Proofs

Lemma 1. *The constrained optimization of* (19) *is equivalent to:*

$$\underset{\{f(\vec{\mathbf{u}}_{1:T}||\vec{\mathbf{z}}_{1:T},\vec{\mathbf{x}}_{1:T})\}}{\text{argmax}} H(\vec{\mathbf{U}}_{1:T}||\vec{\mathbf{Z}}_{1:T},\vec{\mathbf{X}}_{1:T}) \tag{34}$$

$$\textit{where: } f(\vec{\mathbf{u}}_{1:t}||\vec{\mathbf{z}}_{1:T},\vec{\mathbf{x}}_{1:T}) = \prod_{t=1}^{T} f(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t},\vec{\mathbf{x}}_{1:t}); \tag{35}$$

$$\forall t \in \{1\cdots T\}, \vec{\mathbf{u}}_{1:t} \in \vec{\mathcal{U}}_{1:t}, \vec{\mathbf{z}}_{1:t} \in \vec{\mathcal{Z}}_{1:t}, \vec{\mathbf{x}}_{1:t} \in \vec{\mathcal{X}}_{1:t}, \vec{\mathbf{x}}'_{1:t} \in \vec{\mathcal{X}}_{1:t},$$

$$\textit{such that } f(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t},\vec{\mathbf{x}}_{1:t}) \geq 0, \int_{\vec{u}_t\in\vec{\mathcal{U}}_t} f(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t},\vec{\mathbf{x}}_{1:t}) = 1, \tag{36}$$

$$f(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t},\vec{\mathbf{x}}_{1:t}) = f(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t},\vec{\mathbf{x}}'_{1:t}). \tag{37}$$

Proof of Lemma 1. The previously developed theory of maximum causal entropy [28] shows the causally conditioned probability distribution defined according to affine constraint (15),(16) and (18) are equivalent to it defined by the decomposition into a product of conditional probabilities (35),(36). Then, we show partial observability constraint (17) implies (37).

$$\forall \vec{\mathbf{u}}_{1:T} \in \vec{\mathcal{U}}_{1:T}, \vec{\mathbf{x}}_{1:T}, \mathbf{x}'_{1:T} \in \vec{\mathcal{X}}_{1:T}, \vec{\mathbf{z}}_{1:T} \in \vec{\mathcal{Z}}_{1:T},$$

$$\prod_{t=1}^{T} f(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t},\vec{\mathbf{x}}_{1:t}) = \prod_{t=1}^{T} f(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t},\vec{\mathbf{x}}'_{1:t})$$

It is possible, $f(\vec{u}_1|\vec{z}_1,\vec{x}_1)\cdots f(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t},\vec{\mathbf{x}}_{1:t})\cdots f(\vec{u}_T|\vec{\mathbf{u}}_{1:T-1},\vec{\mathbf{z}}_{1:T},\vec{\mathbf{x}}_{1:T})$

$= f(\vec{u}_1|\vec{z}_1,\vec{x}_1)\cdots f(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t},\vec{\mathbf{x}}'_{1:t})\cdots f(\vec{u}_T|\vec{\mathbf{u}}_{1:T-1},\vec{\mathbf{z}}_{1:T},\vec{\mathbf{x}}_{1:T})$

Thus, $f(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t},\vec{\mathbf{x}}_{1:t}) = f(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t},\vec{\mathbf{x}}'_{1:t})$.

It is easy to show (37) implies (17). □

Lemma 2. *The constrained optimization defined in Lemma 1 is equivalent to:*

$$\underset{\{f(\vec{\mathbf{u}}_{1:T}||\vec{\mathbf{z}}_{1:T})\}}{\text{argmax}} H(\vec{\mathbf{U}}_{1:T}||\vec{\mathbf{Z}}_{1:T}) \tag{38}$$

$$\forall \vec{\mathbf{u}}_{1:T} \in \vec{\mathcal{U}}_{1:T}, \vec{\mathbf{z}}_{1:T}, \mathbf{z}'_{1:T} \in \vec{\mathcal{Z}}_{1:T},$$

$$f(\vec{\mathbf{u}}_{1:T}||\vec{\mathbf{z}}_{1:T} \geq 0, \int_{\vec{\mathbf{u}}'_{1:T}\in\vec{\mathcal{U}}_{1:T}} f(\vec{\mathbf{u}}'_{1:T}||\vec{\mathbf{z}}_{1:T})\, d\vec{\mathbf{u}}'_{1:T} = 1, \tag{39}$$

$$\forall \tau \in \{1,\cdots,T\} \textit{ such that } \vec{\mathbf{z}}_{1:\tau} = \vec{\mathbf{z}}'_{1:\tau},$$

$$\int_{\vec{\mathbf{u}}_{\tau+1:T}\in\vec{\mathcal{U}}_{\tau+1:T}} f(\vec{\mathbf{u}}_{1:T}||\vec{\mathbf{z}}_{1:T})\, d\vec{\mathbf{u}}_{\tau+1:T} = \int_{\vec{\mathbf{u}}_{\tau+1:T}\in\vec{\mathcal{U}}_{\tau+1:T}} f(\vec{\mathbf{u}}_{1:T}||\vec{\mathbf{z}}'_{1:T})\, d\vec{\mathbf{u}}_{\tau+1:T}. \tag{40}$$

Proof of Lemma 2.

$$\forall t \in \{1,\cdots,T\}, \vec{\mathbf{u}}_{1:t} \in \vec{\mathcal{U}}_{1:t}, \vec{\mathbf{z}}_{1:t} \in \vec{\mathcal{Z}}_{1:t}, \vec{\mathbf{x}}_{1:t} \in \vec{\mathcal{X}}_{1:t}, \vec{\mathbf{x}}'_{1:t} \in \vec{\mathcal{X}}_{1:t},$$

$$f(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t},\vec{\mathbf{x}}_{1:t}) = f(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t},\vec{\mathbf{x}}'_{1:t}) = f(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t})$$

$$\text{Then, } \prod_{t=1}^{T} f(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t},\vec{\mathbf{x}}_{1:t}) = \prod_{t=1}^{T} f(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t}) = f(\vec{\mathbf{u}}_{1:T}||\vec{\mathbf{z}}_{1:T})$$

Similar to the proof of Lemma 1, the causally conditioned probability distribution defined by a product of conditional probabilities are equivalent to the affine constraint (39), (40).

To show the object function (38) is equivalent to (19), we first show

$$\begin{aligned}
&\int_{\vec{\mathbf{x}}_{1:T}\in\vec{\mathcal{X}}_{1:T}} f(\vec{\mathbf{z}}_{1:T},\vec{\mathbf{x}}_{1:T}||\vec{\mathbf{u}}_{1:T-1})\\
&=\frac{\int_{\vec{\mathbf{x}}_{1:T}\in\vec{\mathcal{X}}_{1:T}}\prod_{t=1}^{T} f(\vec{z}_t,\vec{x}_t|\vec{\mathbf{z}}_{1:t-1},\vec{\mathbf{x}}_{1:t-1},\vec{\mathbf{u}}_{1:t-1})\prod_{t=1}^{T} f(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t},\vec{\mathbf{x}}_{1:t})}{\prod_{t=1}^{T} f(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t})}\\
&=\frac{\int_{\vec{\mathbf{x}}_{1:T}\in\vec{\mathcal{X}}_{1:T}} f(\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T},\vec{\mathbf{x}}_{1:T})}{\prod_{t=1}^{T} f(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t})}=\frac{f(\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T})}{\prod_{t=1}^{T} f(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t})}\\
&=\frac{\prod_{t=1}^{T} f(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t})\prod_{t=1}^{T} f(\vec{z}_t|\vec{\mathbf{z}}_{1:t-1},\vec{\mathbf{u}}_{1:t-1})}{\prod_{t=1}^{T} f(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t})}\\
&=\prod_{t=1}^{T} f(\vec{z}_t|\vec{\mathbf{z}}_{1:t-1},\vec{\mathbf{u}}_{1:t-1})=f(\vec{\mathbf{z}}_{1:T}||\vec{\mathbf{u}}_{1:T-1})
\end{aligned}$$

$$\begin{aligned}
&\text{Then, } H(\vec{\mathbf{U}}_{1:T}||\vec{\mathbf{Z}}_{1:T},\vec{\mathbf{X}}_{1:T})\\
&=\int_{\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T},\vec{\mathbf{x}}_{1:T}} f(\vec{\mathbf{u}}_{1:T}||\vec{\mathbf{z}}_{1:T},\vec{\mathbf{x}}_{1:T})f(\vec{\mathbf{z}}_{1:T},\vec{\mathbf{x}}_{1:T}||\vec{\mathbf{u}}_{1:T-1})\log f(\vec{\mathbf{u}}_{1:T}||\vec{\mathbf{z}}_{1:T},\vec{\mathbf{x}}_{1:T})\\
&=\int_{\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T}} f(\vec{\mathbf{u}}_{1:T}||\vec{\mathbf{z}}_{1:T})\log f(\vec{\mathbf{u}}_{1:T}||\vec{\mathbf{z}}_{1:T})\int_{\vec{\mathbf{x}}_{1:T}} f(\vec{\mathbf{z}}_{1:T},\vec{\mathbf{x}}_{1:T}||\vec{\mathbf{u}}_{1:T-1})\\
&=\int_{\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T}} f(\vec{\mathbf{u}}_{1:T}||\vec{\mathbf{z}}_{1:T})\log f(\vec{\mathbf{u}}_{1:T}||\vec{\mathbf{z}}_{1:T})f(\vec{\mathbf{z}}_{1:T}||\vec{\mathbf{u}}_{1:T-1})\\
&=H(\vec{\mathbf{U}}_{1:T}||\vec{\mathbf{Z}}_{1:T})
\end{aligned}$$

□

Lemma 3. *Suppose the constrained optimization problem in Lemma 2 has the following additional constraint:* $(F:\vec{\boldsymbol{\mathcal{U}}}_{1:T}\times\vec{\boldsymbol{\mathcal{Z}}}_{1:T}\to R^N,\vec{\mathbf{c}}\in R^N)$

$$\mathbb{E}_{f(\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T})}\left[F(\vec{\mathbf{U}}_{1:T},\vec{\mathbf{Z}}_{1:T})\right]=\vec{\mathbf{c}} \tag{41}$$

Then the solution to this optimization problem has the form:

$$\hat{f}(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t})=e^{Q(\vec{\mathbf{u}}_{1:t},\vec{\mathbf{z}}_{1:t})-V(\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t})}$$

where Q and V functions take the following recursive form:

$$\begin{aligned}
Q(\vec{\mathbf{u}}_{1:t},\vec{\mathbf{z}}_{1:t})&=\begin{cases}\lambda^T F(\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T}), & t=T;\\ \mathbb{E}[V(\vec{\mathbf{U}}_{1:t},\vec{\mathbf{Z}}_{1:t+1})|\vec{\mathbf{u}}_{1:t},\vec{\mathbf{z}}_{1:t}], & t<T\end{cases}\\
V(\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t})&=\operatorname*{softmax}_{\vec{u}_t} Q(\vec{\mathbf{u}}_{1:t},\vec{\mathbf{z}}_{1:t})\triangleq\log\int_{\vec{u}_t} e^{Q(\vec{\mathbf{u}}_{1:t},\vec{\mathbf{z}}_{1:t})}d\vec{u}_t
\end{aligned}$$

Proof of Lemma 3. We first show for any joint distribution $g(\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T})$, the following equation holds:

$$\mathbb{E}_g\left[-\log\hat{f}(\vec{\mathbf{U}}_{1:T}||\vec{\mathbf{Z}}_{1:T})\right]=\int_{\vec{z}_1} f(\vec{z}_1)V(\vec{z}_1)-\mathbb{E}_g\left[\lambda^T F(\vec{\mathbf{U}}_{1:T},\vec{\mathbf{Z}}_{1:T})\right] \tag{42}$$

$$\mathbb{E}_g\left[\sum_{t=1}^{T} -\log \hat{f}(\vec{U}_t|\vec{\mathbf{U}}_{1:t-1},\vec{\mathbf{Z}}_{1:t})\right]$$
$$= \mathbb{E}_g\left[-\lambda^T F(\vec{\mathbf{U}}_{1:T},\vec{\mathbf{Z}}_{1:T}) - \sum_{t=1}^{T-1} Q(\vec{\mathbf{U}}_{1:t},\vec{\mathbf{Z}}_{1:t}) + \sum_{t=1}^{T} V(\vec{\mathbf{U}}_{1:t-1},\vec{\mathbf{Z}}_{1:t})\right]$$
$$= \mathbb{E}_g\left[-\lambda^T F(\vec{\mathbf{U}}_{1:T},\vec{\mathbf{Z}}_{1:T})\right] - \int_{\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T}} g(\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T}) \sum_{t=1}^{T-1} \int_{\vec{z}_{t+1}} f(\vec{z}_{t+1}|\vec{\mathbf{u}}_{1:t},\vec{\mathbf{z}}_{1:t}) V(\vec{\mathbf{u}}_{1:t},\vec{\mathbf{z}}_{1:t+1})$$
$$+ \int_{\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T}} g(\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T}) \sum_{t=1}^{T} V(\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t})$$
$$= \mathbb{E}_g\left[-\lambda^T F(\vec{\mathbf{U}}_{1:T},\vec{\mathbf{Z}}_{1:T})\right] - \sum_{t=1}^{T-1} \int_{\vec{\mathbf{u}}_{1:t},\vec{\mathbf{z}}_{1:t+1}} g(\vec{\mathbf{u}}_{1:t},\vec{\mathbf{z}}_{1:t+1}) V(\vec{\mathbf{u}}_{1:t},\vec{\mathbf{z}}_{1:t+1})$$
$$+ \int_{\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T}} g(\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T}) \sum_{t=1}^{T} V(\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t})$$

which implies equation (42).

For any arbitrary causally conditional probability distribution $g(\vec{\mathbf{u}}_{1:T}||\vec{\mathbf{z}}_{1:T})$ satisfies with expectation constraint (41), we show:

$$H_g(\vec{\mathbf{U}}_{1:T}||\vec{\mathbf{Z}}_{1:T}) \le H_{\hat{f}}(\vec{\mathbf{U}}_{1:T}||\vec{\mathbf{Z}}_{1:T})$$

$$\mathbb{E}_g\left[-\log g(\vec{\mathbf{U}}_{1:T}||\vec{\mathbf{Z}}_{1:T})\right]$$
$$= -\int_{\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T}} g(\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T}) \log\left(\frac{g(\vec{\mathbf{u}}_{1:T}||\vec{\mathbf{z}}_{1:T}) f(\vec{\mathbf{z}}_{1:T}||\vec{\mathbf{u}}_{1:T-1})}{\hat{f}(\vec{\mathbf{u}}_{1:T}||\vec{\mathbf{z}}_{1:T}) f(\vec{\mathbf{z}}_{1:T}||\vec{\mathbf{u}}_{1:T-1})} \hat{f}(\vec{\mathbf{u}}_{1:T}||\vec{\mathbf{z}}_{1:T})\right)$$
$$= -D_{KL}\left(g(\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T})||\hat{f}(\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T})\right) - \int_{\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T}} g(\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T}) \log \hat{f}(\vec{\mathbf{u}}_{1:T}||\vec{\mathbf{z}}_{1:T})$$
$$\le -\int_{\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T}} g(\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T}) \log \hat{f}(\vec{\mathbf{u}}_{1:T}||\vec{\mathbf{z}}_{1:T})$$
$$= \int_{\vec{z}_1} f(\vec{z}_1) V(\vec{z}_1) - \mathbb{E}_g\left[\lambda^T F(\vec{\mathbf{U}}_{1:T},\vec{\mathbf{Z}}_{1:T})\right]$$
$$= \int_{\vec{z}_1} f(\vec{z}_1) V(\vec{z}_1) - \mathbb{E}_{\hat{f}}\left[\lambda^T F(\vec{\mathbf{U}}_{1:T},\vec{\mathbf{Z}}_{1:T})\right]$$
$$= H_{\hat{f}}(\vec{\mathbf{U}}_{1:T}||\vec{\mathbf{Z}}_{1:T})$$

D_{KL} is the Kullback-Leibler divergence which is non-negative[8]. Thus, $\hat{f}(\vec{u}_t|\vec{\mathbf{u}}_{1:t-1},\vec{\mathbf{z}}_{1:t})$ is the solution to the optimization problem in Lemma 2 incorporates with expectation constraint (41). □

Proof of Theorem 1. We first incorporate the expectation constraint (20) into the constrained optimization problem defined in Lemma 2

$$\mathbb{E}_{f(\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T+1},\vec{\mathbf{x}}_{1:T+1})}\left[\sum_{t=1}^{T+1} \vec{\mathbf{X}}_t \vec{\mathbf{X}}_t^T\right]$$
$$= \int_{\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T+1},\vec{\mathbf{x}}_{1:T+1}} f(\vec{\mathbf{u}}_{1:T}||\vec{\mathbf{z}}_{1:T},\vec{\mathbf{x}}_{1:T}) f(\vec{\mathbf{z}}_{1:T+1},\vec{\mathbf{x}}_{1:T+1}||\vec{\mathbf{u}}_{1:T}) \sum_{t=1}^{T+1} \vec{x}_t \vec{x}_t^T$$
$$= \int_{\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T}} f(\vec{\mathbf{u}}_{1:T}||\vec{\mathbf{z}}_{1:T}) f(\vec{\mathbf{z}}_{1:T}||\vec{\mathbf{u}}_{1:T-1}) \frac{\int_{\vec{\mathbf{x}}_{1:T+1},\vec{z}_{T+1}} f(\vec{\mathbf{z}}_{1:T+1},\vec{\mathbf{x}}_{1:T+1}||\vec{\mathbf{u}}_{1:T}) \sum_{t=1}^{T+1} \vec{x}_t \vec{x}_t^T}{f(\vec{\mathbf{z}}_{1:T}||\vec{\mathbf{u}}_{1:T-1})}$$
$$= \mathbb{E}_{f(\vec{\mathbf{u}}_{1:T},\vec{\mathbf{z}}_{1:T})}\left[\frac{\int_{\vec{\mathbf{X}}_{1:T+1},\vec{Z}_{T+1}} f(\vec{\mathbf{Z}}_{1:T+1},\vec{\mathbf{X}}_{1:T+1}||\vec{\mathbf{U}}_{1:T}) \sum_{t=1}^{T+1} \vec{\mathbf{X}}_t \vec{\mathbf{X}}_t^T}{f(\vec{\mathbf{Z}}_{1:T}||\vec{\mathbf{U}}_{1:T-1})}\right]$$

According to Lemma 3, the solution to the constrained problem defined in Lemma 2 incorporates with the expected constraint (41) takes the following recursive form:

$$Q(\vec{\mathbf{u}}_{1:t}, \vec{\mathbf{z}}_{1:t}) = \begin{cases} \frac{\int_{\vec{\mathbf{x}}_{1:T+1}, \vec{z}_{T+1}} f(\vec{\mathbf{z}}_{1:T+1}, \vec{\mathbf{x}}_{1:T+1} || \vec{\mathbf{u}}_{1:T}) \sum_{t=1}^{T+1} \vec{x}_t^T \mathbf{M} \vec{x}_t}{f(\vec{\mathbf{z}}_{1:T} || \vec{\mathbf{u}}_{1:T-1})}, & t = T; \\ \mathbb{E}[V(\vec{\mathbf{U}}_{1:t}, \vec{\mathbf{Z}}_{1:t+1}) | \vec{\mathbf{u}}_{1:t}, \vec{\mathbf{z}}_{1:t}], & t < T \end{cases}$$

$$V(\vec{\mathbf{u}}_{1:t-1}, \vec{\mathbf{z}}_{1:t}) = \operatorname*{softmax}_{\vec{u}_t} Q(\vec{\mathbf{u}}_{1:t}, \vec{\mathbf{z}}_{1:t}) \triangleq \log \int_{\vec{u}_t} e^{Q(\vec{\mathbf{u}}_{1:t}, \vec{\mathbf{z}}_{1:t})} d\vec{u}_t$$

$$Q(\vec{\mathbf{u}}_{1:T}, \vec{\mathbf{z}}_{1:T}) = \underbrace{\frac{\int_{\vec{\mathbf{x}}_{1:T}} f(\vec{\mathbf{z}}_{1:T}, \vec{\mathbf{x}}_{1:T} || \vec{\mathbf{u}}_{1:T-1}) \mathbb{E}\left[\vec{\mathbf{X}}_{T+1}^T \mathbf{M} \vec{\mathbf{X}}_{T+1} | \vec{\mathbf{X}}_T, \vec{u}_T\right]}{f(\vec{\mathbf{z}}_{1:T} || \vec{\mathbf{u}}_{1:T-1})}}_{\text{We define it } Q'(\vec{\mathbf{u}}_{1:T}, \vec{\mathbf{z}}_{1:T})}$$

$$+ \underbrace{\frac{\int_{\vec{\mathbf{x}}_{1:T}} f(\vec{\mathbf{z}}_{1:T}, \vec{\mathbf{x}}_{1:T} || \vec{\mathbf{u}}_{1:T-1}) \sum_{t=1}^{T} \vec{x}_t^T \mathbf{M} \vec{x}_t}{f(\vec{\mathbf{z}}_{1:T} || \vec{\mathbf{u}}_{1:T-1})}}_{\text{This is a constant term with respect to } \vec{u}_T \text{,we define it } W_T}$$

$$Q'(\vec{\mathbf{u}}_{1:T}, \vec{\mathbf{z}}_{1:T}) =$$

$$\frac{\int_{\vec{\mathbf{x}}_{1:T}} f(\vec{\mathbf{z}}_{1:T}, \vec{\mathbf{x}}_{1:T} || \vec{\mathbf{u}}_{1:T-1}) f(\vec{\mathbf{u}}_{1:T} || \vec{\mathbf{x}}_{1:T}, \vec{\mathbf{z}}_{1:T}) \mathbb{E}\left[\vec{\mathbf{X}}_{T+1}^T \mathbf{M} \vec{\mathbf{X}}_{T+1} | \vec{\mathbf{X}}_T, \vec{u}_T\right]}{f(\vec{\mathbf{z}}_{1:T} || \vec{\mathbf{u}}_{1:T-1}) f(\vec{\mathbf{u}}_{1:T} || \vec{\mathbf{z}}_{1:T})}$$

$$= \frac{\int_{\vec{x}_{T+1}} f(\vec{x}_{T+1}, \vec{\mathbf{u}}_{1:T}, \vec{\mathbf{z}}_{1:T}) \vec{x}_{T+1}^T \mathbf{M} \vec{x}_{T+1}}{f(\vec{\mathbf{u}}_{1:T}, \vec{\mathbf{z}}_{1:T})}$$

$$= \mathbb{E}\left[\vec{\mathbf{X}}_{T+1}^T \mathbf{M} \vec{\mathbf{X}}_{T+1} | \vec{\mathbf{u}}_{1:T}, \vec{\mathbf{z}}_{1:T}\right]$$

Let $V'(\vec{\mathbf{u}}_{1:T-1}, \vec{\mathbf{z}}_{1:T}) = \log \int_{\vec{u}_T} Q'(\vec{\mathbf{u}}_{1:T}, \vec{\mathbf{z}}_{1:T})$

$$Q(\vec{\mathbf{u}}_{1:T-1}, \vec{\mathbf{z}}_{1:T-1}) = \int_{\vec{z}_T} f(\vec{z}_T | \vec{\mathbf{u}}_{1:T-1}, \vec{\mathbf{z}}_{1:T-1})(W_T + V'(\vec{\mathbf{u}}_{1:T-1}, \vec{\mathbf{z}}_{1:T}))$$

$$= \frac{\int_{\vec{z}_T, \vec{\mathbf{x}}_{1:T}} f(\vec{\mathbf{z}}_{1:T}, \vec{\mathbf{x}}_{1:T} || \vec{\mathbf{u}}_{1:T-1}) f(\vec{\mathbf{u}}_{1:T-1} || \vec{\mathbf{x}}_{1:T-1}, \vec{\mathbf{z}}_{1:T-1}) \vec{x}_T^T \mathbf{M} \vec{x}_T}{f(\vec{\mathbf{z}}_{1:T-1} || \vec{\mathbf{u}}_{1:T-2}) f(\vec{\mathbf{u}}_{1:T-1} || \vec{\mathbf{z}}_{1:T-1})}$$

$$+ \mathbb{E}\left[V'(\vec{\mathbf{U}}_{1:T-1}, \vec{\mathbf{Z}}_{1:T}) | \vec{\mathbf{u}}_{1:T-1}, \vec{\mathbf{z}}_{1:T-1}\right] + W_{T-1}$$

$$= \underbrace{\mathbb{E}\left[\vec{\mathbf{X}}_T^T \mathbf{M} \vec{\mathbf{X}}_T + V'(\vec{\mathbf{U}}_{1:T-1}, \vec{\mathbf{Z}}_{1:T}) | \vec{\mathbf{u}}_{1:T-1}, \vec{\mathbf{z}}_{1:T-1}\right]}_{\text{We define it } Q'(\vec{\mathbf{u}}_{1:T-1}, \vec{\mathbf{z}}_{1:T-1})} + W_{T-1}$$

And let $V'(\vec{\mathbf{u}}_{1:T-2}, \vec{\mathbf{z}}_{1:T-1}) = \log \int_{\vec{u}_{T-1}} Q'(\vec{\mathbf{u}}_{1:T-1}, \vec{\mathbf{z}}_{1:T-1})$

For $t < T-1$, the argument to $Q'(\vec{\mathbf{u}}_{1:t}, \vec{\mathbf{z}}_{1:t}), V'(\vec{\mathbf{u}}_{1:t-1}, \vec{\mathbf{z}}_{1:t})$ is similar. We redefine $Q(\vec{\mathbf{u}}_{1:t}, \vec{\mathbf{z}}_{1:t}) = Q'(\vec{\mathbf{u}}_{1:t}, \vec{\mathbf{z}}_{1:t})$ and $V(\vec{\mathbf{u}}_{1:t-1}, \vec{\mathbf{z}}_{1:t}) = V'(\vec{\mathbf{u}}_{1:t-1}, \vec{\mathbf{z}}_{1:t})$ which gives the recursive form in Theorem 1. □

Lemma 4. *The distribution of belief state $\vec{\mathbf{X}}_t | b_t \sim N(\vec{\mu}_{b_t}, \Sigma_{b_t})$ is recursively defined as following and Σ_{b_t} is independent of b_t.*

$$\vec{\mu}_{b_1} = \vec{\mu} + \Sigma_{d_1}^T \mathbf{C}^T (\Sigma_o + \mathbf{C} \Sigma_{d_1}^T \mathbf{C}^T)^{-1} (\vec{\mathbf{Z}}_1 - \mathbf{C} \vec{\mu}) \tag{43}$$

$$\Sigma_{b_1} = \Sigma_{d_1} - \Sigma_{d_1}^T \mathbf{C}^T (\Sigma_o + \mathbf{C} \Sigma_{d_1}^T \mathbf{C}^T)^{-1} \mathbf{C} \Sigma_{d_1} \tag{44}$$

$$\vec{\mu}_{b_{t+1}} = \mathbf{B} \vec{\mathbf{U}}_t + \mathbf{A} \vec{\mu}_{b_t} + (\Sigma_d + \mathbf{A} \Sigma_{b_t}^T \mathbf{A}^T)^T \mathbf{C}^T$$
$$(\Sigma_o + \mathbf{C}(\Sigma_d + \mathbf{A} \Sigma_{b_t}^T \mathbf{A}^T)^T \mathbf{C}^T)^{-1} (\vec{\mathbf{Z}}_{t+1} - \mathbf{C}(\mathbf{B} \vec{\mathbf{U}}_t + \mathbf{A} \vec{\mu}_{b_t})) \tag{45}$$

$$\Sigma_{b_{t+1}} = \Sigma_d + \mathbf{A} \Sigma_{b_t}^T \mathbf{A}^T - (\Sigma_d + \mathbf{A} \Sigma_{b_t}^T \mathbf{A}^T)^T \mathbf{C}^T$$
$$(\Sigma_o + \mathbf{C}(\Sigma_d + \mathbf{A} \Sigma_{b_t}^T \mathbf{A}^T)^T \mathbf{C}^T)^{-1} \mathbf{C}(\Sigma_d + \mathbf{A} \Sigma_{b_t}^T \mathbf{A}^T) \tag{46}$$

Proof of Lemma 4. Since $\vec{\mathbf{Z}}_1|\vec{x}_1 \sim N(\mathbf{C}\vec{x}_1, \Sigma_o)$ and $\vec{\mathbf{X}}_1 \sim N(\vec{\mu}, \Sigma_{d_1})$, applying Gaussian transformation techniques, it is easy to show that the distribution of initial belief state $\vec{\mathbf{X}}_1|b_1$ (that is $\vec{\mathbf{X}}_1|\vec{z}_1$) is a Gaussian distribution with mean (43) and variance (44).

$$\text{Note that } f(\vec{x}_{t+1}|\vec{x}_t, \vec{u}_t, b_t) = f(\vec{x}_{t+1}|\vec{x}_t, \vec{u}_t) \quad \vec{\mathbf{X}}_{t+1}|\vec{x}_t, \vec{u}_t \sim N(\mathbf{A}\vec{x}_t + \mathbf{B}\vec{u}_t, \Sigma_d)$$

$$f(\vec{x}_t|\vec{u}_t, b_t) = f(\vec{x}_t|b_t) \quad \vec{\mathbf{X}}_t|b_t \sim N(\vec{\mu}_{b_t}, \Sigma_{b_t})$$

$$\text{Then } \vec{\mathbf{X}}_{t+1}|\vec{u}_t, b_t \sim N(\mathbf{B}\vec{u}_t + \mathbf{A}\vec{\mu}_{b_t}, \Sigma_d + \mathbf{A}\Sigma_{b_t}^T\mathbf{A}^T)$$

$$\text{Furthermore } f(\vec{z}_{t+1}|\vec{x}_{t+1}, \vec{u}_t, b_t) = f(\vec{z}_{t+1}|\vec{x}_{t+1}) \quad \vec{\mathbf{Z}}_{t+1}|\vec{x}_{t+1} \sim N(\mathbf{C}\vec{x}_{t+1}, \Sigma_o)$$

Thus, it's easy to show the distribution of $\vec{\mathbf{X}}_{t+1}, \vec{\mathbf{Z}}_{t+1}|\vec{u}_t, b_t)$ is:

$$N\begin{pmatrix} \mathbf{B}\vec{u}_t + \mathbf{A}\vec{\mu}_{b_t} & \Sigma_d + \mathbf{A}\Sigma_{b_t}^T\mathbf{A}^T & (\Sigma_d + \mathbf{A}\Sigma_{b_t}^T\mathbf{A}^T)^T\mathbf{C}^T \\ \mathbf{C}(\mathbf{B}\vec{u}_t + \mathbf{A}\vec{\mu}_{b_t}) ,& \mathbf{C}(\Sigma_d + \mathbf{A}\Sigma_{b_t}^T\mathbf{A}^T) & \Sigma_o + \mathbf{C}(\Sigma_d + \mathbf{A}\Sigma_{b_t}^T\mathbf{A}^T)^T\mathbf{C}^T \end{pmatrix}$$

Finally, $f(\vec{x}_{t+1}|b_{t+1}) = f(\vec{x}_{t+1}|\vec{z}_{t+1}, \vec{u}_t, b_t) = f(\vec{x}_{t+1}, \vec{z}_{t+1}|\vec{u}_t, b_t)/f(\vec{z}_{t+1}|\vec{u}_t, b_t)$ which gives the distribution of $\vec{\mathbf{X}}_{t+1}|b_{t+1}$ with mean (45) and variance (46). □

Proof of Theorem 2.

$$\begin{aligned}
&\mathbb{E}[\vec{\mathbf{X}}_{t+1}^T\mathbf{M}\vec{\mathbf{X}}_{t+1}|\vec{\mathbf{u}}_{1:t}, \vec{\mathbf{z}}_{1:t}] = \mathbb{E}[\vec{\mathbf{X}}_{t+1}^T\mathbf{M}\vec{\mathbf{X}}_{t+1}|\vec{u}_t, b_t] \\
&= (\mathbf{B}\vec{u}_t + \mathbf{A}\vec{\mu}_{b_t})^T\mathbf{M}(\mathbf{B}\vec{u}_t + \mathbf{A}\vec{\mu}_{b_t}) + tr(\mathbf{M}(\Sigma_d + \mathbf{A}\Sigma_{b_t}^T\mathbf{A}^T)) \\
&= \begin{bmatrix} \vec{u}_t \\ \vec{\mu}_{b_t} \end{bmatrix}^T \begin{bmatrix} \mathbf{B} & \mathbf{A} \end{bmatrix}^T \mathbf{M} \begin{bmatrix} \mathbf{B} & \mathbf{A} \end{bmatrix} \begin{bmatrix} \vec{u}_t \\ \vec{\mu}_{b_t} \end{bmatrix} + constant
\end{aligned}$$

Thus $Q(\vec{\mathbf{u}}_{1:T}, \vec{\mathbf{z}}_{1:T}) = Q(\vec{u}_T, \vec{\mu}_{b_T}) = \mathbb{E}[\vec{\mathbf{X}}_{t+1}^T\mathbf{M}\vec{\mathbf{X}}_{t+1}|\vec{u}_t, b_t]$ gives $\mathbf{W}_T$.

$$\begin{aligned}
&V(\vec{\mathbf{u}}_{1:T-1}, \vec{\mathbf{z}}_{1:T}) = V(\vec{\mu}_{b_T}) = V(\vec{z}_T, \vec{u}_{T-1}, \vec{\mu}_{T-1}) = \log \int_{\vec{u}_T} e^{Q(\vec{u}_T, \vec{\mu}_{b_T})} \\
&= \vec{\mu}_{b_T}^T(\mathbf{W}_{T(\mu,\mu)} - \mathbf{W}_{T(U,\mu)}^T\mathbf{W}_{T(U,U)}^{-1}\mathbf{W}_{T(U,\mu)})\vec{\mu}_{b_T} + constant \\
&= \begin{bmatrix} \vec{z}_T \\ \vec{u}_{T-1} \\ \vec{\mu}_{b_{T-1}} \end{bmatrix}^T \mathbf{P}_T^T(\mathbf{W}_{T(\mu,\mu)} - \mathbf{W}_{T(U,\mu)}^T\mathbf{W}_{T(U,U)}^{-1}\mathbf{W}_{T(U,\mu)})\mathbf{P}_T \begin{bmatrix} \vec{z}_T \\ \vec{u}_{T-1} \\ \vec{\mu}_{b_{T-1}} \end{bmatrix} + constant
\end{aligned}$$

which gives $\mathbf{D}_T$.

Thus $\mathbb{E}[V(\vec{\mathbf{U}}_{1:t}, \vec{\mathbf{Z}}_{1:t+1})|\vec{\mathbf{u}}_{1:t}, \vec{\mathbf{z}}_{1:t}] = \mathbb{E}[V(\vec{\mathbf{Z}}_{t+1}, \vec{\mathbf{U}}_t, \vec{\mu}_{b_t})|\vec{u}_t, \vec{\mu}_{b_t}]$

$$\begin{aligned}
&= \mathbb{E}\left[\begin{bmatrix} \vec{u}_t \\ \vec{\mu}_{b_t} \end{bmatrix}^T \mathbf{D}_{t+1(u\mu,z)}\vec{\mathbf{Z}}_{t+1} + \vec{\mathbf{Z}}_{t+1}^T\mathbf{D}_{t+1(z,u\mu)} \begin{bmatrix} \vec{u}_t \\ \vec{\mu}_{b_t} \end{bmatrix} \middle| \begin{bmatrix} \vec{u}_t \\ \vec{\mu}_{b_t} \end{bmatrix}\right] \\
&\quad + \mathbb{E}\left[\vec{\mathbf{Z}}_{t+1}^T\mathbf{D}_{t+1(z,z)}\vec{\mathbf{Z}}_{t+1} \middle| \begin{bmatrix} \vec{u}_t \\ \vec{\mu}_{b_t} \end{bmatrix}\right] + \begin{bmatrix} \vec{u}_t \\ \vec{\mu}_{b_t} \end{bmatrix}^T \mathbf{D}_{t+1(u\mu,u\mu)} \begin{bmatrix} \vec{u}_t \\ \vec{\mu}_{b_t} \end{bmatrix} + constant \\
&= \begin{bmatrix} \vec{u}_t \\ \vec{\mu}_{b_t} \end{bmatrix}^T \mathbf{D}_{t+1(u\mu,z)}\mathbf{C}_{BA} + \mathbf{C}_{BA}^T\mathbf{D}_{t+1(z,u\mu)} \\
&\qquad + \mathbf{C}_{BA}^T\mathbf{D}_{t+1(u,u)}\mathbf{C}_{BA} + \mathbf{D}_{t+1(u\mu,u\mu)} \begin{bmatrix} \vec{u}_t \\ \vec{\mu}_{b_t} \end{bmatrix}
\end{aligned}$$

$Q(\vec{u}_t, \vec{\mu}_{b_t}) = \mathbb{E}[\vec{\mathbf{X}}_{t+1}^T\mathbf{M}\vec{\mathbf{X}}_{t+1} + V(\vec{\mathbf{Z}}_{t+1}, \vec{\mathbf{U}}_t, \vec{\mu}_{b_t})|\vec{\mathbf{u}}_{1:t}, \vec{\mathbf{z}}_{1:t}]$ which gives $\mathbf{W}_t$(26)

The quadratic form of $V(\vec{z}_t, \vec{u}_{t-1}, \vec{\mu}_{t-1})$ is similar to $V(\vec{z}_T, \vec{u}_{T-1}, \vec{\mu}_{T-1})$ which gives $\mathbf{D}_t$(27)

□

Proof of Theorem 3. It's easy to check the initial setting $\mathbf{W}_T = \begin{bmatrix}\mathbf{B} & \mathbf{A}\end{bmatrix}^T \mathbf{M} \begin{bmatrix}\mathbf{B} & \mathbf{A}\end{bmatrix}$ matches (5). For general case, we plug $\mathbf{D}_{t+1}$(27) into $\mathbf{W}_t$ (26) and check with $\mathbf{W}_{t(U,U)}$ first. To simplify proof, let's define

$$\phi_t = \mathbf{W}_{t(\mu,\mu)} - \mathbf{W}^T_{t(U,\mu)}\mathbf{W}^{-1}_{t(U,U)}\mathbf{W}_{t(U,\mu)}.$$

Then from (26),(27)

$$\begin{aligned}\mathbf{W}_t = \begin{bmatrix}\mathbf{B} & \mathbf{A}\end{bmatrix}^T \mathbf{M} \begin{bmatrix}\mathbf{B} & \mathbf{A}\end{bmatrix} + \begin{bmatrix}\mathbf{B}-\mathbf{E}_{t+1}\mathbf{CB} & \mathbf{A}-\mathbf{E}_{t+1}\mathbf{CA}\end{bmatrix}^T \phi_{t+1}\mathbf{E}_{t+1}\begin{bmatrix}\mathbf{CB} & \mathbf{CA}\end{bmatrix} + \\ \begin{bmatrix}\mathbf{CB} & \mathbf{CA}\end{bmatrix}^T \mathbf{E}^T_{t+1}\phi_{t+1}\begin{bmatrix}\mathbf{B}-\mathbf{E}_{t+1}\mathbf{CB} & \mathbf{A}-\mathbf{E}_{t+1}\mathbf{CA}\end{bmatrix} + \\ \begin{bmatrix}\mathbf{CB} & \mathbf{CA}\end{bmatrix}^T \mathbf{E}^T_{t+1}\phi_{t+1}\mathbf{E}_{t+1}\begin{bmatrix}\mathbf{CB} & \mathbf{CA}\end{bmatrix} + \\ \begin{bmatrix}\mathbf{B}-\mathbf{E}_{t+1}\mathbf{CB} & \mathbf{A}-\mathbf{E}_{t+1}\mathbf{CA}\end{bmatrix}^T \phi_{t+1}\begin{bmatrix}\mathbf{B}-\mathbf{E}_{t+1}\mathbf{CB} & \mathbf{A}-\mathbf{E}_{t+1}\mathbf{CA}\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{W}_{t(U,U)} &= \mathbf{B}^T\mathbf{MB} + (\mathbf{B}-\mathbf{E}_{t+1}\mathbf{CB})^T\phi_{t+1}\mathbf{E}_{t+1}\mathbf{CB} + (\mathbf{E}_{t+1}\mathbf{CB})^T\phi_{t+1}(\mathbf{B}-\mathbf{E}_{t+1}\mathbf{CB}) + \\ &\quad (\mathbf{E}_{t+1}\mathbf{CB})^T\phi_{t+1}\mathbf{E}_{t+1}\mathbf{CB} + (\mathbf{B}-\mathbf{E}_{t+1}\mathbf{CB})^T\phi_{t+1}(\mathbf{B}-\mathbf{E}_{t+1}\mathbf{CB}) \\ &= \mathbf{B}^T\mathbf{MB} + \mathbf{B}^T\phi_{t+1}\mathbf{B} \\ \text{That is } \mathbf{B}^T\mathbf{F}_{t+1}\mathbf{B} &= \mathbf{B}^T\mathbf{MB} + \mathbf{B}^T\phi_{t+1}\mathbf{B}. \end{aligned} \tag{47}$$

By plugging out ϕ_{t+1}, the equation(47) matches equation(5). $\mathbf{W}_{t(U,\mu)}$, $\mathbf{W}_{t(\mu,U)}$, $\mathbf{W}_{t(\mu,\mu)}$ follow similar argument. □