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Volumetric Spanners: an Efficient Exploration Basis for Learning

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Abstract

Numerous machine learning problems require an *exploration basis* - a mechanism to explore the action space. We define a novel geometric notion of exploration basis with low variance called volumetric spanners, and give efficient algorithms to construct such bases.

We show how efficient volumetric spanners give rise to an efficient and near-optimal regret algorithm for bandit linear optimization over general convex sets. Previously such results were known only for specific convex sets, or under special conditions such as the existence of an efficient self-concordant barrier for the underlying set.

Keywords: Learning basis, Multi-Armed Bandit, Active Learning, Spanners, Convex Geometry

1. Introduction

A fundamental difficulty in machine learning is environment exploration. A prominent example is the famed multi-armed bandit (MAB) problem, in which a decision maker iteratively chooses an action from a set of available actions and receives a payoff, without observing the payoff of all other actions she could have taken. The MAB problem displays an exploration-exploitation tradeoff, in which the decision maker trades exploring the action space vs. exploiting the knowledge already obtained to pick the best arm.

Another example in which environment exploration is crucial, or perhaps the main point, is active learning and experiment design. In these fields it is important to correctly identify the most informative queries so as to efficiently construct a solution.

Exploration is hardly summarized by picking an action uniformly at random [1]. Indeed, sophisticated techniques from various areas of optimization, statistics and convex geometry have been applied to designing ever better exploration algorithms. To mention a few: Awerbuch and Kleinberg (2008) devise the notion of *barycentric spanners*, and use this construction to give the first low-regret algorithms for complex decision problems such as online routing. Abernethy et al. (2012) use self-concordant barriers to build an efficient exploration strategy for convex sets in Euclidean space. Bubeck et al. (2012) apply tools from convex geometry, namely the John ellipsoid to construct optimal-regret algorithms for bandit linear optimization , albeit not always efficiently.

1. although uniform sampling does at times work fantastically well, i.e. Cesa-Bianchi and Lugosi (2012)

In this paper we consider a generic approach to exploration, and quantify what efficient exploration with *low variance* requires in general. Given a set in Euclidean space, a low-variance exploration basis is a subset with the following property: given noisy estimates of a linear function over the basis, one can construct an estimate for the linear function over the entire set without increasing the variance of the estimates.

By definition, such low variance exploration bases are immediately applicable to noisy linear regression: given a low-variance exploration basis, it suffices to learn the function values only over the basis in order to interpolate the value of the underlying linear regressor over the entire decision set. This fact can be used for active learning as well as for the exploration component of a bandit linear optimization algorithm.

Henceforth we define a novel construction for a low variance exploration basis called **volumetric spanners** and give efficient algorithms to construct them. We further investigate the convex geometry implications of our construction, and define the notion of a **minimal volumetric ellipsoid** of a convex body. We give structural theorems on the existence and properties of these ellipsoids, as well as constructive algorithms to compute them in several cases.

We complement our findings with an application to machine learning, in which we advance a well-studied open problem that has exploration as its core difficulty: an efficient and near-optimal regret algorithm for bandit linear optimization (BLO). We expect that volumetric spanners and volumetric ellipsoids can be useful elsewhere in experiment design and active learning. We briefly discuss the application to BLO next.

Bandit Linear Optimization Bandit linear optimization (BLO) is a fundamental problem in decision making under uncertainty that efficiently captures structured action sets. The canonical example is that of online routing in graphs: a decision maker iteratively chooses a path in a given graph from source to destination, the adversary chooses lengths of the edges of the graph, and the decision maker receives as feedback the length of the path she chose but no other information (see Awerbuch and Kleinberg (2008)). Her goal over many iterations is to attain an average travel time as short as that of the best fixed shortest path in the graph.

This decision problem is readily modeled in the "experts" framework, albeit with efficiency issues: the number of possible paths is potentially exponential in the graph representation. The BLO framework gives an efficient model for capturing such structured decision problems: iteratively a decision maker chooses a point in a convex set and receives as a payoff an adversarially chosen linear cost function. In the particular case of online routing, the decision set is taken to be the s-t-flow polytope, which captures the convex hull of all source-destination shortest paths in a given graph, and has a succinct representation with polynomially many constraints and low dimensionality. The linear cost function corresponds to a weight function on the graphs edges, where the length of a path is defined as the sum of weights of its edges.

The BLO framework captures many other structured problems efficiently, e.g., learning permutations, rankings and other examples (see Abernethy et al. (2012)). As such, it has been the focus of much research in the past few years. The reader is referred to the recent survey of Bubeck and Cesa-Bianchi (2012) for more details on algorithmic results for BLO. Let us remark that certain online bandit problems do not immediately fall into the convex BLO model that we address, such as combinatorial bandits studied in Cesa-Bianchi and Lugosi (2012).

In this paper we contribute to the large literature on the BLO model by giving the first polynomial-time and near optimal-regret algorithm for BLO over general convex decision sets; see Section 6 for a formal statement . Previously efficient algorithms, with non-optimal-regret, were known over convex sets that admit an efficient self-concordant barrier (Abernethy et al., 2012), and optimal-regret algorithms were known over general sets (Bubeck et al., 2012) but these algorithms were not computationally efficient. Our result, based on volumetric spanners, is able to attain the best of both worlds.

1.1. Volumetric Ellipsoids and Spanners

We now describe the main convex geometric concepts we introduce and use for low variance exploration. To do so we first review some basic notions from convex geometry.

Let $\mathbb{R}^d$ be the d-dimensional vector space over the reals. Given a set of vectors $S = \{v_1, \ldots, v_t\} \subset \mathbb{R}^d$, we denote by $\mathcal{E}(S)$ the ellipsoid defined by S:

$$\mathcal{E}(S) = \left\{ \sum_{i \in S} \alpha_i v_i \ : \ \sum_i \alpha_i^2 \leq 1 \right\}.$$

By abuse of notation, we also say that $\mathcal{E}(S)$ is *supported* on the set S.

Ellipsoids play an important role in convex geometry and specific ellipsoids associated with a convex body have been used in previous works in machine learning for designing good exploration bases for convex sets $\mathcal{K} \subseteq \mathbb{R}^d$. For example, the notion of *barycentric spanners* which were introduced in the seminal work of Awerbuch and Kleinberg (2008) corresponds to looking at the ellipsoid of maximum volume supported by exactly d points from $\mathcal{K}$[2]. Barycentric Spanners have since been used as an exploration basis in several works: In (Dani et al., 2007) for online bandit linear optimization, in (Bartlett et al., 2008) for a high probability counterpart of the online bandit linear optimization, in (Kakade et al., 2009) for repeated decision making of approximable functions and in (Dani et al., 2008) for a stochastic version of bandit linear optimization. Another example is the work of Bubeck et al. (2012) which looks at the minimum volume enclosing ellipsoid (MVEE) also known as the John ellipsoid (see Section 2 for more background on this fundamental object from convex geometry).

As will be clear soon, our definition of a *minimal volumetric ellipsoid* enjoys the best properties of the examples above enabling us to get more efficient algorithms. Similar to barycentric spanners, it is supported by a small (linear) set of points of $\mathcal{K}$. Simultaneously and unlike the barycentric counterpart, the volumetric ellipsoid contains the body $\mathcal{K}$, a property shared with the John ellipsoid.

Definition 1 (Volumetric Ellipsoids) *Let $\mathcal{K} \subseteq \mathbb{R}^d$ be a set in Euclidean space. For $S \subseteq \mathcal{K}$, we say that $\mathcal{E}(S)$ is a* volumetric ellipsoid *for $\mathcal{K}$ if it contains $\mathcal{K}$. We say that $\mathcal{E}_{\mathcal{K}} = \mathcal{E}(S)$ is a* minimal volumetric ellipsoid *if it is a containing ellipsoid defined by a set of minimal cardinality*

$$\mathcal{E}_{\mathcal{K}} \in \min_{|S|} \left\{ \mathcal{E}(S) \ \textit{ such that } \ S \subseteq \mathcal{K} \subseteq \mathcal{E}(S) \right\}.$$

We say that $|S|$ is the order *of the minimal volumetric ellipsoid or of*[3] *$\mathcal{K}$ denoted* $\mathbf{order}(\mathcal{K})$.

2. While the definition of Awerbuch and Kleinberg (2008) is not phrased as such, their analysis shows the existence of barycentric spanners by looking at the maximum volume ellipsoid.
3. We note that our definition allows for multi-sets, meaning that S may contain the same vector more than once

We discuss various geometric properties of volumetric ellipsoids later. For now, we focus on their utility in designing efficient exploration bases. To make this concrete and to simplify some terminology later on (and also to draw an analogy to barycentric spanners), we introduce the notion of *volumetric spanners*. Informally, these correspond to sets S that span all points in a given set with coefficients having Euclidean norm at most one. Formally:

Definition 2 *Let $\mathcal{K} \subseteq \mathbb{R}^d$ and let $S \subseteq \mathcal{K}$. We say that S is a* volumetric spanner *for $\mathcal{K}$ if $\mathcal{K} \subseteq \mathcal{E}(S)$.*

Clearly, a set $\mathcal{K}$ has a volumetric spanner of cardinality at most t if and only if $\mathbf{order}(\mathcal{K}) \leq t$.

Our goal in this work is to bound the order of arbitrary sets. A priori, it is not even clear if there is a universal bound (depending only on the dimension and not on the set) on the order S for compact sets $\mathcal{K}$. However, barycentric spanners and the John ellipsoid show that the order of any compact set in $\mathbb{R}^d$ is at most $O(d^2)$. Our main structural result in convex geometry gives a nearly optimal linear bound on the order of all sets.

Theorem 3 *Any compact set $\mathcal{K} \subseteq \mathbb{R}^d$ admits a volumetric ellipsoid of order at most $12d$. Further, if $\mathcal{K} = \{v_1, \ldots, v_n\}$ is a discrete set, then a volumetric ellipsoid for $\mathcal{K}$ of order at most $12d$ can be constructed in time $O(n^{3.5} + dn^3 + d^5)$.*

We emphasize the last part of the above theorem giving an algorithm for finding volumetric spanners of small size; this could be useful in using our structural results for algorithmic purposes. We also give a different algorithmic construction for the discrete case (a set of n vectors) in Section 5. While being sub-optimal by logarithmic factors (gives an ellipsoid of order $O(d(\log d)(\log n))$ this alternate construction has the advantage of being simpler and more efficient to compute.

1.2. Approximate Volumetric Spanners

Theorem 3 shows the existence of good volumetric spanners and also gives an efficient algorithm for finding such a spanner in the discrete case, i.e. when $\mathcal{K}$ is finite and given explicitly. However, the existence proof uses the John ellipsoid in a fundamental way and it is not known how to compute (even approximately) the John ellipsoid efficiently for the case of general convex bodies. For such computationally difficult cases, we introduce a natural relaxation of volumetric ellipsoids which can be computed efficiently for a bigger class of bodies and is similarly useful. The relaxation comes from requiring that the ellipsoid of small support contain all but an ε fraction of the points in $\mathcal{K}$ (under some distribution). In addition, we also require that the measure of points decays exponentially fast w.r.t their $\mathcal{E}(S)$-norm (see precise definition in next section); this property gives us tighter control on the set of points not contained in the ellipsoid. When discussing a measure over the points of a body the most natural one is the uniform distribution over the body. However, it makes sense to consider other measures as well and our approximation results in fact hold for a wide class of distributions.

Definition 4 *Let $S \subseteq \mathbb{R}^d$ be a set of vectors and let V be the matrix whose columns are the vectors of S. We define the semi-norm*

$$\|x\|_{\mathcal{E}(S)} = \sqrt{x^\top (VV^\top)^{-1} x}\,,$$

where $(VV^\top)^{-1}$ *is the Moore-Penrose pseudo-inverse of* $VV^\top$*. Let* $\mathcal{K}$ *be a convex set in* $\mathbb{R}^d$ *and* p *a distribution over it. Let* $\varepsilon > 0$*. A* (p, ε)-exp-volumetric spanner *of* $\mathcal{K}$ *is a set* $S \subseteq \mathcal{K}$ *such that for any* $\theta > 1$

$$\Pr_{x \sim p}[\|x\|_{\mathcal{E}(S)} \geq \theta] \leq \varepsilon^{-\theta}.$$

We prove that spanners as above can be efficiently obtained for any log-concave distribution:

Theorem 5 *Let* $\mathcal{K}$ *be a convex set in* $\mathbb{R}^d$ *and* p *a log-concave distribution over it. By sampling* $O(d + \log^2(1/\varepsilon))$ *i.i.d. points from* p *one obtains, w.p. at least* $1 - \exp\left(-\sqrt{\max\{\log(1/\varepsilon), d\}}\right)$*, a* (p, ε)*-exp-volumetric spanner for* $\mathcal{K}$*.*

1.3. Structure of the paper

In the next section we list the preliminaries and known results from measure concentration, convex geometry and online learning that we need. In Section 3 we show the existence of small size volumetric spanners. In sections 4 and 5 we give efficient constructions of volumetric spanners for continuous and discrete sets, respectively. We then proceed to describe the application of our geometric results to bandit linear optimization in Section 6. Due to space limitations, most of the proofs are omitted from this extended abstract and the reader is referred to Hazan et al. (2013) for full details.

2. Preliminaries

We now describe several preliminary results we need from convex geometry and linear algebra. We start with some notation:

- A matrix $A \in \mathbb{R}^{d \times d}$ is positive semi-definite (PSD) when for all $x \in \mathbb{R}^d$ it holds that $x^\top A x \geq 0$. Alternatively, when all of its eigenvalues are non-negative. We say that $A \succeq B$ if $A - B$ is PSD.
- Given an ellipsoid $\mathcal{E}(S) = \{\sum_i \alpha_i v_i \,:\, \sum_i \alpha_i^2 \leq 1\}$, we shall use the notation $\|x\|_{\mathcal{E}(S)} \triangleq \sqrt{x^\top (VV^\top)^{-1} x}$ to denote the (Minkowski) semi-norm defined by the ellipsoid, where V is the matrix with the vectors v_i's as columns.
- Throughout, we denote by I_d the $d \times d$ identity matrix.

We next describe properties of the John ellipsoid which plays an important role in our proofs.

2.1. The Fritz John Ellipsoid

Let $\mathcal{K} \subseteq \mathbb{R}^n$ be an arbitrary convex body. Then, the **John ellipsoid** of $\mathcal{K}$ is the minimum volume ellipsoid containing $\mathcal{K}$. This ellipsoid is unique and its properties have been the subject of important study in convex geometry since the seminal work of John (1948) (see (Ball, 1997) and (Henk, 2012) for historic information).

Suppose that we have linearly transformed $\mathcal{K}$ such that its minimum volume enclosing ellipsoid (MVEE) is the unit sphere; in convex geometric terms, $\mathcal{K}$ is in *John's position*. The celebrated decomposition theorem of John (1948) gives a characterization of when a body is in John's position and will play an important role in our constructions (the version here is from (Ball, 1997)).

Below we consider only symmetric convex sets, i.e. sets that admit the following property: if $x \in \mathcal{K}$ then also $-x \in \mathcal{K}$. The sets encountered in machine learning applications are most always symmetric, since estimating a linear function on a point x is equivalent to estimating it on its polar $-x$, and negating the outcome.

Theorem 6 (Ball (1997)) *Let $\mathcal{K} \in \mathbb{R}^d$ be a symmetric set such that its MVEE is the unit sphere. Then there exist $m \le d(d+1)/2-1$ contact points of $\mathcal{K}$ and the sphere $u_1, \dots, u_m$ and non-negative weights $c_1, \dots, c_m$ such that $\sum_i c_i u_i = 0$ and $\sum c_i u_i u_i^T = I_d$.*

The contact points of a convex body have been extensively studied in convex geometry and they also make for an appealing exploration basis in our context. Indeed, Bubeck et al. (2012) use these contact points to attain an optimal-regret algorithm for BLO. Unfortunately we know of no efficient algorithm to compute, or even approximate, the John ellipsoid for a general convex set, thus the results of Bubeck et al. (2012) do not yield efficient algorithms for BLO.

For our construction of volumetric spanners we need to compute the MVEE of a discrete symmetric set, which can be done efficiently. We make use of the following (folklore) result:

Theorem 7 (folklore, see e.g. Khachiyan (1996); Damla Ahipasaoglu et al. (2008)) *Let $\mathcal{K} \subseteq \mathbb{R}^d$ be a set of n points. It is possible to compute an ε-approximate MVEE for $\mathcal{K}$ (an enclosing ellipsoid of volume at most $(1+\varepsilon)$ that of the MVEE) that is also supported on a subset of $\mathcal{K}$ in time $O(n^{3.5} \log \frac{1}{\varepsilon})$.*

The run-time above is attainable via the ellipsoid method or path-following interior point methods (see references in theorem statement). An approximation algorithm rather than an exact one is necessary in a real-valued computation model, and the logarithmic dependence on the approximation guarantee is as good as one can hope for in general.

The above theorem allows us to efficiently compute a linear transformation such that the MVEE of $\mathcal{K}$ is essentially the unit sphere. We can then use linear programming to compute an approximate decomposition like in John's theorem as follows (details given in full version of this paper Hazan et al. (2013)):

Theorem 8 *Let $\{x_1, \dots, x_n\} = \mathcal{K} \subseteq \mathbb{R}^d$ be a set of n points and assume that:*

1. *$\mathcal{K}$ is symmetric.*
2. *The John Ellipsoid of $\mathcal{K}$ is the unit sphere.*

Then it is possible, in $O((\sqrt{n}+d)n^3)$ time, to compute non-negative weights $c_1, \dots, c_n$ such that (1) $\sum_i c_i x_i = 0$ and (2) $\sum_{i=1}^n c_i x_i x_i^\top = I_d$.

We next state the results from probability theory that we need.

2.2. Distributions and Measure Concentration

For a set $\mathcal{K}$, let $x \sim \mathcal{K}$ denote a uniformly random vector from $\mathcal{K}$.

Definition 9 *A distribution over $\mathbb{R}^d$ is log-concave if its probability distribution function (pdf) p is such that for all $x, y \in \mathbb{R}^d$ and $\lambda \in [0,1]$,*

$$p(\lambda x + (1-\lambda)y) \ge p(x)^\lambda p(y)^{1-\lambda}$$

Two log-concave distributions of interest to us are (1) the uniform distribution over a convex body and (2) a distribution over a convex body where $p(x) \propto \exp(L^\top x)$, where L is some vector in $\mathbb{R}^d$. The following result shows that given oracle access to the pdf of a log-concave distribution we can sample from it efficiently. An oracle to a pdf accepts as input a point $x \in \mathbb{R}^d$ and returns the value $p(x)$.

Lemma 10 (Lovász and Vempala (2007), Theorems 2.1 and 2.2) *Let p be a log-concave distribution over $\mathbb{R}^d$ and let $\delta > 0$. Then, given oracle access to p, i.e. and oracle computing its pdf for any point in $\mathbb{R}^d$, there is an algorithm which approximately samples from p such that:*

1. *The total variation distance between the produced distribution and the distribution defined by p is no more than δ. That is, the difference between the probabilities of any event in the produced and actual distribution is bounded by δ.*
2. *The algorithm requires a pre-processing time of $\tilde{O}(d^5)$.*
3. *After pre-processing, each sample can be produced in time $\tilde{O}(d^4/\delta^4)$, or amortized time of $\tilde{O}(d^3/\delta^4)$ if more than d samples are needed.*

Definition 11 (Isotropic position) *A random variable x is said to be in isotropic position (or isotropic) if*

$$\mathbf{E}[x] = 0, \qquad \mathbf{E}[xx^\top] = I_d.$$

A set $\mathcal{K} \subseteq \mathbb{R}^d$ is said to be in isotropic position if $x \sim \mathcal{K}$ is isotropic. Similarly, a distribution p is isotropic if $x \sim p$ is isotropic.

The following theorem provides a concentration bound for random vectors originating from an arbitrary distribution.

Theorem 12 (Rudelson (1999)) *Let x be a vector-valued random variable over $\mathbb{R}^d$ with $\mathbf{E}[xx^\top] = \Sigma$ and $\|\Sigma^{-1/2}x\|^2 \le R$. Then, for independent copies $x_1, \ldots, x_n$ of x, and $n \ge CR\log(R/\varepsilon)/\epsilon^2$ the following holds with probability at least $1/2$:*

$$\left\| \frac{1}{n}\sum_{i=1}^{n} x_i x_i^\top - \Sigma \right\| \le \epsilon \|\Sigma\|.$$

Finally, we also make use of barycentric spanners in our application to BLO and we briefly describe them next.

2.3. Barycentric Spanners

Definition 13 (Awerbuch and Kleinberg (2008)) *A barycentric spanner of $\mathcal{K} \subseteq \mathbb{R}^d$ is a set of d points $S = \{u_1, \ldots, u_d\} \subseteq \mathcal{K}$ such that any point in $\mathcal{K}$ may be expressed as a linear combination of the elements of S using coefficients in $[-1, 1]$. For $C > 1$, S is a C-approximate barycentric spanner of $\mathcal{K}$ if any point in $\mathcal{K}$ may be expressed as a linear combination of the elements of S using coefficients in $[-C, C]$*

Figure 1: In $\mathbb{R}^2$ the order of the volumetric ellipsoid of the equilateral triangle centered at the origin is at least 3. If the vertices are $[0, 1], [-\frac{\sqrt{3}}{2}, -\frac{1}{2}], [\frac{\sqrt{3}}{2}, -\frac{1}{2}]$, then the eigenpoles of the ellipsoid of the bottom two vertices are $[0.\frac{2}{3}], [2, 0]$. The second figure shows one possibility for a volumetric ellipsoid by adding $\frac{3}{4}$ of the first vertex to the previous ellipsoid. This shows the ellipsoid to be non-unique, as it can be rotated three ways.

In (Awerbuch and Kleinberg, 2008) it is shown that any compact set has a barycentric spanner. Moreover, they show that given an oracle with the ability to solve linear optimization problems over $\mathcal{K}$, an approximate barycentric spanner can be efficiently obtained. In the following sections we will use this constructive result.

Theorem 14 (Proposition 2.5 in (Awerbuch and Kleinberg, 2008)) *Let $\mathcal{K}$ be a compact set in $\mathbb{R}^d$ that is not contained in any proper linear subspace. Given an oracle for optimizing linear functions over $\mathcal{K}$, for any $C > 1$, it is possible to compute a C-approximate barycentric spanner for $\mathcal{K}$, using $O(d^2 \log_C(d))$ calls to the optimization oracle.*

3. Existence of Volumetric Ellipsoids and Spanners

In this section we show the existence of low order volumetric ellipsoids proving our main structural result, Theorem 3. Before we do so, we first state a few simple properties of volumetric ellipsoids (recall the definition of $\mathbf{order}$ from Definition 1):

- The definition of $\mathbf{order}$ is linear invariant: for any invertible linear transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ and $K \subseteq \mathbb{R}^d$, $\mathbf{order}(\mathcal{K}) = \mathbf{order}(T\mathcal{K})$.
- The minimal volumetric ellipsoid is not unique in general; see example in figure 1. Further, it is in general different from the John ellipsoid.
- For non-degenerate bodies $\mathcal{K}$, their order is naturally lower bounded by d, and there are examples in which it is strictly more than d (e.g., figure 1).

To prove Theorem 3 we shall use John's decomposition theorem, Theorem 6.

Proof of Theorem 3

Let $\mathcal{K} \subseteq \mathbb{R}^d$ be a compact set. Without loss of generality assume that $\mathcal{K}$ is symmetric and contains the origin; we can do so as in the following we only look at outer products of the form $vv^\top$ for vectors $v \in \mathcal{K}$. Further, as $\mathbf{order}(\mathcal{K})$ is invariant under linear transformations, we can also assume that $\mathcal{K}$ has been linearly transformed to be in John's position.

Now, by Theorem 6, there are $m = O(d^2)$ points $S = \{u_1, \ldots, u_m\}$ on the enclosing unit ball that intersect $\mathcal{K}$ (on the boundary of $\mathcal{K}$), and non-negative weights $c_1, \ldots, c_m$ such that $\sum_{i \in S} c_i u_i u_i^\top = I_d$. This implies by taking trace on both sides that $\sum_i c_i = d$.

Our idea is to start with the vectors $u_1, \ldots, u_m$ as a starting point for a volumetric spanner. However, this set has $O(d^2)$ points which is larger than what we want. We now prune these contact points via the sparsification methods of Batson et al. (2012). We use the following lemma that is a corollary of their Theorem 3.1. A full derivation is given in Hazan et al. (2013)

Lemma 15 (Batson et al. (2012)) *Let $u_1, \ldots, u_m$ be unit vectors and let $p \in \Delta(m)$ be some distribution over $[m]$ such that $d \sum_{i=1}^m p_i u_i u_i^\top = I_d$. There exist some (multi) sub set S of $\{u_1, \ldots, u_m\}$ such that $\sum_{v \in S} vv^\top \succeq I_d$ and $|S| \leq 12d$. This set can be computed in time $O(md^3)$.*

Lemma 15 proves the existence of a spanner of size at most $12d$ for any compact set in $\mathbb{R}^d$. Furthermore, the running time to find the set is $O(md^3) = O(d^5)$, using the bound of $m = O(d^2)$. This along with the running time for obtaining the contact points leads to a total running time of $O(n^{3.5} + dn^3 + d^5)$. ■

4. Approximate Volumetric Spanners

Due to space restrictions we present the formal proof of Theorem 5 in Hazan et al. (2013). In our application of volumetric spanners to BLO, we also need the following relaxation of volumetric spanners where we allow ourselves the flexibility to scale the ellipsoid:

Definition 16 *A ρ-ratio-volumetric spanner S of $\mathcal{K}$ is a subset $S \subseteq \mathcal{K}$ such that for all $x \in \mathcal{K}$, $\|x\|_{\mathcal{E}(S)} \leq \rho$.*

One example for such an approximate spanner with $\rho = \sqrt{d}$ is a barycentric spanner (Definition 13). In fact, it is easy to see that a C-approximate barycentric spanner is a $C\sqrt{d}$-ratio-volumetric spanner . The following is immediate from Theorem 14.

Corollary 17 *Let $\mathcal{K}$ be a compact set in $\mathbb{R}^d$ that is not contained in any proper linear subspace. Given an oracle for optimizing linear functions over $\mathcal{K}$, for any $C > 1$, it is possible to compute a $C\sqrt{d}$-ratio-volumetric spanner S of $\mathcal{K}$ of cardinality $|S| = d$, using $O(d^2 \log_C(d))$ calls to the optimization oracle.*

5. Fast Volumetric Spanners for Discrete Sets

In this section we describe a different algorithm that constructs volumetric spanners for discrete sets. The order of the spanners we construct here is suboptimal (in particular, there is a dependence on the size of the set $\mathcal{K}$ which we didn't have before). However, the algorithm is particularly simple and efficient to implement (takes time linear in the size of the set).

Theorem 18 *Given a set of vectors $\mathcal{K} = \{x_1, \ldots, x_n\} \in \mathbb{R}^d$, Algorithm 1 outputs a volumetric spanner of size $O((d \log d)(\log n))$ and has an expected running time of $O(nd^2)$.*

The proof of this theorem is deferred to the appendix.

Algorithm 1

Input $\mathcal{K} = \{x_1, ..., x_n\} \subseteq \mathbb{R}^d$.
if $n < Cd \log d$ **then**
  return $S \leftarrow \mathcal{K}$
end if
Compute $\Sigma = \sum_i x_i x_i^\top$ and let $u_i = \Sigma^{-1/2} x_i$.
For $i \in [n]$, let $p_i = 1/2n + \|u_i\|^2/2d$. Let S be a random set obtained by drawing $M = Cd \log d$ samples with replacement from $[n]$ according to the distribution $p_1, \ldots, p_n$.
Verify that for at least $n/2$ vectors from $\{x_1, \ldots, x_n\}$, it holds that $\|x_i\|_{\mathcal{E}(S)} \leq 1$. If that is not the case discard S and repeat the above step.
Apply the algorithm recursively on the data points for which $\|x_i\|_{\mathcal{E}(S)} > 1$.

6. Bandit Linear Optimization

In this section we describe our efficient algorithm for Bandit Linear Optimization (BLO) using the aforementioned techniques, and show it attains nearly tight expected regret bounds.

Recall the setting of BLO. The decision maker is given an underling convex decision set $\mathcal{K} \subseteq \mathbb{R}^d$. Iteratively at each time sequence t, the environment chooses a loss vector $L_t \in \mathbb{R}^d$ that is not revealed to the player. The player chooses a vector $x_t \in \mathcal{K}$ and once she commits to her choice, the loss $\ell_t = x_t^\top L_t$ (a single real number) is revealed. The objective is to minimize the (expected) loss after T rounds–the regret–defined as the strategy's loss minus the loss of the best fixed strategy of choosing some $x^* \in \mathcal{K}$, where the expectation is taken with respect to the algorithms internal randomization.

$$\text{Regret} = \mathbf{E}\left[\sum_{t=1}^{T} x_t^\top L_t\right] - \min_{x^* \in \mathcal{K}} \sum_{t=1}^{T} (x^*)^\top L_t.$$

The setting of BLO is a natural generalization of the classical Multi-Armed Bandit problem and extremely useful for efficiently modeling decision making under partial feedback for structured problems. As such the research literature is rich with algorithms and insights into this fundamental problem. For a brief historical survey please refer to earlier sections of this manuscript. In this section we focus on the first efficient and optimal-regret algorithm, and thus immediately jump to Algorithm 2. We make the following assumptions about the decision set $\mathcal{K}$:

1. The set $\mathcal{K}$ is equipped with a membership oracle. This implies via (Lovász and Vempala, 2007) (Lemma 10) that there exists an efficient algorithm for sampling from a given log-concave distribution over $\mathcal{K}$. Via the discussion in previous sections, this also implies that we can construct approximate (both types of approximations, see Definitions 4 and 16) volumetric spanners efficiently over $\mathcal{K}$.

2. The losses are bounded in absolute values by 1. That is, the loss functions are always chosen (by an oblivious adversary) from a convex set $\mathcal{Z}$ such that $\mathcal{K}$ is contained in its polar, i.e. $\forall L \in \mathcal{Z}, x \in \mathcal{K}$, $|L^\top x| \leq 1$ (this is referred to as the 'L_2 assumption', see (Audibert et al., 2011)). This implies that the set $\mathcal{K}$ admits for any $\varepsilon > 0$ an ε-net with respect to the norm defined by $\mathcal{Z}$ of size $(\varepsilon/2)^{-d}$. Henceforth, we denote the size of the net by $|K|_\varepsilon$.

For Algorithm 2 we prove the following regret bound. It was shown in (Bubeck and Cesa-Bianchi, 2012) that this bound is tight up to poly-logarithmic factors for some convex sets.

Algorithm 2 GeometricHedge with Volumetric Spanners Exploration

$\mathcal{K}$, parameters γ, η, horizon T.
$p_1(x)$ uniform distribution over $\mathcal{K}$.
for $t = 1$ to T **do**
  Let S'_t be a $(p_t, \exp(-(4\sqrt{d} + \log(2T))))$-exp-volumetric spanner of $\mathcal{K}$.
  Let S''_t be a $2\sqrt{d}$-ratio-volumetric spanner of $\mathcal{K}$
  Set S_t as the union of S'_t, S''_t.
  $\hat{p}_t(x) = (1-\gamma)p_t(x) + \frac{\gamma}{|S_t|}1_{x \in S_t}$
  sample x_t according to $\hat{p}_t$
  observe loss $\ell_t \triangleq L_t^\top x_t$
  Let $C_t \triangleq \mathbf{E}_{x \sim \hat{p}_t}[xx^\top]$
  $\hat{L}_t \triangleq \ell_t C_t^{-1} x_t$
  $p_{t+1}(x) \propto p_t(x) e^{-\eta \hat{L}_t^\top x}$
end for

Theorem 19 *Under the assumptions stated above, and let* $s = \max_t |S_t|$, $\eta = \sqrt{\frac{\log |\mathcal{K}|_{1/T}}{dT}}$ *and let* $\gamma = s\sqrt{\frac{\log(|\mathcal{K}|_{1/T})}{dT}}$. *Algorithm 2 given parameters* γ, η *suffers a regret bounded by*

$$O\left((s+d)\sqrt{\frac{T \log |\mathcal{K}|_{1/T}}{d}}\right)$$

We note that while the size $\log(|\mathcal{K}|_{1/T})$ can be bounded by $d\log(T)$, in certain scenarios such as s-t paths in graphs it is possible to obtain sharper upper bounds that immediately imply better regret via Theorem 19.

To prove the theorem we follow the general methodology used in analyzing the performance of the geometric hedge algorithm. The major deviation from standard technique is the following sub-exponential tail bound, which we use to replace the the standard absolute bound for $|\hat{L}_t x|$. Due to space restrictions we state the tail bound and a helpful auxiliary lemma, their full proofs appear in Hazan et al. (2013).

Lemma 20 *Let* $x \sim p_t$, $x_t \sim \hat{p}_t$ *and let* $\hat{L}_t$ *be defined according to* x_t. *It holds, for any* $\theta > 1$ *that*

$$\Pr\left[\left|\hat{L}_t^\top x\right| > \frac{\theta s}{\gamma}\right] \leq \exp(-2\theta)/T$$

Lemma 21 *For all* $x \in \mathcal{K}$ *it holds that* $\left|\hat{L}_t^\top x\right| \leq \frac{|S_t| \|x\|_{\mathcal{E}(S_t)} \|x_t\|_{\mathcal{E}(S_t)}}{\gamma}$.

Implementation To analyze the running time of the algorithm, there are two computations that should be pointed out. One is a preprocessing step of computing the $2\sqrt{d}$-ratio-volumetric spanner . According to Corollary 17, it can be efficiently constructed assuming we can solve a linear optimization problem in the convex body $\mathcal{K}$. The second computation is that of sampling a vector from the distribution $\hat{p}_t$, which is in fact reducible to the problem of sampling a vector from the distribution p_t, which is a log-concave distribution over $\mathcal{K}$ whose pdf can be efficiently computed for any point in $\mathbb{R}^d$, assuming access to a membership oracle for $\mathcal{K}$. Here, we could use Lemma 10 in order to approximately sample from the required distribution.

A naive analysis results in a running time polynomial in both d and T for each such step, completing the proof of the existence of a solution with polynomial running time. A more refined analysis for exactly such processes can be found in (Dick et al., 2014). Their analysis of the continuous exponential weights algorithm (i.e., the geometric hedge algorithm) in their section 3.2 and 3.3, gives rise to a sampling technique that approximates the required log-concave distribution in a manner that incurs an additional $O(\sqrt{T})$ regret, at the running time cost of $\text{poly}(d, \log(T))$ per sample. As the purpose of this paper is to provide an algorithm of polynomial running time, we defer the complete analysis of the running time to the full version of the paper.

Discussion: Notice that to obtain a (p, ε)-exp-volumetric spanner for a log-concave distribution p over a body $\mathcal{K}$ we simply choose sufficiently many i.i.d samples from p. Since in the above algorithm p_t is always log-concave, it follows that S'_t consists of i.i.d samples from p_t, meaning that if we would not have required S''_t, the exploration and exploration strategies would be the same! Since we still require the set S''_t, there exists a need for a separate exploration strategy. Interestingly, the $2\sqrt{d}$-ratio-volumetric spanner is obtained by taking a barycentric spanner, which is the exploration strategy of Dani et al. (2007).

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Appendix A. Proof of Theorem 18

Proof [of Theorem 18] Consider a single iteration of the algorithm with input $v_1, \ldots, v_n \in \mathbb{R}^d$. We claim that the random set S obtained in step 6 satisfies the following condition with constant probability:

$$\Pr_{x \in \mathcal{K}} \left[\|x\|_{\mathcal{E}(S)} \le 1 \right] \ge 1/2 \tag{1}$$

Suppose the above statement is true. Then, the lemma follows easily as it implies that for the next iteration there are fewer than $n/2$ vectors. Hence, after $(\log n)$ recursive calls we will have a volumetric spanner. The total size of the set will be $O((d \log d)(\log n))$. To see the time complexity, consider a single run of the algorithm. The most computationally intensive steps are computing Σ and $\Sigma^{-1/2}$ which take time $O(nd^2)$ and $O(d^3)$ respectively. We also need to compute $(\sum_{v \in S} vv^\top)^{-1}$ (to compute the $\mathcal{E}(S)$ norm) which takes time $O(d^3 \log d)$, and compute the $\mathcal{E}(S)$ norm of all the vectors which requires $O(nd^2)$. As $n = \Omega(d \log(d))$, it follows that a single iteration runs of a total expected time of $O(nd^2)$. Since the size of n is split in half between iterations, the claim follows.

We now prove that Equation 1 holds with constant probability

$$x_j^\top \left(\sum_{v \in S} vv^\top \right)^{-1} x_j = u_j^\top \left(\sum_{v \in S'} vv^\top \right)^{-1} u_j. \tag{2}$$

where $S' = \{\Sigma^{-1/2} v | v \in S\}$ is the (linearly) shifted version of S. Therefore, it suffices to show that with sufficiently high probability, the right hand side of the above equation is bounded by 1 for at least $n/2$ indices $j \in [n]$.

Note that $p_i = 1/2n + \|u_i\|^2/2d$ form a probability distribution: $\sum_i p_i = 1/2 + (\sum_i \|u_i\|^2)/2d = 1$. Let $X \in \mathbb{R}^d$ be a random variable with $X = u_i/\sqrt{p_i}$ with probability p_i for $i \in [n]$. Then, $\mathbf{E}[XX^\top] = I_d$. Further, for any $i \in [n]$

$$\|u_i\|^2/p_i \le 2d.$$

Therefore, by Theorem 12, if we take $M = Cd(\log d)$ samples $X_1, \ldots, X_M$ for C sufficiently large, then with probability of at least $1/2$, it holds that

$$\sum_{i=1}^{M} X_i X_i^\top \succeq (M/2) I_d.$$

Let $T \subseteq [n]$ be the multiset corresponding to the indices of the sampled vectors $X_1, \ldots, X_M$. The above inequality implies that

$$\sum_{i \in T} \frac{1}{p_i} u_i u_i^\top \succeq (M/2) I_d.$$

Now,

$$\sum_{v \in S'} vv^\top \succeq (\min_i p_i) \sum_{v \in S'} \frac{1}{p_i} vv^\top \succeq (\min_i p_i)(M/2) I_d \succeq (M/4n) I_d.$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{n} u_i^\top \left(\sum_{v\in S'} vv^\top\right)^{-1} u_i &= \sum_{i=1}^{n} \operatorname{Tr}\left(\left(\sum_{v\in S'} vv^\top\right)^{-1} \left(u_i u_i^\top\right)\right) \\ &= \operatorname{Tr}\left(\left(\sum_{v\in S'} vv^\top\right)^{-1} \left(\sum_{i=1}^{n} u_i u_i^\top\right)\right) \\ &= \operatorname{Tr}\left(\left(\sum_{v\in S'} vv^\top\right)^{-1}\right) \\ &\le \frac{4nd}{M} \le \frac{4n}{C\log d} \le \frac{n}{2\log d},\end{aligned}$$

for C sufficiently large. Therefore, by Markov's inequality and Equation 2, it follows that Equation 1 holds with high probability. The theorem now follows. ■