SUPPLEMENTARY MATERIAL

Proof of Lemma 1. For any $u_t \in \mathcal{X}$, it holds that

$$\langle x_t - u_t, \nabla_t \rangle = \langle x_t - \hat{x}_t, \nabla_t - M_t \rangle + \langle x_t - \hat{x}_t, M_t \rangle + \langle \hat{x}_t - u_t, \nabla_t \rangle \,. \tag{15}$$

First, observe that for any primal-dual norm pair we have

$$\langle x_t - \hat{x}_t, \nabla_t - M_t \rangle \leq \| x_t - \hat{x}_t \| \, \| \nabla_t - M_t \|_* \ .$$

Any update of the form $a^* = \arg\min_{a \in \mathcal{X}} \langle a, x \rangle + \mathcal{D}_{\mathcal{R}}(a, c)$ satisfies for any $d \in \mathcal{X}$,

$$\langle a^* - d, x \rangle \leq \mathcal{D}_{\mathcal{R}}(d, c) - \mathcal{D}_{\mathcal{R}}(d, a^*) - \mathcal{D}_{\mathcal{R}}(a^*, c) \ .$$

This entails

$$\langle x_t - \hat{x}_t, M_t \rangle \leq \frac{1}{\eta_t} \left\{ \mathcal{D}_{\mathcal{R}}(\hat{x}_t, \hat{x}_{t-1}) - \mathcal{D}_{\mathcal{R}}(\hat{x}_t, x_t) - \mathcal{D}_{\mathcal{R}}(x_t, \hat{x}_{t-1}) \right\}$$

and

$$\langle \hat{x}_t - u_t, \nabla_t \rangle \leq \frac{1}{\eta_t} \left\{ \mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_{t-1}) - \mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_t) - \mathcal{D}_{\mathcal{R}}(\hat{x}_t, \hat{x}_{t-1}) \right\} \ .$$

Combining the preceding relations and returning to (15), we obtain

$$\begin{aligned}
\langle x_t - u_t, \nabla_t \rangle &\leq \frac{1}{\eta_t} \left\{ \mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_{t-1}) - \mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_t) - \mathcal{D}_{\mathcal{R}}(\hat{x}_t, x_t) - \mathcal{D}_{\mathcal{R}}(x_t, \hat{x}_{t-1}) \right\} \\
&\qquad + \| \nabla_t - M_t \|_* \, \| x_t - \hat{x}_t \| \\
&\leq \frac{1}{\eta_t} \left\{ \mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_{t-1}) - \mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_t) - \frac{1}{2} \| \hat{x}_t - x_t \|^2 - \frac{1}{2} \| \hat{x}_{t-1} - x_t \|^2 \right\} \\
&\qquad + \| \nabla_t - M_t \|_* \, \| x_t - \hat{x}_t \| \,,
\end{aligned} \tag{16}$$

where in the last step we appealed to strong convexity: $\mathcal{D}_{\mathcal{R}}(x, y) \geq \frac{1}{2} \| x - y \|^2$ for any $x, y \in \mathcal{X}$. Using the simple inequality $ab \leq \frac{\rho a^2}{2} + \frac{b^2}{2\rho}$ for any $\rho > 0$ to split the product term, we get

$$\begin{aligned}
\langle x_t - u_t, \nabla_t \rangle &\leq \frac{1}{\eta_t} \left\{ \mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_{t-1}) - \mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_t) - \frac{1}{2} \| \hat{x}_t - x_t \|^2 - \frac{1}{2} \| \hat{x}_{t-1} - x_t \|^2 \right\} \\
&\qquad + \frac{\eta_{t+1}}{2} \| \nabla_t - M_t \|_*^2 + \frac{1}{2\eta_{t+1}} \| x_t - \hat{x}_t \|^2 \,,
\end{aligned}$$

Applying the bound

$$\frac{1}{2\eta_{t+1}} \| x_t - \hat{x}_t \|^2 - \frac{1}{2\eta_t} \| x_t - \hat{x}_t \|^2 \leq R_{\max}^2 \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) ,$$

and summing over $t \in [T]$ yields ,

$$\sum_{t=1}^{T} \langle x_t - u_t, \nabla_t \rangle \leq \sum_{t=1}^{T} \frac{\eta_{t+1}}{2} \|\nabla_t - M_t\|_*^2 + \sum_{t=1}^{T} \frac{1}{\eta_t} \Big\{ \mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_{t-1}) - \mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_t) \Big\} + \frac{R_{\max}^2}{\eta_{T+1}}$$
$$\leq \sum_{t=1}^{T} \frac{\eta_{t+1}}{2} \|\nabla_t - M_t\|_*^2 + R_{\max}^2 \left(\frac{1}{\eta_1} + \frac{1}{\eta_{T+1}} \right)$$
$$+ \sum_{t=2}^{T} \left\{ \frac{\mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_{t-1})}{\eta_t} - \frac{\mathcal{D}_{\mathcal{R}}(u_{t-1}, \hat{x}_{t-1})}{\eta_{t-1}} \right\}$$
$$\leq \sum_{t=2}^{T} \left\{ \frac{\mathcal{D}_{\mathcal{R}}(u_t, \hat{x}_{t-1})}{\eta_t} - \frac{\mathcal{D}_{\mathcal{R}}(u_{t-1}, \hat{x}_{t-1})}{\eta_t} + \frac{\mathcal{D}_{\mathcal{R}}(u_{t-1}, \hat{x}_{t-1})}{\eta_t} - \frac{\mathcal{D}_{\mathcal{R}}(u_{t-1}, \hat{x}_{t-1})}{\eta_{t-1}} \right\}$$
$$+ \sum_{t=1}^{T} \frac{\eta_{t+1}}{2} \|\nabla_t - M_t\|_*^2 + \frac{2R_{\max}^2}{\eta_{T+1}}$$
$$\leq \sum_{t=1}^{T} \frac{\eta_{t+1}}{2} \|\nabla_t - M_t\|_*^2 + \gamma \sum_{t=2}^{T} \frac{\|u_t - u_{t-1}\|}{\eta_t}$$
$$+ \sum_{t=2}^{T} \mathcal{D}_{\mathcal{R}}(u_{t-1}, \hat{x}_{t-1}) \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}} \right) + \frac{2R_{\max}^2}{\eta_{T+1}}$$
$$\leq \sum_{t=1}^{T} \frac{\eta_{t+1}}{2} \|\nabla_t - M_t\|_*^2 + \gamma \sum_{t=2}^{T} \frac{\|u_t - u_{t-1}\|}{\eta_t} + \frac{4R_{\max}^2}{\eta_{T+1}} \ ,$$

where we used the Lipschitz continuity of $\mathcal{D}_{\mathcal{R}}$ in the penultimate step. Now let us set

$$\eta_t = \frac{L}{\sqrt{\sum_{s=0}^{t-1} \|\nabla_s - M_s\|_*^2} + \sqrt{\sum_{s=0}^{t-2} \|\nabla_s - M_s\|_*^2}} = \frac{L \left(\sqrt{\sum_{s=0}^{t-1} \|\nabla_s - M_s\|_*^2} - \sqrt{\sum_{s=0}^{t-2} \|\nabla_s - M_s\|_*^2} \right)}{\|\nabla_{t-1} - M_{t-1}\|_*^2},$$

and $\|\nabla_0 - M_0\|_*^2 = 1$ to have

$$\sum_{t=1}^{T} \langle x_t - u_t, \nabla_t \rangle \leq \frac{L}{2} \sum_{t=1}^{T} \left\{ \sqrt{\sum_{s=0}^{t} \|\nabla_s - M_s\|_*^2} - \sqrt{\sum_{s=0}^{t-1} \|\nabla_s - M_s\|_*^2} \right\}$$
$$+ \frac{2\gamma \sqrt{1 + \sum_{t=1}^{T} \|\nabla_t - M_t\|_*^2}}{L} \sum_{t=2}^{T} \|u_t - u_{t-1}\| + \frac{8R_{\max}^2 \sqrt{1 + \sum_{t=1}^{T} \|\nabla_t - M_t\|_*^2}}{L}$$
$$\leq 2 \sqrt{1 + \sum_{t=1}^{T} \|\nabla_t - M_t\|_*^2} \left(L + \frac{\gamma \sum_{t=1}^{T} \|u_t - u_{t-1}\| + 4R_{\max}^2}{L} \right) \ .$$

Appealing to convexity of $\{f_t\}_{t=1}^{T}$, and replacing C_T (3) and D_T (4) in above, completes the proof . ■

Proof of Lemma 2. We define

$$U_T \triangleq \left\{ u_1, ..., u_T \in \mathcal{X} : \gamma \sum_{t=1}^{T} \|u_t - u_{t-1}\| \leq L^2 - 4R_{\max}^2 \right\}, \tag{17}$$

and

$$(u_1^*, ..., u_T^*) \triangleq \text{argmin}_{u_1,...,u_T \in U_T} \sum_{t=1}^{T} f_t(u_t).$$

Our choice of $L > 2R_{\max}$ guarantees that any sequence of fixed comparators $u_t = u$ for $t \in [T]$ belongs to U_T, and hence, $(u_1^*, ..., u_T^*)$ exists. Noting that $(u_1^*, ..., u_T^*)$ is an element of U_T, we have $\gamma \sum_{t=1}^T \left\|u_t^* - u_{t-1}^*\right\| + 4R_{\max}^2 \leq L^2$. We now apply Lemma 1 to $\{u_t^*\}_{t=1}^T$ to bound the dynamic regret for arbitrary comparator sequence $\{u_t\}_{t=1}^T$ as follows,

$$\begin{aligned}
\mathbf{Reg}_T^d(u_1, ..., u_T) &= \sum_{t=1}^T \left\{ f_t(x_t) - f_t(u_t^*) \right\} + \sum_{t=1}^T \left\{ f_t(u_t^*) - f_t(u_t) \right\} \\
&\leq 4\sqrt{1 + D_T} L + \sum_{t=1}^T \left\{ f_t(u_t^*) - f_t(u_t) \right\} \\
&\leq 4\sqrt{1 + D_T} L + \mathbf{1} \left\{ \gamma \sum_{t=1}^T \|u_t - u_{t-1}\| > L^2 - 4R_{\max}^2 \right\} \left(\sum_{t=1}^T \left\{ f_t(u_t^*) - f_t(u_t) \right\} \right), \qquad (18)
\end{aligned}$$

where the last step follows from the fact that

$$\sum_{t=1}^T f_t(u_t^*) - \sum_{t=1}^T f_t(u_t) \leq 0 \quad \text{ if } \quad (u_1, ..., u_T) \in U_T.$$

Given the definition of $R_{\max}^2$, by strong convexity of $\mathcal{D}_{\mathcal{R}}(x, y)$, we get that $\|x - y\| \leq \sqrt{2} R_{\max}$, for any $x, y \in \mathcal{X}$. This entails that once we divide the horizon into B number of batches and use a single, fixed point as a comparator along each batch, we have

$$\sum_{t=1}^T \|u_t - u_{t-1}\| \leq B\sqrt{2} R_{\max}, \qquad (19)$$

since there are at most B number of changes in the comparator sequence along the horizon. Now let $B = \frac{L^2 - 4R_{\max}^2}{\gamma\sqrt{2}R_{\max}}$, and for ease of notation, assume that T is divisible by B. Noting that $f_t(x_t^*) \leq f_t(u_t)$, we use an argument similar to that of [14] to get for any fixed $t_i \in [(i-1)(T/B) + 1, i(T/B)]$,

$$\begin{aligned}
\sum_{t=1}^T \left\{ f_t(u_t^*) - f_t(u_t) \right\} &\leq \sum_{t=1}^T \left\{ f_t(u_t^*) - f_t(x_t^*) \right\} &(20) \\
&= \sum_{i=1}^B \sum_{t=(i-1)(T/B)+1}^{i(T/B)} \left\{ f_t(u_t^*) - f_t(x_t^*) \right\} \\
&\leq \sum_{i=1}^B \sum_{t=(i-1)(T/B)+1}^{i(T/B)} \left\{ f_t(x_{t_i}^*) - f_t(x_t^*) \right\} &(21) \\
&\leq \left(\frac{T}{B} \right) \sum_{i=1}^B \max_{t \in [(i-1)(T/B)+1, i(T/B)]} \left\{ f_t(x_{t_i}^*) - f_t(x_t^*) \right\}. &(22)
\end{aligned}$$

Note that $x_{t_i}^*$ is fixed for each batch i. Substituting our choice of $B = \frac{L^2 - 4R_{\max}^2}{\gamma\sqrt{2}R_{\max}}$ in (19) implies that the comparator sequence $u_t = x_{t_i}^* \mathbf{1} \left\{ \frac{(i-1)T}{B} + 1 \leq t \leq \frac{iT}{B} \right\}$ belongs to U_T, and (21) follows by optimality of $(u_1^*, ..., u_T^*)$. We now claim that for any $t \in [(i-1)(T/B) + 1, i(T/B)]$, we have,

$$f_t(x_{t_i}^*) - f_t(x_t^*) \leq 2 \sum_{s=(i-1)(T/B)+1}^{i(T/B)} \sup_{x \in \mathcal{X}} |f_s(x) - f_{s-1}(x)|. \qquad (23)$$

Assuming otherwise, there must exist a $\hat{t}_i \in [(i-1)(T/B)+1, i(T/B)]$ such that

$$f_{\hat{t}_i}(x^*_{t_i}) - f_{\hat{t}_i}(x^*_{\hat{t}_i}) > 2 \sum_{t=(i-1)(T/B)+1}^{i(T/B)} \sup_{x \in \mathcal{X}} |f_t(x) - f_{t-1}(x)|,$$

which results in

$$\begin{aligned} f_t(x^*_{\hat{t}_i}) &\le f_{\hat{t}_i}(x^*_{\hat{t}_i}) + \sum_{t=(i-1)(T/B)+1}^{i(T/B)} \sup_{x \in \mathcal{X}} |f_t(x) - f_{t-1}(x)| \\ &< f_{\hat{t}_i}(x^*_{t_i}) - \sum_{t=(i-1)(T/B)+1}^{i(T/B)} \sup_{x \in \mathcal{X}} |f_t(x) - f_{t-1}(x)| \le f_t(x^*_{t_i}), \end{aligned}$$

The preceding relation for $t = t_i$ violates the optimality of $x^*_{t_i}$, which is a contradiction. Therefore, Equation (23) holds for any $t \in [(i-1)(T/B)+1, i(T/B)]$ Combining (20), (22) and (23) we have

$$\begin{aligned} \sum_{t=1}^{T} \left\{ f_t(u^*_t) - f_t(u_t) \right\} &\le \frac{2T}{B} \sum_{i=1}^{B} \sum_{t=(i-1)(T/B)+1}^{i(T/B)} \sup_{x \in \mathcal{X}} |f_t(x) - f_{t-1}(x)| \\ &= \frac{2TV_T}{B} = \frac{2\gamma\sqrt{2}R_{\max}TV_T}{L^2 - 4R^2_{\max}}. \end{aligned} \tag{24}$$

Using the above in Equation (18) we conclude the following upper bound

$$\mathbf{Reg}^d_T(u_1, ..., u_T) \le 4\sqrt{1+D_T}L + \mathbf{1}\left\{ \gamma \sum_{t=1}^{T} \|u_t - u_{t-1}\| > L^2 - 4R^2_{\max} \right\} \frac{4\gamma R_{\max}TV_T}{L^2 - 4R^2_{\max}},$$

thereby completing the proof. ■

Proof of Proposition 5. Assume that the player I uses the prescribed strategy. This corresponds to using the optimistic mirror descent update with $\mathcal{R}(x) = \sum_{i=1}^n x_i \log(x_i)$ as the function that is strongly convex w.r.t. $\|\cdot\|_1$. Correspondingly, $\nabla_t = f_t^\top A_t$ and $M_t = f_{t-1}^\top A_{t-1}$. Following the line of proof in Lemma 1, in particular, using Equation 16 for the specific case with $\mathcal{D}_\mathcal{R}$ as KL divergence, we get that for any t and any $u_t \in \Delta_n$,

$$\begin{aligned} f_t^\top A_t x_t - f_t^\top A_t u_t &\le \frac{1}{\eta_t}\left\{ \sum_{i=1}^{n} u_t[i] \log\left(\frac{\hat{x}_t[i]}{\hat{x}'_{t-1}[i]}\right) - \frac{1}{2}\|\hat{x}_t - x_t\|_1^2 - \frac{1}{2}\left\|\hat{x}'_{t-1} - x_t\right\|_1^2 \right\} \\ &\quad + \left\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\right\|_\infty \|x_t - \hat{x}_t\|_1 \\ &\le \frac{1}{\eta_t}\left\{ \sum_{i=1}^{n} u_t[i] \log\left(\frac{\hat{x}'_t[i]}{\hat{x}'_{t-1}[i]}\right) - \frac{1}{2}\|\hat{x}_t - x_t\|_1^2 - \frac{1}{2}\left\|\hat{x}'_{t-1} - x_t\right\|_1^2 \right\} \\ &\quad + \left\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\right\|_\infty \|x_t - \hat{x}_t\|_1 + \frac{1}{\eta_t}\max_{i \in [n]} \log\left(\frac{\hat{x}_t[i]}{\hat{x}'_t[i]}\right). \end{aligned}$$

Now let us bound for some i the term, $\log\left(\frac{\hat{x}_t[i]}{\hat{x}'_t[i]}\right)$. Notice that if $\hat{x}_t[i] \le \hat{x}'_t[i]$ then the term is anyway bounded by 0. Now assume $\hat{x}_t[i] > \hat{x}'_t[i]$. Letting $\beta = 1/T^2$, since $\hat{x}'_t[i] = (1-T^{-2})\hat{x}_t[i] + 1/(nT^2)$, we can have $\hat{x}_t[i] > \hat{x}'_t[i]$ only when $\hat{x}_t[i] > 1/n$. Hence,

$$\log\left(\frac{\hat{x}_t[i]}{\hat{x}'_t[i]}\right) = \log\left(\frac{\hat{x}_t[i]}{(1-T^{-2})\hat{x}_t[i] + 1/(nT^2)}\right) \le \frac{2}{T^2}.$$

Using this we can conclude that :

$$f_t^\top A_t x_t - f_t^\top A_t u_t \leq \frac{1}{\eta_t}\left\{ \sum_{i=1}^n u_t[i]\log\left(\frac{\hat{x}_t'[i]}{\hat{x}_{t-1}'[i]}\right) - \frac{1}{2}\left\|\hat{x}_t - x_t\right\|_1^2 - \frac{1}{2}\left\|\hat{x}_{t-1}' - x_t\right\|_1^2 \right\}$$
$$+ \left\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\right\|_\infty \left\|x_t - \hat{x}_t\right\|_1 + \frac{2}{T^2}\frac{1}{\eta_t}.$$

Summing over $t \in [T]$ we obtain that :

$$\sum_{t=1}^T \left(f_t^\top A_t x_t - f_t^\top A_t u_t\right) \leq \sum_{t=1}^T \frac{1}{\eta_t}\left\{ \sum_{i=1}^n u_t[i]\log\left(\frac{\hat{x}_t'[i]}{\hat{x}_{t-1}'[i]}\right) - \frac{1}{2}\left\|\hat{x}_t - x_t\right\|_1^2 - \frac{1}{2}\left\|\hat{x}_{t-1}' - x_t\right\|_1^2 \right\}$$
$$+ \sum_{t=1}^T \left\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\right\|_\infty \left\|x_t - \hat{x}_t\right\|_1 + \frac{2}{T^2}\sum_{t=1}^T \frac{1}{\eta_t}.$$

Note that $\frac{1}{\eta_t} \leq \mathcal{O}\left(\sqrt{T}\right)$ and so assuming T is large enough, $\frac{1}{T^2}\sum_{t=1}^T \frac{1}{\eta_t} \leq 1$ and so,

$$\sum_{t=1}^T \left(f_t^\top A_t x_t - f_t^\top A_t u_t\right) \leq \sum_{t=1}^T \frac{1}{\eta_t}\left\{ \sum_{i=1}^n u_t[i]\log\left(\frac{\hat{x}_t'[i]}{\hat{x}_{t-1}'[i]}\right) - \frac{1}{2}\left\|\hat{x}_t - x_t\right\|_1^2 - \frac{1}{2}\left\|\hat{x}_{t-1}' - x_t\right\|_1^2 \right\}$$
$$+ \sum_{t=1}^T \left\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\right\|_\infty \left\|x_t - \hat{x}_t\right\|_1 + 1. \tag{25}$$

Now note that we can rewrite the first sum in the above bound and get :

$$\sum_{t=1}^T \frac{1}{\eta_t}\sum_{i=1}^n u_t[i]\log\left(\frac{\hat{x}_t'[i]}{\hat{x}_{t-1}'[i]}\right) \leq \sum_{t=2}^T \frac{\sum_{i=1}^n u_t[i]\log\left(\frac{1}{\hat{x}_{t-1}'[i]}\right)}{\eta_t} - \frac{\sum_{i=1}^n u_{t-1}[i]\log\left(\frac{1}{\hat{x}_{t-1}'[i]}\right)}{\eta_{t-1}} + \frac{\log(T^2 n)}{\eta_1}$$
$$\leq \sum_{t=2}^T \frac{\sum_{i=1}^n \left(u_t[i] - u_{t-1}[i]\right)\log\left(\frac{1}{\hat{x}_{t-1}'[i]}\right)}{\eta_t}$$
$$+ \sum_{t=2}^T \sum_{i=1}^n u_{t-1}[i]\log\left(\frac{1}{\hat{x}_{t-1}'[i]}\right)\left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}\right) + \frac{\log(T^2 n)}{\eta_1}.$$

Since by definition of $\hat{x}_{t-1}'$, we are mixing in $1/T^2$ of the uniform distribution we have that for any i, $\hat{x}_{t-1}'[i] > \frac{1}{T^2 n}$ and, since η_t's are non-increasing, we continue bounding above as

$$\sum_{t=1}^T \frac{1}{\eta_t}\sum_{i=1}^n u_t[i]\log\left(\frac{\hat{x}_t'[i]}{\hat{x}_{t-1}'[i]}\right) \leq \log(T^2 n)\sum_{t=2}^T \frac{\left\|u_{t-1} - u_t\right\|_1}{\eta_t} + \log(T^2 n)\sum_{t=2}^T \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}\right) + \frac{\log(T^2 n)}{\eta_1}$$
$$\leq \log(T^2 n)\left(\sum_{t=2}^T \frac{\left\|u_{t-1} - u_t\right\|_1}{\eta_t} + \frac{1}{\eta_T} - \frac{1}{\eta_1}\right) + \frac{\log(T^2 n)}{\eta_1}$$
$$\leq \log(T^2 n)\left(\sum_{t=2}^T \frac{\left\|u_{t-1} - u_t\right\|_1}{\eta_t} + \frac{1}{\eta_T}\right),$$

using the above in Equation 25 we get

$$\begin{aligned}
&\sum_{t=1}^{T} f_t^\top A_t x_t - f_t^\top A_t u_t \\
&\qquad \le \log(T^2 n) \sum_{t=2}^{T} \frac{\|u_{t-1} - u_t\|_1}{\eta_t} - \frac{1}{2}\sum_{t=1}^{T} \frac{1}{\eta_t}\|\hat{x}_t - x_t\|_1^2 - \frac{1}{2}\sum_{t=1}^{T}\frac{1}{\eta_t}\left\|\hat{x}'_{t-1} - x_t\right\|_1^2 + 1 \\
&\qquad\qquad + \sum_{t=1}^{T}\left\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\right\|_\infty \|x_t - \hat{x}_t\|_1 + \frac{\log(T^2 n)}{\eta_T} \\
&\qquad \le \frac{\log(T^2 n)\left(C_T(u_1, \dots, u_T) + 2\right)}{\eta_T} - \frac{1}{2}\sum_{t=1}^{T}\frac{1}{\eta_t}\|\hat{x}_t - x_t\|_1^2 - \frac{1}{2}\sum_{t=1}^{T}\frac{1}{\eta_t}\left\|\hat{x}'_{t-1} - x_t\right\|_1^2 \\
&\qquad\qquad + \sum_{t=1}^{T}\left\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\right\|_\infty \|x_t - \hat{x}_t\|_1 . \qquad (26)
\end{aligned}$$

Notice that our choice of step size given by,

$$\begin{aligned}
\eta_t &= \min\left\{\log(T^2 n)\frac{L}{\sqrt{\sum_{i=1}^{t-1}\left\|f_i^\top A_i - f_{i-1}^\top A_{i-1}\right\|_\infty^2} + \sqrt{\sum_{i=1}^{t-2}\left\|f_i^\top A_i - f_{i-1}^\top A_{i-1}\right\|_\infty^2}}, \frac{1}{32L}\right\} \\
&= \min\left\{\log(T^2 n)\frac{L\left(\sqrt{\sum_{i=1}^{t-1}\left\|f_i^\top A_i - f_{i-1}^\top A_{i-1}\right\|_\infty^2} - \sqrt{\sum_{i=1}^{t-2}\left\|f_i^\top A_i - f_{i-1}^\top A_{i-1}\right\|_\infty^2}\right)}{\left\|f_{t-1}^\top A_{t-1} - f_{t-2}^\top A_{t-2}\right\|_\infty^2}, \frac{1}{32L}\right\}, \qquad (27)
\end{aligned}$$

guarantees that

$$\eta_t^{-1} = \max\left\{\frac{\sqrt{\sum_{i=1}^{t-1}\left\|f_i^\top A_i - f_{i-1}^\top A_{i-1}\right\|_\infty^2} + \sqrt{\sum_{i=1}^{t-2}\left\|f_i^\top A_i - f_{i-1}^\top A_{i-1}\right\|_\infty^2}}{\log(T^2 n)L}, 32L\right\}.$$

Using the step-size specified above in the bound 26, we get

$$\begin{aligned}
&\sum_{t=1}^{T} f_t^\top A_t x_t - \sum_{t=1}^{T} f_t^\top A_t u_t \\
&\qquad \le \log(T^2 n)\left(C_T(u_1, \dots, u_T) + 2\right)\left(\frac{2\sqrt{\sum_{t=1}^{T}\left\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\right\|_\infty^2}}{\log(T^2 n)L} + 32L\right) \\
&\qquad + \sum_{t=1}^{T}\left\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\right\|_\infty \|x_t - \hat{x}_t\|_1 - 16L\sum_{t=1}^{T}\|\hat{x}_t - x_t\|_1^2 - 16L\sum_{t=1}^{T}\left\|\hat{x}'_{t-1} - x_t\right\|_1^2 . \qquad (28)
\end{aligned}$$

Now note that by triangle inequality, we have

$$\begin{aligned}
\left\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\right\|_\infty &= \left\|f_t^\top A_t - f_t^\top A_{t-1} + f_t^\top A_{t-1} - f_{t-1}^\top A_{t-1}\right\|_\infty \\
&\le \|A_{t-1} - A_t\|_\infty + \|f_t - f_{t-1}\|_1 \\
&\le \|A_{t-1} - A_t\|_\infty + \left\|f_t - \hat{f}_{t-1}\right\|_1 + \left\|\hat{f}_{t-1} - f_{t-1}\right\|_1 ,
\end{aligned}$$

since the entries of matrix sequence $\{A_t\}_{t=1}^{T}$ are bounded by one. Using the bound above in (28) and splitting the product term, we see that

$$\sum_{t=1}^{T}\left(f_t^\top A_t x_t - f_t^\top A_t u_t\right) \le \log(T^2 n)\left(C_T(u_1,\ldots,u_T)+2\right)\left(\frac{2\sqrt{\sum_{t=1}^{T}\left\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\right\|_\infty^2}}{\log(T^2 n)L} + 32L\right)$$
$$+ 2\sum_{t=1}^{T}\|A_t - A_{t-1}\|_\infty - 8L\sum_{t=1}^{T}\|\hat{x}_t - x_t\|_1^2 - 16L\sum_{t=1}^{T}\left\|\hat{x}'_{t-1} - x_t\right\|_1^2$$
$$+ \frac{1}{16L}\sum_{t=1}^{T}\left\|f_t - \hat{f}_{t-1}\right\|_1^2 + \frac{1}{16L}\sum_{t=1}^{T}\left\|\hat{f}_{t-1} - f_{t-1}\right\|_1^2, \tag{29}$$

where we used the simple inequality $ab \le \frac{\rho}{2}a^2 + \frac{1}{2\rho}b^2$ for $\rho > 0$.

a) When Player II follows prescribed strategy: In this case we would like to get convergence of payoffs to the average value of the games. To get this, using the notation $x_t^* = \underset{x_t\in\Delta_n}{\operatorname{argmin}}\ f_t^\top A_t x_t$ and denoting the corresponding sequence regularity for Player I by C_T, we get

$$\sum_{t=1}^{T}\left(f_t^\top A_t x_t - f_t^\top A_t x_t^*\right) \le \log(T^2 n)\left(C_T+2\right)\left(\frac{2\sqrt{\sum_{t=1}^{T}\left\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\right\|_\infty^2}}{\log(T^2 n)L} + 32L\right)$$
$$+ 2\sum_{t=1}^{T}\|A_t - A_{t-1}\|_\infty - 8L\sum_{t=1}^{T}\|\hat{x}_t - x_t\|_1^2 - 16L\sum_{t=1}^{T}\left\|\hat{x}'_{t-1} - x_t\right\|_1^2$$
$$+ \frac{1}{16L}\sum_{t=1}^{T}\left\|f_t - \hat{f}_{t-1}\right\|_1^2 + \frac{1}{16L}\sum_{t=1}^{T}\left\|\hat{f}_t - f_t\right\|_1^2 + \frac{1}{4L},$$

where the term $\frac{1}{4L}$ appeared in the last line comparing to (29) is due to

$$\frac{1}{16L}\sum_{t=1}^{T}\left\|\hat{f}_{t-1} - f_{t-1}\right\|_1^2 - \frac{1}{16L}\sum_{t=1}^{T}\left\|\hat{f}_t - f_t\right\|_1^2 \le \frac{1}{4L}.$$

Using the same bound for Player 2 (using loss as $-f_t^\top A_t x_t$ on round t), as well as using $f_t^* = \underset{f_t\in\Delta_m}{\operatorname{argmin}}\ -f_t^\top A_t x_t$ and denoting the corresponding sequence regularity by C'_T, we have that

$$\sum_{t=1}^{T}\left(f_t^\top A_t x_t - {f_t^*}^\top A_t x_t\right) \ge -\log(T^2 m)\left(C'_T+2\right)\left(\frac{2\sqrt{\sum_{t=1}^{T}\|A_t x_t - A_{t-1}x_{t-1}\|_\infty^2}}{\log(T^2 m)L} + 32L\right)$$
$$- 2\sum_{t=1}^{T}\|A_t - A_{t-1}\|_\infty + 8L\sum_{t=1}^{T}\left\|\hat{f}_t - f_t\right\|_1^2 + 16L\sum_{t=1}^{T}\left\|\hat{f}'_{t-1} - f_t\right\|_1^2$$
$$- \frac{1}{16L}\sum_{t=1}^{T}\|x_t - \hat{x}_{t-1}\|_1^2 - \frac{1}{16L}\sum_{t=1}^{T}\|\hat{x}_t - x_t\|_1^2 - \frac{1}{4L}.$$

Combining the two and noting that

$$\begin{aligned} {f_t^*}^\top A_t x_t &= \sup_{f_t\in\Delta_m} f_t^\top A_t x_t \ge \inf_{x_t\in\Delta_n}\sup_{f_t\in\Delta_m} f_t^\top A_t x_t \\ &= \sup_{f_t\in\Delta_m}\inf_{x_t\in\Delta_n} f_t^\top A_t x_t \ge \inf_{x_t\in\Delta_n} f_t^\top A_t x_t = f_t^\top A_t x_t^*, \end{aligned}$$

we get

$$\sum_{t=1}^{T} \sup_{f_t \in \Delta_m} f_t^\top A_t x_t \le \sum_{t=1}^{T} \inf_{x_t \in \Delta_n} \sup_{f_t \in \Delta_m} f_t^\top A_t x_t + \frac{256L}{T} + \frac{1}{2L} + 4\sum_{t=1}^{T} \|A_t - A_{t-1}\|_\infty$$
$$+ \log(T^2 n)\,(C_T + 2)\left(\frac{2\sqrt{\sum_{t=1}^{T} \left\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\right\|_\infty^2}}{\log(T^2 n)L} + 32L\right)$$
$$+ \log(T^2 m)\,(C_T' + 2)\left(\frac{2\sqrt{\sum_{t=1}^{T} \|A_t x_t - A_{t-1}x_{t-1}\|_\infty^2}}{\log(T^2 m)L} + 32L\right)$$
$$+ \left(\frac{1}{16L} - 8L\right)\sum_{t=1}^{T} \|\hat{x}_t - x_t\|_1^2 + \left(\frac{1}{16L} - 16L\right)\sum_{t=1}^{T} \|\hat{x}_{t-1} - x_t\|_1^2$$
$$+ \left(\frac{1}{16L} - 8L\right)\sum_{t=1}^{T} \left\|\hat{f}_t - f_t\right\|_1^2 + \left(\frac{1}{16L} - 16L\right)\sum_{t=1}^{T} \left\|\hat{f}_{t-1} - f_t\right\|_1^2, \tag{30}$$

where the constant $256L/T$ appeared in the first line accounts for the identities

$$\|\hat{x}_{t-1} - x_t\|_1^2 - \left\|\hat{x}_{t-1}' - x_t\right\|_1^2 \le \frac{8}{T^2} \qquad \left\|\hat{f}_{t-1} - f_t\right\|_1^2 - \left\|\hat{f}_{t-1}' - f_t\right\|_1^2 \le \frac{8}{T^2}.$$

Using the triangle inequality again,

$$\sum_{t=1}^{T} \left\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\right\|_\infty^2 = \sum_{t=1}^{T} \left\|f_t^\top A_t - f_t^\top A_{t-1} + f_t^\top A_{t-1} - f_{t-1}^\top A_{t-1}\right\|_\infty^2$$
$$\le 2\sum_{t=1}^{T} \|A_{t-1} - A_t\|_\infty^2 + 2\sum_{t=1}^{T} \|f_t - f_{t-1}\|_1^2$$
$$\le 2\sum_{t=1}^{T} \|A_{t-1} - A_t\|_\infty^2 + 4\sum_{t=1}^{T} \left\|f_t - \hat{f}_{t-1}\right\|_1^2 + 4\sum_{t=1}^{T} \left\|\hat{f}_{t-1} - f_{t-1}\right\|_1^2, \tag{31}$$

which also implies

$$\sqrt{\sum_{t=1}^{T} \left\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\right\|_\infty^2} \le \sqrt{2\sum_{t=1}^{T} \|A_{t-1} - A_t\|_\infty^2 + 4\sum_{t=1}^{T} \left\|f_t - \hat{f}_{t-1}\right\|_1^2 + 4\sum_{t=1}^{T} \left\|\hat{f}_{t-1} - f_{t-1}\right\|_1^2}$$
$$\le 2\sqrt{\sum_{t=1}^{T} \|A_{t-1} - A_t\|_\infty^2} + 2\sqrt{\sum_{t=1}^{T} \left\|f_t - \hat{f}_{t-1}\right\|_1^2 + \sum_{t=1}^{T} \left\|\hat{f}_{t-1} - f_{t-1}\right\|_1^2}$$
$$\le 2\sqrt{\sum_{t=1}^{T} \|A_{t-1} - A_t\|_\infty^2} + 2 + 2\sum_{t=1}^{T} \left\|f_t - \hat{f}_{t-1}\right\|_1^2 + 2\sum_{t=1}^{T} \left\|\hat{f}_{t-1} - f_{t-1}\right\|_1^2$$
$$\le 2\sqrt{\sum_{t=1}^{T} \|A_{t-1} - A_t\|_\infty^2} + 10 + 2\sum_{t=1}^{T} \left\|f_t - \hat{f}_{t-1}\right\|_1^2 + 2\sum_{t=1}^{T} \left\|\hat{f}_t - f_t\right\|_1^2, \tag{32}$$

where we used the bound $\sqrt{c} \leq c+1$ for any $c \geq 0$ in the penultimate line. Similar bounds as Equations (31) and (32) hold for the other player as well. Using them in Equation 30 after some calculations, we conclude that

$$\begin{aligned}
\sum_{t=1}^{T} \sup_{f_t \in \Delta_m} f_t^\top A_t x_t \leq \sum_{t=1}^{T} \inf_{x_t \in \Delta_n} \sup_{f_t \in \Delta_m} f_t^\top A_t x_t + \frac{256L}{T} + \frac{1}{2L} + 4\sum_{t=1}^{T} \|A_{t-1} - A_t\|_\infty \\
+ 32L\big(\log(T^2 n) C_T + \log(T^2 m) C'_T + 2\log(T^4 nm)\big) + \big(C_T + C'_T + 4\big) \frac{20 + 4\sqrt{\sum_{t=1}^{T} \|A_{t-1} - A_t\|_\infty^2}}{L} \\
+ 4\left(\frac{C_T + 3}{L} - 2L\right)\left(\sum_{t=1}^{T} \left\|\hat{f}_t - f_t\right\|_1^2 + 2\sum_{t=1}^{T} \left\|\hat{f}_{t-1} - f_t\right\|_1^2\right) \\
+ 4\left(\frac{C'_T + 3}{L} - 2L\right)\left(\sum_{t=1}^{T} \|\hat{x}_t - x_t\|_1^2 + 2\sum_{t=1}^{T} \|\hat{x}_{t-1} - x_t\|_1^2\right).
\end{aligned}$$

b) When Player II is dishonest: In this case we would like to bound Player I's regret regardless of the strategy adopted by Player II. Dropping one of the negative terms in Equation 26, we get :

$$\begin{aligned}
\sum_{t=1}^{T} \left(f_t^\top A_t x_t - f_t^\top A_t u_t\right) &\leq \frac{\log(T^2 n)\left(C_T(u_1, \ldots, u_T) + 2\right)}{\eta_T} - \frac{1}{2}\sum_{t=1}^{T} \frac{1}{\eta_t} \|\hat{x}_t - x_t\|_1^2 \\
&\quad + \sum_{t=1}^{T} \left\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\right\|_\infty \|x_t - \hat{x}_t\|_1 \\
&\leq \frac{\log(T^2 n)\left(C_T(u_1, \ldots, u_T) + 2\right)}{\eta_T} - \frac{1}{2}\sum_{t=1}^{T} \frac{1}{\eta_t} \|\hat{x}_t - x_t\|_1^2 \\
&\quad + \sum_{t=1}^{T} \frac{\eta_{t+1}}{2} \left\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\right\|_\infty^2 + \frac{1}{2}\sum_{t=1}^{T} \frac{1}{\eta_{t+1}} \|x_t - \hat{x}_t\|_1^2. \qquad (33)
\end{aligned}$$

Noting to the telescoping sum

$$\frac{1}{2}\sum_{t=1}^{T} \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t}\right) \|x_t - \hat{x}_t\|_1^2 \leq 2\sum_{t=1}^{T} \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t}\right) \leq \frac{2}{\eta_{T+1}},$$

as well as the choice of step-size (27) which entails

$$\begin{aligned}
\sum_{t=1}^{T} \frac{\eta_{t+1}}{2} \left\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\right\|_\infty^2 &\leq \log(T^2 n)\frac{L}{2}\sum_{t=1}^{T} \sqrt{\sum_{i=1}^{t} \left\|f_i^\top A_i - f_{i-1}^\top A_{i-1}\right\|_\infty^2} - \sqrt{\sum_{i=1}^{t-1} \left\|f_i^\top A_i - f_{i-1}^\top A_{i-1}\right\|_\infty^2} \\
&\leq \log(T^2 n)\frac{L}{2}\sqrt{\sum_{t=1}^{T} \left\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\right\|_\infty^2},
\end{aligned}$$

we bound (33) to obtain

$$\begin{aligned}
\sum_{t=1}^{T} \left(f_t^\top A_t x_t - f_t^\top A_t u_t\right) &\leq \frac{\log(T^2 n)\left(C_T(u_1, \ldots, u_T) + 2\right)}{\eta_T} + \frac{2}{\eta_{T+1}} + \log(T^2 n)\frac{L}{2}\sqrt{\sum_{t=1}^{T} \left\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\right\|_\infty^2} \\
&\leq 2\log(T^2 n)\left(C_T(u_1, \ldots, u_T) + 2\right)\left(32L + \frac{2\sqrt{\sum_{t=1}^{T} \left\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\right\|_\infty^2}}{\log(T^2 n) L}\right) \\
&\quad + \log(T^2 n)\frac{L}{2}\sqrt{\sum_{t=1}^{T} \left\|f_t^\top A_t - f_{t-1}^\top A_{t-1}\right\|_\infty^2}.
\end{aligned}$$

A similar statement holds for Player II that her/his pay off converges at the provided rate to the average minimax equilibrium value. ■