Supplemental Materials

Lemmata

Lemma 1 Let X be a non-negative r.v. and $\mathcal{C}$ be an measurable event, then $\mathbb{E}[X|\mathcal{C}]\,\mathbb{P}(\mathcal{C}) \leq \mathbb{E}[X]$.

Proof.

$$\mathbb{E}[X] = \mathbb{E}[X|\mathcal{C}]\,\mathbb{P}(\mathcal{C}) + \mathbb{E}[X|\mathcal{C}^c]\,\mathbb{P}(\mathcal{C}^c) \geq \mathbb{E}[X|\mathcal{C}]\,\mathbb{P}(\mathcal{C})$$

□

Lemma 2: $\frac{1}{n}\sum_{k=1}^{n} \varphi_m(k/n)\varphi_l(k/n) = \mathbb{I}\{l = m\}$, for $1 \leq l, m \leq n-1$.

Proof. See Lemma 1.7 in [11].

Lemma 3: Let H_j be the $N \times M_n$ matrix with entries $H_j(i,m) = \eta_{jm}^{(i)} = \frac{1}{n}\vec{\varphi}_m^T \xi_j^{(i)}$, then its rows $H_j^{(i)} \overset{iid}{\sim} \mathcal{N}(0, \frac{\sigma_\xi^2}{n} I)$.

Proof. $H_j^{(i)} = \frac{1}{n}[\vec{\varphi}_1 \ldots \vec{\varphi}_{M_n}]^T \xi_j^{(i)}$, hence it is clearly Gaussian with mean 0. Furthermore,

$$\begin{aligned}\mathbb{E}[H_{jl}^{(i)} H_{jm}^{(i)}] =& \mathbb{E}[(\tfrac{1}{n}\vec{\varphi}_l^T \xi_j^{(i)})(\tfrac{1}{n}\vec{\varphi}_m^T \xi_j^{(i)})] \\ =& \tfrac{1}{n}\vec{\varphi}_l^T \mathbb{E}[(\xi_j^{(i)})(\xi_j^{(i)})^T]\tfrac{1}{n}\vec{\varphi}_m \\ =& \tfrac{1}{n^2}\vec{\varphi}_l^T(\sigma_\xi^2 I)\vec{\varphi}_m \\ =& \tfrac{\sigma_\xi^2}{n}(\tfrac{1}{n}\vec{\varphi}_l^T \vec{\varphi}_m) \\ =& \tfrac{\sigma_\xi^2}{n}\mathbb{I}\{l = m\},\end{aligned}$$

where the last line follows from Lemma 2. Furthermore, $H_S^{(i)} \overset{iid}{\sim} \mathcal{N}(0, \frac{\sigma_\xi^2}{n} I)$ directly follows using Lemma 3 and the fact that $\xi_j^{(i)}$ are independent over j as well as i indices. □

Lemma 4 $\mathbb{P}(\|H\|_{\max} \geq n^a) \leq 2\sigma_\xi p M_n n^{a-1/2} e^{-\frac{n^{1-2a}}{2\sigma_\xi^2}}$

Proof.

$$\begin{aligned}\mathbb{P}(\|H\|_{\max} \geq n^a) \leq& \mathbb{P}\left(\cup_{ij}\{|H_{Sj}^{(i)}| \geq n^{-a}\}\right) \\ \leq& \sum_{ij} \mathbb{P}\left(|H_j^{(i)}| \geq n^{-a}\right) \\ =& pM_n \mathbb{P}\left(\frac{\sigma_\xi}{\sqrt{n}}|Z| \geq n^{-a}\right) \\ \leq& 2\sigma_\xi p M_n n^{a-1/2} e^{-\frac{n^{1-2a}}{2\sigma_\xi^2}},\end{aligned}$$

□

where $Z \sim \mathcal{N}(0,1)$, and the last line follows from a Gaussian Tail inequality.

Lemma 5 $\|E_j\|_{\max} \leq C_Q n^{-\gamma+1/2}$, where $C_Q \in (0, \infty)$ is a constant depending on Q.

Proof. See Lemma 1.8 in [11].

Lemma 6 $\|\beta_S^*\|_2^2 \leq Qs$.

Proof.

$$\begin{aligned}\|\beta_S^*\|_2^2 =& \sum_{j\in S}\|\beta_j^*\|_2^2 = \sum_{j\in S}\sum_{m=1}^{M_n}\beta_{jm}^{*2} \leq \sum_{j\in S}\sum_{m=1}^{M_n} c_k^2 \beta_{jm}^{*2} \\ \leq& Qs\end{aligned}$$

□

Lemma 7: $\exists N_0, n_0, \tilde{C}_{\min}, \tilde{C}_{\max}$, $0 < \tilde{C}_{\min} \leq \tilde{C}_{\max} < \infty$, $0 < \tilde{\delta} \leq 1$ s.t. if $\|H\|_{\max} < n^{-a}$, and $N > N_0$, $n > n_0$ then

$$\Lambda_{\max}\left(\tfrac{1}{N}\tilde{A}_S^T\tilde{A}_S\right) \leq \tilde{C}_{\max} < \infty \tag{34}$$

$$\Lambda_{\min}\left(\tfrac{1}{N}\tilde{A}_S^T\tilde{A}_S\right) \geq \tilde{C}_{\min} > 0 \tag{35}$$

$$\forall j \in S^c,\ \left\|(\tfrac{1}{N}\tilde{A}_j^T\tilde{A}_S)(\tfrac{1}{N}\tilde{A}_S^T\tilde{A}_S)^{-1}\right\|_2 \leq \frac{1-\tilde{\delta}}{\sqrt{s}} \tag{36}$$

Proof. First, note that by the Courant-Fischer-Weyl min-max principle (e.g. [1]), for symmetric real matrices B, C we have that:

$$\begin{aligned}\Lambda_{\max}(B+C) =& \max_{\|x\|=1} x^T(B+C)x \\ =& \max_{\|x\|=1} x^TBx + x^TCx \\ \leq& \max_{\|x\|=1} x^TBx + \max_{\|x\|=1} x^TCx \\ =& \Lambda_{\max}(B) + \Lambda_{\max}(C)\end{aligned}$$

and,

$$\begin{aligned}\Lambda_{\min}(B+C) =& \min_{\|x\|=1} x^T(B+C)x \\ =& \min_{\|x\|=1} x^TBx + x^TCx \\ \geq& \min_{\|x\|=1} x^TBx + \min_{\|x\|=1} x^TCx \\ =& \Lambda_{\min}(B) + \Lambda_{\min}(C).\end{aligned}$$

Thus,

$$\Lambda_{\max}\left(\tfrac{1}{N}\tilde{A}_S^T\tilde{A}_S\right)$$

$$\leq \Lambda_{\max}(\tfrac{1}{N}A_S^TA_S) \tag{37}$$

$$+\,\Lambda_{\max}\left(\tfrac{1}{N}((E_S+H_S)^TA_S + A_S^T(E_S+H_S))\right) \tag{38}$$

$$+\,\Lambda_{\max}\left(\tfrac{1}{N}(E_S+H_S)^T(E_S+H_S)\right). \tag{39}$$

Since the term in (37) is bounded by (16), we need only show that (38) and (39) are bounded for large enough N, n. Similarly, we have that:

$$\begin{aligned}
&\Lambda_{\min}\left(\tfrac{1}{N}\tilde{A}_S^T\tilde{A}_S\right)\\
&\geq \Lambda_{\min}(\tfrac{1}{N}A_S^TA_S) &(40)\\
&+\Lambda_{\min}\left(\tfrac{1}{N}((E_S+H_S)^TA_S+A_S^T(E_S+H_S))\right) &(41)\\
&+\Lambda_{\min}\left(\tfrac{1}{N}(E_S+H_S)^T(E_S+H_S)\right). &(42)
\end{aligned}$$

Since the term in (40) is bounded by (17) and (42) is a positive semi-definite matrices, it suffices to show that

$$\Lambda_{\min}\left(\tfrac{1}{N}((E_S+H_S)^TA_S+A_S^T(E_S+H_S))\right)>-C_{\min}$$

for large enough N, n. Note further, that for symmetric matrix B:

$$\Lambda_{\max}(B)\leq\max_{\|x\|=1}|x^TBx| \text{ and}$$
$$\Lambda_{\min}(B)\geq-\max_{\|x\|=1}|x^TBx|.$$

Hence we will use the maximum absolute Rayleigh quotient (x^TBx) control bounds on the expected eigenvalues. Note that:

$$\begin{aligned}
&\max_{\|x\|=1}|x^T((E_S+H_S)^TA_S+A_S^T(E_S+H_S))x|\leq\\
&\max_{\|x\|=1}|((E_S+H_S)x)^T(A_Sx)|+|(A_Sx)^T((E_S+H_S)x)|\\
&=2\max_{\|x\|=1}|((E_S+H_S)x)^T(A_Sx)|\\
&\leq 2\max_{\|x\|=1}\|(E_S+H_S)x\|_2\|A_Sx\|_2\\
&\leq 2\left(\max_{\|x\|=1}\|A_Sx\|_2\right)\left(\max_{\|x\|=1}\|(E_S+H_S)x\|_2\right)\\
&\leq 2\left(\Lambda_{\max}(A_S^TA_S)\right)^{\frac{1}{2}}\left(\sqrt{N}\max_{\|x\|=1}\|(E_S+H_S)x\|_\infty\right)\\
&\leq 2\sqrt{NC_{\max}}\left(\sqrt{N}\max_{\|x\|=1,i}|(E_S^{(i)}+H_S^{(i)})^Tx|\right)\\
&\leq 2N\sqrt{C_{\max}}\left(\max_{\|x\|=1,i}\|E_S^{(i)}+H_S^{(i)}\|_2\|x\|_2\right)\\
&\leq 2N\sqrt{C_{\max}}\sqrt{sM_n}(C_Qn^{-\gamma+1/2}+n^{-a}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\max_{\|x\|=1}|x^T(E_S+H_S)^T(E_S+H_S)x|\leq\\
&sM_nN(C_Qn^{-\gamma+1/2}+n^{-a})^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Lambda_{\max}\left(\tfrac{1}{N}\tilde{A}_S^T\tilde{A}_S\right)&\leq C_{\max}\\
&+2\sqrt{C_{\max}sM_n}(C_Qn^{-\gamma+1/2}+n^{-a})\\
&+sM_n(C_Qn^{-\gamma+1/2}+n^{-a})^2\\
&\leq\tilde{C}_{\max},
\end{aligned}$$

and

$$\begin{aligned}
\Lambda_{\min}\left(\tfrac{1}{N}\tilde{A}_S^T\tilde{A}_S\right)&\geq C_{\min}\\
&-2\sqrt{C_{\max}sM_n}(C_Qn^{-\gamma+1/2}+n^{-a})\\
&\geq\tilde{C}_{\min},
\end{aligned}$$

for large enough n, N and appropriate $\tilde{C}_{\max}, \tilde{C}_{\min}$ using our assumptions. Let $\|\cdot\|=\|\cdot\|_2$ below. Hence,

$$\begin{aligned}
&\left\|(\tfrac{1}{N}\tilde{A}_j^T\tilde{A}_S)(\tfrac{1}{N}\tilde{A}_S^T\tilde{A}_S)^{-1}\right\|\\
&=\left\|(\tfrac{1}{N}\tilde{A}_j^T\tilde{A}_S)(\tfrac{1}{N}A_S^TA_S)^{-1}(\tfrac{1}{N}A_S^TA_S)(\tfrac{1}{N}\tilde{A}_S^T\tilde{A}_S)^{-1}\right\|\\
&\leq\left\|(\tfrac{1}{N}\tilde{A}_j^T\tilde{A}_S)(\tfrac{1}{N}A_S^TA_S)^{-1}\right\|\left\|(\tfrac{1}{N}A_S^TA_S)(\tfrac{1}{N}\tilde{A}_S^T\tilde{A}_S)^{-1}\right\|.
\end{aligned}$$

Also,

$$\begin{aligned}
&\left\|(\tfrac{1}{N}\tilde{A}_j^T\tilde{A}_S)(\tfrac{1}{N}A_S^TA_S)^{-1}\right\|\\
&=\left\|\tfrac{1}{N}(A_j+E_j+H_j)^T(A_S+E_S+H_S)(\tfrac{1}{N}A_S^TA_S)^{-1}\right\|\\
&\leq\left\|\tfrac{1}{N}A_j^TA_S(\tfrac{1}{N}A_S^TA_S)^{-1}\right\|\\
&+\left\|\tfrac{1}{N}A_j^T(E_S+H_S)(\tfrac{1}{N}A_S^TA_S)^{-1}\right\|\\
&+\left\|\tfrac{1}{N}(E_j+H_j)^TA_S(\tfrac{1}{N}A_S^TA_S)^{-1}\right\|\\
&+\left\|\tfrac{1}{N}(E_j+H_j)^T(E_S+H_S)(\tfrac{1}{N}A_S^TA_S)^{-1}\right\|\\
&\leq\frac{1-\delta}{\sqrt{s}}+\left\|\tfrac{1}{\sqrt{N}}A_j^T\right\|\left\|\tfrac{1}{\sqrt{N}}(E_S+H_S)\right\|\|(\tfrac{1}{N}A_S^TA_S)^{-1}\|\\
&+\left\|\tfrac{1}{\sqrt{N}}(E_j+H_j)\right\|\left\|\tfrac{1}{\sqrt{N}}A_S\right\|\|(\tfrac{1}{N}A_S^TA_S)^{-1}\|\\
&+\|\tfrac{1}{N}(E_j+H_j)^T(E_S+H_S)\|\|(\tfrac{1}{N}A_S^TA_S)^{-1}\|\\
&\leq\frac{1-\delta}{\sqrt{s}}+\frac{\sqrt{C_{\max}}}{C_{\min}}\sqrt{sM_n}(C_Qn^{-\gamma+1/2}+n^{-a}) &(43)\\
&+\frac{\sqrt{C_{\max}}}{C_{\min}}\sqrt{M_n}(C_Qn^{-\gamma+1/2}+n^{-a}) &(44)\\
&+\frac{1}{C_{\min}}\sqrt{s}M_n(C_Qn^{-\gamma+1/2}+n^{-a})^2. &(45)
\end{aligned}$$

Also,

$$\begin{aligned}
&\tfrac{1}{N}A_S^TA_S\\
&=\tfrac{1}{N}\tilde{A}_S^T\tilde{A}_S-\tfrac{1}{N}A_S^T(E_S+H_S)\\
&-\tfrac{1}{N}(E_S+H_S)^TA_S-\tfrac{1}{N}(E_S+H_S)^T(E_S+H_S).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left\|(\tfrac{1}{N}A_S^TA_S)(\tfrac{1}{N}\tilde{A}_S^T\tilde{A}_S)^{-1}\right\|\\
&\leq 1+\left\|\tfrac{1}{N}A_S^T(E_S+H_S)(\tfrac{1}{N}\tilde{A}_S^T\tilde{A}_S)^{-1}\right\|\\
&+\left\|\tfrac{1}{N}(E_S+H_S)^TA_S(\tfrac{1}{N}\tilde{A}_S^T\tilde{A}_S)^{-1}\right\|\\
&+\left\|\tfrac{1}{N}(E_S+H_S)^T(E_S+H_S)(\tfrac{1}{N}\tilde{A}_S^T\tilde{A}_S)^{-1}\right\|
\end{aligned}$$

$$\leq 1 + \frac{\sqrt{C_{\max}}}{C_{\min}}\sqrt{sM_n}(C_Q n^{-\gamma+1/2} + n^{-a}) \quad (46)$$

$$+ \frac{\sqrt{C_{\max}}}{C_{\min}}\sqrt{sM_n}(C_Q n^{-\gamma+1/2} + n^{-a}) \quad (47)$$

$$+ \frac{1}{C_{\min}} sM_n(C_Q n^{-\gamma+1/2} + n^{-a})^2. \quad (48)$$

By our assumptions all terms in (43)-(48), except $\frac{1-\delta}{\sqrt{s}}$, are going to zero. Hence, keeping leading terms, one may see that

$$\left\|(\tfrac{1}{N}\tilde{A}_j^T\tilde{A}_S)(\tfrac{1}{N}\tilde{A}_S^T\tilde{A}_S)^{-1}\right\|$$
$$\leq \frac{1-\delta}{\sqrt{s}} + O\left(\sqrt{sM_n}(n^{-\gamma+1/2}+n^{-a})\right)$$
$$+ O\left(\sqrt{s}M_n(n^{-\gamma+1/2}+n^{-a})^2\right)$$
$$\leq \frac{1-\tilde{\delta}}{\sqrt{s}}$$

for large enough n and appropriate $\tilde{\delta}$. □

Lemma 8 $\mathbb{E}\left[\|\tilde{\Sigma}_{SS}^{-1}(\frac{1}{N}\tilde{A}_S^T)V\|_\infty\Big|\mathcal{B}\right]\mathbb{P}(\mathcal{B}) = O\left(\frac{s^{3/2}}{M_n^{2\gamma-1/2}}\right)$.

Proof. First note that:

$$\|\tilde{\Sigma}_{SS}^{-1}(\tfrac{1}{N}\tilde{A}_S^T)V\|_\infty \leq \|\tilde{\Sigma}_{SS}^{-1}\|_\infty\|(\tfrac{1}{N}\tilde{A}_S^T)\|_\infty\|V\|_\infty.$$

We have that

$$|V_i| = \left|\sum_{j\in S}\sum_{m=M_n+1}^{\infty}\alpha_{jm}^{(i)}\beta_{jm}^*\right| \leq \sum_{j\in S}\sum_{m=M_n+1}^{\infty}\left|\alpha_{jm}^{(i)}\beta_{jm}^*\right|$$
$$\leq \sum_{j\in S}\left(\sum_{m=M_n+1}^{\infty}\alpha_{jm}^{(i)2}\right)^{\frac{1}{2}}\left(\sum_{m=M_n+1}^{\infty}\beta_{jm}^{*2}\right)^{\frac{1}{2}},$$

and

$$\frac{1}{M_n^{2\gamma}}\left(\sum_{m=M_n+1}^{\infty}M_n^{2\gamma}\alpha_{jm}^{(i)2}\right)^{\frac{1}{2}}\left(\sum_{m=M_n+1}^{\infty}M_n^{2\gamma}\beta_{jm}^{*2}\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{M_n^{2\gamma}}\left(\sum_{m=1}^{\infty}c_k^2\alpha_{jm}^{(i)2}\right)^{\frac{1}{2}}\left(\sum_{m=1}^{\infty}c_k^2\beta_{jm}^{*2}\right)^{\frac{1}{2}}$$
$$\leq \frac{Q}{M_n^{2\gamma}}.$$

Thus $|V_i| \leq \frac{Qs}{M_n^{2\gamma}}$. Also,

$$\|(\tfrac{1}{N}\tilde{A}_S^T)\|_\infty \leq \frac{1}{N}(\|A_S^T\|_\infty + \|E_S^T\|_\infty + \|H_S^T\|_\infty)$$
$$\leq Q + C_Q n^{-\gamma+1/2} + n^{-a}.$$

Hence,

$$\mathbb{E}\left[\|\tilde{\Sigma}_{SS}^{-1}(\tfrac{1}{N}\tilde{A}_S^T)V\|_\infty\Big|\mathcal{B}\right]$$
$$\leq \frac{\sqrt{sM_n}}{\tilde{C}_{\min}}\left(Q + C_Q n^{-\gamma+1/2} + n^{-a}\right)\frac{Qs}{M_n^{2\gamma}}. \quad (49)$$

□

Lemma 9 $\mathbb{E}\left[\|\tilde{\Sigma}_{SS}^{-1}(\frac{1}{N}\tilde{A}_S^T)\epsilon\|_\infty\Big|\mathcal{B}\right]\mathbb{P}(\mathcal{B}) = O\left(\sqrt{\log(sM_n)/N}\right)$.

Proof. Note that given H_S, $Z = \tilde{\Sigma}_{SS}^{-1}(\frac{1}{N}\tilde{A}_S^T)\epsilon$ is normal with mean 0 and co-variance matrix:

$$\mathbb{E}\left[\tilde{\Sigma}_{SS}^{-1}(\tfrac{1}{N}\tilde{A}_S^T)\epsilon\epsilon^T(\tfrac{1}{N}\tilde{A}_S)\tilde{\Sigma}_{SS}^{-1}|H_S\right]$$
$$= \tilde{\Sigma}_{SS}^{-1}(\tfrac{1}{N}\tilde{A}_S^T)(\sigma_\epsilon^2 I)(\tfrac{1}{N}\tilde{A}_S)\tilde{\Sigma}_{SS}^{-1}$$
$$= \frac{\sigma_\epsilon^2}{N}\tilde{\Sigma}_{SS}^{-1}(\tfrac{1}{N}\tilde{A}_S^T\tilde{A}_S)\tilde{\Sigma}_{SS}^{-1}$$
$$= \frac{\sigma_\epsilon^2}{N}\tilde{\Sigma}_{SS}^{-1}.$$

Hence, given H_S and $\mathcal{B}$

$$\max_i \mathrm{Var}[Z_i] = \max_i \frac{\sigma_\epsilon^2}{N}e_i^T\tilde{\Sigma}_{SS}^{-1}e_i \leq \frac{\sigma_\epsilon^2}{N}\left(\Lambda_{\min}(\tilde{\Sigma}_{SS})\right)^{-1}$$
$$\leq \frac{\sigma_\epsilon^2}{N}\tilde{C}_{\min}^{-1}.$$

And so [4],

$$\mathbb{E}\left[\|Z\|_\infty\Big|\mathcal{B}\right]$$
$$= \mathbb{E}\left[\mathbb{E}\left[\|Z\|_\infty\Big|\mathcal{B}, H_S\right]\Big|\mathcal{B}\right]$$
$$\leq \mathbb{E}\left[\mathbb{E}\left[3\sqrt{\log(sM_n)\|\mathrm{Var}[Z]\|_\infty}\Big|\mathcal{B}, H_S\right]\Big|\mathcal{B}\right]$$
$$\leq \mathbb{E}\left[3\sigma_\epsilon\sqrt{\frac{\log(sM_n)}{N\tilde{C}_{\min}}}\Big|\mathcal{B}\right]$$
$$= 3\sigma_\epsilon\sqrt{\frac{\log(sM_n)}{N\tilde{C}_{\min}}}.$$

□

Lemma 10 $\mathbb{P}\left(\max_{j\in S^c}\|\mu_j^H\|_2 \geq 1 - \frac{\tilde{\delta}}{2}\right) \to 0$

Proof.

$$\mathbb{P}\left(\max_{j\in S^c}\|\mu_j^H\|_2 \geq 1 - \frac{\tilde{\delta}}{2}\right)$$
$$= \mathbb{P}\left(\max_{j\in S^c}\|\mu_j^H\|_2 - (1-\tilde{\delta}) \geq \frac{\tilde{\delta}}{2}\right)$$

$$\leq \mathbb{P}\left(\max_{j\in S^c}\|\mu_j^H\|_2 - (1-\tilde{\delta}) \geq \frac{\tilde{\delta}}{2}\Big|\mathcal{B}\right)\mathbb{P}\left(\mathcal{B}\right) + \mathbb{P}\left(\mathcal{B}^c\right)$$
$$\leq \frac{2}{\tilde{\delta}}\mathbb{E}\left[\max_{j\in S^c}\|\mu_j^H\|_2 - (1-\tilde{\delta})\Big|\mathcal{B}\right]\mathbb{P}\left(\mathcal{B}\right) + \mathbb{P}\left(\mathcal{B}^c\right),$$

Recall that

$$\hat{u}_j = \tfrac{1}{\lambda_N}\tilde{\Sigma}_{jS}\tilde{\Sigma}_{SS}^{-1}\Big(\tfrac{1}{N}\tilde{A}_S^T(E_S+H_S)\beta_S^* - \tfrac{1}{N}\tilde{A}_S^T V$$
$$- \tfrac{1}{N}\tilde{A}_S^T\epsilon + \lambda_N\hat{u}_S\Big) - \tfrac{1}{\lambda_N N}\tilde{A}_j^T(E_S+H_S)\beta_S^*$$
$$+ \tfrac{1}{\lambda_N N}\tilde{A}_j^T(V+\epsilon),$$

where $\tilde{\Sigma}_{jS} = \frac{1}{N}\tilde{A}_j^T\tilde{A}_S$. Let $\mu_j^H \equiv \mathbb{E}\left[\hat{u}_j\big|H\right] =$

$$\tilde{\Sigma}_{jS}\tilde{\Sigma}_{SS}^{-1}\left(\tfrac{1}{\lambda_N N}\tilde{A}_S^T(E_S+H_S)\beta_S^* - \tfrac{1}{\lambda_N N}\tilde{A}_S^T V + \hat{u}_S\right)$$
$$- \tfrac{1}{\lambda_N N}\tilde{A}_j^T(E_S+H_S)\beta_S^* + \tfrac{1}{\lambda_N N}\tilde{A}_j^T V.$$

Note that

$$\|\mu_j^H\|_2 \leq \|\tilde{\Sigma}_{jS}\tilde{\Sigma}_{SS}^{-1}\|_2\Big(\|\tfrac{1}{\lambda_N N}\tilde{A}_S^T(E_S+H_S)\beta_S^*\|_2$$
$$+ \|\tfrac{1}{\lambda_N N}\tilde{A}_S^T V\|_2 + \|\hat{u}_S\|_2\Big)$$
$$+ \|\tfrac{1}{\lambda_N N}\tilde{A}_j^T(E_S+H_S)\beta_S^*\|_2 + \|\tfrac{1}{\lambda_N N}\tilde{A}_j^T V\|_2.$$

Given $\mathcal{B}$,

$$\|\mu_j^H\|_2 \leq \frac{1-\tilde{\delta}}{\sqrt{s}}\Big(\|\tfrac{1}{\lambda_N N}\tilde{A}_S^T(E_S+H_S)\beta_S^*\|_2$$
$$+ \|\tfrac{1}{\lambda_N N}\tilde{A}_S^T V\|_2 + \sqrt{s}\Big)$$
$$+ \|\tfrac{1}{\lambda_N N}\tilde{A}_j^T(E_S-H_S)\beta_S^*\|_2 + \|\tfrac{1}{\lambda_N N}\tilde{A}_j^T V\|_2$$
$$= 1-\tilde{\delta} + \tfrac{1-\tilde{\delta}}{\sqrt{s}}\|\tfrac{1}{\lambda_N N}\tilde{A}_S^T(E_S+H_S)\beta_S^*\|_2$$
$$+ \tfrac{1-\tilde{\delta}}{\sqrt{s}}\|\tfrac{1}{\lambda_N N}\tilde{A}_S^T V\|_2$$
$$+ \|\tfrac{1}{\lambda_N N}\tilde{A}_j^T(E_S+H_S)\beta_S^*\|_2 + \|\tfrac{1}{\lambda_N N}\tilde{A}_j^T V\|_2$$

and so:

$$\mathbb{E}\left[\max_{j\in S^c}\|\mu_j^H\|_2 - (1-\tilde{\delta})\Big|\mathcal{B}\right]\mathbb{P}(\mathcal{B})$$
$$\leq \tfrac{1-\tilde{\delta}}{\sqrt{s}}\mathbb{E}\left[\|\tfrac{1}{\lambda_N N}\tilde{A}_S^T(E_S+H_S)\beta_S^*\|_2\Big|\mathcal{B}\right]\mathbb{P}(\mathcal{B})$$
$$+ \tfrac{1-\tilde{\delta}}{\sqrt{s}}\mathbb{E}\left[\|\tfrac{1}{\lambda_N N}\tilde{A}_S^T V\|_2\Big|\mathcal{B}\right]\mathbb{P}(\mathcal{B})$$
$$+ \mathbb{E}\left[\max_{j\in S^c}\|\tfrac{1}{\lambda_N N}\tilde{A}_j^T(E_S+H_S)\beta_S^*\|_2\Big|\mathcal{B}\right]\mathbb{P}(\mathcal{B})$$
$$+ \mathbb{E}\left[\max_{j\in S^c}\|\tfrac{1}{\lambda_N N}\tilde{A}_j^T V\|_2\Big|\mathcal{B}\right]\mathbb{P}(\mathcal{B}).$$

First, note that

$$\mathbb{E}\left[\|\tfrac{1}{\lambda_N N}A_S^T(E_S+H_S)\beta_S^*\|_2\right]\mathbb{P}(\mathcal{B})$$
$$\leq \tfrac{1}{\lambda_N\sqrt{N}}\mathbb{E}\left[\|\tfrac{1}{\sqrt{N}}A_S^T\|_2\|(E_S+H_S)\beta_S^*\|_2\right]\mathbb{P}(\mathcal{B})$$
$$\leq \tfrac{\sqrt{C_{max}}}{\lambda_N\sqrt{N}}\mathbb{E}\left[\sqrt{N}\|(E_S+H_S)\beta_S^*\|_\infty\right]\mathbb{P}(\mathcal{B})$$
$$= O\left(\frac{1}{\lambda_N}\left(s\sqrt{M_n}n^{-\gamma+1/2} + \sqrt{\frac{s\log(N)}{n}}\right)\right).$$

Moreover,

$$\mathbb{E}\left[\|\tfrac{1}{\lambda_N N}(E_S+H_S)^T(E_S+H_S)\beta_S^*\|_2\Big|\mathcal{B}\right]\mathbb{P}(\mathcal{B})$$
$$\leq \tfrac{\mathbb{P}(\mathcal{B})}{\lambda_N N}\mathbb{E}\left[\sqrt{sM_n}\|(E_S+H_S)^T(E_S+H_S)\beta_S^*\|_\infty\Big|\mathcal{B}\right]$$
$$= O\Bigg(\frac{\sqrt{sM_n}}{\lambda_N}\left(s\sqrt{M_n}n^{-\gamma+1/2} + \sqrt{\frac{s\log(N)}{n}}\right)$$
$$\left(n^{-\gamma+1/2} + n^{-a}\right)\Bigg).$$

Thus,

$$\tfrac{1-\tilde{\delta}}{\sqrt{s}}\mathbb{E}\left[\|\tfrac{1}{\lambda_N N}\tilde{A}_S^T(E_S+H_S)\beta_S^*\|_2\Big|\mathcal{B}\right]\mathbb{P}(B)$$
$$= O\left(\frac{1}{\lambda_N}\sqrt{sM_n}n^{-\gamma+1/2} + \frac{1}{\lambda_N}\sqrt{\frac{\log(N)}{n}}\right)$$
$$+ O\left(\frac{sM_n}{\lambda_N}n^{-2\gamma+1} + \frac{\sqrt{sM_n\log(N)}}{\lambda_N n^{\gamma}}\right)$$
$$+ O\left(\frac{sM_n}{\lambda_N n^{\gamma+a-1/2}} + \frac{\sqrt{sM_n\log(N)}}{\lambda_N n^{a+1/2}}\right).$$

Similarly,

$$\mathbb{E}\left[\|\tfrac{1}{\lambda_N N}A_j^T(E_S+H_S)\beta_S^*\|_2\Big|\mathcal{B}\right]\mathbb{P}(\mathcal{B})$$
$$\leq \frac{\sqrt{N}}{\lambda_N\sqrt{N}}\|\tfrac{1}{\sqrt{N}}A_j^T\|_2\mathbb{E}\left[\|(E_S+H_S)\beta_S^*\|_\infty\Big|\mathcal{B}\right]\mathbb{P}(\mathcal{B})$$
$$= O\left(\frac{1}{\lambda_N}\left(s\sqrt{M_n}n^{-\gamma+1/2} + \sqrt{\frac{s\log(N)}{n}}\right)\right),$$

and,

$$\mathbb{E}\left[\|\tfrac{1}{\lambda_N N}(E_j+H_j)^T(E_S+H_S)\beta_S^*\|_2\Big|\mathcal{B}\right]\mathbb{P}(\mathcal{B})$$
$$\leq \frac{\sqrt{M_n}}{\lambda_N N}\mathbb{E}\left[\|(E_j+H_j)^T(E_S+H_S)\beta_S^*\|_\infty\Big|\mathcal{B}\right]\mathbb{P}(\mathcal{B})$$
$$= O\Bigg(\frac{\sqrt{M_n}}{\lambda_N}\left(s\sqrt{M_n}n^{-\gamma+1/2} + \sqrt{\frac{s\log(N)}{n}}\right)$$
$$\left(n^{-\gamma+1/2} + n^{-a}\right)\Bigg).$$

Hence,

$$\mathbb{E}\left[\|\tfrac{1}{\lambda_N N}\tilde{A}_j^T(E_S+H_S)\beta_S^*\|_2\Big|\mathcal{B}\right]\mathbb{P}(B)$$
$$= O\left(\frac{s\sqrt{M_n}}{\lambda_N}n^{-\gamma+1/2} + \frac{1}{\lambda_N}\sqrt{\frac{s\log(N)}{n}}\right)$$

$$+O\left(\frac{sM_n}{\lambda_N}n^{-2\gamma+1}+\frac{\sqrt{sM_n\log(N)}}{\lambda_N n^{\gamma}}\right)$$
$$+O\left(\frac{sM_n}{\lambda_N n^{\gamma+a-1/2}}+\frac{\sqrt{sM_n\log(N)}}{\lambda_N n^{a+1/2}}\right)$$

Also, $\mathbb{E}\left[\|\frac{1}{\lambda_N N}\tilde{A}_S^T V\|_2\middle|\mathcal{B}\right]\mathbb{P}(\mathcal{B})$

$$\begin{aligned}&\leq \frac{\sqrt{sM_n}}{\lambda_N}\mathbb{E}\left[\|\tfrac{1}{N}\tilde{A}_S^T V\|_\infty\middle|\mathcal{B}\right]\mathbb{P}(\mathcal{B})\\ &= O\left(\frac{\sqrt{sM_n}}{\lambda_N}\left(Q+C_Q n^{-\gamma+1/2}+n^{-a}\right)\frac{Qs}{M_n^{2\gamma}}\right)\\ &= O\left(\frac{s^{3/2}}{\lambda_N M_n^{2\gamma-1/2}}\left(1+n^{-\gamma+1/2}+n^{-a}\right)\right)\\ &= O\left(\frac{s^{3/2}}{\lambda_N M_n^{2\gamma-1/2}}\right).\end{aligned}$$

So,

$$\tfrac{1-\tilde{\delta}}{\sqrt{s}}\mathbb{E}\left[\|\tfrac{1}{\lambda_N N}\tilde{A}_S^T V\|_2\middle|\mathcal{B}\right]\mathbb{P}(\mathcal{B})=O\left(\frac{s}{\lambda_N M_n^{2\gamma-1/2}}\right).$$

Similarly, $\mathbb{E}\left[\|\frac{1}{\lambda_N N}\tilde{A}_j^T V\|_2\middle|\mathcal{B}\right]\mathbb{P}(\mathcal{B})$

$$\begin{aligned}&\leq \frac{\sqrt{M_n}}{\lambda_N}\mathbb{E}\left[\|\tfrac{1}{N}\tilde{A}_j^T V\|_\infty\middle|\mathcal{B}\right]\mathbb{P}(\mathcal{B})\\ &= O\left(\frac{s}{\lambda_N M_n^{2\gamma-1/2}}\right).\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{2}{\tilde{\delta}}\mathbb{E}\left[\max_{j\in S^c}\|\mu_j^H\|_2-(1-\tilde{\delta})\middle|\mathcal{B}\right]\mathbb{P}(\mathcal{B})\\ &= O\left(\frac{s\sqrt{M_n}}{\lambda_N}n^{-\gamma+1/2}+\frac{1}{\lambda_N}\sqrt{\frac{s\log(N)}{n}}\right)\\ &+O\left(\frac{sM_n}{\lambda_N n^{\gamma+a-1/2}}+\frac{\sqrt{sM_n\log(N)}}{\lambda_N n^{a+1/2}}\right)\\ &+O\left(\frac{s}{\lambda_N M_n^{2\gamma-1/2}}\right),\end{aligned}$$

where we used that fact that $\gamma\geq 1$. Thus, with assumptions (20)-(28), $\mathbb{P}\left(\max_{j\in S^c}\|\mu_j^H\|_2\geq 1-\frac{\tilde{\delta}}{2}\right)\to 0$ □

Lemma 11 $\mathbb{P}\left(\max_{j\in S^c}\|\hat{u}_j-\mu_j^H\|_\infty\geq\frac{\tilde{\delta}}{2\sqrt{M_n}}\right)\to 0$
Note that

$$\begin{aligned}&\mathbb{P}\left(\max_{j\in S^c}\|\hat{u}_j-\mu_j^H\|_\infty\geq\frac{\tilde{\delta}}{2\sqrt{M_n}}\right)\\ &\leq \mathbb{P}\left(\max_{j\in S^c}\|\hat{u}_j-\mu_j^H\|_\infty\geq\frac{\tilde{\delta}}{2\sqrt{M_n}}\middle|\mathcal{B}\right)\mathbb{P}(\mathcal{B})+\mathbb{P}(\mathcal{B}^c),\end{aligned}$$

and

$$\begin{aligned}&\frac{2\sqrt{M_n}}{\tilde{\delta}}\mathbb{E}\left[\max_{j\in S^c}\|\hat{u}_j-\mu_j^H\|_\infty\middle|\mathcal{B}\right]\\ &= \frac{2\sqrt{M_n}}{\tilde{\delta}}\mathbb{E}\left[\mathbb{E}\left[\max_{j\in S^c}\|\hat{u}_j-\mu_j^H\|_\infty\middle|\mathcal{B},H\right]\middle|\mathcal{B}\right].\end{aligned}$$

Let

$$\begin{aligned}Z_j \equiv&\lambda_N(\hat{u}_j-\mu_j^H)\\ =&\tilde{A}_j^T(I-\tilde{A}_S(\tilde{A}_S^T\tilde{A}_S)^{-1}\tilde{A}_S^T)\frac{\epsilon}{N}.\end{aligned}$$

Thus, given H, Z_j is a zero mean Gaussian random variable. Furthermore, given $\mathcal{B}$, $\max_k\operatorname{Var}[Z_{jk}]\leq\sigma_\epsilon^2/N$.

$$\begin{aligned}\mathbb{E}\left[Z_j^TZ_j\right]=&\frac{1}{N^2}\tilde{A}_j^T(I-\tilde{A}_S(\tilde{A}_S^T\tilde{A}_S)^{-1}\tilde{A}_S^T)\mathbb{E}\left[\epsilon\epsilon^T\right]\\ &(I-\tilde{A}_S(\tilde{A}_S^T\tilde{A}_S)^{-1}\tilde{A}_S^T)\tilde{A}_j\\ =&\frac{\sigma^2}{N^2}\tilde{A}_j^T(I-\tilde{A}_S(\tilde{A}_S^T\tilde{A}_S)^{-1}\tilde{A}_S^T)\\ &(I-\tilde{A}_S(\tilde{A}_S^T\tilde{A}_S)^{-1}\tilde{A}_S^T)\tilde{A}_j\\ =&\frac{\sigma^2}{N^2}\tilde{A}_j^T\Big(I-\tilde{A}_S(\tilde{A}_S^T\tilde{A}_S)^{-1}\tilde{A}_S^T\\ &-\tilde{A}_S(\tilde{A}_S^T\tilde{A}_S)^{-1}\tilde{A}_S^T\\ &+\tilde{A}_S(\tilde{A}_S^T\tilde{A}_S)^{-1}\tilde{A}_S^T\tilde{A}_S(\tilde{A}_S^T\tilde{A}_S)^{-1}\tilde{A}_S^T\Big)\tilde{A}_j\\ =&\frac{\sigma^2}{N^2}\tilde{A}_j^T\Big(I-\tilde{A}_S(\tilde{A}_S^T\tilde{A}_S)^{-1}\tilde{A}_S^T\\ &-\tilde{A}_S(\tilde{A}_S^T\tilde{A}_S)^{-1}\tilde{A}_S^T+\tilde{A}_S(\tilde{A}_S^T\tilde{A}_S)^{-1}\tilde{A}_S^T\Big)\tilde{A}_j\\ =&\frac{\sigma^2}{N^2}\tilde{A}_j^T\left(I-\tilde{A}_S(\tilde{A}_S^T\tilde{A}_S)^{-1}\tilde{A}_S^T\right)\tilde{A}_j\\ =&\frac{\sigma^2}{N}\left(\frac{1}{N}\tilde{A}_j^T\tilde{A}_j-\frac{1}{N}\tilde{A}_j^T\tilde{A}_S(\tilde{A}_S^T\tilde{A}_S)^{-1}\tilde{A}_S^T\tilde{A}_j\right).\end{aligned}$$

So, given $\mathcal{B}$

$$\begin{aligned}\operatorname{Var}[Z_{jk}]=&\frac{\sigma^2}{N}\left(\frac{1}{N}e_k^T\tilde{A}_j^T\tilde{A}_je_k\right)\\ &-\frac{\sigma^2}{N^2}e_k^T\tilde{A}_j^T\tilde{A}_S(\tilde{A}_S^T\tilde{A}_S)^{-1}\tilde{A}_S^T\tilde{A}_je_k\\ =&O\left(\frac{1}{N}\right),\end{aligned}$$

where the last line follows from the fact that $\tilde{A}_j^T\tilde{A}_S(\tilde{A}_S^T\tilde{A}_S)^{-1}\tilde{A}_S^T\tilde{A}_j$ is PSD and $\|\frac{1}{N}\tilde{A}_j^T\tilde{A}_j\|_{\max}\leq(Q+C_Qn^{-\gamma+1/2}+n^{-a})^2$.

Hence,

$$\frac{1}{\lambda_N}\mathbb{E}\left[\max_{j\in S^c}\|Z_j\|_\infty\middle|\mathcal{B},H\right]=O\left(\frac{1}{\lambda_N}\sqrt{\frac{\log((p-s)M_n)}{N}}\right).$$

Hence,

$$\begin{aligned}&\frac{2\sqrt{M_n}}{\tilde{\delta}}\mathbb{E}\left[\max_{j\in S^c}\|\hat{u}_j-\mu_j^H\|_\infty\right]\\ \leq &O\left(\frac{1}{\lambda_{\text{N}}}\sqrt{M_n\frac{\log((p-s)M_n)}{N}}\right),\end{aligned}$$

and so $\mathbb{P}\left(\max_{j\in S^c}\|\hat{u}_j-\mu_j^H\|_\infty\geq\frac{\tilde{\delta}}{2\sqrt{M_n}}\right)\to 0$