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Reconstruction from Anisotropic Random Measurements

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Abstract

Random matrices are widely used in sparse recovery problems, and the relevant properties of matrices with i.i.d. entries are well understood. The current paper discusses the recently introduced Restricted Eigenvalue (RE) condition, which is among the most general assumptions on the matrix, guaranteeing recovery. We prove a reduction principle showing that the RE condition can be guaranteed by checking the restricted isometry on a certain family of low-dimensional subspaces. This principle allows us to establish the RE condition for several broad classes of random matrices with dependent entries, including random matrices with subgaussian rows and non-trivial covariance structure, as well as matrices with independent rows, and uniformly bounded entries.

Keywords: ℓ_1 minimization, Sparsity, Restricted Eigenvalue conditions, Subgaussian random matrices, Design matrices with uniformly bounded entries.

1. Introduction

In a typical high dimensional setting, the number of variables p is much larger than the number of observations n. This challenging setting appears in statistics and signal processing, for example, in regression, covariance selection on Gaussian graphical models, signal reconstruction, and sparse approximation. Consider a simple setting, where we try to recover a vector $\beta \in \mathbb{R}^p$ in the following linear model:

$$Y = X\beta + \epsilon. \tag{1}$$

Here X is an $n \times p$ design matrix, Y is a vector of noisy observations, and ϵ is the noise term. Even in the noiseless case, recovering β (or its support) from (X, Y) seems impossible when $n \ll p$, given that we have more variables than observations.

A line of recent research shows that when β is sparse, that is, when it has a relatively small number of nonzero coefficients, it is possible to recover β from an underdetermined system of equations. In order to ensure reconstruction, the design matrix X needs to behave sufficiently nicely in a sense that it satisfies certain incoherence conditions. One notion of the incoherence which has been formulated in the sparse reconstruction literature (Candès and Tao, 2005, 2006, 2007) bears the name of Uniform Uncertainty Principle (UUP). It states that for all s-sparse sets T, the matrix X restricted to the columns from T acts as an almost isometry. Let X_T, where $T \subset \{1, \ldots, p\}$ be the

$n \times |T|$ submatrix obtained by extracting columns of X indexed by T. For each integer $s = 1, 2, \ldots$ such that $s < p$, the s-restricted isometry constant θ_s of X is the smallest quantity such that

$$(1 - \theta_s) \|c\|_2^2 \leq \|X_T c\|_2^2 / n \leq (1 + \theta_s) \|c\|_2^2, \tag{2}$$

for all $T \subset \{1, \ldots, p\}$ with $|T| \leq s$ and coefficients sequences $(c_j)_{j \in T}$. Throughout this paper, we refer to a vector $\beta \in \mathbb{R}^p$ with at most s non-zero entries, where $s \leq p$, as a s-**sparse** vector.

To understand the formulation of the UUP, consider the simplest noiseless case as mentioned earlier, where we assume $\epsilon = 0$ in (1). Given a set of values $(\langle X^i, \beta \rangle)_{i=1}^n$, where $X^1, X^2, \ldots, X^n$ are independent random vectors in $\mathbb{R}^p$, the basis pursuit program (Chen et al., 1998) finds $\widehat{\beta}$ which minimizes the ℓ_1-norm of β' among all β' satisfying $X\beta' = X\beta$, where X is a $n \times p$ matrix with rows $X^1, X^2, \ldots, X^n$. This can be cast as a linear program and thus is computationally efficient. Under variants of such conditions, the exact recovery or approximate reconstruction of a sparse β using the basis pursuit program has been shown in a series of powerful results (Donoho, 2006b; Candès et al., 2006; Candès and Tao, 2005, 2006; Donoho, 2006a; Rudelson and Vershynin, 2006, 2008; Candès and Tao, 2007). We refer to these papers for further references on earlier results for sparse recovery.

In other words, under the UUP, the design matrix X is taken as a $n \times p$ measurement ensemble through which one aims to recover both the unknown non-zero positions and the magnitude of a s-sparse signal β in $\mathbb{R}^p$ efficiently (thus the name for compressed sensing). Naturally, we wish n to be as small as possible for given values of p and s. It is well known that for random matrices, UUP holds for $s = O(n/\log(p/n))$ with i.i.d. Gaussian random entries, Bernoulli, and in general subgaussian entries (Candès and Tao, 2005; Rudelson and Vershynin, 2005; Candès and Tao, 2006; Donoho, 2006a; Baraniuk et al., 2008; Mendelson et al., 2008). Recently, it has been shown (Adamczak et al., 2009) that UUP holds for $s = O(n/\log^2(p/n))$ when X is a random matrix composed of columns that are independent isotropic vectors with log-concave densities. For a random Fourier ensemble, or randomly sampled rows of orthonormal matrices, it is shown that (Rudelson and Vershynin, 2006, 2008) the UUP holds for $s = O(n/\log^c p)$ for $c = 4$, which improves upon the earlier result of Candès and Tao (2006) where $c = 6$. To be able to prove UUP for random measurements or design matrix, the isotropicity condition (cf. Definition 5) has been assumed in all literature cited above. This assumption is not always reasonable in statistics and machine learning, where we often come across high dimensional data with correlated entries.

The work of Bickel et al. (2009) formulated the restricted eigenvalue (RE) condition and showed that it is among the weakest and hence the most general conditions in literature imposed on the Gram matrix in order to guarantee nice statistical properties for the Lasso estimator (Tibshirani, 1996) as well as the Dantzig selector (Candès and Tao, 2007). In particular, it is shown to be a relaxation of the UUP under suitable choices of parameters involved in each condition; see Bickel et al. (2009). We now state one version of the *Restricted Eigenvalue condition* as formulated in (Bickel et al., 2009). For some integer $0 < s_0 < p$ and a positive number k_0, $\mathsf{RE}(s_0, k_0, X)$ for matrix X requires that the following holds:

$$\forall \upsilon \neq 0, \quad \min_{\substack{J \subset \{1,\ldots,p\},\ \|\upsilon_{J^c}\|_1 \leq k_0 \|\upsilon_J\|_1 \\ |J| \leq s_0}} \frac{\|X\upsilon\|_2}{\|\upsilon_J\|_2} > 0, \tag{3}$$

where υ_J represents the subvector of $\upsilon \in \mathbb{R}^p$ confined to a subset J of $\{1, \ldots, p\}$. In the context of compressed sensing, RE condition can also be taken as a way to guarantee recovery for anisotropic

measurements. We refer to van de Geer and Buhlmann (2009) for other conditions which are closely related to the RE condition.

Consider now the linear regression model in (1). For a chosen penalization parameter $\lambda_n \geq 0$, regularized estimation with the ℓ_1-norm penalty, also known as the Lasso (Tibshirani, 1996) refers to the following convex optimization problem

$$\widehat{\beta} = \arg\min_{\beta} \frac{1}{2n}\|Y - X\beta\|_2^2 + \lambda_n \|\beta\|_1, \tag{4}$$

where the scaling factor $1/(2n)$ is chosen for convenience. Under i.i.d Gaussian noise and the RE condition, bounds on ℓ_2 prediction loss and on ℓ_q, $1 \leq q \leq 2$, loss for estimating the parameter β in (1) for both the Lasso and the Dantzig selector have all been derived in Bickel et al. (2009). For a given $\lambda_n \geq 0$, the Dantzig selector is defined as:

$$\text{(DS)} \quad \arg\min_{\widehat{\beta} \in \mathbb{R}^p} \left\|\widehat{\beta}\right\|_1 \quad \text{subject to} \quad \left\|\frac{1}{n} X^T (Y - X\widehat{\beta})\right\|_\infty \leq \lambda_n$$

In particular, ℓ_2 loss of $\Theta(\lambda\sigma\sqrt{s})$ were obtained for the Lasso under $\mathsf{RE}(s, 3, X)$ and the Dantzig selector under $\mathsf{RE}(s, 1, X)$ respectively in Bickel et al. (2009), where it is shown that $\mathsf{RE}(s, 1, X)$ condition is weaker than the UUP used in Candès and Tao (2007).

RE condition with parameters s_0 and k_0 was shown to hold for random Gaussian measurements / design matrix which consists of $n = O(s_0 \log p)$ independent copies of a p-dimensional Gaussian random vector Y with covariance matrix Σ in Raskutti et al. (2010), assuming that condition (3) holds for the square root of Σ. The matrix Σ is called the population covariance matrix in this context. As we show below, the bound $n = O(s_0 \log p)$ can be improved to the optimal one $n = O(s_0 \log(p/s_0))$ when $\mathsf{RE}(s_0, k_0, \Sigma^{1/2})$ is replaced with $\mathsf{RE}(s_0, (1+\varepsilon)k_0, \Sigma^{1/2})$ for any $\varepsilon > 0$. The work by Raskutti et al. (2010) has motivated the investigation for a non-iid subgaussian random design by Zhou (2009), as well as the present work. The proof of Raskutti et al. (2010) relies on a deep result from the theory of Gaussian random processes – Gordon's Minimax Lemma Gordon (1985). However, this result relies on the properties of the normal random variables, and is not available beyond the Gaussian setting. To establish the RE condition for more general classes of random matrices we had to introduce a new approach based on geometric functional analysis. We defer the comparison of the present paper with Zhou (2009) to Section 1.2. Both Zhou et al. (2009b) and van de Geer and Buhlmann (2009) obtained weaker (but ealier) results which are based on bounding the maximum entry-wise difference between sample and the population covariance matrices. We refer to Raskutti et al. (2010) for a more elaborate comparison.

1.1. Notation and definitions

Let $e_1, \ldots, e_p$ be the canonical basis of $\mathbb{R}^p$. For a set $J \subset \{1, \ldots, p\}$, denote $E_J = \text{span}\{e_j : j \in J\}$. For a matrix A, we use $\|A\|_2$ to denote its operator norm. For a set $V \subset \mathbb{R}^p$, we let conv V denote the convex hull of V. For a finite set Y, the cardinality is denoted by $|Y|$. Let B_2^p and S^{p-1} be the unit Euclidean ball and the unit sphere respectively. For a vector $u \in \mathbb{R}^p$, let T_0 denote the locations of the s_0 largest coefficients of u in absolute values, and u_{T_0} be the subvector of u confined to the locations of its s_0 largest coefficients in absolute values. In this paper, C, c, etc, denote various absolute constants which may change line by line. Occasionally, we use $u_T \in \mathbb{R}^{|T|}$,

where $T \subseteq \{1, \dots, p\}$, to also represent its 0-extended version $u' \in \mathbb{R}^p$ such that $u'_{T^c} = 0$ and $u'_T = u_T$.

We define $\mathcal{C}(s_0, k_0)$, where $0 < s_0 < p$ and k_0 is a positive number, as the set of vectors in $\mathbb{R}^p$ which satisfy the following cone constraint:

$$\mathcal{C}(s_0, k_0) = \{x \in \mathbb{R}^p \mid \exists I \in \{1, \dots, p\}, |I| = s_0 \text{ s.t. } \|x_{I^c}\|_1 \le k_0 \|x_I\|_1\}. \tag{5}$$

Let β be a s-sparse vector and $\widehat{\beta}$ be the solution from either the Lasso or the Dantzig selector. One of the common properties of the Lasso and the Dantzig selector is: for an appropriately chosen λ_n and under i.i.d. Gaussian noise, the condition

$$\upsilon := \widehat{\beta} - \beta \in \mathcal{C}(s, k_0) \tag{6}$$

holds with high probability. Here $k_0 = 1$ for the Dantzig selector, and $k_0 = 3$ for the Lasso; see Bickel et al. (2009) and Candès and Tao (2007) for example. The combination of the cone property (6) and the RE condition leads to various nice convergence results as stated earlier.

We now define some parameters related to the RE and sparse eigenvalue conditions.

Definition 1 *Let $1 \le s_0 \le p$, and let k_0 be a positive number. We say that a $q \times p$ matrix A satisfies $\mathsf{RE}(s_0, k_0, A)$ condition with parameter $K(s_0, k_0, A)$ if for any $\upsilon \ne 0$,*

$$\frac{1}{K(s_0, k_0, A)} := \min_{\substack{J \subseteq \{1, \dots, p\}, \\ |J| \le s_0}} \min_{\|\upsilon_{J^c}\|_1 \le k_0 \|\upsilon_J\|_1} \frac{\|A\upsilon\|_2}{\|\upsilon_J\|_2} > 0. \tag{7}$$

It is clear that when s_0 and k_0 become smaller, this condition is easier to satisfy.

Definition 2 *For $m \le p$, we define the largest and smallest m-sparse eigenvalue of a $q \times p$ matrix A to be*

$$\rho_{\max}(m, A) := \max_{t \in \mathbb{R}^p, t \ne 0; m-sparse} \|At\|_2^2 / \|t\|_2^2, \tag{8}$$

$$\rho_{\min}(m, A) := \min_{t \in \mathbb{R}^p, t \ne 0; m-sparse} \|At\|_2^2 / \|t\|_2^2. \tag{9}$$

1.2. Main results

The main purpose of this paper is to show that the RE condition holds with high probability for systems of random measurements/random design matrices of a general nature. To establish such result with high probability, one has to assume that it holds in average. So, our problem boils down to showing that, under some assumptions on random variables, the RE condition on the covariance matrix implies a similar condition on a random design matrix with high probability when n is sufficiently large (cf. Theorems 6 and Theorem 8). This generalizes the results on UUP mentioned above, where the covariance matrix is assumed to be identity. Denote by A a fixed $q \times p$ matrix. We consider the design matrix

$$X = \Psi A, \tag{10}$$

where the rows of the matrix Ψ are isotropic random vectors. An example of such a random matrix X consists of independent rows, each being a random vector in $\mathbb{R}^p$ that follows a multivariate normal

distribution $N(0, \Sigma)$, when we take $A = \Sigma^{1/2}$ in (10). Our first main result is related to this setup. We consider a matrix represented as $\tilde{X} = \tilde{\Psi}A$, where the matrix A satisfies the RE condition. The result is purely geometric, so we consider a *deterministic* matrix $\tilde{\Psi}$.

We prove a general reduction principle showing that if the matrix $\tilde{\Psi}$ acts as almost isometry on the images of the sparse vectors under A, then the product $\tilde{\Psi}A$ satisfies the RE condition with a smaller parameter k_0. More precisely, we prove Theorem 3.

Theorem 3 *Let $1/5 > \delta > 0$. Let $0 < s_0 < p$ and $k_0 > 0$. Let A be a $q \times p$ matrix such that $\mathsf{RE}(s_0, 3k_0, A)$ holds for $0 < K(s_0, 3k_0, A) < \infty$. Set*

$$d \;=\; s_0 + s_0 \max_j \|Ae_j\|_2^2 \, \frac{16K^2(s_0, 3k_0, A)(3k_0)^2(3k_0+1)}{\delta^2}, \tag{11}$$

and let $E = \cup_{|J|=d} E_J$ for $d < p$ and E denotes $\mathbb{R}^p$ otherwise. Let $\tilde{\Psi}$ be a matrix such that

$$\forall x \in AE \quad (1-\delta)\|x\|_2 \le \left\|\tilde{\Psi}x\right\|_2 \le (1+\delta)\|x\|_2 . \tag{12}$$

Then $\mathsf{RE}(s_0, k_0, \tilde{\Psi}A)$ condition holds with $0 < K(s_0, k_0, \tilde{\Psi}A) \le K(s_0, k_0, A)/(1-5\delta)$.

Remark 4 *We note that this result does not involve $\rho_{\max}(s_0, A)$, nor the global parameters of the matrices A and $\tilde{\Psi}$, such as the norm or the smallest singular value. We refer to Raskutti et al. (2010) for an example of matrix A satisfying the RE condition, such that $\rho_{\max}(s_0, A)$ grows linearly with s_0 while the maximum of $\|Ae_j\|_2$ is bounded above.*

The assumption $\mathsf{RE}(s_0, 3k_0, A)$ can be replaced by $\mathsf{RE}(s_0, (1+\varepsilon)k_0, A)$ for any $\varepsilon > 0$ by appropriately increasing d. See Remark 14 for details.

We apply the reduction principle to analyze different classes of random design matrices. This analysis is reduced to checking that the almost isometry property holds for all vectors from some low-dimensional subspaces, which is easier than checking the RE property directly. The first example is the matrix Ψ whose rows are independent isotropic vectors with *subgaussian* marginals as in Definition 5. This result extends a theorem of Raskutti et al. (2010) to a non-Gaussian setting, in which the entries of the design matrix may even not have a density.

Definition 5 *Let Y be a random vector in $\mathbb{R}^p$*

1. *Y is called isotropic if for every $y \in \mathbb{R}^p$, $\mathbb{E}\,|\langle Y, y\rangle|^2 = \|y\|_2^2$.*
2. *Y is ψ_2 with a constant α if for every $y \in \mathbb{R}^p$,*

$$\|\langle Y, y\rangle\|_{\psi_2} := \inf\{t : \mathbb{E}\exp(\langle Y, y\rangle^2/t^2) \le 2\} \;\le\; \alpha\|y\|_2 . \tag{13}$$

The ψ_2 condition on a scalar random variable V is equivalent to the subgaussian tail decay of V, which means for some constant c,

$$\mathbb{P}(|V| > t) \le 2\exp(-t^2/c^2), \quad \text{for all } \; t > 0.$$

We use ψ_2, vector with subgaussian marginals and subgaussian vector interchangeably. Examples of isotropic random vectors with subgaussian marginals are:

- The random vector Y with i.i.d $N(0,1)$ random coordinates.
- Discrete Gaussian vector, which is a random vector taking values on the integer lattice $\mathbb{Z}^p$ with distribution $\mathbb{P}(X = m) = C\exp(-\|m\|_2^2/2)$ for $m \in \mathbb{Z}^p$.
- A vector with independent centered bounded random coordinates. The subgaussian property here follows from the Hoeffding inequality for sums of independent random variables. This example includes, in particular, vectors with random symmetric Bernoulli coordinates, in other words, random vertices of the discrete cube.

It is hard to argue that such multivariate Gaussian or Bernoulli random designs are not relevant for statistical applications.

Theorem 6 *Set $0 < \delta < 1$, $k_0 > 0$, and $0 < s_0 < p$. Let A be a $q \times p$ matrix satisfying $\mathsf{RE}(s_0, 3k_0, A)$ condition as in Definition 1. Let d be as defined in (11), and let $m = \min(d, p)$. Let Ψ be an $n \times q$ matrix whose rows are independent isotropic ψ_2 random vectors in $\mathbb{R}^q$ with constant α. Suppose the sample size satisfies*

$$n \geq \frac{2000 m\alpha^4}{\delta^2} \log\left(\frac{60ep}{m\delta}\right). \tag{14}$$

Then with probability at least $1 - 2\exp(-\delta^2 n/2000\alpha^4)$, $\mathsf{RE}(s_0, k_0, (1/\sqrt{n})\Psi A)$ condition holds for matrix $(1/\sqrt{n})\Psi A$ with

$$0 < K(s_0, k_0, (1/\sqrt{n})\Psi A) \leq \frac{K(s_0, k_0, A)}{1-\delta}. \tag{15}$$

Remark 7 *We note that all constants in Theorem 6 are explicit, although they are not optimized.*

The reconstruction of sparse signals by subgaussian design matrices was analyzed in Mendelson et al. (2008) and Baraniuk et al. (2008). Note however that both papers used the RIP assumptions and estimate the deviation of the restricted operator from identity. These methods are not applicable in our contexts since the matrix A may be far from identity.

Theorem 6 is applicable in various contexts. We describe two examples. The first example concerns cases which have been considered in Raskutti et al. (2010); Zhou (2009). They show that the RE condition on the covariance matrix Σ implies a similar condition on a random design matrix $X = \Psi\Sigma^{1/2}$ with high probability when n is sufficiently large. In particular, in Zhou (2009), the author considered subgaussian random matrices of the form $X = \Psi\Sigma^{1/2}$ where Σ is a $p \times p$ positive semidefinite matrix satisfying $\mathsf{RE}(s_0, k_0, \Sigma^{1/2})$ condition, and Ψ is as in Theorem 6. Unlike the current paper, the author allowed $\rho_{\max}(s_0, \Sigma^{1/2})$ as well as $K^2(s_0, k_0, \Sigma^{1/2})$ to appear in the lower bound on n, and showed that $X/\sqrt{n}$ satisfies the RE condition as in (15) with overwhelming probability whenever

$$n > \frac{9c'\alpha^4}{\delta^2}(2+k_0)^2 K^2(s_0, k_0, \Sigma^{1/2}) \min(4\rho_{\max}(s_0, \Sigma^{1/2}) s_0 \log(5ep/s_0), s_0 \log p) \tag{16}$$

where the first term was given in Zhou (2009, Theorem 1.6) explicitly, and the second term is an easy consequence by combining arguments in Zhou (2009) and Raskutti et al. (2010). Analysis there used Corollary 2.7 in Mendelson et al. (2007) crucially. In the present work, we get rid of the dependency

of the sample size on $\rho_{\max}(s_0, \Sigma^{1/2})$, although under a slightly stronger $\mathsf{RE}(s_0, 3k_0, \Sigma^{1/2})$ (See Remarks 4 and 14). More precisely, let Σ be a $p \times p$ covariance matrix satisfying $\mathsf{RE}(s_0, 3k_0, \Sigma^{1/2})$ condition. Then, (15) implies that with probability at least $1 - 2\exp(-\delta^2 n/2000\alpha^4)$,

$$0 < K(s_0, k_0, (1/\sqrt{n})\Psi\Sigma^{1/2}) \le \frac{K(s_0, k_0, \Sigma^{1/2})}{1-\delta} \tag{17}$$

where n satisfies (14) for d defined in (11), with A replaced by $\Sigma^{1/2}$. In particular, bounds developed in the present paper can be applied to obtain tight convergence results for covariance estimation for a multivariate Gaussian model Zhou et al. (2011).

Another application of Theorem 6 is given in Zhou et al. (2009a). The $q \times p$ matrix A can be taken as a data matrix with p attributes (e.g., weight, height, age, etc), and q individual records. The data are compressed by a random linear transformation $X = \Psi A$. Such transformations have have been called "matrix masking" in the privacy literature (Duncan and Pearson, 1991). We think of X as "public," while Ψ, which is a $n \times q$ random matrix, is private and only needed at the time of compression. However, even with Ψ known, recovering A from Ψ requires solving a highly underdetermined linear system and comes with information theoretic privacy guarantees when $n \ll q$, as demonstrated in Zhou et al. (2009a). On the other hand, sparse recovery using X is highly feasible given that the RE conditions are guaranteed to hold by Theorem 6 with a small n. We refer to Zhou et al. (2009a) for a detailed setup on regression using compressed data as in (10).

The second application of the reduction principle is to the design matrices with uniformly bounded entries. As we mentioned above, if the entries of such matrix are independent, then its rows are subgaussian. However, the independence of entries is not assumed, so the decay of the marginals can be arbitrary slow. Indeed, if all coordinates of the vector equal to the same symmetric Bernoulli random variable, then the maximal ψ_2-norm of the marginals is of the order $\sqrt{p}$.

A natural example for compressed sensing would be measurements of random Fourier coefficients, when some of the coefficients cannot be measured.

Theorem 8 *Let $0 < \delta < 1$ and $0 < s_0 < p$. Let $Y \in \mathbb{R}^p$ be a random vector such that $\|Y\|_\infty \le M$ a.s and denote $\Sigma = \mathbb{E}YY^T$. Let X be an $n \times p$ matrix, whose rows $X_1, \dots, X_n$ are independent copies of Y. Let Σ satisfy the $\mathsf{RE}(s_0, 3k_0, \Sigma^{1/2})$ condition as in Definition 1. Let d be as defined in (11), where we replace A with $\Sigma^{1/2}$. Assume that $d \le p$ and $\rho = \rho_{\min}(d, \Sigma^{1/2}) > 0$. Suppose the sample size satisfies for some absolute constant C*

$$n \ge \frac{CM^2 d \cdot \log p}{\rho\delta^2} \cdot \log^3\left(\frac{CM^2 d \cdot \log p}{\rho\delta^2}\right).$$

Then with probability at least $1 - \exp\left(-\delta\rho n/(6M^2 d)\right)$, $\mathsf{RE}(s_0, k_0, X)$ condition holds for matrix $X/\sqrt{n}$ with $0 < K(s_0, k_0, X/\sqrt{n})) \le K(s_0, k_0, \Sigma^{1/2})/(1-\delta)$.

Remark 9 *Note that unlike the case of a random matrix with subgaussian marginals, the estimate of Theorem 8 contains the minimal sparse singular value ρ. We will provide an example illustrating that this is necessary in Remark 20.*

We will prove Theorems 3, 6, and 8 in Sections 2, 3, and 4 respectively.

We note that the reduction principle can be applied to other types of random variables. One can consider the case of heavy-tailed marginals. In this case the estimate for the images of sparse vectors

can be proved using the technique developed by Vershynin (2011a,b). One can also consider random vectors with log-concave densities, and obtain similar estimates following the methods of Adamczak et al. (2009, 2011).

To make our exposition complete, we will show some immediate consequences in terms of statistical inference on high dimensional data that satisfy such RE and sparse eigenvalue conditions. As mentioned, the restricted eigenvalue (RE) condition as formulated by Bickel et al. (2009) are among the weakest and hence the most general conditions in literature imposed on the Gram matrix in order to guarantee nice statistical properties for the Lasso and the Dantzig selector. For random design as considered in the present paper, one can show that various oracle inequalities in terms of ℓ_2 convergence hold for the Lasso and the Dantzig selector as long as n satisfies the lower bounds above. Let $s = |\mathrm{supp}\ \beta|$ for β in (1). Under $\mathsf{RE}(s, 9, \Sigma^{1/2})$, a sample size of $n = O(s \log(p/s))$ is sufficient for us to derive bounds corresponding to those in Bickel et al. (2009, Theorem 7.2). As a consequence, we see that this setup requires only $\Theta(\log(p/s))$ observations per nonzero value in β where Θ hides a constant depending on $K^2(s, 9, \Sigma^{1/2})$ for the family of random matrices with subgaussian marginals which satisfies $\mathsf{RE}(s, 9, \Sigma^{1/2})$ condition. Similarly, we note that for random matrix X with a.s. bounded entries of size M, $n = O(sM^2 \log p \log^3(s \log p))$ samples are sufficient in order to achieve accurate statistical estimation. We say this is a *linear or sublinear sparsity*. For $p \gg n$, this is a desirable property as it implies that accurate statistical estimation is feasible given a very limited amount of data.

2. Reduction principle

We first reformulate the reduction principle in the form of restrictive isometry: we show that if the matrix $\tilde{\Psi}$ acts as almost isometry on the images of the sparse vectors under A, then it acts the same way on the images of a set of vectors which satisfy the cone constraint (5). We then prove Theorem 3 as a corollary of Theorem 10 in Section A. The proof of Theorem 10 itself uses several auxiliary results, which will be established in the next two subsections.

Theorem 10 *Let $1/5 > \delta > 0$. Let $0 < s_0 < p$ and $k_0 > 0$. Let A be a $q \times p$ matrix such that $\mathsf{RE}(s_0, 3k_0, A)$ condition holds for $0 < K(s_0, 3k_0, A) < \infty$. Set*

$$d = s_0 + s_0 \max_j \|Ae_j\|_2^2 \left(\frac{16K^2(s_0, 3k_0, A)(3k_0)^2(3k_0 + 1)}{\delta^2} \right),$$

and let $E = \cup_{|J|=d} E_J$ for $d < p$ and $E = \mathbb{R}^p$ otherwise. Let $\tilde{\Psi}$ be a matrix such that

$$\forall x \in AE \quad (1 - \delta)\|x\|_2 \le \left\|\tilde{\Psi}x\right\|_2 \le (1 + \delta)\|x\|_2 . \tag{18}$$

Then for any $x \in A\Big(\mathcal{C}(s_0, k_0)\Big) \cap S^{q-1}$,

$$(1 - 5\delta) \le \left\|\tilde{\Psi}x\right\|_2 \le (1 + 3\delta) \tag{19}$$

2.1. Preliminary results

Our first lemma is based on Maurey's empirical approximation argument Pisier (1981). We show that any vector belonging to the convex hull of many vectors can be approximated by a convex combination of a few of them.

Lemma 11 *Let $u_1, \ldots, u_M \in \mathbb{R}^q$. Let $y \in \operatorname{conv}(u_1, \ldots, u_M)$. There exists a set $L \subset \{1, 2, \ldots, M\}$ such that*

$$|L| \leq m = \frac{4 \max_{j \in \{1,\ldots,M\}} \|u_j\|_2^2}{\varepsilon^2}$$

and a vector $y' \in \operatorname{conv}(u_j, j \in L)$ such that

$$\left\|y' - y\right\|_2 \leq \varepsilon.$$

For each vector $x \in \mathbb{R}^p$, let T_0 denote the locations of the s_0 largest coefficients of x in absolute values. Any vector $x \in \mathcal{C}(s_0, k_0) \cap S^{p-1}$ satisfies:

$$\left\|x_{T_0^c}\right\|_\infty \leq \|x_{T_0}\|_1 / s_0 \quad \leq \quad \frac{\|x_{T_0}\|_2}{\sqrt{s_0}} \tag{20}$$

$$\left\|x_{T_0^c}\right\|_1 \leq k_0 \sqrt{s_0}\, \|x_{T_0}\|_2 \quad \leq \quad k_0 \sqrt{s_0}; \text{ and } \left\|x_{T_0^c}\right\|_2 \leq 1. \tag{21}$$

The next elementary estimate will be used in conjunction with the RE condition.

Lemma 12 *For each vector $v \in \mathcal{C}(s_0, k_0)$, let T_0 denotes the locations of the s_0 largest coefficients of v in absolute values. Then*

$$\|v_{T_0}\|_2 \geq \frac{\|v\|_2}{\sqrt{1 + k_0}}. \tag{22}$$

We prove lemma 11 and 12 in Section B and Section C respectively.

2.2. Convex hull of sparse vectors

For a set $J \subset \{1, \ldots, p\}$, denote $E_J = \operatorname{span}\{e_j : j \in J\}$. In order to prove the restricted isometry property of Ψ over the set of vectors in $A\Big(\mathcal{C}(s_0, k_0)\Big) \cap S^{q-1}$, we first show that this set is contained in the convex hull of the images of the sparse vectors with norms not exceeding $(1 - \delta)^{-1}$. More precisely, we state the following lemma, the proof of which appears in Section D.

Lemma 13 *Let $1 > \delta > 0$. Let $0 < s_0 < p$ and $k_0 > 0$. Let A be a $q \times p$ matrix such that* $\mathsf{RE}(s_0, k_0, A)$ *condition holds for $0 < K(s_0, k_0, A) < \infty$. Define*

$$d = d(k_0, A) = s_0 + s_0 \max_j \|Ae_j\|_2^2 \left(\frac{16 K^2(s_0, k_0, A) k_0^2 (k_0 + 1)}{\delta^2}\right). \tag{23}$$

Then

$$A\Big(\mathcal{C}(s_0, k_0)\Big) \cap S^{q-1} \subset (1 - \delta)^{-1} \operatorname{conv}\left(\bigcup_{|J| \leq d} AE_J \cap S^{q-1}\right) \tag{24}$$

where for $d \geq p$, E_J is understood to be $\mathbb{R}^p$.

2.3. Proof of the reduction principle

To prove the restricted isomorphism condition (19), we apply Lemma 13 with k_0 being replaced by $3k_0$. The upper bound in (19) follows immediately from the lemma. To prove the lower bound, we consider a vector $x \in \mathcal{C}(s_0, k_0)$ as an endpoint of an interval, whose midpoint is a sparse vector from the same cone. Then the other endpoint of the interval will be contained in the larger cone $\mathcal{C}(s_0, 3k_0)$. Comparison between the upper estimate for the norm of the image of this endpoint with the lower estimate for the midpoint will yield the required lower estimate for the point x. The complete proof appears in Section E.

Remark 14 *Let $\varepsilon > 0$. Instead of v defined in (44), one can consider the vector*

$$v_\varepsilon = x_I + y - \varepsilon(x - y) \in \mathcal{C}\big(s_0, (1+\varepsilon)k_0\big).$$

Then replacing v by v_ε throughout the proof, we can establish Theorem 10 under the assumption $\mathsf{RE}(s_0, (1+\varepsilon)k_0, A)$ instead of $\mathsf{RE}(s_0, 3k_0, A)$, if we increase the dimension $d(3k_0)$ by a factor depending on ε.

3. Subgaussian random design

Theorem 6 can be reformulated as an almost isometry condition for the matrix $X = \Psi A$ acting on the set $\mathcal{C}(s_0, k_0)$. Recall that

$$d(3k_0, A) = s_0 + s_0 \max_j \|Ae_j\|_2^2 \left(\frac{16K^2(s_0, 3k_0, A)(3k_0)^2(3k_0+1)}{\delta^2} \right).$$

Theorem 15 *Set $0 < \delta < 1$, $0 < s_0 < p$, and $k_0 > 0$. Let A be a $q \times p$ matrix satisfying $\mathsf{RE}(s_0, 3k_0, A)$ condition as in Definition 1. Let $m = \min(d(3k_0, A), p) < p$. Let Ψ be an $n \times q$ matrix whose rows are independent isotropic ψ_2 random vectors in $\mathbb{R}^q$ with constant α. Assume that the sample size satisfies*

$$n \geq \frac{2000 m \alpha^4}{\delta^2} \log\left(\frac{60ep}{m\delta}\right). \tag{25}$$

Then with probability at least $1 - 2\exp(\delta^2 n/2000\alpha^4)$, for all $v \in \mathcal{C}(s_0, k_0)$ such that $v \neq 0$,

$$1 - \delta \;\leq\; \frac{1}{\sqrt{n}} \frac{\|\Psi A v\|_2}{\|Av\|_2} \;\leq\; 1 + \delta. \tag{26}$$

Theorem 6 follows immediately from Theorem 15. Indeed, by (26),

$$\forall u \in \mathcal{C}(s_0, k_0) \;\text{ s.t. }\; u \neq 0 \quad \left\| \frac{1}{\sqrt{n}} \Psi A u \right\|_2 \geq (1-\delta)\|Au\|_2 \geq (1-\delta)\frac{\|u_{T_0}\|_2}{K(s_0, k_0, A)} > 0.$$

To derive Theorem 15 from Theorem 10 we need a lower estimate for the norm of the image of a sparse vector. Such estimate relies on the standard ε-net argument similarly to Mendelson et al. (2008, Section 3). A complete proof of Theorem 15 appears in Section F.

Theorem 16 *Set $0 < \delta < 1$. Let A be a $q \times p$ matrix, and let Ψ be an $n \times q$, matrix whose rows are independent isotropic ψ_2 random vectors in $\mathbb{R}^q$ with constant α. For $m \leq p$, assume that*

$$n \geq \frac{80m\alpha^4}{\tau^2} \log\left(\frac{12ep}{m\tau}\right). \tag{27}$$

Then with probability at least $1 - 2\exp(-\tau^2 n/80\alpha^4)$, for all m-sparse vectors u in $\mathbb{R}^p$,

$$(1-\tau)\|Au\|_2 \;\leq\; \frac{1}{\sqrt{n}}\|\Psi Au\|_2 \;\leq\; (1+\tau)\|Au\|_2. \tag{28}$$

We note that Theorem 16 does not require the RE condition to hold. No particular upper bound on $\rho_{\max}(m, A)$ is imposed here either. Proof of Theorem 16 appears in Section G.

We now state a large deviation bound for m-sparse eigenvalues $\rho_{\min}(m, \tilde{X})$ and $\rho_{\max}(m, \tilde{X})$ for random design $\tilde{X} = n^{-1/2}\Psi A$ which follows from Theorem 16 directly.

Corollary 17 *Suppose conditions in Theorem 16 hold. Then with probability $\geq 1-2\exp(-\tau^2 n/80\alpha^4)$,*

$$(1-\tau)\sqrt{\rho_{\min}(m,A)} \leq \sqrt{\rho_{\min}(m,\tilde{X})} \leq \sqrt{\rho_{\max}(m,\tilde{X})} \leq (1+\tau)\sqrt{\rho_{\max}(m,A)}. \tag{29}$$

4. RE condition for random matrices with bounded entries

We next consider the case of design matrix X consisting of independent identically distributed rows with bounded entries. As in the previous section, we reformulate Theorem 8 in the form of an almost isometry condition.

Theorem 18 *Let $0 < \delta < 1$ and $0 < s_0 < p$. Let $Y \in \mathbb{R}^p$ be a random vector such that $\|Y\|_\infty \leq M$ a.s., and denote $\Sigma = \mathbb{E}YY^T$. Let X be an $n \times p$ matrix, whose rows $X_1, \ldots, X_n$ are independent copies of Y. Let Σ satisfy the $\mathsf{RE}(s_0, 3k_0, \Sigma^{1/2})$ condition as in Definition 1. Set*

$$d = d(3k_0, \Sigma^{1/2}) = s_0 + s_0 \max_j \left\|\Sigma^{1/2} e_j\right\|_2^2 \left(\frac{16K^2(s_0, 3k_0, \Sigma^{1/2})(3k_0)^2(3k_0+1)}{\delta^2}\right).$$

Assume that $d \leq p$ and $\rho = \rho_{\min}(d, \Sigma^{1/2}) > 0$. If for some absolute constant C

$$n \geq \frac{CM^2 d \cdot \log p}{\rho\delta^2} \cdot \log^3\left(\frac{CM^2 d \cdot \log p}{\rho\delta^2}\right),$$

then with probability at least $1 - \exp\left(-\delta\rho n/(6M^2 d)\right)$ all vectors $u \in \mathcal{C}(s_0, k_0)$ satisfy

$$(1-\delta)\|u\|_2 \leq \frac{\|Xu\|_2}{\sqrt{n}} \leq (1+\delta)\|u\|_2.$$

Similarly to Theorem 15, Theorem 18 can be derived from Theorem 10, and the corresponding bound for d-sparse vector, the proof of which appears in Section H.

Theorem 19 *Let $Y \in \mathbb{R}^p$ be a random vector such that $\|Y\|_\infty \le M$ a.s., and denote $\Sigma = \mathbb{E}YY^T$. Let X be an $n \times p$ matrix, whose rows $X_1, \dots, X_n$ are independent copies of Y. Let $0 < m \le p$. If $\rho = \rho_{\min}(m, \Sigma^{1/2}) > 0$ and*

$$n \ge \frac{CM^2 m \cdot \log p}{\rho \delta^2} \cdot \log^3 \left(\frac{CM^2 m \cdot \log p}{\rho \delta^2} \right), \tag{30}$$

then with probability at least $1 - 2\exp\left(-\frac{\varepsilon \rho n}{6M^2 m}\right)$ all m-sparse vectors u satisfy

$$1 - \delta \le \frac{1}{\sqrt{n}} \cdot \left\| \frac{Xu}{\left\|\Sigma^{1/2} u\right\|_2} \right\|_2 \le 1 + \delta.$$

To prove Theorem 19 we consider random variables $Z_u = \|Xu\|_2 / (\sqrt{n} \left\|\Sigma^{1/2} u\right\|_2) - 1$, and estimate the expectation of the supremum of Z_u over the set of sparse vectors using Dudley's entropy integral. The proof of this part closely follows Rudelson and Vershynin (2008), so we will only sketch it. To derive the large deviation estimate from the bound on the expectation we use Talagrand's measure concentration theorem for empirical processes Talagrand (1996), which provides a sharper estimate, than the method used in Rudelson and Vershynin (2008).

Remark 20 *Note that unlike the case of a random matrix with subgaussian marginals, the estimate of Theorem 19 contains the minimal sparse singular value ρ. This is, however, necessary, as the following example shows.*

Let $m = 2^l$, and assume that $p = k \cdot m$, for some $k \in \mathbb{N}$. For $j = 1, \dots, k$ let D_j be the $m \times m$ Walsh matrix. Let A be a $p \times p$ block-diagonal matrix with blocks $D_1, \dots, D_k$ on the diagonal, and let $Y \in \mathbb{R}^p$ be a random vector, whose values are the rows of the matrix A taken with probabilities $1/p$. Then $\|Y\|_\infty = 1$ and $\mathbb{E}YY^T = (m/p) \cdot id$, so $\rho = m/p$. Hence, the right-hand side of (30) reduces to

$$\frac{Cp \cdot \log p}{\delta^2} \cdot \log^3 \left(\frac{Cp \cdot \log p}{\delta^2} \right)$$

From the other side, if the matrix X satisfies the conditions of Theorem 19 with, say, $\delta = 1/2$, then all rows of the matrix A should be present among the rows of the matrix X. An elementary calculation shows that in this case it is necessary to assume that $n \ge Cp \log p$, so the estimate (30) is exact up to a power of the logarithm.

Unlike the matrix Σ, the matrix A is not symmetric. However, the example above can be easily modified by considering a $2p \times 2p$ matrix

$$\tilde{A} = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}.$$

This shows that the estimate (30) is tight under the symmetry assumption as well.

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Appendix A. Proof of Theorem 3

Theorem 3 By the $\mathsf{RE}(s_0, 3k_0, A)$ condition, $\mathsf{RE}(s_0, k_0, A)$ condition holds as well. Hence for $u \in \mathcal{C}(s_0, k_0)$ such that $u \neq 0$,

$$\|Au\|_2 \geq \frac{\|u_{T_0}\|_2}{K(s_0, k_0, A)} > 0,$$

and by (19)

$$\left\|\tilde{\Psi} Au\right\|_2 \geq (1 - 5\delta) \|Au\|_2 \geq (1 - 5\delta) \frac{\|u_{T_0}\|_2}{K(s_0, k_0, A)} > 0.$$

□

Appendix B. Proof of Lemma 11

Lemma 11 Assume that

$$y = \sum_{j \in \{1,\dots,M\}} \alpha_j u_j \quad \text{where} \quad \alpha_j \geq 0, \text{ and } \sum_j \alpha_j = 1.$$

Let Y be a random vector in $\mathbb{R}^q$ such that

$$\mathbb{P}(Y = u_\ell) = \alpha_\ell, \ \ell \in \{1, \dots, M\}$$

Then

$$\mathbb{E}Y = \sum_{\ell \in \{1,\dots,M\}} \alpha_\ell u_\ell = y.$$

Let $Y_1, \dots, Y_m$ be independent copies of Y and let $\varepsilon_1, \dots, \varepsilon_m$ be ± 1 i.i.d. mean zero Bernoulli random variables, chosen independently of $Y_1, \dots, Y_m$. By the standard symmetrization argument Ledoux and Talagrand (1991, Section 6.1), we have

$$\mathbb{E}\left\|y - \frac{1}{m}\sum_{j=1}^m Y_j\right\|_2^2 \leq 4\mathbb{E}\left\|\frac{1}{m}\sum_{j=1}^m \varepsilon_j Y_j\right\|_2^2 = \frac{4}{m^2}\sum_{j=1}^m \mathbb{E}\|Y_j\|_2^2 \leq \frac{4\max_{\ell \in \{1,\dots,M\}} \|u_\ell\|_2^2}{m} \leq \varepsilon^2 \quad (31)$$

where

$$\mathbb{E}\|Y_j\|_2^2 \leq \sup \|Y_j\|_2^2 \leq \max_{\ell \in \{1,\dots,M\}} \|u_\ell\|_2^2$$

and the last inequality in (31) follows from the definition of m.

Fix a realization $Y_j = u_{k_j}$, $j = 1, \dots, m$ for which

$$\left\|y - \frac{1}{m}\sum_{j=1}^m Y_j\right\|_2 \leq \varepsilon.$$

The vector $\frac{1}{m}\sum_{j=1}^m Y_j$ belongs to the convex hull of $\{u_\ell \ : \ \ell \in L\}$, where L is the set of different elements from the sequence $k_1, \dots, k_m$. Obviously $|L| \leq m$ and the lemma is proved. □

Appendix C. Proof of Lemma 12

Lemma 12 By definition of $\mathcal{C}(s_0, k_0)$, by (20)

$$\left\|\upsilon_{T_0^c}\right\|_2^2 \le \left\|\upsilon_{T_0^c}\right\|_1 \left\|\upsilon_{T_0^c}\right\|_\infty \le k_0 \left\|\upsilon_{T_0}\right\|_1 \cdot \left\|\upsilon_{T_0}\right\|_1 / s_0 \le k_0 \left\|\upsilon_{T_0}\right\|_2^2 .$$

Therefore $\|\upsilon\|_2^2 = \left\|\upsilon_{T_0^c}\right\|_2^2 + \|\upsilon_{T_0}\|_2^2 \le (k_0+1)\|\upsilon_{T_0}\|_2^2$. □

Appendix D. Proof of Lemma 13

. We first state the following lemma, which concerns the extremum of a linear functional on a big circle of a q-dimensional sphere. We consider a line passing through the extreme point, and show that the value of the functional on a point of the line, which is relatively close to the extreme point, provides a good bound for the extremum.

Lemma 21 *let $u, \theta, x \in \mathbb{R}^q$ be vectors such that*

1. $\|\theta\|_2 = 1$.
2. $\langle x, \theta \rangle \neq 0$.
3. *Vector u is not parallel to x.*

Define $\phi : \mathbb{R} \to \mathbb{R}$ by:

$$\phi(\lambda) = \frac{\langle x + \lambda u, \theta \rangle}{\|x + \lambda u\|_2}. \tag{32}$$

Assume $\phi(\lambda)$ has a local maximum at 0, then

$$\frac{\langle x + u, \theta \rangle}{\langle x, \theta \rangle} \ge 1 - \frac{\|u\|_2}{\|x\|_2}.$$

Proof of Lemma 13. Without loss of generality, assume that $d(k_0, A) < p$, otherwise the lemma is vacuously true. For each vector $x \in \mathbb{R}^p$, let T_0 denote the locations of the s_0 largest coefficients of x in absolute values. Decompose a vector $x \in \mathcal{C}(s_0, k_0) \cap S^{p-1}$ as

$$x = x_{T_0} + x_{T_0^c} \in x_{T_0} + k_0 \|x_{T_0}\|_1 \operatorname{absconv}(e_j \mid j \in T_0^c), \text{ where } \|x_{T_0}\|_2 \ge \frac{1}{\sqrt{k_0+1}} \text{ by (22)}$$

and hence

$$Ax \in Ax_{T_0} + k_0 \|x_{T_0}\|_1 \operatorname{absconv}(Ae_j \mid j \in T_0^c).$$

Since the set $A\mathcal{C}(s_0, k_0) \cap S^{q-1}$ is not easy to analyze, we introduce set of a simpler structure instead. Define

$$V = \left\{ x_{T_0} + k_0 \|x_{T_0}\|_1 \operatorname{absconv}(e_j \mid j \in T_0^c) | x \in \mathcal{C}(s_0, k_0) \cap S^{p-1} \right\}.$$

For a given $x \in \mathcal{C}(s_0, k_0) \cap S^{p-1}$, if T_0 is not uniquely defined, we include all possible sets of T_0 in the definition of V. Clearly $V \subset \mathcal{C}(s_0, k_0)$ is a compact set. Moreover, V contains a base of $\mathcal{C}(s_0, k_0)$, that is, for any $y \in \mathcal{C}(s_0, k_0) \setminus \{0\}$ there exists $\lambda > 0$ such that $\lambda y \in V$.

For any $v \in \mathbb{R}^p$ such that $\|Av\|_2 \neq 0$, define

$$F(v) = \frac{Av}{\|Av\|_2}.$$

By condition $\mathsf{RE}(s_0, k_0, A)$, the function F is well-defined and continuous on $\mathcal{C}(s_0, k_0) \setminus \{0\}$, and, in particular, on V. Hence,

$$A\mathcal{C}(s_0, k_0) \cap S^{q-1} = F\big(\mathcal{C}(s_0, k_0) \setminus \{0\}\big) = F(V).$$

By duality, inclusion (24) can be derived from the fact that the supremum of any linear functional over the left side of (24) does not exceed the supremum over the right side of it. By the equality above, it is enough to show that for any $\theta \in S^{q-1}$, there exists $z' \in \mathbb{R}^p \setminus \{0\}$ such that $|\operatorname{supp}(z')| \le d$ and $F(z')$ is well defined, which satisfies

$$\max_{v \in V} \langle F(v), \theta \rangle \quad \le \quad (1-\delta)^{-1} \langle F(z'), \theta \rangle. \tag{33}$$

For a given θ, we construct a d-sparse vector z' which satisfies (33). Let

$$z := \arg\max_{v \in V} \langle F(v), \theta \rangle.$$

By definition of V there exists $I \subset \{1, \dots, p\}$ such that $|I| = s_0$, and for some $\varepsilon_j \in \{1, -1\}$,

$$z = z_I + \|z_I\|_1 k_0 \sum_{j \in I^c} \alpha_j \varepsilon_j e_j, \text{ where } \alpha_j \in [0,1], \sum_{j \in I^c} \alpha_j \le 1, \text{ and } 1 \ge \|z_I\|_2 \ge \frac{1}{\sqrt{k_0 + 1}}. \tag{34}$$

Note if $\alpha_i = 1$ for some $i \in I^c$, then z is a sparse vector itself, and we can set $z' = z$ in order for (33) to hold. We proceed assuming $\alpha_i \in [0, 1)$ for all $i \in I^c$ in (34) from now on, in which case, we construct a required sparse vector z' via Lemma 11. To satisfy the assumptions of this lemma, denote $e_{p+1} = \mathbf{0}$, $\varepsilon_{p+1} = 1$ and set

$$\alpha_{p+1} = 1 - \sum_{j \in I^c} \alpha_j, \text{ hence } \alpha_{p+1} \in [0,1].$$

Let

$$y := Az_{I^c} = \|z_I\|_1 k_0 \sum_{j \in I^c} \alpha_j \varepsilon_j Ae_j = \|z_I\|_1 k_0 \sum_{j \in I^c \cup \{p+1\}} \alpha_j \varepsilon_j Ae_j$$

and denote $\mathcal{M} := \{j \in I^c \cup \{p+1\} : \alpha_j > 0\}$. Let $\varepsilon > 0$ be specified later. Applying Lemma 11 with vectors $u_j = k_0 \|z_I\|_1 \varepsilon_j Ae_j$ for $j \in \mathcal{M}$, construct a set $J' \subset \mathcal{M}$ satisfying

$$|J'| \le m := \frac{4 \max_{j \in I^c} k_0^2 \|z_I\|_1^2 \|Ae_j\|_2^2}{\varepsilon^2} \le \frac{4 k_0^2 s_0 \max_{j \in I^c} \|Ae_j\|_2^2}{\varepsilon^2} \tag{35}$$

and a vector

$$y' = k_0 \|z_I\|_1 \sum_{j \in J'} \beta_j \varepsilon_j Ae_j \text{ where for } J' \subset \mathcal{M}, \beta_j \in [0,1] \text{ and } \sum_{j \in J'} \beta_j = 1$$

such that $\|y' - y\|_2 \le \varepsilon$.

Set $u := k_0 \|z_I\|_1 \sum_{j \in J'} \beta_j \varepsilon_j e_j$ and let

$$z' \quad = \quad z_I + u.$$

By construction, $Az' \in AE_J$, where $J := (I \cup J') \cap \{1, \dots, p\}$ and

$$|J| \le |I| + |J'| \le s_0 + m. \tag{36}$$

Furthermore, we have

$$\big\|Az - Az'\big\|_2 = \|A(z_{I^c} - u)\|_2 = \big\|y - y'\big\|_2 \le \varepsilon$$

For $\{\beta_j, j \in J'\}$ as above, we extend it to $\{\beta_j, j \in I^c \cup \{p+1\}\}$ setting $\beta_j = 0$ for all $j \in I^c \cup \{p+1\} \setminus J'$ and write

$$z' = z_I + k_0 \|z_I\|_1 \sum_{j \in I^c \cup \{p+1\}} \beta_j \varepsilon_j e_j \ \text{ where } \ \beta_j \in [0,1] \ \text{ and } \sum_{j \in I^c \cup \{p+1\}} \beta_j = 1.$$

If $z' = z$, we are done. Otherwise, for some λ to be specified, consider the vector

$$z + \lambda(z' - z) = z_I + k_0 \|z_I\|_1 \sum_{j \in I^c \cup \{p+1\}} [(1-\lambda)\alpha_j + \lambda\beta_j] \, \varepsilon_j e_j.$$

We have $\sum_{j \in I^c \cup \{p+1\}} [(1-\lambda)\alpha_j + \lambda\beta_j] = 1$ and

$$\exists \, \delta_0 > 0 \ \text{ s. t. } \ \forall j \in I^c \cup \{p+1\}, \quad (1-\lambda)\alpha_j + \lambda\beta_j \in [0,1] \text{ if } |\lambda| < \delta_0.$$

To see this, we note that

- This condition holds by continuity for all j such that $\alpha_i \in (0,1)$.
- If $\alpha_j = 0$ for some j, then $\beta_j = 0$ by construction.

Thus $\sum_{j \in I^c} [(1-\lambda)\alpha_j + \lambda\beta_j] \le 1$ and $z + \lambda(z' - z) = z_I + k_0 \|z_I\|_1 \sum_{j \in I^c} [(1-\lambda)\alpha_j + \lambda\beta_j] \, \varepsilon_j e_j \in V$ whenever $|\lambda| < \delta_0$.

Consider now a function $\phi : (-\delta_0, \delta_0) \to \mathbb{R}$,

$$\phi(\lambda) := \langle \, F(z + \lambda(z' - z)), \theta \, \rangle = \frac{\langle \, Az + \lambda(Az' - Az), \theta \, \rangle}{\|Az + \lambda(Az' - Az)\|_2}$$

Since z maximizes $\langle \, F(v), \theta \, \rangle$ for all $v \in V$, $\phi(\lambda)$ attains the local maximum at 0. Then by Lemma 21, we have

$$\frac{\langle \, Az', \theta \, \rangle}{\langle \, Az, \theta \, \rangle} = \frac{\langle \, Az + (Az' - Az), \theta \, \rangle}{\langle \, Az, \theta \, \rangle} \ge 1 - \frac{\|(Az' - Az)\|_2}{\|Az\|_2} = \frac{\|Az\|_2 - \|(Az' - Az)\|_2}{\|Az\|_2}$$

hence

$$\begin{aligned}\frac{\langle F(z'),\theta\rangle}{\langle F(z),\theta\rangle} &= \frac{\langle Az'/\|Az'\|_2,\theta\rangle}{\langle Az/\|Az\|_2,\theta\rangle} = \frac{\|Az\|_2}{\|Az'\|_2}\times\frac{\langle Az',\theta\rangle}{\langle Az,\theta\rangle}\\ &\geq \frac{\|Az\|_2}{\|Az\|_2+\|(Az'-Az)\|_2}\times\frac{\|Az\|_2-\|(Az'-Az)\|_2}{\|Az\|_2}\\ &= \frac{\|Az\|_2-\|(Az'-Az)\|_2}{\|Az\|_2+\|(Az'-Az)\|_2}\\ &= \frac{\|Az\|_2-\varepsilon}{\|Az\|_2+\varepsilon} = 1-\frac{2\varepsilon}{\|Az\|_2+\varepsilon}.\end{aligned}$$

By definition, $z\in\mathcal{C}(s_0,k_0)$. Hence we apply $\mathsf{RE}(k_0,s_0,A)$ condition and (34) to obtain

$$\|Az\|_2 \geq \frac{\|z_I\|_2}{K(s_0,k_0,A)} \geq \frac{1}{\sqrt{1+k_0}K(s_0,k_0,A)}.$$

Now we can set $\varepsilon=\frac{\delta}{2\sqrt{1+k_0}K(s_0,k_0,A)}$ which yields

$$\frac{\langle F(z'),\theta\rangle}{\langle F(z),\theta\rangle}\geq 1-\delta \tag{37}$$

and thus (33) holds. Finally, by (35), we have

$$m\leq s_0\max_{j\in I^c}\|Ae_j\|_2^2\left(\frac{16K^2(s_0,k_0,A)k_0^2(k_0+1)}{\delta^2}\right)$$

and hence the inclusion (24) holds in view of (36) and (37). □

It remains to prove Lemma 21.

D.1. Proof of Lemma 21

Lemma 21 Let $v=\frac{x}{\|x\|_2}$. Also let

$$\begin{aligned}\theta &= \beta v+\gamma t,\ \text{where } t\perp v, \|t\|_2=1 \ \text{ and } \beta^2+\gamma^2=1, \beta\neq 0\\ \text{and } \ u &= \eta v+\mu t+s \text{ where } s\perp v \ \text{ and } s\perp t\end{aligned}$$

Define $f:\mathbb{R}\to\mathbb{R}$ by:

$$f(\lambda)=\frac{\lambda}{\|x\|_2+\lambda\eta},\ \lambda\neq-\frac{\eta}{\|x\|_2}. \tag{38}$$

Then

$$\begin{aligned}\phi(\lambda) &= \frac{\langle x+\lambda u,\theta\rangle}{\|x+\lambda u\|_2} = \frac{\langle(\|x\|_2+\lambda\eta)v+\lambda\mu t+\lambda s,\beta v+\gamma t\rangle}{\|(\|x\|_2+\lambda\eta)v+\lambda\mu t+\lambda s\|_2}\\ &= \frac{\beta(\|x\|_2+\lambda\eta)+\lambda\mu\gamma}{\sqrt{(\|x\|_2+\lambda\eta)^2+(\lambda\mu)^2+\lambda^2\|s\|_2^2}}\\ &= \frac{\beta+\mu\gamma f(\lambda)}{\sqrt{1+(\mu^2+\|s\|_2^2)f^2(\lambda)}}\end{aligned}$$

Since $f(\lambda) = \frac{\lambda}{\|x\|_2} + O(\lambda^2)$ we have $\phi(\lambda) = \beta + \mu\gamma\frac{\lambda}{\|x\|_2} + O(\lambda^2)$ in the neighborhood of 0, Hence, in order to for $\phi(\lambda)$ to have a local maximum at 0, μ or γ must be 0. Consider these cases separately.

- First suppose $\gamma = 0$, then $\beta^2 = 1$ and $|\langle x, \theta \rangle| = \|x\|_2$. Hence,

$$\frac{\langle x+u, \theta \rangle}{\langle x, \theta \rangle} = 1 + \frac{\langle u, \theta \rangle}{\langle x, \theta \rangle} \geq 1 - \frac{|\langle u, \theta \rangle|}{|\langle x, \theta \rangle|} \geq 1 - \frac{\|u\|_2}{\|x\|_2}$$

  where $|\langle u, \theta \rangle| \leq \|u\|_2$.

- Otherwise, suppose that $\mu = 0$. Then we have $|\eta| = |\langle u, v \rangle| \leq \|u\|_2$ and

$$\frac{\langle x+u, \theta \rangle}{\langle x, \theta \rangle} = 1 + \frac{\langle \eta v + s, \beta v + \gamma t \rangle}{\langle v \|x\|_2, \beta v + \gamma t \rangle} = 1 + \frac{\eta\beta}{\|x\|_2 \beta} = 1 + \frac{\eta}{\|x\|_2} \geq 1 - \frac{\|u\|_2}{\|x\|_2}$$

  where we used the fact that $\beta \neq 0$ given $\langle x, \theta \rangle \neq 0$.

□

Appendix E. Proof of Theorem 10

Theorem 10 Let $v \in \mathcal{C}(s_0, 3k_0) \setminus \{0\}$, and so $\|Av\|_2 > 0$ by $\mathsf{RE}(s_0, 3k_0, A)$ condition. Let $d(3k_0, A)$ be defined as in (23). As in the proof of Lemma 13, we may assume that $d(3k_0, A) < p$. By Lemma 13, applied with k_0 replaced with $3k_0$, we have

$$\frac{Av}{\|Av\|_2} \in A\Big(\mathcal{C}(s_0, 3k_0)\Big) \cap S^{q-1} \subset (1-\delta)^{-1} \operatorname{conv}\left(\bigcup_{|J| = d(3k_0, A)} AE_J \cap S^{q-1}\right)$$

$$\text{and} \quad \left\|\frac{\tilde{\Psi} Av}{\|Av\|_2}\right\|_2 \leq \frac{1}{1-\delta} \max_{u \in \operatorname{conv}(AE \cap S^{q-1})} \left\|\tilde{\Psi} u\right\|_2 = \frac{1}{1-\delta} \max_{u \in AE \cap S^{q-1}} \left\|\tilde{\Psi} u\right\|_2.$$

The last equality holds, since the maximum of $\|\tilde{\Psi} u\|_2$ occurs at an extreme point of the set $\operatorname{conv}(AE \cap S^{q-1})$, because of convexity of the function $f(x) = \|\tilde{\Psi} x\|_2$. Hence, by (18)

$$\forall x \in A\Big(\mathcal{C}(s_0, 3k_0)\Big) \cap S^{q-1}, \quad \left\|\tilde{\Psi} x\right\|_2 \leq (1+\delta)(1-\delta)^{-1} \leq 1 + 3\delta \tag{39}$$

where the last inequality is satisfied once $\delta < 1/3$, which proves the upper estimate in (19).

We have to prove the opposite inequality. Let $x = x_I + x_{I^c} \in \mathcal{C}(s_0, k_0) \cap S^{p-1}$, where the set I contains the locations of the s_0 largest coefficients of x in absolute values. We have

$$x = x_I + \|x_{I^c}\|_1 \sum_{j \in I^c} \frac{|x_j|}{\|x_{I^c}\|_1} \operatorname{sgn}(x_j) e_j, \text{ where } \quad 1 \geq \|x_I\|_2 \geq \frac{1}{\sqrt{k_0 + 1}} \text{ by (22)} \tag{40}$$

Let $\varepsilon > 0$ be specified later. We now construct a $d(3k_0, A)$-sparse vector $y = x_I + u \in \mathcal{C}(s_0, k_0)$, where u is supported on I^c which satisfies

$$\|u\|_1 = \|y_{I^c}\|_1 = \|x_{I^c}\|_1 \quad \text{and} \quad \|Ax - Ay\|_2 = \|A(x_{I^c} - y_{I^c})\|_2 \leq \varepsilon \tag{41}$$

To do so, set

$$w := Ax_{I^c} = \|x_{I^c}\|_1 \sum_{j\in I^c} \frac{|x_j|}{\|x_{I^c}\|_1} \mathrm{sgn}(x_j) Ae_j.$$

Let $\mathcal{M} := \{j \in I^c : x_j \neq 0\}$. Applying Lemma 11 with vectors $u_j = \|x_{I^c}\|_1 \,\mathrm{sgn}(x_j) Ae_j$ for $j \in \mathcal{M}$, construct a set $J' \subset \mathcal{M}$ satisfying

$$|J'| \leq m := \frac{4 \max_{j\in\mathcal{M}} \|x_{I^c}\|_1^2 \|Ae_j\|_2^2}{\varepsilon^2} \leq \frac{4k_0^2 s_0 \max_{j\in\mathcal{M}} \|Ae_j\|_2^2}{\varepsilon^2} \tag{42}$$

and a vector

$$w' = \|x_{I^c}\|_1 \sum_{j\in J'} \beta_j \mathrm{sgn}(x_j) Ae_j, \text{ where for } J' \subset \mathcal{M}, \beta_j \in [0,1] \text{ and } \sum_{j\in J'} \beta_j = 1$$

such that $\|Ax - Ay\|_2 = \|w' - w\|_2 \leq \varepsilon$. Set $u := \|x_{I^c}\|_1 \sum_{j\in J'} \beta_j \mathrm{sgn}(x_j) e_j$ and let

$$y = x_I + u = x_I + \|x_{I^c}\|_1 \sum_{j\in J'} \beta_j \mathrm{sgn}(x_j) e_j \text{ where } \beta_j \in [0,1] \text{ and } \sum_{j\in J'} \beta_j = 1.$$

By construction, $y \in \mathcal{C}(s_0, k_0) \cap E_J$, where $J := I \cup J'$ and

$$|J| = |I| + |J'| \leq s_0 + m. \tag{43}$$

This, in particular, implies that $\|Ay\|_2 > 0$. Assume that ε is chosen so that $s_0 + m \leq d(3k_0, A)$, and so by (18)

$$\left\| \frac{\tilde{\Psi} Ay}{\|Ay\|_2} \right\|_2 \geq 1 - \delta.$$

Set

$$v = x_I + 2y_{I^c} - x_{I^c} = y + (y_{I^c} - x_{I^c}). \tag{44}$$

Then (41) implies

$$\|Av\|_2 \leq \|Ay\|_2 + \|A(y_{I^c} - x_{I^c})\| \leq \|Ay\|_2 + \varepsilon, \tag{45}$$

and $v \in \mathcal{C}(s_0, 3k_0)$ as

$$\|v_{I^c}\|_1 \leq 2\|y_{I^c}\|_1 + \|x_{I^c}\|_1 = 3\|x_{I^c}\|_1 \leq 3k_0 \|x_I\|_1 = 3k_0 \|v_I\|_1$$

where we use the fact that $\|x_{I^c}\|_1 = \|y_{I^c}\|_1$. Hence, by the upper estimate (39), we have

$$\left\| \frac{\tilde{\Psi} Av}{\|Av\|_2} \right\|_2 \leq (1+\delta)(1-\delta)^{-1} \tag{46}$$

Since $y = \frac{1}{2}(x+v)$, where $y_I = x_I$, we have by the lower bound in (18) and the triangle inequality,

$$\begin{aligned} 1-\delta &\le \left\| \frac{\tilde\Psi Ay}{\|Ay\|_2} \right\|_2 \le \frac{1}{2}\left(\left\| \frac{\tilde\Psi Ax}{\|Ay\|_2} \right\|_2 + \left\| \frac{\tilde\Psi Av}{\|Ay\|_2} \right\|_2 \right) \\ &\le \frac{1}{2}\left(\left\| \frac{\tilde\Psi Ax}{\|Ax\|_2} \right\|_2 + \left\| \frac{\tilde\Psi Av}{\|Av\|_2} \right\|_2 \right) \cdot \frac{\|Ay\|_2 + \varepsilon}{\|Ay\|_2} \\ &\le \frac{1}{2}\left(\left\| \frac{\tilde\Psi Ax}{\|Ax\|_2} \right\|_2 + \frac{1+\delta}{1-\delta} \right) \cdot \big(1+\delta/6\big) \end{aligned}$$

where in the second line, we apply (45) and (41), and in the third line, (46). By the $\mathsf{RE}(s_0, k_0, A)$ condition and (40) we have

$$\|Ay\|_2 \ge \frac{\|y_I\|_2}{K(s_0,k_0,A)} = \frac{\|x_I\|_2}{K(s_0,k_0,A)} \ge \frac{1}{K(s_0,k_0,A)\cdot\sqrt{k_0+1}}.$$

Set

$$\varepsilon = \frac{\delta}{6\sqrt{1+k_0}K(s_0,k_0,A)} \quad \text{so that} \quad \frac{\|Ay\|_2+\varepsilon}{\|Ay\|_2} \le (1+\delta/6\big).$$

Then for $\delta < 1/5$

$$\left\| \frac{\tilde\Psi Ax}{\|Ax\|_2} \right\|_2 \ge 2\frac{1-\delta}{1+\delta/6} - (1+\delta)(1-\delta)^{-1} \ge 1-5\delta.$$

This verifies the lower estimate. It remains to check the bound for the cardinality of J. By (42) and (43), we have for $k_0 > 0$,

$$|J| \le s_0 + m \le s_0 + s_0 \max_{j\in\mathcal{M}} \|Ae_j\|_2^2 \left(\frac{16K^2(s_0,k_0,a)(3k_0)^2(k_0+1)}{\delta^2} \right) < d(3k_0,A)$$

as desired. This completes the proof of Theorem 10.

□

Appendix F. Proof of Theorem 15

For n as bounded in (25), where $m = \min(d(3k_0,A),p)$, we have (27) holds with $\tau = \delta/5$. Then by Theorem 16, we have with probability at least $1 - 2\exp\left(-n\delta^2/(2000\alpha^4)\right)$,

$$\forall m\text{-sparse vectors } u, \quad \left(1-\frac{\delta}{5}\right)\|Au\|_2 \le \frac{1}{\sqrt{n}}\left\|\tilde\Psi Au\right\|_2 \le \left(1+\frac{\delta}{5}\right)\|Au\|_2 .$$

The proof finishes by application of Theorem 10.

Appendix G. Proof of Theorem 16

We start with a definition.

Definition 22 *Given a subset $U \subset \mathbb{R}^p$ and a number $\varepsilon > 0$, an ε-net Π of U with respect to the Euclidean metric is a subset of points of U such that ε-balls centered at Π covers U:*

$$U \subset \bigcup_{x \in \Pi} (x + \varepsilon B_2^p),$$

where $A + B := \{a + b : a \in A, b \in B\}$ is the Minkowski sum of the sets A and B. The covering number $\mathcal{N}(U, \varepsilon)$ is the smallest cardinality of an ε-net of U.

The proof of Theorem 16 uses two well-known results. The first one is the *volumetric estimate*; see e.g. Milman and Schechtman (1986).

Lemma 23 *Given $m \geq 1$ and $\varepsilon > 0$. There exists an ε-net $\Pi \subset B_2^m$ of B_2^m with respect to the Euclidean metric such that $B_2^m \subset (1-\varepsilon)^{-1} \operatorname{conv} \Pi$ and $|\Pi| \leq (1 + 2/\varepsilon)^m$. Similarly, there exists an ε-net of the sphere S^{m-1}, $\Pi' \subset S^{m-1}$ such that $|\Pi'| \leq (1 + 2/\varepsilon)^m$.*

The second lemma with a worse constant can be derived from Bernstein's inequality for subexponential random variables. Since we are interested in the numerical value of the constant, we provide a proof below.

Lemma 24 *Let $Y_1, \ldots, Y_n$ be independent random variables such that $\mathbb{E} Y_j^2 = 1$ and $\|Y_j\|_{\psi_2} \leq \alpha$ for all $j = 1, \ldots, n$. Then for any $\theta \in (0, 1)$*

$$\mathbb{P}\left(\left| \frac{1}{n} \sum_{j=1}^n Y_j^2 - 1 \right| > \theta \right) \leq 2 \exp\left(-\frac{\theta^2 n}{10 \alpha^4} \right).$$

For a set $J \subset \{1, \ldots, p\}$, denote $E_J = \operatorname{span}\{e_j : j \in J\}$, and set $F_J = A E_J$. For each subset $F_J \cap S^{q-1}$, construct an ε-net Π_J, which satisfies

$$\Pi_J \subset F_J \cap S^{q-1} \quad \text{and} \quad |\Pi_J| \leq (1 + 2/\varepsilon)^m.$$

The existence of such Π_J is guaranteed by Lemma 23. If

$$\Pi = \bigcup_{|J| = m} \Pi_J,$$

then the previous estimate implies

$$|\Pi| = (3/\varepsilon)^m \binom{p}{m} \leq \left(\frac{3ep}{m\varepsilon} \right)^m = \exp\left(m \log\left(\frac{3ep}{m\varepsilon} \right) \right)$$

For $y \in S^{q-1} \cap F_J \subset F$, let $\pi(y)$ be one of the closest point in the ε-cover Π_J. Then

$$\frac{y - \pi(y)}{\|y - \pi(y)\|_2} \in F_J \cap S^{q-1} \quad \text{where} \quad \|y - \pi(y)\|_2 \leq \varepsilon.$$

Denote by $\Psi_1, \dots, \Psi_n$ the rows of the matrix Ψ, and set $\Gamma = n^{-1/2}\Psi$. Let $x \in S^{q-1}$. Applying Lemma 24 to the random variables $\langle \Psi_1, x \rangle^2, \dots, \langle \Psi_n, x \rangle^2$, we have that for every $\theta < 1$

$$\mathbb{P}\left(\left|\|\Gamma x\|_2^2 - 1\right| > \theta\right) \quad = \quad \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n}\langle \Psi_i, x \rangle^2 - 1\right| > \theta\right) \le 2\exp\left(-\frac{n\theta^2}{10\alpha^4}\right). \tag{47}$$

For

$$n \ge \frac{20m\alpha^4}{\theta^2}\log\left(\frac{3ep}{m\varepsilon}\right),$$

the union bound implies

$$\mathbb{P}\left(\exists x \in \Pi \text{ s. t. } \left|\|\Gamma x\|_2^2 - 1\right| > \theta\right) \le 2\left|\Pi\right|\exp\left(-\frac{n\theta^2}{10\alpha^4}\right) \le 2\exp\left(-\frac{n\theta^2}{20\alpha^4}\right)$$

Then for all $y_0 \in \Pi$

$$1 - \theta \le \|\Gamma y_0\|_2^2 \le 1 + \theta \text{ and so}$$
$$1 - \theta \le \|\Gamma y_0\|_2 \le 1 + \frac{\theta}{2}$$

with probability at least $1 - 2\exp\left(-\frac{n\theta^2}{20\alpha^4}\right)$, The bound over the entire $S^{q-1} \cap F_J$ is obtained by approximation. We have

$$\|\Gamma\pi(y)\|_2 - \|\Gamma(y - \pi(y))\|_2 \le \|\Gamma y\|_2 \le \|\Gamma\pi(y)\|_2 + \|\Gamma(y - \pi(y))\|_2 \tag{48}$$

Define

$$\|\Gamma\|_{2,F_J} := \sup_{y \in S^{q-1}\cap F_J}\|\Gamma y\|_2.$$

The RHS of (48) is upper bounded by $1 + \frac{\theta}{2} + \varepsilon\|\Gamma\|_{2,F_J}$. By taking the supremum over all $y \in S^{q-1} \cap F_J$, we have

$$\|\Gamma\|_{2,F_J} \le 1 + \frac{\theta}{2} + \varepsilon\|\Gamma\|_{2,F_J} \quad \text{and hence} \quad \|\Gamma\|_{2,F_J} \le \frac{1 + \theta/2}{1 - \varepsilon}.$$

The LHS of (48) is lower bounded by $1 - \theta - \varepsilon\|\Gamma\|_{2,F_J}$, and hence for all $y \in S^{q-1} \cap F_J$

$$\|\Gamma y\|_2 \quad \ge \quad 1 - \theta - \varepsilon\|\Gamma\|_{2,F_J} \ge 1 - \theta - \frac{\varepsilon(1 + \theta/2)}{1 - \varepsilon}$$

Putting these together, we have for all $y \in S^{q-1} \cap F_J$

$$1 - \theta - \frac{\varepsilon(1 + \theta/2)}{1 - \varepsilon} \le \|\Gamma y\|_2 \le \frac{1 + \theta/2}{1 - \varepsilon}$$

which holds for all sets J. Thus for $\theta < 1/2$ and $\varepsilon = \frac{\theta}{1+2\theta}$,

$$1 - 2\theta < \|\Gamma y\|_2 < 1 + 2\theta.$$

For any m-sparse vector $u \in S^{p-1}$

$$\frac{Au}{\|Au\|_2} \in F_J \quad \text{for } J = \operatorname{supp}(u),$$

and so

$$(1 - 2\theta)\|Au\|_2 \le \|\Gamma Au\|_2 \le (1 + 2\theta)\|Au\|_2.$$

Taking $\tau = \theta/2$ finishes the proof for Theorem 16.

G.1. Proof of Lemma 24

Note that $\alpha \geq \|Y_1\|_{\psi_2} \geq \|Y_1\|_2 = 1$. Using the elementary inequality $t^k \leq k! s^k e^{t/s}$, which holds for all $t, s > 0$, we obtain

$$|\mathbb{E}(Y_j^2 - 1)^k| \leq \max(\mathbb{E}Y_j^{2k}, 1) \leq \max(k!\alpha^{2k} \cdot \mathbb{E}e^{Y_j^2/\alpha^2}, 1) \leq 2k!\alpha^{2k}$$

for any $k \geq 2$. Since for any j $\mathbb{E}Y_j^2 = 1$, for any $\tau \in \mathbb{R}$ with $|\tau|\alpha^2 < 1$

$$\begin{aligned}\mathbb{E}\exp\left[\tau(Y_j^2 - 1)\right] &\leq 1 + \sum_{k=2} \frac{1}{k!}|\tau|^k \cdot |\mathbb{E}(Y_j^2 - 1)^k| \leq 1 + \sum_{k=2} |\tau|^k \cdot 2\alpha^{2k} \\ &\leq 1 + \frac{2\tau^2\alpha^4}{1 - |\tau|\alpha^2} \leq \exp\left(\frac{2\tau^2\alpha^4}{1 - |\tau|\alpha^2}\right).\end{aligned}$$

By Markov's inequality, for $\tau \in (0, \alpha^{-2})$

$$\begin{aligned}\mathbb{P}\left(\frac{1}{n}\sum_{j=1}^n Y_j^2 - 1 > \theta\right) &\leq \mathbb{E}\exp\left(\tau\sum_{j=1}^n (Y_j^2 - 1) - \tau\theta n\right) \\ &= e^{-\tau\theta n} \cdot \left(\mathbb{E}\exp\left[\tau(Y^2 - 1)\right]\right)^n \leq \exp\left(-\tau\theta n + \frac{2\tau^2\alpha^4 n}{1 - |\tau|\alpha^2}\right).\end{aligned}$$

Set $\tau = \frac{\theta}{5\alpha^4}$, so $\tau\alpha^2 \leq 1/5$. Then the previous inequality implies

$$\mathbb{P}\left(\frac{1}{n}\sum_{j=1}^n Y_j^2 - 1 > \theta\right) \leq \exp\left(-\frac{\theta^2 n}{10\alpha^4}\right).$$

Similarly, considering $\tau < 0$, we obtain

$$\mathbb{P}\left(1 - \frac{1}{n}\sum_{j=1}^n Y_j^2 > \theta\right) \leq \exp\left(-\frac{\theta^2 n}{10\alpha^4}\right).$$

□

Appendix H. Proof of Theorem 19

Proof of Theorem 19.

For $J \subset \{1, \ldots, p\}$, let E_J be the coordinate subspace spanned by the vectors e_j, $j \in J$. Set

$$F = \bigcup_{|J|=m} \Sigma^{1/2} E_J \cap S^{p-1}.$$

Denote $\Psi = \Sigma^{-1/2} X$ so $\mathbb{E}\Psi\Psi^T = id$, and let $\Psi_1, \ldots, \Psi_n$ be independent copies of Ψ. It is enough to show that with probability at least $1 - \exp\left(-\frac{\varepsilon\rho n}{6M^2 m}\right)$ for any $y \in F$

$$\left|1 - \frac{1}{n}\sum_{j=1}^n \langle \Psi_j, y\rangle^2\right| \leq \delta.$$

To this end we estimate

$$\Delta := \mathbb{E} \sup_{y \in F} \left| 1 - \frac{1}{n} \sum_{j=1}^{n} \langle \Psi_j, y \rangle^2 \right| .$$

The standard symmetrization argument implies that

$$\mathbb{E} \sup_{y \in F} \left| 1 - \frac{1}{n} \sum_{j=1}^{n} \langle \Psi_j, y \rangle^2 \right| \le \frac{2}{n} \mathbb{E} \sup_{y \in F} \left| \sum_{j=1}^{n} \varepsilon_j \langle \Psi_j, y \rangle^2 \right| ,$$

where $\varepsilon_1, \dots, \varepsilon_n$ are independent Bernoulli random variables taking values ± 1 with probability $1/2$. The estimate of the last quantity is based on the following Lemma, which is similar to Lemma 3.6 Rudelson and Vershynin (2008).

Lemma 25 *Let F be as above, and let $\psi_1, \dots, \psi_n \in \mathbb{R}^p$. Set*

$$Q = \max_{j=1,\dots,n} \left\| \Sigma^{1/2} \psi_j \right\|_\infty .$$

Then

$$\mathbb{E} \sup_{y \in F} \left| \sum_{j=1}^{n} \varepsilon_j \langle \psi_j, y \rangle^2 \right| \le \sqrt{\frac{CmQ^2 \cdot \log n \cdot \log p}{\rho}} \cdot \log \left(\frac{CmQ^2}{\rho} \right) \cdot \sup_{y \in F} \left(\sum_{j=1}^{n} \langle \psi_j, y \rangle^2 \right)^{1/2} .$$

Assuming Lemma 25, we finish the proof of the Theorem. First, note that by the definition of Ψ_j,

$$\max_{j=1,\dots,n} \left\| \Sigma^{1/2} \Psi_j \right\|_\infty \le M \text{ a.s.}$$

Hence, conditioning on $\Psi_1, \dots, \Psi_n$ and applying Lemma 25, we obtain

$$\Delta \le \frac{2}{n} \cdot \sqrt{\frac{CmM^2 \cdot \log n \cdot \log p}{\rho}} \cdot \log \left(\frac{CmM^2}{\rho} \right) \cdot \mathbb{E} \sup_{y \in F} \left(\sum_{j=1}^{n} \langle \Psi_j, y \rangle^2 \right)^{1/2} ,$$

and by Cauchy–Schwartz inequality,

$$\mathbb{E} \sup_{y \in F} \left(\sum_{j=1}^{n} \langle \Psi_j, y \rangle^2 \right)^{1/2} \le \left(\mathbb{E} \sup_{y \in F} \sum_{j=1}^{n} \langle \Psi_j, y \rangle^2 \right)^{1/2} ,$$

so

$$\Delta \le \frac{2}{\sqrt{n}} \cdot \sqrt{\frac{CmM^2 \cdot \log n \cdot \log p}{\rho}} \cdot \log \left(\frac{CmM^2}{\rho} \right) \cdot (\Delta + 1)^{1/2} .$$

If n satisfies (30), then

$$\Delta \le \delta \cdot (\Delta + 1)^{1/2} , \text{ and thus } \Delta \le 2\delta.$$

For $y \in F$ define a random variable $f(y) = \langle \Psi, y\rangle^2 - 1$. Then $|f(y)| \le \langle X, \Sigma^{-1/2} y\rangle^2 + 1 \le M^2\rho^{-1}m + 1 := a$ a.s., because $\Sigma^{-1/2}y$ is an m-sparse vector, whose norm does not exceed $\rho^{-1/2}$. Set

$$Z = \sup_{y\in F} \sum_{j=1}^n f_j(y),$$

where $f_1(y), \dots, f_n(y)$ are independent copies of $f(y)$. The argument above shows that $\mathbb{E}Z \le 2\delta n$. Then Talagrand's concentration inequality for empirical processes Talagrand (1996); Ledoux (2001) reads

$$\mathbb{P}(Z \ge t) \le \exp\left(-\frac{t}{6a}\right) \le \exp\left(-\frac{t\rho}{6M^2m}\right)$$

for all $t \ge 2\mathbb{E}Z$. Setting $t = 4\delta n$, we have

$$\mathbb{P}(\sup_{y\in F} \sum_{j=1}^n \left(\langle \Psi_j, y\rangle^2 - 1\right) \ge 4\delta n) \le \exp\left(-\frac{4\delta n\rho}{6M^2m}\right).$$

Similarly, considering random variables $g(y) = 1 - \langle \Psi, y\rangle^2$, we show that

$$\mathbb{P}(\sup_{y\in F} \sum_{j=1}^n \left(1 - \langle \Psi_j, y\rangle^2\right) \ge 4\delta n) \le \exp\left(-\frac{4\delta n\rho}{6M^2m}\right),$$

which completes the proof of the theorem. □

H.1. Proof of Lemma 25

It remains to prove Lemma 25. By Dudley's inequality, see e.g. Talagrand (2000, Eq. (1.18))

$$\mathbb{E}\sup_{y\in F}\left|\sum_{j=1}^n \varepsilon_j \langle \psi_j, y\rangle^2\right| \le C\int_0^\infty \log^{1/2} N(F, d, u)\, du$$

where $N(F, d, u)$ is the covering number, which is the minimal number of balls of radius u in the metric d covering the set F. Here d is the natural metric of the related Gaussian process defined as

$$\begin{aligned} d(x,y) &= \left[\sum_{j=1}^n \left(\langle \psi_j, x\rangle^2 - \langle \psi_j, y\rangle^2\right)^2\right]^{1/2} \\ &\le \left[\sum_{j=1}^n \left(\langle \psi_j, x\rangle + \langle \psi_j, y\rangle\right)^2\right]^{1/2} \cdot \max_{j=1,\dots,n} |\langle \psi_j, x - y\rangle| \\ &\le 2R \cdot \|x - y\|_Y, \end{aligned}$$

where

$$R = \sup_{y\in F}\left(\sum_{j=1}^n \langle \psi_j, y\rangle^2\right)^{1/2}, \quad \text{and } \|z\|_Y = \max_{j=1,\dots,n} |\langle \psi_j, z\rangle|.$$

The inclusion $\sqrt{m}B_1^p \supset \bigcup_{|J|=m} E_J \cap S^{p-1}$ implies

$$\sqrt{m}\Sigma^{1/2}B_1^p \supset \Sigma^{1/2}\operatorname{conv}\Big(\bigcup_{|J|=m} E_J \cap S^{p-1}\Big) \supset \rho^{1/2}F.$$

Hence, for any $y \in F$

$$\|z\|_Y \le \rho^{-1/2}\sqrt{m}\max_{j=1,\dots,n}\left\|\Sigma^{1/2}\psi_j\right\|_\infty = \rho^{-1/2}\sqrt{m}Q. \tag{49}$$

Replacing the metric d with the norm $\|\cdot\|_Y$, we obtain

$$\mathbb{E}\sup_{y\in F}\left|\sum_{j=1}^n \varepsilon_j\langle\psi_j, y\rangle^2\right| \le CR\int_0^{\rho^{-1/2}\sqrt{m}Q}\log^{1/2}N(F,\|\cdot\|_Y,u)\,du.$$

The upper limit of integration is greater or equal than the diameter of F in the norm $\|\cdot\|_Y$, so for $u > \rho^{-1/2}\sqrt{m}Q$ the integrand is 0. Arguing as in Lemma 3.7 Rudelson and Vershynin (2008), we can show that

$$N(F,\|\cdot\|_Y,u) \le N(\rho^{-1/2}\sqrt{m}\Sigma^{1/2}B_1^p,\|\cdot\|_Y,u) \le (2p)^l, \tag{50}$$

where

$$l = \frac{C\rho^{-1}m\left(\max_{i=1,\dots,p}\max_{j=1,\dots,n}|\langle\Sigma^{1/2}e_i,\psi_j\rangle|\right)^2}{u^2}\cdot\log n = \frac{CmQ^2\cdot\log n}{\rho u^2}$$

Also, since F consists of the union $\binom{p}{m}$ Euclidean spheres, the inclusion (49) and the volumetric estimate yield

$$N(F,\|\cdot\|_Y,u) \le \binom{p}{m}\cdot\left(1+\frac{2\rho^{-1/2}\sqrt{m}Q}{u}\right)^m \le \left(\frac{ep}{m}\right)^m\cdot\left(1+\frac{2\rho^{-1/2}\sqrt{m}Q}{u}\right)^m \tag{51}$$

Estimating the covering number of F as in (50) for $u \ge 1$, and as in (51) for $0 < u < 1$, we obtain

$$\begin{aligned}
&\mathbb{E}\sup_{y\in F}\left|\sum_{j=1}^n \varepsilon_j\langle\psi_j,y\rangle^2\right| \\
\le\ & CR\int_0^1\sqrt{m}\cdot\left(\log\left(\frac{ep}{m}\right)+\log\left(1+\frac{2\rho^{-1/2}\sqrt{m}Q}{u}\right)\right)^{1/2}du \\
&+CR\int_1^{\rho^{-1/2}\sqrt{m}Q}\sqrt{\frac{CmQ^2\cdot\log n}{\rho u^2}}\cdot\sqrt{\log 2p}\,du \\
\le\ & CR\sqrt{\frac{mQ^2\cdot\log n\cdot\log p}{\rho}}\cdot\log\left(\frac{CmQ^2}{\rho}\right).\square
\end{aligned}$$