A. Assumptions on Degree of Correlation

We introduce a parameterized condition called α-*strong correlation* in this paper. Note that Albert et al. (2017a) also provides a condition in their paper, known as γ-*separation*, to measure the degree of correlation between the valuation types and external signals. However, α-strong correlation and γ-separation are distinct notions.

Example. Let π_1 and π_2 be the joint distributions with matrices Γ_1 and Γ_2 respectively, where Γ_1 and Γ_2 are defined as follows:

$$\Gamma_1 = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix} \quad \Gamma_2 = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}.$$

We denote by $\alpha(\cdot)$ and $\gamma(\cdot)$ the degree of correlation measured by α-strong correlation and γ-separation, respectively. For these two distributions π_1 and π_2, $\alpha(\pi_1) = \alpha(\pi_2) = \frac{1}{2}$, however, $\gamma(\pi_1) = \frac{\sqrt{2}}{2} < 1 = \gamma(\pi_2)$.

B. Missing Proofs in Section 3

We provide proofs of Lemma 3, Lemma 5 and Lemma 6.

B.1. Proof of Lemma 3

Fix any $\theta \in \Theta$ and denote by $m(\theta)$ the number of samples containing type θ. Apply Lemma 1 with $X = m(\theta)$, $t = \frac{1}{2}$, $n = m$ and $p = \eta \leq \pi(\theta)$. By our assumption that

$$\begin{aligned} m &\geq 90K\eta^{-3}\alpha^{-2}\epsilon^{-2}\max\{5\ln(6K\delta^{-1}), 8K\} \\ &\geq 8\eta^{-1}\ln(2K\delta^{-1}), \end{aligned}$$

for the fixed θ, we have

$$\Pr\left[m(\theta) < \frac{\eta m}{2}\right] \leq \frac{\delta}{2K}.$$

For any $\Omega' \subseteq \Omega$, let $q = \Pr[\omega \notin \Omega' | \theta]$ denote the probability that an ω which does not belong to the subset Ω' occurs conditioned on that the bidder's valuation type is θ, and denote by $\hat{q}$ the estimation of q with $m(\theta)$ samples. If $m(\theta) \geq \frac{\eta m}{2}$, apply Lemma 2 with $n = m(\theta)$ and $t = \zeta$, we have that $\frac{K+1}{m(\theta)} \leq \frac{\zeta^2}{20}$, and the following bound

$$\sum_{\omega \in \Omega'} |\hat{\pi}(\omega|\theta) - \pi(\omega|\theta)| + |\hat{q} - q| \leq \zeta$$

fails with probability at most $\frac{\delta}{2K}$.

Therefore, for a fixed θ, the probability that

$$\sum_{\omega \in \Omega'} |\hat{\pi}(\omega|\theta) - \pi(\omega|\theta)| \leq \zeta$$

fails is upper bounded by $\frac{\delta}{2K} + (1 - \frac{\delta}{2K})\frac{\delta}{2K} \leq \frac{\delta}{K}$. Taking union bound over all $\theta \in \Theta$, we obtain that this difference bound holds for all $\theta \in \Theta$ with probability at least $1 - \delta$.

B.2. Proof of Lemma 5

By our assumption that the prior distribution π is α-strongly correlated, there exists a subset Ω' of Ω with size K, such that the submatrix Γ' defined on it is nonsingular and with singular values at least α. We would like to show that when ζ is small enough, the matrix $\hat{\Gamma}'$ determined by the same subset Ω' is nonsingular as well. Moreover, the singular values of this $\hat{\Gamma}'$ are all greater than $\frac{2}{3}\alpha$.

We denote by $\delta\Gamma'$ the difference matrix $\hat{\Gamma}' - \Gamma'$. Since Eq. 5 holds for all $\theta \in \Theta$ by Lemma 3, we have $\|\delta\Gamma'\| \leq \sqrt{K}\zeta$. With $\left\|(\Gamma')^{-1}\right\| \leq \alpha^{-1}$ by assumption, we can bound the norm of the matrix $(\Gamma')^{-1}(\delta\Gamma')$ by

$$\left\|(\Gamma')^{-1}(\delta\Gamma')\right\| \leq \left\|(\Gamma')^{-1}\right\| \cdot \|\delta\Gamma'\| \leq \frac{\sqrt{K}\zeta}{\alpha}.$$

With the assumption on the size of m, we have $\frac{\sqrt{K}\zeta}{\alpha} < \frac{1}{3}$. By Lemma 4, the matrix $I + (\Gamma')^{-1}(\delta\Gamma')$ is invertible, and its inverse is the series $\sum_{k=0}^{\infty}[-(\Gamma')^{-1}(\delta\Gamma')]^k$. Thus we have

$$\left\|\left[I + (\Gamma')^{-1}(\delta\Gamma')\right]^{-1}\right\| \leq \sum_{k=0}^{\infty}\left\|(\Gamma')^{-1}(\delta\Gamma')\right\|^k < \frac{3}{2}.$$

This implies $\sigma_K(I + (\Gamma')^{-1}(\delta\Gamma')) > \frac{2}{3}$, where $\sigma_K(\cdot)$ represents the minimum (K-th) singular value.

As a result, the matrix $\hat{\Gamma}' = \Gamma'(I + (\Gamma')^{-1}(\delta\Gamma'))$ is nonsingular, and its minimum singular value $\sigma_K(\hat{\Gamma}')$ is lower bounded by

$$\sigma_K(\hat{\Gamma}') \geq \sigma_K(\Gamma')\sigma_K(I + (\Gamma')^{-1}(\delta\Gamma')) > \frac{2}{3}\alpha.$$

B.3. Proof of Lemma 6

Suppose now we have found $\hat{\Gamma}'$ at Step 2 of Algorithm 1. We claim that the matrix Γ', whose entries are indexed by the same way as $\hat{\Gamma}'$ but constituted by the true probabilities, is nonsingular. Otherwise, with the assumption on the size of m, we have

$$\sigma_K(\hat{\Gamma}') \leq \sigma_K(\Gamma') + \|\delta\Gamma'\| \leq \sqrt{K}\zeta < \frac{\alpha}{3},$$

which contradicts with how we find $\hat{\Gamma}'$ in Algorithm 1.

Denote by $\boldsymbol{v}'$ the vector $(v'(1), v'(2), \ldots, v'(K))$. Let $\boldsymbol{p}$ be the vector $(\Gamma')^{-1}\boldsymbol{v}'$, and $\hat{\boldsymbol{p}}$ be the vector $(\hat{\Gamma}')^{-1}\boldsymbol{v}'$.Then left-multiply the equation $\hat{\Gamma}'\hat{\boldsymbol{p}} = \boldsymbol{v}'$ by $(\Gamma')^{-1}$ and obtain

$$\left[I + (\Gamma')^{-1}(\delta\Gamma')\right]\hat{\boldsymbol{p}} = \boldsymbol{p}.$$

In the proof of Lemma 5, the norm of $\left[I + (\Gamma')^{-1}(\delta\Gamma')\right]^{-1}$ is bounded by $\frac{3}{2}$. Therefore, we have

$$\|\hat{\boldsymbol{p}}\| \leq \left\|\left[I + (\Gamma')^{-1}(\delta\Gamma')\right]^{-1}\right\| \|\boldsymbol{p}\| \leq \frac{3}{2}\|\boldsymbol{p}\|.$$

By the definition of $\hat{\boldsymbol{p}}$, we have

$$\Gamma'\hat{\boldsymbol{p}} - \boldsymbol{v}' = \Gamma'\hat{\boldsymbol{p}} - \hat{\Gamma}'\hat{\boldsymbol{p}} = -(\delta\Gamma')\hat{\boldsymbol{p}}.$$

Hence the norm of $\Gamma'\hat{\boldsymbol{p}} - \boldsymbol{v}'$ can be bounded by

$$\|\Gamma'\hat{\boldsymbol{p}} - \boldsymbol{v}'\| \le \|\delta\Gamma'\| \, \|\hat{\boldsymbol{p}}\| \le \frac{3\sqrt{K}\,\|\boldsymbol{p}\|\,\zeta}{2} \le \frac{3\sqrt{K}\,\|\boldsymbol{v}\|\,\zeta}{2\alpha},$$

where we use $\|\boldsymbol{p}\| \le \left\|(\Gamma')^{-1}\right\| \|\boldsymbol{v}'\| \le \|\boldsymbol{v}\|/\alpha$.

Therefore, for any $\theta \in \Theta$,

$$v'(\theta) - \frac{3\sqrt{K}\,\|\boldsymbol{v}\|\,\zeta}{2\alpha} \le \sum_{\omega \in \Omega'} \pi(\omega|\theta)\hat{p}(\omega) \le v(\theta),$$

which indicates that the auction returned by Algorithm 2 is interim IR.

C. Missing Proofs in Section 4

In this appendix, we provide the proofs that we omitted from Section 4, and show the validness of our hard instances.

C.1. Proof of Lemma 7

For an ϵ that is small enough, the allocation rule follows directly from interim IR and that the auction extracts revenue at least $1-\epsilon$ social surplus as revenue.

As for the payment rule, by the definition of ex-post IC, for all $\theta, \theta' \in \Theta$ and $\omega \in \Omega$:

$$v(\theta) - p(\theta, \omega) \ge v(\theta) - p(\theta', \omega).$$

Hence for all $\theta, \theta' \in \Theta$ and $\omega \in \Omega$,

$$p(\theta, \omega) = p(\theta', \omega) = p^*(\omega).$$

C.2. Validity of Hard Instances

It suffices to show that the minimum singular value of the following matrix Γ

$$\begin{bmatrix} \frac{1+\sqrt{K}\alpha}{2} & 0 & \dots & 0 & \frac{1-\sqrt{K}\alpha}{2} \\ 0 & \frac{1+\sqrt{K}\alpha}{2} & \dots & 0 & \frac{1-\sqrt{K}\alpha}{2} \\ \vdots & & & & \\ 0 & 0 & \dots & \frac{1+\sqrt{K}\alpha}{2} & \frac{1-\sqrt{K}\alpha}{2} \\ 0 & 0 & \dots & \frac{1-\sqrt{K}\alpha}{2} & \frac{1+\sqrt{K}\alpha}{2} \end{bmatrix}$$

is at least α.

We factorize Γ into a symmetric matrix Φ and an upper triangular matrix Λ as below:

$$\Gamma = \begin{bmatrix} \frac{1+\sqrt{K}\alpha}{2} & 0 & \dots & 0 & 0 \\ 0 & \frac{1+\sqrt{K}\alpha}{2} & \dots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & \frac{1+\sqrt{K}\alpha}{2} & \frac{1-\sqrt{K}\alpha}{2} \\ 0 & 0 & \dots & \frac{1-\sqrt{K}\alpha}{2} & \frac{1+\sqrt{K}\alpha}{2} \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 & 0 & \dots & 0 & \frac{1-\sqrt{K}\alpha}{1+\sqrt{K}\alpha} \\ 0 & 1 & \dots & 0 & \frac{1-\sqrt{K}\alpha}{1+\sqrt{K}\alpha} \\ \vdots & & & & \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix} \triangleq \Phi \cdot \Lambda.$$

Let β be $\frac{1-\sqrt{K}\alpha}{1+\sqrt{K}\alpha}$ ($\beta < 1$). From straightforward calculations, we get

$$\lambda_K(\Lambda^\top \Lambda) = \frac{2}{K\beta^2 + \sqrt{K^2\beta^4 - 4}} > \frac{1}{K},$$

where $\lambda_K(\cdot)$ means the minimum (K-th) eigenvalue of the matrix. This implies $\sigma_K(\Lambda) > \frac{1}{\sqrt{K}}$. Moreover, it is clear that $\sigma_K(\Phi) = \sqrt{K}\alpha$. Therefore, we have

$$\sigma_K(\Gamma) \ge \sigma_K(\Phi)\sigma_K(\Lambda) > \sqrt{K}\alpha \cdot \frac{1}{\sqrt{K}} = \alpha.$$

C.3. Proof of Lemma 10

Suppose M is an auction in $\mathcal{M}^1$ whose payment rule is p. Then M is also in $\mathcal{M}$, so we have

$$v(K) - \frac{1-\sqrt{K}\alpha}{2}p(\omega_{K-1}) - \frac{1+\sqrt{K}\alpha}{2}p(\omega_K) \le \frac{\epsilon}{\eta}. \tag{10}$$

On the other hand, by interim IR of auction M, we have

$$v(K-1) - \frac{1+\sqrt{K}\alpha}{2}p(\omega_{K-1}) - \frac{1-\sqrt{K}\alpha}{2}p(\omega_K) \ge 0. \tag{11}$$

We can get the following inequality by subtracting Eq. 10 from Eq. 11:

$$p(\omega_K) - p(\omega_{K-1}) \ge \frac{v(K) - v(K-1) - \frac{\epsilon}{\eta}}{\sqrt{K}\alpha} > \frac{1}{5\sqrt{K}\alpha\eta}.$$

Similar to the steps above, we have $\Delta_{M,\pi^1}(i) \ge 0$ due to interim IR and $\Delta_{M,\pi^1}(K-1) \le \frac{\epsilon}{1-(K-1)\eta}$ by definition of $\mathcal{M}$. Putting these two inequalities together gives

$$p(\omega_{K-1}) - p(\omega_i) \ge 0,$$

which is directly followed by

$$p(\omega_K) - p(\omega_i) \ge p(\omega_K) - p(\omega_{K-1}) > \frac{1}{5\sqrt{K}\alpha\eta}.$$

Adding $\epsilon' \cdot [p(w_K) - p(w_i)]$ to the equation $\Delta_{M,\boldsymbol{\pi}^1}(i) \geq 0$, we have

$$v(i) - \Big(\frac{1+\sqrt{K}\alpha}{2} + \epsilon'\Big)p(\omega_i) - \Big(\frac{1-\sqrt{K}\alpha}{2} - \epsilon'\Big)p(\omega_K) > \frac{125\alpha\epsilon}{\sqrt{K}} \cdot \frac{1}{5\sqrt{K}\alpha\eta} = \frac{25\epsilon}{\eta K},$$

which indicates that M is not in $\mathcal{M}^2$.

C.4. Proof of Lemma 11

Consider the following algorithm for distinguishing the two distributions π^1 and π^2. Given an unknown distribution $\pi \in \{\pi^1, \pi^2\}$, run algorithm A with m samples from π. If $A(\pi)$ is in $\mathcal{M}^1$, the algorithm will return π^1; if $A(\pi)$ is in $\mathcal{M}\backslash\mathcal{M}^1$, the algorithm will return π^2; otherwise, return *unknown*.

By our assumption that the algorithm A takes less than $m < c \cdot D_{\mathsf{SKL}}(\pi^1, \pi^2)^{-1}$ samples, it cannot distinguish the two distributions correctly. That is, either

$$\Pr[A(\pi^1) \in \mathcal{M}^1] < \frac{2}{3},$$

or

$$\Pr[A(\pi^2) \in \mathcal{M}\backslash\mathcal{M}^1] < \frac{2}{3},$$

or both holds. Then we have at least one of the following inequalities:

$$\begin{aligned}&\Pr[A(\pi^1) \in \mathcal{M}\backslash\mathcal{M}^1] \\ =&\Pr[A(\pi^1) \in \mathcal{M}] - \Pr[A(\pi^1) \in \mathcal{M}^1] > \frac{3}{10}, \\ &\Pr[A(\pi^2) \in \mathcal{M}\backslash\mathcal{M}^2] \geq \Pr[A(\pi^2) \in \mathcal{M}^1] \\ =&\Pr[A(\pi^2) \in \mathcal{M}] - \Pr[A(\pi^2) \in \mathcal{M}\backslash\mathcal{M}^1] > \frac{3}{10}.\end{aligned}$$

And the lemma naturally follows.

C.5. Proof of Lemma 12

We first show that when a distribution π is drawn uniformly at random from $\mathcal{H}$, for any $1 \leq i \leq K-2$, we must have $\Pr[i \in \mathcal{B}_\pi] \geq \frac{1}{2}$.

Note that Lemma 11 holds for any π^1 and π^2. Enumerating over every $1 \leq i \leq K-2$ and all possible $S' \subseteq \{1, \ldots, K-2\}\backslash\{i\}$, $\pi^1 = \pi_{S'}$ and $\pi^2 = \pi_{S' \cup \{i\}}$ together covers all distributions in $\mathcal{H}$. We get that at least half of the distributions $\pi \in \mathcal{H}$ satisfy that $i \in \mathcal{B}_\pi$.

Then through a simple counting argument, we know that there must exist some distribution $\pi^* \in \mathcal{H}$, such that $|\mathcal{B}_{\pi^*}| \geq \frac{K}{2} - 1$.

C.6. Proof of Lemma 13

Let $q = \Pr[|\mathcal{B}^*{}_{A(\pi^*)}| \geq \frac{K}{25}]$ denote the probability that the conclusion of the lemma holds.

On the one hand, each $i \in \mathcal{B}_{\pi^*}$ is in $\mathcal{B}^*{}_{A(\pi^*)}$ with probability at least $\frac{3}{10}$ by definition. The expected size of $\mathcal{B}^*{}_{A(\pi^*)}$ is therefore lower bounded by

$$\mathbb{E}_{A(\pi^*)}\left[|\mathcal{B}^*{}_{A(\pi^*)}|\right] \geq \frac{3}{10} \cdot \left(\frac{K}{2} - 1\right) = \frac{3(K-2)}{20} \geq \frac{K}{20}.$$

On the other hand, we have:

$$\begin{aligned}&\mathbb{E}_{A(\pi^*)}[|\mathcal{B}^*{}_{A(\pi^*)}|] \\ \leq&\frac{K}{25} \cdot \Pr\left[|\mathcal{B}^*{}_{A(\pi^*)}| < \frac{K}{25}\right] + K \cdot \Pr\left[|\mathcal{B}^*{}_{A(\pi^*)}| \geq \frac{K}{25}\right] \\ \leq&\frac{K}{25}(1-q) + K \cdot q \leq \frac{K}{25} + K \cdot q\ .\end{aligned}$$

Putting these two inequalities together gives $q \geq 0.01$.

D. Missing Proofs in Section 5

In this appendix, we provide the proofs of Lemma 14 and Theorem 3 and show the validness of our hard instances in 2-bidder case.

D.1. Proof of Lemma 14

This proof is similar to the proof of Lemma 3.

Fixing any bidder i and any $\theta_i \in \Theta$, we denote by $m(\theta_i)$ the number of samples containing type θ_i. Apply Lemma 1 with $X = m(\theta_i)$, $t = \frac{1}{2}$, $n = m$ and $p = \eta \leq \pi(\theta_i)$. By our assumption that

$$\begin{aligned}m &\geq 250n^2K^2\eta^{-3}\alpha^{-2}\epsilon^{-2}\max\{5\ln(12nK\delta^{-1}), 8K\} \\ &\geq 8\eta^{-1}\ln(4nK\delta^{-1}),\end{aligned}$$

for the fixed θ_i, we have

$$\Pr\left[m(\theta_i) < \frac{\eta m}{2}\right] \leq \frac{\delta}{4nK}.$$

For the fixed θ_i, there are at most θ_i possible values of $v(\theta_i)x_i^*(\theta_i, \boldsymbol{\theta}_{-i}) - p_i^*(\theta_i, \boldsymbol{\theta}_{-i})$. They are $v(\theta_i) - v(1)$, $v(\theta_i) - v(2)$, $\ldots$, $v(\theta_i) - v(\theta_i - 1)$ and 0. We divide the external signal set Θ_{-i} into θ_i subsets Ω_1, Ω_2, $\ldots$, Ω_{θ_i} corresponding to the highest valuation by others from $v(1)$ to no less than $v(\theta_i)$. For each $j \in \{1, 2, \ldots, \theta_i\}$, let $q_j = \Pr[\boldsymbol{\theta_{-i}} \in \Omega_j | \theta_i]$ denote the conditional probability that $\boldsymbol{\theta}_{-i}$ falls into Ω_j.

If $m(\theta_i) \geq \frac{\eta m}{2}$, apply Lemma 2 with $n = m(\theta_i)$ and $t = \zeta$, we have $\frac{K}{m(\theta_i)} \leq \frac{\zeta^2}{20}$, and the bound

$$\sum_{j \in [\theta_i]} |\hat{q}_j - q_j| \leq \zeta \tag{12}$$

fails with probability at most $\frac{\delta}{4nK}$. Therefore, for the fixed θ, the probability that Eq. 12 fails is upper bounded by

$$\frac{\delta}{4nK} + (1 - \frac{\delta}{4nK})\frac{\delta}{4nK} \le \frac{\delta}{2nK}.$$

Taking union bound over all bidder i and all $\theta_i \in \Theta$, we obtain that the difference bound Eq. 12 holds with probability at least $1 - \frac{\delta}{2}$.

Hence we can bound the mis-estimation by

$$|u_i(\theta_i) - \hat{u}_i(\theta_i)| \le v(K) \cdot \sum_{j \in [\theta_i]} |q_j - \hat{q}_j| \le v(K)\zeta,$$

which holds for all bidder i and all $\theta_i \in \Theta$ with probability at least $1 - \frac{\delta}{2}$.

D.2. Proof of Theorem 3

In Algorithm 3, there are two sources of error which may lead to the additional payments of the bidders failing to fully extract the expected utilities. The first is the the inaccurate estimation of the conditional probabilities $\pi(\boldsymbol{\theta}_{-i}|\theta)$ ($\boldsymbol{\theta}_{-i} \in \Theta'_{-i}$). The revenue loss of bidder i caused by the inaccurate estimation is denoted as

$$\Delta_i^1(\pi) = \sum_{\theta_i \in \Theta} \pi_i(\theta_i) \cdot \Big[\hat{u}_i(\theta_i) - \sum_{\boldsymbol{\theta}_{-i} \in \Theta_{-i}} \pi(\boldsymbol{\theta}_{-i}|\theta_i)\hat{p}_i(\boldsymbol{\theta}_{-i})\Big].$$

By Lemma 6, we have that

$$\Big|\hat{u}'_i(\theta_i) - \sum_{\boldsymbol{\theta}_{-i} \in \Theta_{-i}} \pi(\boldsymbol{\theta}_{-i}|\theta_i)\hat{p}_i(\boldsymbol{\theta}_{-i})\Big| \le \frac{3Kv(K)\zeta}{2\alpha}$$

holds for all bidder i and all $\theta_i \in \Theta$ with probability at least $1 - \frac{\delta}{2}$. As we set

$$\hat{u}'_i(\theta_i) = \max\Big\{0,\ \hat{u}_i(\theta_i) - \frac{5Kv(K)\zeta}{2\alpha}\Big\},$$

we can bound $\Delta_i^1(\pi)$ by $\frac{4Kv(K)\zeta}{\alpha}$.

The other source of revenue loss is the mis-estimation of the expected utilities $u_i(\theta_i)$. We denote the expected mis-estimation as $\Delta_i^2(\pi)$. By Lemma 14, for all bidder $i \in [n]$, $\Delta_i^2(\pi)$ is bounded by

$$\Delta_i^2(\pi) = \sum_{\theta_i \in \Theta} \pi_i(\theta_i) \cdot [u_i(\theta_i) - \hat{u}_i(\theta_i)] \le v(K)\zeta$$

with probability at least $1 - \frac{\delta}{2}$.

Therefore, when the number of samples m satisfies Eq. 8, with probability at least $1 - \delta$, for all bidder $i \in [n]$ and all $\theta_i \in \Theta$, we have

$$\begin{aligned}\sum_{\boldsymbol{\theta}_{-i} \in \Theta_{-i}} \pi(\boldsymbol{\theta}_{-i}|\theta_i)\hat{p}_i(\boldsymbol{\theta}_{-i}) &\le \hat{u}'_i(\theta_i) + \frac{3Kv(K)\zeta}{2\alpha} \\ &\le \hat{u}_i(\theta_i) - \frac{Kv(K)\zeta}{\alpha} \le u_i(\theta_i).\end{aligned}$$

This shows the auction returned by Algorithm 3 satisfies interim IR.

Finally, we show the near-optimal revenue guarantee of the auction. When the number of samples m satisfies Eq. 8, with probability at least $1 - \delta$, we have

$$\frac{\Delta(\pi)}{\textsc{Opt}(\pi)} \le \frac{\sum_{i \in [n]} \big(\Delta_i^1(\pi) + \Delta_i^2(\pi)\big)}{v(K)\eta} \le \frac{5nK\zeta}{\alpha\eta} \le \epsilon.$$

This completes the proof of Theorem 3.

D.3. Validity of Hard Instances in the 2-Bidder Case

By letting the external signals of bidder 1 be the valuation types of bidder 2, we rewrite the hard instances we construct in Section 4 as a family of 2-bidder joint distributions. What we need to verify is that the smallest marginal probability of bidder 2 is $\Omega(\eta)$ and the minimum singular value of Γ_2 is $\Omega(\alpha)$.

The marginal probabilities of any distribution π in the set $\mathcal{H}$ are at least $(\frac{1+\sqrt{K}\alpha}{2})\eta \ge \frac{\eta}{2}$. Then it suffices to show the minimum singular value of the following matrix

$$\Gamma_2 = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & & & & \vdots & \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & \gamma_1 & \gamma_2 \\ \beta & \beta & \dots & \beta & \gamma_3 & \gamma_4 \end{bmatrix}$$

is at least α, where β and γ_i $(i = 1, 2, 3, 4)$ are

$$\begin{aligned}\beta &= \frac{(1 - \sqrt{K}\alpha)\eta}{1 - \sqrt{K}\alpha(1 - 2\eta)}, \\ \gamma_1 &= \frac{(1 + \sqrt{K}\alpha)[1 - (K-1)\eta]}{1 + \sqrt{K}\alpha - \eta[K^{\frac{3}{2}}\alpha + (K-2)]}, \\ \gamma_2 &= \frac{(1 - \sqrt{K}\alpha)\eta}{1 + \sqrt{K}\alpha - \eta[K^{\frac{3}{2}}\alpha + (K-2)]}, \\ \gamma_3 &= \frac{(1 - \sqrt{K}\alpha)[1 - (K-1)\eta]}{1 - \sqrt{K}\alpha(1 - 2\eta)} \text{ and} \\ \gamma_4 &= \frac{(1 + \sqrt{K}\alpha)\eta}{1 - \sqrt{K}\alpha(1 - 2\eta)},\end{aligned}$$

respectively. We factorize Γ_2 into a lower triangular matrix Λ and a matrix Φ as below:

$$\begin{aligned}\Gamma_2 = &\begin{bmatrix} 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & \dots & 0 & 1 & 0 \\ \beta & \beta & \dots & \beta & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & \gamma_1 & \gamma_2 \\ 0 & 0 & \dots & \gamma_3 & \gamma_4 \end{bmatrix} \\ &\triangleq \Lambda \cdot \Phi.\end{aligned}$$

From straightforward calculations, we get

$$\lambda_K(\Lambda^\top \Lambda) = \frac{2}{K\beta^2 + \sqrt{K^2\beta^4 - 4}} > \frac{1}{K\beta^2},$$

where $\lambda_K(\cdot)$ means the minimum (K-th) eigenvalue of the matrix. This implies

$$\sigma_K(\Lambda) > \frac{1}{\sqrt{K}\beta} = \frac{1 - \sqrt{K}\alpha(1-2\eta)}{\sqrt{K}(1-\sqrt{K}\alpha)\eta} \geq \frac{1}{\sqrt{K}\eta}.$$

Moreover, we have

$$\lambda_K(\Phi^\top \Phi) \geq \frac{(\gamma_1\gamma_4 - \gamma_2\gamma_3)^2}{\gamma_1^2 + \gamma_2^2 + \gamma_3^2 + \gamma_4^2},$$

which is followed by

$$\sigma_K(\Phi) \geq \frac{\gamma_1\gamma_4 - \gamma_2\gamma_3}{\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4} \geq \sqrt{K}\alpha\eta.$$

Therefore, we have

$$\sigma_K(\Gamma_2) \geq \sigma_K(\Phi)\sigma_K(\Lambda) \geq \sqrt{K}\alpha\eta \cdot \frac{1}{\sqrt{K}\eta} = \alpha.$$