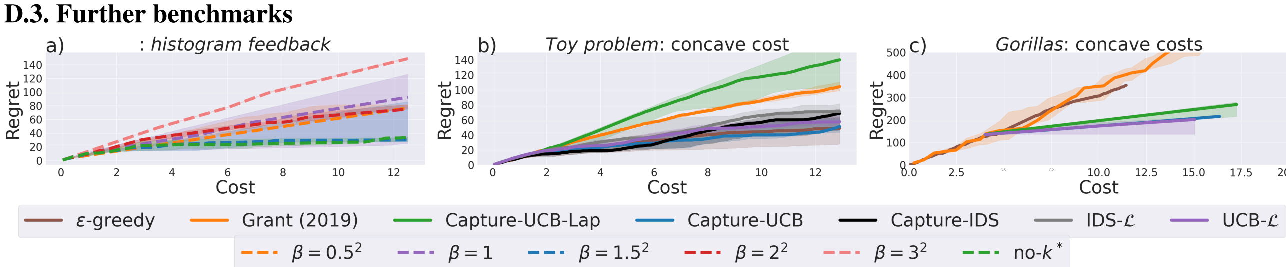


Figure 4. Numerical experiments on additional benchmarks: Each experiment was repeated $5-10$ times. Standard quantiles are depicted as shaded regions. in a) we investigate choice of empirical β. It seems that the best choice seems to lie around $\beta = 1.5$ to 2. The value at 1.5 is even comparable to *heuristic* of not using correction k^* at all b) we report *toy-problem* with concave costs. In c) we report another *Gorilla* with different sensing duration.

D.4. Hyper-parameters and Detail

In our experiments, we consider three cases of cost

- uniform $w(A) = |A|$
- fixed $w(A) = |A| + 0.005$ for *Gorillas* and $w(A) = |A| + 0.002$ for toy problem.
- one-off as in the main text.

In the table below, ℓ designates the lengthscale of the RBF kernel as $k(r) = \exp(-\frac{1}{2\ell^2} r^2)$. Also, in the table below depth of $\mathcal{A}$ or $\mathcal{B}$ signifies how deep was the quadtree for the action and partition $\mathcal{B}$, respectively. Note that the number of actions grows as 4^{depth} for 2D problems.

Experiment name	**Domain**	**Kernel**	**Cost**
toy-problem	$[-1,1]$	RBF	both
Taxis	$[-1,1]^2$	RBF	uniform
San Francisco dataset	$[-1,1]^2$	RBF	uniform
Gorillas dataset	$[-1,1]^2$	RBF	fixed
Gorillas dataset (Appendix)	$[-1,1]^2$	RBF	fixed
Beilschmiedia dataset	$[-1,1]^2$	(see below)	one-off

Experiment name	**Params.**	l	u	U	m	Δ	**Depth** $\mathcal{A}$	**Depth** $\mathcal{B}$	**Repeats**	T
toy-problem	$\ell = 0.1$	0	4	1	64	5	6	7	10	200
Taxis	$\ell = 0.075$	0	5.5	1	20^2	60 mins	3	N/A	10	100
San Francisco dataset	$\ell = 0.1$	0	7.8	1	30^2	30 days	4	5	10	120
Gorillas dataset	$\ell = 0.1$	0	12.95	1	25^2	1	4	5	5	400
Gorillas dataset (Appendix)	$\ell = 0.1$	0	12.95	1	25^2	5	4	5	5	200
Beilschmiedia dataset	$\ell = 0.1$	0	≈ 800	N/A	10×10	30	4	5	N/A	64

Beilschmiedia This benchmarks differs from other mainly by using different cost model. We use the real dataset and try to simulate a sensing process from a satellite. Since trees do not have ability to move sensing multiple times does not lead to any different information and this mode of operation is justified. To model simillarity of the sensing regions we use slope magnitude and height of the spatial profile as we believe these might be predictive of the habitat.

Namely, we create a smooth model of slope $s(x, y)$ and height $h(x, y)$ with Gaussian process fit and then use these to define the additive kernel:

$$k((x, y), (x', y')) = \exp\left(\frac{(s(x, y) - s(x', y'))^2}{2\ell^2}\right) + \exp\left(\frac{(h(x, y) - h(x', y'))^2}{2\ell^2}\right).$$

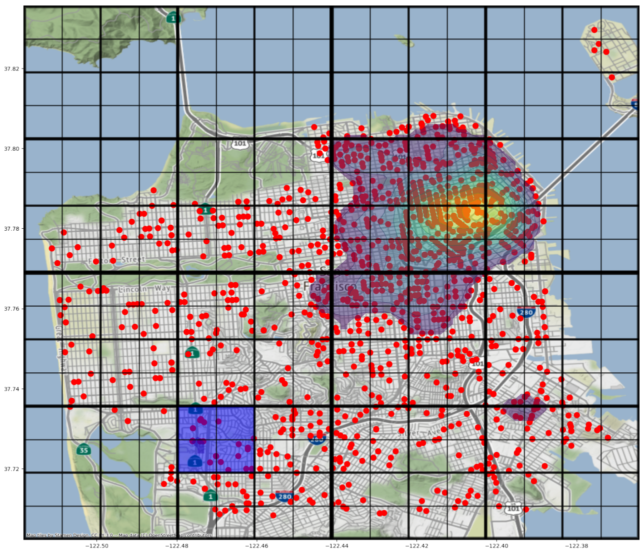
We do not use directly using this kernel, instead we approximate each additive component with Quadrature Fourier Features with $m = 10$ stack them together and then use the trace regression formulation. The trace regression formulation due to its feature to increase capacity from k to k^2 automatically induces cross-terms between the additive components. We found this model to be sufficiently good in terms of descriptive power and small enough such that the positive definite constraint could be efficiently enforced. The visual fit of this model can be seen in Figures below.

D.5. Infrastructure

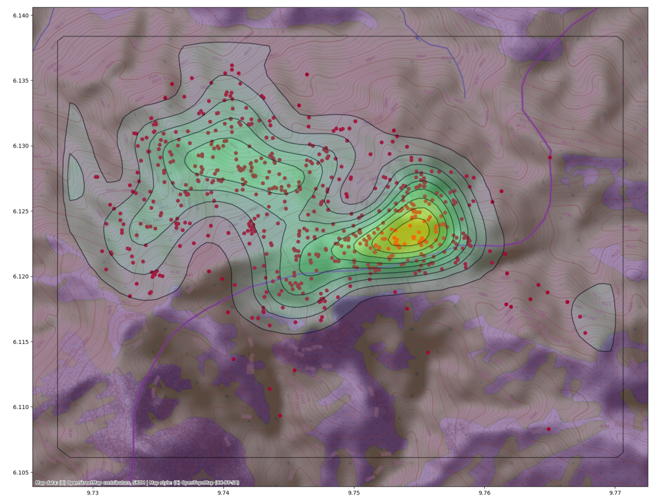
The average runtine of each benchmark is below 6 hours. We run the experiments on a medium size CPU server with 28 cores and used approximately $2GB$ of memory.

D.6. Pictorial depictions of the problems

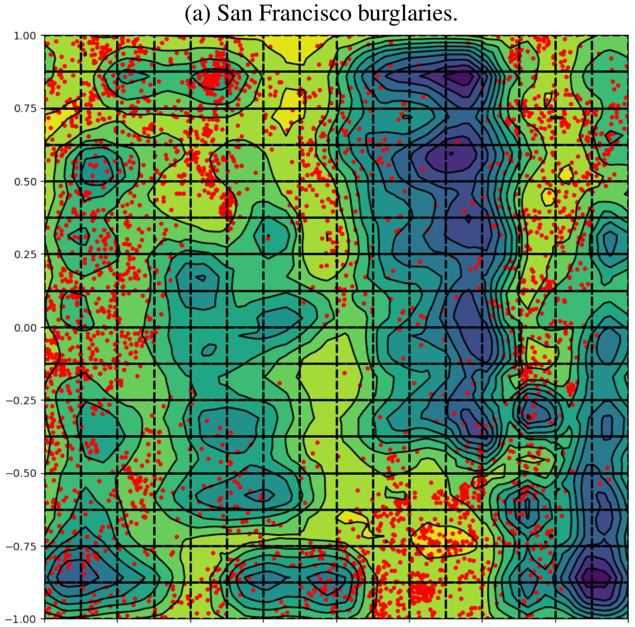
In Figure 5 we can see the intensity functions of the benchmarks we used in the experimental section.



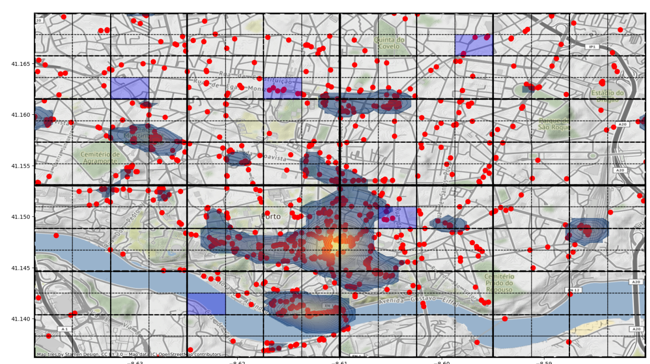
(a) San Francisco burglaries.



(b) Gorillas nesting spots in Cameroon rain forest.



(c) Beilschmiedia - African tree shrub



(d) Porto spontaneous taxi pickups

Figure 5. Experiments and their rates.